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# A $C^\infty$ NOWHERE ANALYTIC FUNCTION
## A FORGOTTEN EXAMPLE FROM 1888

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The subtitle of this note is slightly contradictory. Obviously the example has not been completely forgotten and I found it browsing through a Czech collection of problems in analysis [3].

The question of a simple function with a divergent Maclaurin series was raised in *James Cook Mathematical Notes*. An example appeared in the same notes [5]; it reads

$$F(x) = \int_0^\infty \frac{e^{-t}}{1+x^2t^2}\,dt, \tag{1}$$

and the Maclaurin expansion is

$$1 - 2!\,x^2 + 4!\,x^4 - 6!\,x^6 + \cdots . \tag{2}$$

The idea behind this example is simple: (1) can be obtained from (2) by the Euler summation method [1]. This example begs a natural question, namely, can the coefficients of a divergent Maclaurin series be all positive? The answer is contained in a theorem of Borel which states that for a given sequence of real numbers $\{a_n\}$ there always exists a $C^\infty$ function $f$ such that $f^{(n)}(0) = a_n$. Stromberg [4] gives an elegant proof of this which he attributes to L. Garding.

Neither (1) nor Borel's theorems seems to be suitable for an undergraduate class. A strikingly simple example of $f$,

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos a^n x}{n!}, \tag{3}$$

with $a$ an odd positive integer was published in 1888 by a Czech mathematician M. Lerch [2]. The function $f\colon \mathbf{R} \to \mathbf{R}$ from (3) is obviously $C^\infty$ because of uniform convergence of all series obtained by term by term differentiation. To prove that the Maclaurin series diverges for all $x \neq 0$ we have first $|f^{(n)}(0)| = \exp a^n$ for even $n$ and $n! < n^n$. Consequently, for $x > 0$ we have

$$\frac{\exp a^n}{n!}x^n \geq \exp(a^n + n\log x - n\log n) > \exp(a^n - n^2) \to \infty.$$

Hence the Maclaurin series must diverge for every $x \neq 0$. So far we needed only $a > 1$; that $a$ is an odd positive integer is needed for the proof that the Taylor series centred at any point diverges (except at the centre). This proof is based on the fact that similar growth of derivatives occurs at any point of the form $b\pi/a^m$, with $b$ odd and $m$ a positive integer. Lerch's paper was concerned with the study of series of the form

$$\sum_{n=0}^{\infty} c_n \cos a_n \pi x, \tag{4}$$

$x \in \mathbf{R}$, $c_n \in \mathbf{R}$, $a_n \in \mathbf{N}$, and was a natural extension of Weierstrass' investigation of a continuous nowhere differentiable function based also on a series of the form (4). Lerch proved, among other things, that under suitable but fairly general conditions as far as the existence of the derivative is concerned series (4) behaves similarly on a dense subset of the reals as at zero.

The series (3) deserves perhaps to be better known and should complement the standard example $f(x) = e^{-1/x^2}$, $f(0) = 0$ of a function for which the Maclaurin series does not converge to $f$.

## References


1. G. H. Hardy, *Divergent Series*, Oxford University Press, London (1949).
2. M. Lerch, "Ueber die Nichtdifferentiirbarkeit gewisser Functionen", *Crelles Journal für Reine und Angewandte Mathematik* **103** (1888), 126–138.
3. J. Lukeš and a collective, *Problems in Mathematical Analysis*, Charles University, Prague (1982) (in Czech).
4. K. R. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, Belmont, California (1981).
5. R. Vyborny, "Divergent Maclaurin series", *James Cook Mathematical Notes* (1984), 4040.



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