EQUADIFF 3

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In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 115--118.

Persistent URL: http://dml.cz/dmlcz/700062

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THREE-POINT BOUNDARY VALUE PROBLEM IN A DIFFERENTIAL EQUATION OF THE THIRD ORDER

by M. GREGUŠ

1. Consider differential equation of the third order

$$y''' + q(x, \lambda, \mu)\, y = 0, \tag{a}$$

where $q = q(x, \lambda, \mu)$ is a continuous function of $x \in \langle a, c\rangle$, $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$.

The problem will be to determine sufficient conditions on q such that it will be possible to choose parameters λ and μ so that there exists a non-trivial solution y of the differential equation (a) satisfying the boundary conditions

$$y(a) = y'(a) = y(b) = y(c) = 0, \tag{1}$$

where $a < b < c$.

We will show that under certain conditions on q there exists such a number N and an infinite number of couples $(\lambda_{N+p}, \mu_{N+p})$, $p = 0, 1, 2, \ldots$, to which the sequence of functions $\{y_{N+p}\}_{p=0}^{\infty}$ belongs, where $y_{N+p} = y(x, \lambda_{N+p}, \mu_{N+p})$ is a solution of the equation (a) and fulfils the boundary conditions (1).

Similar problem for an equation of the second order is solved in paper [1]. The reconstruction of the function q and the method of the proof are differing, concerning the specific character of solutions of the equation of the second order.

2. Given differential equation of the third order of the form

$$y''' + q(x, \lambda)\, y = 0 \tag{a$_1$}$$

where $q = q(x, \lambda)$ is a continuous function of $x \in \langle a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and let $q(x, \lambda) \geqq 0$ for $x \in \langle a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$.

Lemma 1. Let $y(x, \lambda)$ be the solution of the differential equation (a_1) with the property $y(\alpha, \lambda) = 0$, where $a \leqq \alpha < \infty$.

Then the null-points of the solution y lying right of α are the continuous functions of parameter $\lambda \in (\Lambda_1, \Lambda_2)$.

The proof is given in paper [2].

Oscillation theorem. Let $q = q(x, \lambda)$ be a continuous function of $x \in \langle a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$. Further let $\lim_{\lambda \to \Lambda_2} q(x, \lambda) = +\infty$ uniformly for all $x \in \langle a, \infty)$. Let $a < b$ be a given number and $y(x, \lambda)$ be the solution of the differential equation (a_1) of the property $y(a, \lambda) = 0$.

Then with the increasing $\lambda \to \Lambda_2$ there grows the number of null-points of the solution $y(x, \lambda)$ on the interval $\langle a, b\rangle$ to infinity and, at the same time, the distance between two neighbouring null-points tends to zero.

The proof is done in paper [2].

Note 1. The assertion of the oscillation theorem is also true when $\lim_{\lambda \to \Lambda_2} q(x, \lambda) = +\infty$ uniformly for all $x \in \langle \alpha, b\rangle$, where $a < \alpha < b$, but the distance of two neighbouring null-points tends now to zero for $\lambda \to \Lambda_2$ only on the interval $\langle \alpha, b\rangle$.

3. Now we are in a position to state and prove the main result concerning the problem (a), (1).

Theorem. *Let $a < b < c$ be real numbers. Let the coefficient $q = q(x, \lambda, \mu)$ of the differential equation (a) be of the form*

$$q(x, \lambda, \mu) = q_\lambda(x, \lambda) + q_\mu(x, \mu),$$

where $q(x, \lambda)$ is a continuous function of $x \in \langle a, c\rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and

$$q_\mu(x, \mu) = \begin{cases} r(x) & \text{for } x \in \langle a, b\rangle \\ s(x, \mu) & \text{for } x \in \langle b, c\rangle, \quad \mu \in (M_1, M_2) \end{cases}$$

and let $s(b, \mu) = r(b)$ for all $\mu \in (M_1, M_2)$. Let $q_\mu(x, \mu)$ be a continuous function of $x \in \langle a, c\rangle$ and $\mu \in (M_1, M_2)$. Further let $q(x, \lambda, \mu) \geqq 0$ for all $x \in \langle a, c\rangle$, $\lambda \in (\Lambda_1, \Lambda_2)$ and $\mu \in (M_1, M_2)$ and let $\lim_{\lambda \to \Lambda_2} q(x, \lambda) = +\infty$ be true uniformly for all $x \in \langle a, c\rangle$ and $\lim_{\mu \to M_2} q_\mu(x, \mu) = +\infty$ be true uniformly for all $x \in \langle \beta, c\rangle$, where $b < \beta < c$.

Then there exists such a number N and such sequences of values of parameters $\{\lambda_{N+p}\}_{p=0}^{\infty}$, $\{\mu_{N+p}\}_{p=0}^{\infty}$, to which there exists the sequence of functions $\{y_{N+p}\}_{p=0}^{\infty}$, where $y_{N+p} = y(x, \lambda_{N+p}, \mu_{N+p})$ is the solution of the equation (a) which fulfils boundary conditions (1) and has on (a, b) exactly $N + p$ null-points.

Proof. We will extend the function q onto the interval $\langle c, \infty\rangle$ as follows:

$$q(x, \lambda, \mu) = q(c, \lambda, \mu) \quad \text{for} \quad \lambda \in (\Lambda_1, \Lambda_2) \quad \text{and} \quad \mu \in (M_1, M_2).$$

Let $y(x, \lambda, \mu)$ be the solution of the differential equation (a) of the property $y(a, \lambda, \mu) = y'(a, \lambda, \mu) = 0$, $y''(a, \lambda, \mu) \neq 0$. For $\lambda = \bar{\lambda} \in (\Lambda_1, \Lambda_2)$ and $\mu = \bar{\mu} \in (M_1, M_2)$ let $y(x, \bar{\lambda}, \bar{\mu})$ have on (a, b) exactly N null-points. Such $\bar{\lambda}$, $\bar{\mu}$, N obviously do exist. Then it is true that

$$x_N(\bar{\lambda}, \bar{\mu}) < b \leqq x_{N+1}(\bar{\lambda}, \bar{\mu})$$

where x_N is the N-th null-point of the solution y on (a, b).

From the oscillation theorem it follows that there exists such $\lambda^* \in (\bar{\lambda}, \Lambda_2)$ for which $x_{N+1}(\lambda^*, \bar{\mu}) < b$. Then it follows from lemma 1 that there exists such λ_N for which $y(b, \lambda_N, \bar{\mu}) = 0$ and $y(x, \lambda_N, \bar{\mu})$ has on (a, b) exactly N null-points.

For $\lambda = \lambda_N$, $\mu = \bar{\mu}$, let $y(x, \lambda_N, \bar{\mu})$ have on (b, c) exactly ν null-points. Then evidently the following inequality is true:

$$\xi_\nu(\lambda_N, \bar{\mu}) < c \leqq \xi_{\nu+1}(\lambda_N, \bar{\mu})$$

where ξ_ν is the ν-th null-point of the solution $y(x, \lambda_N, \bar{\mu})$ on the interval (b, c). From the oscillation theorem it follows that there exists such $\mu^* \in (\mu, M_2)$ for which $\xi_{\nu+1}(\lambda_N, \mu^*) < c$.

From Lemma 1 follows the existence of such $\mu_N \in (\bar{\mu}, \mu^*)$ for which there holds $y(c, \lambda_N, \mu_N) = 0$.

If we denote $y_N = y(x, \lambda_N, \mu_N)$, we thus get the confirmation of the existence of the first members of the sequences

$$\{\lambda_{N+p}\}, \quad \{\mu_{N+p}\}, \quad \{y_{N+p}\}.$$

Continuing in the same way we prove the existence of next members of the mentioned sequences.

Note 2. Let $q = q(x, \lambda, \mu)$ be defined as follows: $q = q_\lambda(x, \lambda) + q_\mu(x, \mu)$.

Let $q_\lambda(x, \lambda) = \begin{cases} r_1(x) & \text{for } x \in \langle b, c \rangle \\ s_1(x, \lambda) & \text{for } x \in (a, b), \quad \lambda \in (\Lambda_1, \Lambda_2) \end{cases}$

and $s_1(b, \lambda) = r_1(b)$ for $\lambda \in (\Lambda_1, \Lambda_2)$ and $q_\lambda(x, \lambda)$ be a continuous function of $x \in \in \langle a, c \rangle$ and $\lambda \in (\Lambda_1, \Lambda_2)$.

Let $q_\mu(x, \mu) = \begin{cases} r_2(x) & \text{for } x \in \langle a, b \rangle \\ s_2(x, \mu) & \text{for } x \in \langle b, c \rangle \text{ and } \mu \in (M_1, M_2) \end{cases}$ and let $r_2(b) = s_2(b, \mu)$ for all $\mu \in (M_1, M_2)$.

Let $\lim_{\lambda \to \Lambda_2} s_1(x, \lambda) = +\infty$ be true uniformly for all $x \in \langle a, \alpha \rangle$ and $\lim_{\mu \to M_2} s_2(x, \mu) = +\infty$ uniformly for $x \in \langle \beta, c \rangle$ where $a < \alpha < b$, $b < \beta < c$.

Then there can be proved the existence of the sequences $\{\lambda_{N+p}\}_{p=0}^\infty$, $\{\mu_{N_1+p}\}_{p=0}^\infty$, $\{y_{N+p,N_1+p}\}_{p=0}^\infty$, where $y_{N+p,N_1+p} = y(x, \lambda_{N+p}, \mu_{N_1+p})$ is the solution of the equation (a) which fulfils the boundary condition (1) and has on (a, b) $N + p$ and on (b, c) $N_1 + p$ null-points, where N, N_1 are suitable natural numbers.

Note 3. By a suitable readjustment of assumptions for coefficient $q(x, \lambda, \mu)$ there can be shown the existence of eigenvalues λ, μ such that the solution $y(x, \lambda, \mu)$ of (a) fulfils the boundary conditions

$$y(a, \lambda, \mu) = y(b, \lambda, \mu) = y(c, \lambda, \mu) = y(d, \lambda, \mu)$$

where $a < b < c < d$.

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