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APPLICATION OF THE RITZ METHOD TO THE SOLUTION OF PARABOLIC BOUNDARY VALUE PROBLEMS OF ARBITRARY ORDER IN THE SPACE VARIABLES

by KAREL REKTORYS

Direct variational methods, commonly used to the solution of elliptic boundary value problems and based on the minimalization of corresponding functionals, cannot be used in their classical form to the solution of parabolic problems, because functionals with similar properties do not exist in this case. In our lecture, we shall show a procedure permitting the application of the Ritz method to the solution of these problems. The method is rather interesting from the teoretical point of view and very effective for numerical solution.

Let E_N be the N-dimensional Euclidean space, $x_1, x_2, \ldots, x_N$ being Cartesian coordinates of the point $x \in E_N$. Let Ω be a bounded region in E_N with a Lipschitz boundary $\dot{\Omega}$ (see e.g. NEČAS [2]), let

$$Q = \Omega \times (0, T).$$

Let $i = (i_1, \ldots, i_N)$ be a vector (the so-called multiindex) the components of which are nonegative integers. Denote

$$|i| = i_1 + \ldots + i_N,$$

$$D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \ldots \partial x_N^{i_N}}.$$

Similarly for $j = (j_1, \ldots, j_N)$ a.s.o.

Let the following boundary value problem be given:

$$Au + \frac{\partial u}{\partial t} = f(x) \qquad \text{in } Q, \tag{1}$$

$$u(x, 0) = u_0(x), \tag{2}$$

$$u = \frac{\partial u}{\partial v} = \ldots = \frac{\partial^{k-1} u}{\partial v^{k-1}} = 0 \quad \text{on} \quad \dot{\Omega} \times (0, T), \tag{3}$$

where v is the outward normal to Ω, $f \in L_2(\Omega)$, $u_0 \in L_2(\Omega)$ and A is a differential operator of order $2k$,

$$A = \sum_{|i|, |j| \leq k} (-1)^{|i|} D^i (a_{ij}(x) D^j), \tag{4}$$

with $a_{ij}(x)$ bounded and measurable in Ω.

Note that the functions $a_{ij}(x)$ and $f(x)$ are assumed to be independent of t. See, however, Remark 3.

Denote

$$A(v, u) = \sum_{|i|, |j| \leqq k} \int_\Omega a_{ij}(x) D^i v D^j u \, dx, \tag{5}$$

the bilinear form associated with the operator A, $W_2^{(m)}(\Omega)$ the Sobolev space of functions which are square integrable in Ω together with their generalized derivatives (taken in the sense of distributions) up to the order m including (m being a nonnegative integer). $W_2^{(m)}(\Omega)$ is a Hilbert space with the scalar product

$$(v, u)_{W_2^{(m)}(\Omega)} = \sum_{|i| \leqq m} \int_\Omega D^i v D^i u \, dx. \tag{6}$$

If $m = 0$, we get the well-known space $L_2(\Omega)$ with the scalar product $(v, u) = \int_\Omega vu \, dx$.

Denote, further,

$$V = \left\{ v;\ v \in W_2^{(k)}(\Omega),\ v = \frac{\partial v}{\partial \nu} = \ldots = \frac{\partial^{k-1} v}{\partial \nu^{k-1}} = 0 \text{ on } \dot{\Omega} \text{ in the sense of traces}^{1)} \right\}. \tag{7}$$

Let the form $A(v, u)$ be V-elliptic, i.e. let a constant $\alpha > 0$ exist such that

$$A(v, v) \geqq \alpha \| v \|^2_{W_2^{(k)}(\Omega)} \text{ holds for every } v \in V. \tag{8}$$

Let p and r be fixed positive integers. Divide the interval $[0, T]$ into p subintervals of the length $h_1 = T/p$. Denote, for the sake of symmetry of the notation, $u_0(x) = z_{10}(x)$. Let

$$z_{11}(x),\ z_{12}(x),\ \ldots,\ z_{1p}(x) \tag{9}$$

be elements of the space V (see (7)), defined, successively, by the following relations:

$$\begin{aligned} & A(v, z_{11}) + \frac{1}{h_1}(v, z_{11}) = (v, f) + \frac{1}{h_1}(v, z_{10}), \\ & A(v, z_{12}) + \frac{1}{h_1}(v, z_{12}) = (v, f) + \frac{1}{h_1}(v, z_{11}), \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & A(v, z_{1p}) + \frac{1}{h_1}(v, z_{1p}) = (v, f) + \frac{1}{h_1}(v, z_{1,p-1}). \end{aligned} \tag{10}$$

All these relations should be fulfilled for every $v \in V$.

In consequence of (8) and of the fact that $1/h_1 > 0$, a unique function $z_{11} \in V$ exists satisfying, for all $v \in V$, the first of the relations (10); it is well known (see e.g. [2]) that this function is the so-called weak solution of the elliptic problem

$$Az + \frac{z - z_{10}}{h_1} = f \text{ in } \Omega, \qquad z = \frac{\partial z}{\partial \nu} = \ldots = \frac{\partial^{k-1} z}{\partial \nu^{k-1}} = 0 \text{ on } \dot{\Omega}. \tag{11}$$

²) See e.g. Nečas [2].

The function $z_{11}(x)$ being found, a unique function $z_{12} \in V$ exists satisfying the second of relations (10) for all $v \in V$ and being the weak solution of the problem

$$Az + \frac{z - z_{11}}{h_1} = f \text{ in } \Omega, \qquad z = \frac{\partial z}{\partial \nu} = \ldots = \frac{\partial^{k-1} z}{\partial \nu^{k-1}} = 0 \text{ on } \dot{\Omega},$$

etc.

Let us denote $t_j = jh_1, j = 0, 1, \ldots, p$ and define, in $\bar{Q} = \bar{\Omega} \times [0, T]$, the function $u_1(x, t)$ in the following way:

$$u_1(x, t) = z_{1j}(x) + \frac{t - t_j}{h_1} \{z_{1,j+1}(x) - z_{1j}(x)\} \quad \text{for} \quad t_j \leqq t \leqq t_{j+1}, \tag{12}$$

$= 0, 1, \ldots, p - 1$. Thus, in $\bar{Q}$, this function is piecewise linear in t for every fixed $x \in \Omega$ and for $t = t_j$ we have $u_1(x, t_j) = z_{1j}(x)$.

Now, instead of dividing the interval $[0, T]$ into p subintervals of the length $h_1 = T/p$, let us divide it into $2p$ subintervals of the length $h_2 = T/2p$. Let

$$z_{21}(x),\ z_{22}(x), \ldots, z_{2,2p}(x) \tag{13}$$

be functions determined by relations analogous to relations (10), and let us construct the function $u_2(x, t)$ in a similar way as we have constructed the function $u_1(x, t)$, with the only difference that we use functions (13) instead of functions (9). In general, divide the interval $[0, T]$ into $2^{n-1}p$ subintervals of the length $h_n = T/(2^{n-1}p)$ and construct, for every positive integer $n = 1, 2, 3, \ldots$, the corresponding function $u_n(x, t)$. Thus, we get a sequence of functions

$$u_1(x, t),\ u_2(x, t), \ldots, u_n(x, t), \ldots. \tag{14}$$

It seems to be plausible that the limiting function of this sequence (if it exists) will be a solution, in a certain sense, of our problem.

Before clarifying this question, let us investigate more thoroughly the case when the coefficients of the operator A are symmetric,

$$a_{ij}(x) = a_{ji}(x).$$

It is well known (see [2]) that in this case $z_{11}(x), \ldots, z_{1p}(x)$ are functions, minimizing, in V, functionals

$$\begin{aligned}
G_1(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(z_{10}, z), \\
G_2(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(z_{11}, z), \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
G_p(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(z_{1,p-1}, z).
\end{aligned} \tag{15}$$

Let us use the Ritz method to this minimalization. Thus let

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_r(x) \tag{16}$$

be the first r terms of a base in V and let us minimize the first of functionals (15) in the subspace V_r of the space V, produced by the functions (16). Denote by $\tilde{z}_{11}(x)$ the function minimizing, in V_r, this functional, and by $\tilde{G}_2(z)$ the second of functionals (15) in which $\tilde{z}_{11}$ is substituted for z_{11}. Let us again minimize this functional in V_r by the Ritz method. Denote the minimizing function by $\tilde{z}_{12}(x)$, the third of functionals (15), in which $\tilde{z}_{12}$ stands for z_{12}, by $\tilde{G}_3(z)$, and repeat the process again, etc. In this way, we get the functions

$$\tilde{z}_{11}(x), \tilde{z}_{12}(x), \ldots, \tilde{z}_{1p}(x), \tag{17}$$

minimizing, in V_r, subsequently the functionals

$$\begin{aligned}
\tilde{G}_1(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(\tilde{z}_{10}, z),\\
\tilde{G}_2(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(\tilde{z}_{11}, z),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\tilde{G}_p(z) &= A(z, z) + \frac{1}{h_1}(z, z) - 2(f, z) - \frac{2}{h_1}(\tilde{z}_{1,p-1}, z).
\end{aligned} \tag{18}$$

(For the sake of symmetry of the notation we write $\tilde{z}_{10}$, or $\tilde{G}_1$ instead of z_{10}, or G_1, respectively; note, further, that the function $\tilde{z}_{10} = z_{10} = u_0$ is given.)

It is to be noted that functions (17) differ from functions (9) not only because of using the Ritz method (and thus because of minimizing the functionals in the subspace V_r instead of in the space V), but also because of the difference between functionals (15) and (18), caused by substituting $\tilde{z}_{11}$ for z_{11}, etc.

Having found the functions (17), let us construct the function $u_1^r(x, t)$, analogous to the function (12), using functions (17) instead of functions (9). In a similar way let us construct the functions $u_2^r(x, t), u_3^r(x, t), \ldots$, corresponding to the division of the interval $[0, T]$ into $2p, 2^2p, \ldots$ subintervals. We thus get a sequence of the "Ritz functions"

$$u_1^r(x, t),\ u_2^r(x, t), \ldots, u_n^r(x, t), \ldots. \tag{19}$$

Now, two questions arise:

1. Does the sequence (14) converge, in an appropriate functional space, to a function $u(x, t)$ which can be declared, in a certain sense, as a solution of problem (1)–(3)?

2. Can $u_n^r(x, t)$ be made arbitrarily close (in an appropriate metric) to this solution if r and n are sufficiently large?

Each of these questions is positively answered in the author's work [1]. It is not possible to present here corresponding theorems in full extent – those, who are interested in details, are referred to the work [1]. We shall prefer to introduce here some remarks concerning these questions:

Remark 1. Existence of solution of parabolic boundary value problems of higher order has been proved by several authors (LIONS, BROWDER, LADYŽENSKAJA, IBRAGIMOV, etc.). The concept of the solution is slightly different in their works, according to problems in question and to methods used by individual authors. An analogy of the "Rothe sequence" (14) has been used, to prove an existence theorem, in the LADYŽENSKAJA paper [3]. However, our concept of the solution is also rather different. Note that in our case no symmetry of the operator A is required (this requirement appears but when the Ritz method is applied). As to the course of the proof of our existence theorem, it is such that it could find use also for the proof of the main result of our paper, i.e. for the proof of convergence of the "Ritz sequence" (19) to the solution $u(x, t)$.

If the initial function $u_0(x)$ is sufficiently smooth (if, for example, $u_0 \in W_2^{(2k)}(\Omega) \cap V$, then $u \in L_2([0, T], V)$, i.e., for almost all $t \in [0, T]$ we have $u \in V$ (and this mapping is square integrable in the interval $[0, T]$). In this sense, the boundary conditions (3) are fulfilled. Moreover, this mapping is continuous (even absolutely) as mapping into the space $L_2(\Omega)$, i.e. $u \in C([0, T], L_2(\Omega))$, and we have $u(0 = u_0$ in this metric. Thus, in this sense, the initial condition (2) is fulfilled.

For other properties of the solution see [1].

Remark 2. To the minimalization of functionals (15), or (18), we have used the Ritz method. Let us note that whichever method can be applied which produces a minimizing sequence with similar properties as that produced by the Ritz method. In general, convergence of sequence (19) to the solution $u(x, t)$ is ensured only in $L_2(Q)$. If the given data of the problem (1) – (3) are sufficiently smooth, then, as is shown in [1], the solution is also sufficiently smooth. In this case, the convergence of the "Ritz sequence" (19) can by examined in a finer metric than in $L_2(Q)$. To improve the convergence, a "finer" method than the Ritz method (for example, the Courant method) can then be applied.

Remark 3. In our work [1], the just explained method has been developped to the construction of an approximate solution of the problem (1) – (3). The method can be applied to more general problems. Especially, the assumption that coefficients a_{ij} of the operator A and the right hand side f of the given equation do not depend on t and are functions of x only, is not essential.

Remark 4. The method is very effective for numerical solution of the given problem. Note that the "quadratic" terms are the same in each of functionals (18), so that, applying the Ritz method to the minimalization of these functionals, the left hand sides of the corresponding Ritz systems remain unchanged. Thus, the numerical procedure and also the programme for its realization if automatic computers are applied, are very simple. For a numerical example see author's monography [4].

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