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ORTHOGONAL BIPOLYNOMIALS

by F. M. ARSCOTT

1. INTRODUCTION

(A) The theory of orthogonal polynomials in a single variable, say x, is of course extremely well known, e.g. [4], [9]. We have a sequence of polynomials $\{p_n(x)\}$, n being the degree of the polynomial, such that

$$\langle p_n(x), p_m(x)\rangle \equiv \int_a^b p_n(x)\, p_m(x)\, w(x)\, \mathrm{d}x = 0 \quad \text{if } n \neq m, \qquad (1.1)$$
$$\neq 0 \quad \text{if } n = m$$

where the "weight function" $w(x) \geqq 0$ on [a, b].

Certain properties of such sequences can be derived from the orthogonality relation (1.1) alone, such as

(a) expansion property; a sufficiently well-behaved function $f(x)$ has a formal expansion $f(x) \sim c_n p_n(x)$ with

$$c_n = \langle f(x), p_n(x)\rangle / \langle p_n(x), p_n(x)\rangle \qquad (1.2)$$

and this series converges, at least in norm, on [a, b].

(b) 3-term recurrence relationship; for suitable constants A_n, B_n, C_n,

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \qquad (1.3)$$

(c) The Christoffel-Darboux formula

$$\sum_0^n h_r^{-1} p_r(x)\, p_r(x') = h_n^{-1} \frac{p_{n+1}(x)\, p_n(x') - p_n(x)\, p_{n+1}(x')}{x - x'} \qquad (1.4)$$

where $h_n = \langle p_n(x), p_n(x)\rangle$

(d) Properties of zeros; the zeros of $p_n(x)$ are all real, distinct and lie in (a, b) and the zeros of $p_{n+1}(x)$ and $p_n(x)$ interlace.

Certain other properties, however, do not hold for all such families, but only for the so-called "classical" polynomials—in effect, the polynomials of Jacobi, Laguerre and Hermite with their special cases. These properties include [4]

(e) Orthogonality of derivatives
(f) Satisfying of a differential equation
(g) Rodrigues-type formulae
(h) Integral representations
(j) Generating functions.

(B) In the late 19th and early 20th centuries, some families of polynomials appeared with a wider orthogonality property—those of Lamé, Heun and Ince ([2], [4], [7]).

Such a family has the form of a set $\{E_n^m(x)\}$, where $m = 0$ to n, $n = 0$ to ∞; there are two simple orthogonality relations, namely

$$\int_{a_1}^{b_1} E_n^{m_1}(x)\, E_n^{m_2}(x)\, \mathrm{d}x = 0 \qquad m_1 \neq m_2, \tag{1.5}$$

and a similar property over the range (a_2, b_2); there is also a "double-orthogonality" relation

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} E_{n_1}^{m_1}(x)\, E_{n_1}^{m_1}(y)\, E_{n_2}^{m_2}(x)\, E_{n_2}^{m_2}(y)\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y = 0 \tag{1.6}$$

$$\text{unless} \quad n_1 = n_2 \quad \text{and} \quad m_1 = m_2\,.$$

These polynomials have some, but not all, of the properties (a) – (j) mentioned above; for instance, though they arise from differential equations, they are given by no known generating functions or integral representations.
(C) Meanwhile, some studies were made of orthogonal polynomials in two or more variables, ([6], [8]) but in the course of generalisation the theory lost much of its elegance. By restricting ourselves, however, to a special class of such polynonials, which we call *bipolynomials*, one can construct a theory which generalises, fairly satisfactorily, properties (a) – (d). The essence of this was presented by the author in 1968 [1], and has subsequently been developed. Very recently, some new families of polynomials have been constructed which, it may be conjectured, will play a part analogous to that of the "classical" polynomials in the single-variable theory.

2. DEFINITIONS AND NOTATION

A bipolynomial $p_n(x, y)$ of degree n in the two variables x, y is a function which, when either of x, y is held constant, becomes a polynomial of degree n in the other. Thus a bipolynomial of degree 1 is of the form $axy + bx + cy + d$. We call such a bipolynomial *symmetric* if $p_n(x, y) = p_n(y, x)$ and *separable* if it can be written as a product $p(x)\, q(y)$.

A matrix notation is useful; if we write $\mathbf{X}_n$, $\mathbf{Y}_n$ for the column vectors $\{1, x, x^2, \dots, x^n\}$, $\{1, y, y^2, \dots, y^n\}$, then $p_n(x, y) = \mathbf{X}_n^T \mathbf{A}_n \mathbf{Y}_n$, where $\mathbf{A}_n$ is a square n-matrix; if $p_n(x, y)$ is symmetric then $\mathbf{A}_n$ is symmetric, and if $p_n(x, y)$ is separable then $\mathbf{A}_n$ is of rank 1.

From here on, we shall confine ourselves to symmetric bipolynomials; the theory to be developed seems capable of extension, at the cost of some elegance, to general bipolynomials.

We now set up the appropriate linear space. Let R be a region of the xy plane and $w(x, y) \geqq 0$ a weight function such that

$$\iint_R x^m y^n\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

exists $\forall\, m, n \geqq 0$. Let $L^2_w(R)$ be the space of all functions $f(x, y)$ such that

$$\iint_R \{f(x, y)\}^2\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

exists, and define

$$\langle f, g\rangle = \iint_R f(x, y)\, g(x, y)\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

$\| f \| = \langle f, f\rangle^{\frac{1}{2}}$. Then $L^2_w(R)$ is an inner product space. We shall write $f \perp g$ if $\langle f, g\rangle = 0$.

3. CONSTRUCTION OF A FAMILY OF ORTHOGONAL BIPOLYNOMIALS

A bipolynomial of degree zero is, of course, a constant, so let us set $p_0 = 1$. Now consider $p_1 = axy + b(x + y) + c$. The condition $p_1 \perp p_0$ gives one linear relation between the constants a, b, c of the form $c = -\alpha a - \beta b$, so

$$p_1(x, y) = a(xy - \alpha) + b(x + y - \beta). \tag{3.1}$$

Thus p_1 is not a uniquely determined bipolynomial but is a manifold spanned by the two bipolynomials

$$\pi_1^0 = xy - \alpha, \quad \pi_1^1 = x + y - \beta\,. \tag{3.2}$$

To maintain the same pattern, we write $p_0 = \pi_0^0$. Now we take $p_2(x, y) = ax^2y^2 + bxy(x + y) + c(x + y)^2 + \mathrm{d}xy + e(x + y) + f$; the conditions $p_2 \perp p_0$, $p_2 \perp p_1$ yield three linear relations, which give d, e, f in terms of a, b, c and hence

$$p_2(x, y) = a\pi_2^0 + b\pi_2^1 + c\pi_2^2\,, \tag{3.3}$$

where $\pi_2^0 = x^2y^2 + \ldots$, $\pi_2^1 = xy(x + y) + \ldots$, $\pi_2^2 = (x + y)^2 + \ldots$, the terms omitted in each case forming a bipolynomial of degree 1.

This process can be continued indefinitely; $p_n(x, y)$ is given in the form of a manifold of dimension $n + 1$, spanned by the polynomials π_n^m, $m = 0$ to n, the leading term in π_n^m being $(xy)^{n-m}\,(x + y)^m$.

For example, if R is the square $(0, 1) \times (0, 1)$ and $w(x, y) = 1$ we have

$$\pi_1^0 = xy - \frac{1}{4}, \qquad \pi_1^1 = x + y - 1,$$

$$\pi_2^0 = x^2y^2 - xy - \frac{1}{6}(x + y) - \frac{1}{36},$$

$$\pi_2^1 = xy(x + y) - 2xy + \frac{1}{6}(x + y),$$

$$\pi_2^2 = (x + y)^2 - 2xy - x - y + \frac{1}{3}.$$

Each bipolynomial π_n^m is orthogonal to each bipolynomial $\pi_{n'}^{m'}$ where $n' \neq n$, but the π_n^m, for the same n but different m, are not orthogonal. However, we can now apply the usual Schmidt orthogonalisation process to the set $\{\pi_n^m\}$ $m = 0$ to n, and obtain from these an orthogonal set $\{\tilde{\pi}_n^m\}$ with $\tilde{\pi}_n^0 = \pi_n^0$, $\tilde{\pi}_n^1 = \tilde{\pi}_n^1 - \gamma_1 \pi_n^0$, etc. such that $\tilde{\pi}_n^{m_1} \perp \tilde{\pi}_n^{m_2} (m_1 \neq m_2)$ and hence $\tilde{\pi}_{n_1}^{m_1} \perp \tilde{\pi}_{n_2}^{m_2}$

unless
$$n_1 = n_2 \quad \text{and} \quad m_1 = m_2 . \tag{3.4}$$

4. MATRIX FORMULATION

The process described above can usefully be put in matrix notation. Let $\pi_n = \{\pi_n^0, \pi_n^1, \ldots, \pi_n^n\}$, so any bipolynomial of degree n is $\mathbf{c}\pi_n$, where $\mathbf{c}$ is an a row-vector. The orthogonality property is thus

$$\langle \pi_n, \pi_{n'}^T \rangle = 0, \qquad n \neq n'. \tag{4.1}$$

Further, let $\tilde{\pi}_n = \{\tilde{\pi}_n^0, \tilde{\pi}_n^1, \ldots, \tilde{\pi}_n^n\}$; then

$$\tilde{\pi}_n = \mathbf{L}_n \pi_n , \tag{4.2}$$

where $\mathbf{L}_n$ is lower triangular. Finally, let us set

$$\langle \pi_n, \pi_n^T \rangle = \mathbf{H}_n, \quad \langle \tilde{\pi}_n, \tilde{\pi}_n^T \rangle = \mathbf{D}_n ; \tag{4.3}$$

thus $\mathbf{D}_n$ is diagonal with positive elements, and $\mathbf{D}_n = \mathbf{L}_n \mathbf{H}_n \mathbf{L}_n^T$, so $\mathbf{H}_n$ is non-singular.

The process of construction of the π_n is as follows: let

$$\mathbf{Z}_n = \{x^n y^n, x^{n-1} y^{n-1} (x + y), \ldots, (x + y)^n\} \tag{4.4}$$

so that $\pi_n = \mathbf{Z}_n +$ a vector of bipolynomials of degree $n - 1$. Write, therefore,

$$\pi_n = \mathbf{Z}_n - \alpha_{n-1}\pi_{n-1} - \alpha_{n-2}\pi_{n-2} - \ldots - \alpha_0 \pi_0 , \tag{4.5}$$

where α_i is an $(n + 1, i + 1)$ matrix. Then the orthogonality conditions to be imposed on π_n give

$$\begin{aligned} 0 = \langle \pi_n, \pi_i^T \rangle = \langle \mathbf{Z}_n, \pi_i^T \rangle - \sum_{s=1}^{n-1} \alpha_s \langle \pi_s, \pi_i^T \rangle &= \\ = \langle \mathbf{Z}_n, \pi_i^T \rangle - \alpha_i \langle \pi_i, \pi_i^T \rangle & \qquad \text{(by (4.1))} \\ = \langle \mathbf{Z}_n, \pi_i^T \rangle - \alpha_i \mathbf{H}_i & \qquad \text{(using (4.3))} \end{aligned}$$

so
$$\alpha_i = \langle \mathbf{Z}_n, \pi^T \rangle \, \mathbf{H}_i^{-1} \tag{4.6}$$

Thus π_n is constructed in the form (4.5). The process is, formally, almost identical with the corresponding construction in functions of a single variable, but vector and matrix quantities have replaced scalars.

5. GENERAL PROPERTIES OF FAMILIES OF ORTHOGONAL BIPOLYNOMIALS

Properties of families of orthogonal bipolynomials, analogous to properties (a) – (c) mentioned in para. 1, will now be given:

(a) Formal expansion. If the symmetric function $F(x, y)$ is expressible as a double series

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \alpha_n^m \pi_n^m(x, y) \tag{5.1}$$

this may be written as

$$F(x, y) = \sum_{n=0}^{\infty} \mathbf{c}_n \pi_n(x, y), \tag{5.2}$$

and application of (4.1), (4.3) gives

$$\mathbf{c}_n = \langle F(x, y), \pi_n^T \rangle \, \mathbf{H}_n^{-1} \tag{5.3}$$

If $F \in L_w^2(R)$ the convergence in norm of this series follows immediately.

(b) Recurrence Relations

We introduce the two $n \times (n+1)$ matrices $\mathbf{J}_n$, $\mathbf{K}_n$ by

$$\mathbf{J}_n = (\mathbf{I}_n, 0), \quad \mathbf{K}_n = (0, \mathbf{I}_n) \tag{5.4}$$

that is, the unit $n \times n$ matrix followed or preceded by a column of zeros.

Then we have

$$xy\,\pi_n = \mathbf{J}_n \pi_{n+1} + \mathbf{B}_n \pi_n + \mathbf{H}_n \mathbf{J}_{n-1}^T \, \mathbf{H}_{n-1}^{-1} \, \pi_{n-1} \tag{5.5}$$

and

$$(x + y)\,\pi_n = \mathbf{K}_n \pi_{n+1} + \mathbf{B}_n' \pi_n + \mathbf{H}_n \mathbf{K}_{n-1}^T \, \mathbf{H}_{n-1}^{-1} \pi_{n-1} \tag{5.6}$$

where

$$\mathbf{B}_n = \langle xy\pi_n, \pi_n^T \rangle \, \mathbf{H}_n^{-1}, \quad \mathbf{B}_n' = \langle (x + y)\,\pi_n, \pi_n^T \rangle \, \mathbf{H}_n^{-1}. \tag{5.7}$$

The analogy between (5.5), (5.6) and (1.3) is obvious. The proof is essentially as given in [6], the simplification to the above forms resulting from the manner in which we have chosen the leading terms of π_n.

(c) Christoffel-Darboux formula

By repeated application of (5.5) we obtain an analogue of the formula (1.4), namely

$$\begin{aligned}(xy - x'y') \sum_0^n \pi_r^T(x', y') \, \mathbf{H}_r^{-1} \pi_r(x, y) = \\ \psi_n(x, y, x', y') - \psi_n^T(x', y', x, y) + \\ + \sum_0^n \pi_r^T(x', y') (\mathbf{H}_r^{-1} \mathbf{B}_r - \mathbf{B}_r \mathbf{H}_r^{-1}) \, \pi_r(x, y),\end{aligned} \tag{5.8}$$

where

$$\psi_n(x, y, x', y') = \pi_n^T(x', y') \, \mathbf{H}_n^{-1} \, \mathbf{J}_n \pi_{n+1}(x, y).$$

6. ZEROS OF ORTHOGONAL BIPOLYNOMIALS

When considering what properties might be possessed by zeros of orthogonal bipolynomials analogous to those for single-variable polynomials, as detailed in (d), para. 1, we immediately meet a difficulty of interpretation. For a bipolynomial is a function of two variables, and the relation $p_n(x, y) = 0$ yields, not a set of points, but a curve, usually with several branches, in the xy plane. To emphasise this difference we shall call the curve given by the relation $p_n(x, y) = 0$ the *zeroid* of $p_n(x, y)$.

No general study of the properties of these zeroids has been made, but an examination of simple cases provoked the following conjectures

(a) Every zeroid – but not necessarily each branch of it – crosses the region R. This can be deduced from the property $p_n \perp p_0$.

(b) No two zeroids corresponding to bipolynomials of the same manifold $p_n(x, y)$ can meet outside the region R.

In certain cases, we find that the zeroids of all the bipolynomials of a manifold $p_n(x, y)$ – i.e. all the curves $\pi_n^m(x, y) = 0$, $m = 0$ to n, have one or more points in common; these points we call *nodes*. It may be conjectured that

(c) No node can lie outside the region R.

7. THE SEPARABILITY PROBLEM

An interesting, and so far quite unresolved, problem is that of determining, in any given system of orthogonal bipolynomials, the number of independent separable bipolynomials. Thus in the case of $R = (0,1) \times (0,1)$, $w = 1$ we find there is only one separable bipolynomial of degree 1, namely $(x - \frac{1}{2})(y - \frac{1}{2})$, while in the corresponding family with $w = x$, there are two such. Separable bipolynomials seem to be exceptional, yet there are families – as we shall see – in which the number of separable bipolynomials of degree n is the maximum possible, namely $n + 1$. The zeroid of a separable bipolynomial is, of course, a set of straight lines parallel to the axes of the xy plane.

8. SEPARABLE BIPOLYNOMIALS AS SOLUTIONS OF DIFFERENTIAL EQUATIONS

The separability problem mentioned in para. 7 above prompted a search for some fairly simple separable bipolynomials, constructed as the solutions of suitable differential equations. The results are published in [9], and can only be sketched here.

We take a differential equation of the form

$$A(x)\, u''(x) + B(x)\, u'(x) + \{\lambda f(x) + \mu g(x)\}\, u(x) = 0 \tag{8.1}$$

where $A(x)$, $B(x)$, $f(x)$, $g(x)$ are polynomials. We also set

$$R(x) = \exp\left[\int B(x)/A(x)\, \mathrm{d}x\right]. \tag{8.2}$$

Then a simple adaptation of the method used in [2] gives the following orthogonality result: if $u(x)$, $\hat{u}(x)$ are polynomial solutions of (8.1) corresponding to different values of λ and/or μ, then

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} u(x)\, u(y)\, \hat{u}(x)\, \hat{u}(y)\, w(x, y)\, \mathrm{d}x\, \mathrm{d}y = 0$$

where

$$w(x, y) = \frac{R(x)\, R(y)}{A(x)\, A(y)} \{f(x)\, g(y) - f(y)\, g(x)\} \tag{8.3}$$

provided

$$\left.\begin{aligned} &[R(x)\,\{u(x)\,\hat{u}'(x) - u'(x)\,\hat{u}(x)\}]_{x_1}^{x_2} = 0 \\ \text{and}\quad &[R(y)\,\{u(y)\,\hat{u}'(y) - u'(y)\,\hat{u}(y)\}]_{y_1}^{y_2} = 0 \end{aligned}\right\} \tag{8.4}$$

and the paths of integration do not pass through a zero of A. (A zero of A may, however, occur at one end of a range provided the resulting integral converges.)

The search, therefore, reduces essentially to choosing A, B, f and g so that (8.1) has polynomial solutions, the conditions (8.4) are satisfied and the paths of integration are so chosen that, for $x \in (x_1, x_2)$, $y \in (y_1, y_2)$, the weight function $w(x, y)$ is non-negative.

A simple case arises when $A(x) = x$. This leads to a differential equation

$$xu'' + (\beta_0 + \beta_1 x + \beta_2 x^2)\, u' + (\lambda - n\beta_2 x)\, u = 0; \tag{8.5}$$

it is found that, for any non-negative integer n there are $n + 1$ distinct real values of λ giving rise to solutions which are polynomials of degree n. The weight function is found to be

$$w(x, y) = (xy)^{\beta_0 - 1}\, \mathrm{e}^{\beta_1(x+y)}\, \mathrm{e}^{\frac{1}{2}\beta_2(x^2+y^2)}\, (y - x) \tag{8.6}$$

and provided $\beta_0 > 0$, $\beta_2 < 0$ the region of integration R can be taken as $x \in (-\infty, 0)$, $y \in (0, \infty)$; that is, the second quadrant of the xy plane. For the simple case $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = -2$ the orthogonality relation becomes

$$\iint_R u(x)\, u(y)\, \hat{u}(x)\, \hat{u}(y)\, \mathrm{e}^{-x^2-y^2}\, (y - x)\, \mathrm{d}x\, \mathrm{d}y = 0, \tag{8.7}$$

the form of which makes it reasonable to regard these as analogous to the Hermite polynomials.

Next taking $A(x) = x(1 - x)$ we consider the equation

$$x(1 - x)\, u'' + (\beta_0 + \beta_1 x + \beta_2 x^2)\, u' + (\lambda - n\beta_2 x)\, u = 0; \tag{8.8}$$

which leads to an orthogonality relation over the infinite strip $(0,1) \times (1,\infty)$ with weight function

$$w(x, y) = (xy)^{\beta_0 - 1}\, \{(1 - x)\,(y - 1)\}^{\delta - 1}\, \mathrm{e}^{-\beta_2(x+y)}\, (y - x), \tag{8.9}$$

where $\delta = -\beta_0 - \beta_1 - \beta_2$, provided β_0, δ, $\beta_2 > 0$. For different values of the parameters, we can have orthogonality over the infinite strip $(-\infty, 0) \times (0, 1)$ with essentially the same $w(x, y)$. The Ince polynomials arise from the case $\beta_0 = \delta = {}^1/_2$.

Finally taking $A(x) = x(x - 1)(x - b)$ leads to the Heun polynomials ([7])—a special case of which is the Lamé polynomials—where the region of integration is a rectangle. The finiteness of the two ranges of integration suggest an analogy with the Jacobi polynomials.

The location of the zeros of these polynomials have recently been investigated by Mr. Josef Rovder of Bratislava, who has shown that, in many cases, the $n + 1$ polynomials $u_n^r(x)$, say, of degree n can be so ordered that $u_n^r(x)$ has r zeros on one side of the region R and $n - r$ zeros on the adjacent side, so that the zeroid of the bipolynomial $u_n^r(x)\, u_n^r(y)$ consists of r straight lines parallel to one side of R and $n - r$ lines parallel to the other.

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