Jan Mařík

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DERIVATIVES AND CLOSED SETS

J. MAŘÍK (East Lansing)

In their article [1] G. Petruska and M. Laczkovich proved (among other things) that a function defined on a perfect set S and differentiable relative to S can be extended to a function differentiable on the whole real line R. This note contains an elementary proof of a more general theorem where the set S is supposed only to be closed in R.

NOTATION. The word function means a mapping to $R=(-\infty, \infty)$. Let $a\in S\subset R$ and let F be a function. If $S\cap(a, b)\neq\emptyset$ for each $b>a$, we define

$$F_S'^+(a)=\lim\big(F(x)-F(a)\big)/(x-a)\ (x\in S,\ x\searrow a)$$

provided that this limit exists. We define analogously the meaning of $F_S'^-(a)$ and $F_S'(a)$. (Note that $F_S'(a)$ may exist even if $F_S'^+(a)$ is undefined.) The symbols $F'^+(a)$, $F'^-(a)$ and $F'(a)$ will have the usual meaning (i.e. $F'^+(a)=F_R'^+(a)$ etc.).
Points in $R\times R$ will be denoted by $\langle\cdot,\cdot\rangle$.

1. *Let $a, b\in R$, $a<b$ and let $J=[a, b]$. Let φ and ψ be functions continuous on J. Let φ be convex, ψ concave, $\varphi=\psi$ on $\{a, b\}$. Set $s=(\varphi(b)-\varphi(a))/(b-a)$. Let $\alpha, \beta, M, N\in R$, $\varphi'^+(a)\leqq\alpha\leqq\psi'^+(a)$, $\psi'^-(b)\leqq\beta\leqq\varphi'^-(b)$, $M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N$. Then there is a function G continuously differentiable on J such that $G'^+(a)=\alpha$, $G'^-(b)=\beta$, $M<G'<N$ on (a, b) and that, for each $x\in(a, b)$, $G(x)=\varphi(a)+s(x-a)$ or $\varphi(x)<G(x)<\psi(x)$.*

PROOF. We may assume that $\varphi=\psi=0$ on $\{a, b\}$. Then $s=0$. Let $c=(a+b)/2$. We construct a function H continuously differentiable on J such that $H'^+(a)=\alpha$, $H=0$ on (c, b), $M<H'<N$ on (a, b) and that, for each $x\in(a, b)$, $H(x)=0$ or $\varphi(x)<H(x)<\psi(x)$. If $\alpha=0$, we choose $H=0$ on J. Now let, e.g., $\alpha>0$. Choose an $\varepsilon\in(0, -M)$ and set $\mu(x)=\psi'^+(x)$ $(x\in[a, b))$. We have $\alpha\leqq\mu(a)=\mu(a^+)$. There is an $a_1\in(a, c)$ such that ψ increases on (a, a_1). There is an $a_2\in(a, a_1)$ and a function p continuous and decreasing on $[a, a_2]$ such that $\alpha(a_2-a)<\varepsilon(a_1-a_2)$, $p(a)=\alpha$, $p<\mu$ on (a, a_2) and $p(a_2)=0$. Since $\int_a^{a_2} p<\alpha(a_2-a)<\varepsilon(a_1-a_2)$, there is a function q continuous on $[a_2, a_1)$ such that $0\leqq q\leqq\varepsilon$, $\int_{a_2}^{a_1} q=\int_a^{a_2} p$ and that $q=0$ on $\{a_2, a_1\}$. Set $h=p$ on $[a, a_2)$, $h=-q$ on $[a_2, a_1]$, $h=0$ on $(a_1, b]$ and $H(x)=\int_a^x h$ for each $x\in J$. It is easy to see that $-\varepsilon\leqq H'(x)<\alpha$ and $0\leqq H(x)<\psi(x)$ for each $x\in(a, b)$.

In an analogous way we construct a function K continuously differentiable on J such that $K=0$ on (a, c), $K'^{-}(b)=\beta$, $M<K'<N$ on (a, b) and that, for each $x\in(a, b), K(x)=0$ or $\varphi(x)<K(x)<\psi(x)$. Now it suffices to take $G=H+K$.

2. *Let a, b and J be as in 1. Let P be a function on J such that the derivatives $\alpha=P'^{+}(a), \beta=P'^{-}(b)$ exist. Set $s=(P(b)-P(a))/(b-a)$. Let $M, N\in R$, $M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N$. Then there is a function G continuously differentiable on J such that the graph of G is contained in the convex hull of the graph of P and that $G'^{+}(a)=\alpha$, $G'^{-}(b)=\beta$, $G=P$ on $\{a, b\}$ and $M<G'<N$ on (a, b).*

PROOF. Let Φ and Ψ be functions continuous on J such that $\Phi=\Psi=P$ on $\{a, b\}$, Φ is convex, Ψ is concave, $\Phi'^{+}(a)=\Psi'^{-}(b)=-\infty$, $\Psi'^{+}(a)=\Phi'^{-}(b)=\infty$. Set $P_0=(P\vee\Phi)\wedge\Psi$. Obviously $\alpha=P_0'^{+}(a)$, $\beta=P_0'^{-}(b)$. Let C and C_0 be the convex hulls of the graphs of P and P_0 respectively. It is easy to see that $C_0\subset C$. Let φ be the greatest convex function on J such that $\varphi\leqq P_0$ and let ψ be the smallest concave function on J such that $P_0\leqq\psi$. Let C_1 be the set of all points $\langle x, y\rangle$ such that $x\in(a, b)$ and that $y=P(a)+s(x-a)$ or $\varphi(x)<y<\psi(x)$. Then $C_1\subset C_0$. Now we apply 1.

3. *Let S be a nonempty set closed in R. Let $A, B\in R\cup\{-\infty, \infty\}$. Let P be a function on R such that $A<P'(x)<B$ for each $x\in S$ and that*

$$A<(P(y)-P(x))/(y-x)<B,$$

whenever $x, y\in S$, $x\neq y$. Then there is a function G differentiable on R such that $G=P, G'=P'$ on S and $A<G'<B$ on R.

PROOF. We may suppose that $\inf S=-\infty$, $\sup S=\infty$. Let (a, b) be a component of $R\setminus S$ and let α, β, s be as in 2. There are $M, N\in R$ such that $A<M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N<B$. Construct a function G according to 2. In this way we define G on $R\setminus S$; further we set $G=P$ on S. It is easy to see that G has the required properties.

4. *Let $x_0, y_0, s\in R$. For each $\gamma\in(0, \infty)$ define*

$$W_\gamma=\{\langle x, y\rangle\in R\times R;\ |y-y_0-s(x-x_0)|<\gamma(x-x_0)\}. \tag{1}$$

Let $\varepsilon\in(0, \infty)$ and let $\langle x_1, y_1\rangle, \langle b, c\rangle\in W_\varepsilon, 3x_1\leqq 4b-x_0$. Then $\langle 2b-x_1, 2c-y_1\rangle\in W_{5\varepsilon}$.

PROOF. We may suppose that $x_0=y_0=0$. Then $6x_1\leqq 8b$ and hence $|2c-y_1-s(2b-x_1)|\leqq 2|c-sb|+|y_1-sx_1|<\varepsilon(2b+x_1)\leqq\varepsilon(10b-5x_1)=5\varepsilon(2b-x_1)$.

REMARK. The geometric meaning of W_γ is obvious. To see the geometric meaning of assertion 4 the reader should realize that $3x_1\leqq 4b-x_0$ means the same as $x_1-x_0\leqq\frac{4}{3}(b-x_0)$ and that $\langle b, c\rangle$ is the center of the segment with end points $\langle x_1, y_1\rangle$ and $\langle 2b-x_1, 2c-y_1\rangle$.

5. *Let $x_0, y_0, s\in R$. For each $\gamma\in(0, \infty)$ define W_γ by (1). Let $\varepsilon\in(0, \infty)$ and let $\langle x_1, y_1\rangle, \langle b, c\rangle, \langle x_2, y_2\rangle\in W_\varepsilon$, $x_1<b<x_2$, $x\in R$, $3|x-b|\leqq b-x_1$. Let $q=(y_2-y_1)/(x_2-x_1)$. Then $\langle x, c+q(x-b)\rangle\in W_{3\varepsilon}$.*

PROOF. We may suppose that $x_0=y_0=0$. Set $y=c+q(x-b)$, $Z=|x-b|(x_1+x_2)/(x_2-x_1)$. As $3|x-b|<\min(x_2-x_1, b)$, we have $3Z<\min(x_1+x_2, b(x_1+x_2)/(x_2-x_1))$. If $x_2\leqq 2b$, then $x_1+x_2<3b$; if $x_2>2b$, then

$$(x_1+x_2)/(x_2-x_1) < (b+x_2)/(x_2-b) < 3.$$

Thus in either case $Z<b$.

Obviously $|q-s|=|y_2-sx_2-(y_1-sx_1)|/(x_2-x_1)\leqq \varepsilon(x_1+x_2)/(x_2-x_1)$; therefore $|y-xs|=|c-sb+(x-b)(q-s)|\leqq \varepsilon b+\varepsilon Z<2\varepsilon b$. Since $x=b-(b-x)>2b/3$, we have $|y-sx|<3\varepsilon x$.

6. *Let S be a set closed in R. Let F be a function on S such that $F'_S(x)$ is finite for each accumulation point x of S. Then there is a function H on R differentiable at each point of S such that $H=F$ on S.*

PROOF. We may suppose that $\inf S=-\infty$, $\sup S=\infty$. Set

$$A^+ = \{x\in S;\ S\cap(x, y) \neq \emptyset \quad \text{for each } y > x\},$$
$$A^- = \{x\in S;\ S\cap(y, x) \neq \emptyset \quad \text{for each } y < x\},$$

$I^+=A^-\setminus A^+$, $I^-=A^+\setminus A^-$, $I=S\setminus(A^+\cup A^-)$. Define a function f on S as follows: If $b\in A^+\cup A^-\ (=S\setminus I)$, set $f(b)=F'_S(b)$. If $b\in I$, find $x_1, x_2\in S$ such that $S\cap(x_1, x_2)=\{b\}$ and set

$$f(b) = \big(F(x_2)-F(x_1)\big)/(x_2-x_1).$$

For each $b\in S$ define a set M_b as follows:

If $b\in A^+\cap A^-$, let $M_b=\{b\}$.

If $b\in I^+\cup I^-$, choose a $d_b>0$ such that either $S\cap(b, b+3d_b)=\emptyset$ or $S\cap(b-3d_b, b)=\emptyset$ and set

$$M_b = \{x;\ 2b-x\in S\cap[b-d_b, b+d_b]\}.$$

If $b\in I$, choose a $d_b>0$ such that $S\cap(b-3d_b, b+3d_b)=\{b\}$ and set $M_b=[b-d_b, b+d_b]$.

Let $M=\bigcup M_b$ $(b\in S)$. Obviously $b\in M_b$ for each $b\in S$ and $M_a\cap M_b=\emptyset$, whenever $a, b\in S$, $a\neq b$. If (a, b) is a component of $R\setminus S$, then $M_c\cap(a, b)=\emptyset$ for each $c\in S\setminus\{a, b\}$. Thus $(a, b)\setminus M=(a, b)\setminus(M_a\cup M_b)$ which is open. Therefore $R\setminus M=(R\setminus S)\setminus M$ is open, M is closed.

There is a unique function G on M with the following properties: $G=F$ on S; if $x\in M_b$, $b\in I^+\cup I^-$, then $G(x)=2F(b)-F(2b-x)$; if $x\in M_b$, $b\in I$, then $G(x)=F(b)+(x-b)f(b)$.

Let $x_0\in S$. We shall prove that

$$G'^{+}_M(x_0) = f(x_0). \tag{2}$$

The case $x_0\notin A^+$ is left to the reader. Now let $x_0\in A^+$ and let $\varepsilon\in(0, \infty)$. Set $s=f(x_0)$ $\big(=F'_S(x_0)\big)$. For each $\gamma\in(0, \infty)$ define W_γ by (1). There is a $z>x_0$ such that $\langle x, F(x)\rangle\in W_\varepsilon$ for each $x\in S\cap(x_0, z)$. There are $z_1, z_2\in S$ such that $x_0<z_2<z$ and that $0<z_1-x_0<\frac{3}{4}(z_2-x_0)$ (so that $x_0<z_1<z_2$). Let $x\in M\cap(x_0, z_1)$. If $x\in S$,

then, obviously, $\langle x, G(x)\rangle \in W_\varepsilon$. Thus, let $x \notin S$ and let (a, b) be the component of $R \setminus S$ containing x. We have $x_0 < a < x < b \leqq z_1$. There are the following four possibilities:

1. $x \in M_b$, $b \in I^-$. Set $x_1 = 2b - x$. Then $x_1 \in S$, $0 < x_1 - b \leqq d_b \leqq (b-a)/3 < < (b-x_0)/3$, therefore $3x_1 < 4b - x_0$, and $x_1 < z_1 + (z_1 - x_0)/3 = x_0 + \frac{4}{3}(z_1 - x_0) < z_2$. Set $c = F(b)$, $y_1 = F(x_1)$. We have $\langle b, c\rangle$, $\langle x_1, y_1\rangle \in W_\varepsilon$, $x = 2b - x_1$, $G(x) = 2c - y_1$ so that, by 4, $\langle x, G(x)\rangle \in W_{5\varepsilon}$.

2. $x \in M_b$, $b \in I$. There is an $x_2 \in S \cap (b, \infty)$ such that $S \cap (b, x_2) = \emptyset$. Obviously $x_2 \leqq z_2$. Then $G(x) = F(b) + (x-b) f(b)$, $0 < b - x \leqq d_b \leqq (b-a)/3$ so that by 5 with $x_1 = a$, $q = f(b)$ etc. we have $\langle x, G(x)\rangle \in W_{3\varepsilon}$.

3. $x \in M_a$, $a \in I^+$. Proceeding as in 1 we get $\langle x, G(x)\rangle \in W_{5\varepsilon}$.

4. $x \in M_a$, $a \in I$. Proceeding as in 2 we get $\langle x, G(x)\rangle \in W_{3\varepsilon}$.

This proves (2). Similarly, it can be shown that $G_M'^-(x_0) = f(x_0)$ for each $x_0 \in S$. Now it suffices to choose for H the function that equals G on M and is linear on the closure of each component of $R \setminus M$.

7. *Let T be a closed set in R, $V = R \setminus T$, $Q \subset V$ and let Q be isolated in V. Let g be a function on Q. Then there is a function K differentiable on R such that $K = 0$ on $T \cup Q$, $K' = 0$ on T and $K' = g$ on Q.*

Proof. Let φ be a function differentiable on R such that $\varphi = 0$ on $\{0\} \cup \cup (R \setminus (-1, 1))$, $\varphi'(0) = 1$, $|\varphi| < 1$ on R. There is a function ω continuous on R such that $\omega = \omega' = 0$ on T and that $\omega > 0$ on V. There are positive numbers ε_q $(q \in Q)$ such that the intervals $J_q = [q - \varepsilon_q, q + \varepsilon_q]$ are pairwise disjoint and that $J_q \subset V$ for each q. Now let $\eta_q = \min \{\omega(x); x \in J_q\}$, $c_q = \max (1/\varepsilon_q, |g(q)|/\eta_q)$ and, for each $x \in R$, let

$$K(x) = \sum_{q \in Q} \frac{g(q)}{c_q} \varphi\big(c_q(x-q)\big).$$

Obviously $|K| \leqq \omega$ on R. It is easy to see that K satisfies our requirements.

Remark. The following assertion is a generalization of Theorem 5.5.3 in [1].

8. *Let S be a nonempty set closed in R. Let F and f be functions on S such that $F_S'(x) = f(x)$ for each accumulation point x of S. Let $A, B \in R \cup \{-\infty, \infty\}$. Suppose that $A < f(x) < B$ for each $x \in S$ and that $A < \big(F(y) - F(x)\big)/(y - x) < B$, whenever $x, y \in S$ and $x \neq y$. Then there is a function G differentiable on R such that $G = F$, $G' = f$ on S and $A < G' < B$ on R.*

Proof. Let T be the set of all accumulation points of S. Let H be as in 6. By 7 there is a function K differentiable on R such that $K = 0$ on S, $K' = 0$ on T and that $K' = f - H'$ on $S \setminus T$. Set $P = H + K$. Obviously $P = F$ and $P' = f$ on S. Now we apply 3.

Remark. It has been mentioned in [1] that there is a perfect set S and a function F on S such that $|F_S'(x)| \leqq 1$ for each $x \in S$ and that G' is unbounded for each function G differentiable on R such that $G = F$ on S. The following example shows a little more.

Let $1=x_0>x_1>\dots$, $x_n\to 0$, $y_n=x_{n-1}-x_n^2(x_{n-1}-x_n)$ $(n=1,2,\dots)$. It is easy to see that $x_n<y_n<x_{n-1}$. Set $S=\left(\bigcup_{n=1}^{\infty}[x_n,y_n]\right)\cup\{0\}$. Define a function F on S setting $F(0)=0$ and $F(x)=x_n^2$ for each $x\in[x_n,y_n]$. Then S is perfect and $F_S'=0$ on S. Now let G be a function differentiable on R such that $G=F$ on S. Then $G(x_{n-1})-G(y_n)=x_{n-1}^2-x_n^2>2x_n(x_{n-1}-x_n)=2(x_{n-1}-y_n)/x_n$ so that $(G(x_{n-1})-G(y_n))/(x_{n-1}-y_n)\to\infty$ $(n\to\infty)$. We see that G' is unbounded on $(0,1)$.

Thus, we have constructed a perfect set S and a function on S twice (actually, infinitely many times) differentiable relative to S that cannot be extended to a function twice differentiable on R.

Reference

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DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824
USA