Nikolai Nikolaevich Yanenko

On some problems of the theory of the difference schemes

In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 43--50.

Persistent URL: http://dml.cz/dmlcz/700055

ON SOME PROBLEMES OF THE THEORY OF THE DIFFERENCE SCHEMES

by N. N. JANENKO

1. This paper is a review of the work results of the author, his pupils and collaborators, achieved in Novosibirsk (Computing Center of the Academy of Sciences and State University) in the last years.

The main topics to be discussed are:

a) The definition of the Ω-consistence (approximation) and the Ω-stability of the difference schemes. These notions will be defined
   α) with regard to a given differential equation,
   β) independently of any differential equation.
   In the last case one may speak about the theory of the difference schemes "per se".
b) The weak Ω-consistence.
c) The existence of the absolutely consistent and stable difference schemes.
d) The differential equations generated by a given difference scheme.
e) The theorem of the consistence for weakly consistent schemes.
f) Group properties of the difference schemes.

2. We shall consider the two-level difference scheme [1–4]

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} = \Lambda_1 U^{n+1}(x) + \Lambda_0 U^n(x) + F^n(x) \tag{1}$$

with the initial condition

$$U^0(x) = \varphi(x). \tag{2}$$

Here $\Lambda_s = \Lambda_s(\tau, h, n)$, $(s = 0, 1)$ are some operators; $U^n(x)$, $F^n(x)$ are vector functions defined in some Banach space B. Generally, Λ_s depends on x, but for the sake of simplicity we shall neglect this dependence. The equations (1), (2) define the Cauchy problem for the difference scheme (1).

The scheme (1) is called *explicit* if $\Lambda_1 = 0$, *implicit* if $\Lambda_1 \neq 0$. We shall assume that the scheme (1) satisfies the condition

$$\| (I - \tau\Lambda_1)^{-1} \| \leqq M, \tag{3}$$

where M is independent of n, τ, h. In this case the scheme (1) can be resolved and takes the form

$$U^{n+1} = \sigma_n U^n + \tau G^n, \tag{4}$$

where

$$\sigma_n = (I - \tau\Lambda_1)^{-1} (I + \tau\Lambda_0), \; G^n = (I - \tau\Lambda_1)^{-1} F^n. \tag{5}$$

The set $\{U^n(x)\}$ is called the solution of the problem (1), (2), corresponding to the fixed τ, h.

We shall introduce now *the transfer operator*

$$\Sigma_{n,m} = \sigma_{n-1} \cdot \sigma_{n-2} \dots \sigma_m \tag{6}$$

The scheme (1) is called *stable* if

$$\| \Sigma_{n,m} \| \leqq M(T), \tag{7}$$

where M is independent of τ, h, n in the time interval $(0, T)$.

The class $U = \{u(t)\}$ is called the class of probe functions if

α) *U is dense in B,*

β) *$u(t)$ is the solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} = \mathscr{L}u + f, \qquad u(0) = u_0. \tag{8}$$

If both the conditions α, β are satisfied, then we have the known theory of the convergence, based on the notions of stability and consistence. If the condition β is omitted, then we consider the difference scheme (1) independently of the equations (8) (the theory of the difference schemes "per se").

Definition 1. *The set Ω of the points τ, h, $(\tau > 0, h > 0)$ is called Ω_a and the scheme* (1) *is called Ω_a-*consistent *if*

i) $(0, 0) \in \bar{\Omega}_a$

ii) $$\lim_{\substack{\tau, h \to 0 \\ n\tau \to t}} \left\{ \frac{u[(n+1)\tau] - u[n\tau]}{\tau} - \Lambda_1 u[(n+1)\tau] - \Lambda_0 u[n\tau] - F^n \right\} = \alpha_u(t),$$

for all τ, $h \in \Omega_a$ and an arbitrary fixed probe function $u(t)$.

If $\Omega_a = Q$, where Q is the set of points τ, $h(\tau > 0, h > 0)$, $\tau^2 + h^2 < \tau^2(\tau_0$-sufficiently small), then the scheme (1) *is called* absolutely consistent; *in the opposite case it is called* conditionally consistent.

Definition 2. *The set Ω of the points τ, $h(\tau > 0, h > 0)$ is called Ω_s and the scheme* (1) *is called Ω_s-*stable *if*

i) $(0, 0) \in \bar{\Omega}_s$

ii) *the condition* (7) *holds for all τ, $h \in \Omega_s$.*

If $\Omega_s = Q$, *then the scheme* (1) *is called* absolutely stable; *in the opposite case the scheme is called* conditionally stable.

*If $\Omega_a = \Omega_s = \Omega$, then we speak simply about Ω-*consistent *and Ω-*stable scheme, *respectively.*

Theorem 1. [3] *If*

i) *the operator* $\mathscr{L}$ *from* (8) *is unbounded,*

ii) *the difference scheme* (1) *is absolutely stable and consistent,*

then the difference scheme (1) *is implicit.*

The theorem 1 can be formulated as follows as well: There exists no absolutely explicit stable scheme (1), which is absolutely consistent with the differential equation (8), where $\mathscr{L}$ is an unbounded operator. A. M. Iljin [5] has proved that

α) there exists an explicit scheme, which is absolutely consistent with the differential equation (8) ($\mathscr{L}$-unbounded) and conditionally stable ($\Omega_s \neq Q$);

β) there exists the absolutely consistent and absolutely stable implicit scheme (1).

As a sequel of the theorem 1 and of the theorems by A. M. Iljin the following statement is true: Among the schemes (1) which are absolutely consistent with the differential equation (8) (L-unbounded)

α) there exists an absolutely stable scheme,

β) it is always implicit.

The simple examples of this theorem are given in [1], where it was formulated as a conjecture.

3. We shall give a few examples illustrating the notion of the Ω-stability and the Ω-consistence.

Example 1. The difference scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} + \frac{U^n(x) - U^n(x-h)}{h} = 0,$$

is conditionally stable and absolutely consistent. Here $\Omega_s = \{\tau, h;\ \tau > 0,\ h > 0,\ \tau \leqq h\}$ (see Fig. 1) and $\Omega_a = Q$ (see Fig. 2).

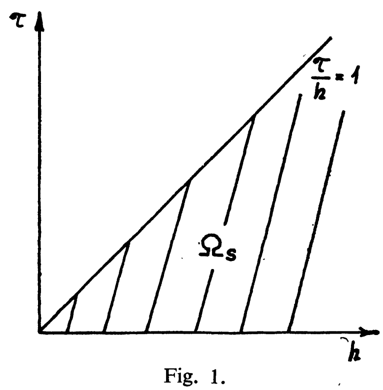

Fig. 1.

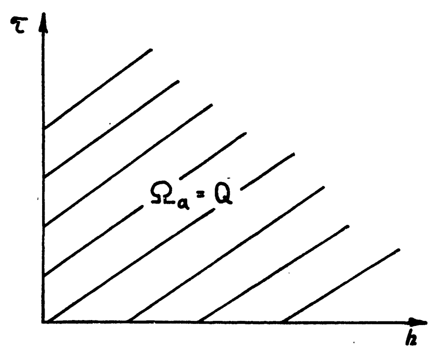

Fig. 2.

This scheme generates the differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

The corresponding implicit scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} + \frac{U^{n+1}(x) - U^{n+1}(x-h)}{h} = 0$$

is absolutely stable and consistent ($\Omega_a = \Omega = Q$) and generates the differential equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$.

Example 2. The Lax's scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} + \frac{U^n(x+h) - U^n(x-h)}{2h} -$$

$$- \frac{h^2}{2\tau} \cdot \frac{U^n(x+h) - 2U^n(x) + U^n(x-h)}{h^2} = 0$$

is conditionally consistent and conditionally stable.

$$\Omega_s = \{\tau, h; \tau > 0, h > 0, \tau \leqq h\} \text{ (see Fig. 1).}$$

If $\tau >$ const. h, then Ω_a is the interior region between two straight lines $\frac{\tau}{h} = c_1$, $\frac{\tau}{h} = c_2$ (see Fig. 3).

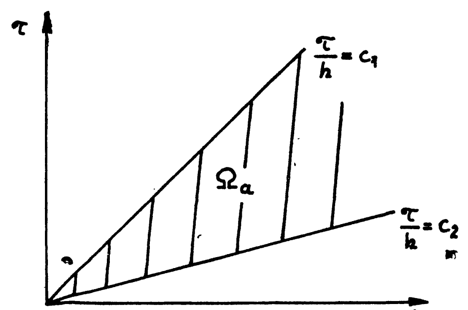

Fig. 3.

The curve $\frac{\tau}{h^2} = c$ is the Ω_a set (see Fig. 4). But the interior region between two curves $\frac{\tau}{h^2} = c_1$, $\frac{\tau}{h^2} = c_2$ (see Fig. 5) is not the Ω_a set. No curve that touches alternatively both these curves can be the Ω_a set (see Fig. 5).

The Lax's scheme generates different partial differential equations according to the choice of the Ω_a set.

If Ω_a is the shaded region on the figure 3, then the Lax's scheme generates the equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$.

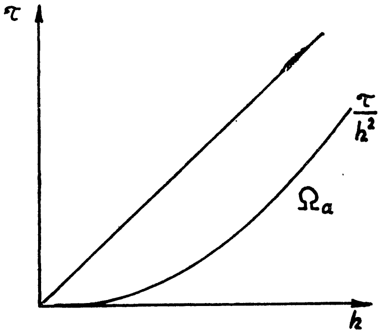

Fig. 4.

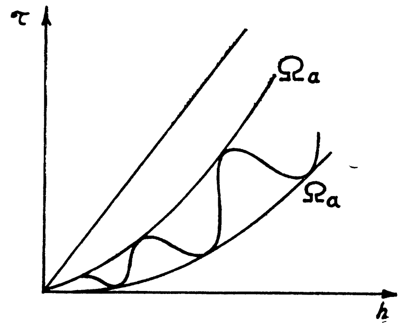

Fig. 5.

If Ω_a is the curve (the parabola) on the figure 4, then the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial^2 u}{\partial x^2} = 0$$

is generated.

If Ω_a is an interior curve on the figure 5, then no differential equation is generated. In the case of the nonlinear difference scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} - \frac{U^n(x+h) - U^n(x-h)}{2h} - \\ - f(U^n)\frac{h^2}{2\tau}\frac{U^n(x+h) - 2U^n(x) + U^n(x-h)}{h^2} = 0 \tag{9}$$

it is impossible to define the sets Ω_a, Ω_s, because they depend on the solution U^n.

Example 3. The scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} - \frac{\tau(h-\alpha)}{\tau^2 + h^2}\frac{U^n(x+h) - U^n(x)}{h} - \\ - \frac{\tau(h+\alpha)}{\tau^2 + h^2}\frac{U^{n+1}(x+h) - U^{n+1}(x)}{h} = 0, \tag{10}$$

is absolutely stable for $\alpha \geqq 0$ and conditionally consistent. It generates the equation $\mathscr{L}u = 0$ of nonregular type that depends on Ω_a.

If $\Omega = \Omega_a = \{\tau, h; \tau > 0, h > 0; h = K . \tau\}$, then we get the equation

$$\mathscr{L}u = \frac{\partial u}{\partial t} - \frac{2K}{1 + K^2} \frac{\partial u}{\partial x} - \frac{\alpha}{1 + K^2} \frac{\partial^2 u}{\partial t \, \partial x} = 0.$$

If $\Omega_a = \{\tau, h; \tau > 0, h > 0; h = o(\tau)\}$, then we get the equation

$$\mathscr{L}u = \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial t \, \partial x} = 0.$$

4. Now we shall introduce the notion of *the weak Ω-consistence* or the *weak Ω-approximation*. The schemes of the fractional steps method have the property of the weak approximation (see [1]). These notions can be generalized as follows (see [6]).

Definition 3. *The scheme* (1) *is weakly* Ω-consistent *if there exists*

$$\lim_{(\tau,h)_\Omega \to 0} \left\{ \tau \sum_{i=n}^{n+k} \frac{u[(i+1)\tau] - u(i\tau)}{\tau} - A_1 u[(i+1)\tau] - A_0 u(i\tau) \right\} = \int_{t_1}^{t_2} \alpha_u(t) \, \mathrm{d}t,$$

uniformly for $t_1, t_2 \in [0, T]$, $n\tau \to t_1$, $(n+k)\tau \to t_2$, $u(t) \in U$.

If the scheme (1) is Ω-consistent, then it is weakly consistent. The contrary is not true.

Example 4. The scheme

$$\frac{U^{n+1} - U^n}{\tau} - \alpha U^n - \cos n\tau^\beta U^n = 0$$

is Ω-consistent for all $u(t) \in C_1$ if $\beta > 1$. It generates the differential equation

$$u'(t) - (\alpha + 1) u(t) = 0.$$

If $\beta < 1$, then the scheme is Ω-consistent only for $u(t) \equiv 0$. In this case, however, the scheme is weakly Ω-consistent for all $u(t) \in C_1$.

Example 5. Let us consider the scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} + 2 \sin^2 \frac{n\pi}{2m} \frac{\Delta_{-1}}{h_1} U^{n+1}(x) + 2 \cos^2 \frac{n\pi}{2m} \frac{\Delta_{-2}}{h_2} U^{n+1}(x) = 0, \tag{11}$$

where the operators Δ_{-1}, Δ_{-2} are defined by relations

$$\Delta_{-1} f(x_1, x_2) = f(x_1, x_2) - f(x_1 - h_1, x_2)$$
$$\Delta_{-2} f(x_1, x_2) = f(x_1, x_2) - f(x_1, x_2 - h_2), \; (m > 0, \text{integer}).$$

This scheme is absolutely stable and absolutely weakly consistent. It generates the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0.$$

For the case $m = 1$ the scheme is splitting up the schemes

$$\left.\begin{aligned} \frac{U^{2k+1} - U^{2k}}{\tau} + 2\frac{\Delta_{-2}U^{2k+1}}{h_2} = 0 \\ \frac{U^{2k+2} - U^{2k+1}}{\tau} + 2\frac{\Delta_{-1}U^{2k+2}}{h_1} = 0 \end{aligned}\right\} \quad k = 0, 1, 2, \ldots$$

For a large class of the difference schemes the following theorem is valid (see [6]).

Theorem 2. *If*

i) *the difference scheme* (1) *is Ω-stable,*

ii) *the Cauchy problem* (8) *is correctly posed, then the weak Ω-consistence is the necessary and sufficient condition of the convergence.*

5. The scheme (1) generates a sequence of differential equations. For the sake of simplicity we shall consider the hyperbolic case. Let the scheme

$$\frac{U^{n+1}(x) - U^n(x)}{\tau} = \Lambda U^n(x), \tag{12}$$

generate the hyperbolic system with constant coefficients

$$\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial x} \quad \text{(A is a constant matrix)}, \tag{13}$$

$\Omega_a = \{\tau, h; \tau > 0, h > 0; \tau = \text{const. } h\}$.

If we expand the scheme (12) in the Taylor series, use the equation (13) and its sequel

$$\frac{\partial^2 u}{\partial t^2} = A^2 \frac{\partial^2 u}{\partial x^2}$$

and retain only the terms of the first order with regard to τ, then we get *the first differential approximation* (FDA)

$$\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial x} + C\frac{\partial^2 u}{\partial x^2}, \tag{14}$$

where the matrix C depends on τ, h, and is responsible for the stability and dissipative properties. For the important classes of the difference schemes the author and Ju. I. Shokin have proved that the scheme (12) is stable if $C \geqq 0$ (see [7, 8]), i.e. FDA is a parabolic system. In an analogous way we can construct the differential approximation of the second and higher order, respectively. The analogous construction can be applied to the case of variable coefficients and to the nonlinear case. Many interesting properties of the difference schemes can be interpreted as the properties of the FDA (14).

6. Now we can formulate the important notion of the group properties of the difference schemes. We shall illustrate it for the case of the difference scheme (12). Let the system (13) (generally nonlinear) be *invariant under some group G* of trasformations in the space t, x, u. We shall call the scheme (12) *group invariant* if the system (14) is invariant under the same group G. In such a way we can distinguish two classes of the difference schemes, ***the invariant*** and ***the noninvariant*** ones. The numerical calculations show that the invariant schemes have an advantage over the noninvariant ones.

I thank Dr. J. Miklíček for the helpful discussion.

REFERENCES

[1] JANENKO, N. N.: *The fractional steps method.* New York: Springer 1971.

[2] JANENKO, N. N.: *Méthodes numériques nouvelles en méchanique du continu.* Actes, Congrés intern. Math., 1970, Tome 3, 297–309.

[3] JANENKO, N. N., DEMIDOFF, G. V.: *O strukture absoljutno approximirujuščikh i absoljutno ustojčivykh skhem.* Sbornik, Problemy prikladnoj matematiki i mekhaniki. Moskva: Nauka 1972.

[4] JANENKO, N. N., DEMIDOFF, G. V.: *Èvolucionnye dvukhslojnye raznostnye skhemy.* DAN SSSR, 1972.

[5] ILJIN, A. M.: *DAN SSSR,* 164 (1965), 491–494.

[6] CANTOR, S. A.: *DAN SSSR* (to appear).

[7] JANENKO, N. N., SHOKIN, YU. I.: *Sibirskij mat. žurnal,* 10, 5, 1969, 1174–1188.

[8] JANENKO, N. N., SHOKIN, YU. I.: *The Physics of Fluids,* 12, part II, 1969, 28–33.

[9] JANENKO, N. N., SHOKIN, YU. I.: *O grupovoj klassifikacii raznostnykh skhem dlja sistemy odnomernykh uravnenij gazovoj dinamiki.* Sbornik, Nekotorye problemy matematiki i mekhaniki, Leningrad, 1970.

[10] JANENKO, N. N., SHOKIN, JU. I.: *On the group classifications of difference schemes of equations in gas dynamics.* Lect. Notes in Phys. 8 (1971), 3–17.

Author's address:

N. N. Janenko
Vyčislitel'nyj centr Sibirskogo otdela AN SSSR
Novosibirsk 90
USSR