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XV.

Some properties of trigonometric and algebraic polynomials.

By Dr. **M. KÖSSLER** in **PRAGUE.**

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1. Factorization of trigon. polynomials. Every trigon. polynomial of n^{-th} degree

$$P(\varphi) = a_0 + \sum_1^{n-1} (a_k \cos k\varphi - b_k \sin k\varphi) + a_n \cos n\varphi, \; a_n > 0 \qquad (1,1)$$

is expressible uniquely in the form of a product. If we put

$$x = e^{i\varphi}, \; \cos k\varphi = \tfrac{1}{2}(x^k + x^{-k}), \; \sin k\varphi = -\frac{i}{2}(x^k - x^{-k})$$

we deduce

$$P(\varphi) = a_0 + \tfrac{1}{2}\sum_1^{n} (A_k x^k + \bar{A}_k x^{-k}), \; A_k = a_k + ib_k,$$

$$2P(\varphi) = x^{-n}\{A_n x^{2n} + A_{n-1}x^{2n-1} + \ldots + A_1 x^{n+1} + 2a_0 x^n + \\ + \bar{A}_1 x^{n-1} + \ldots + \bar{A}_{n-1} x + A_n\}.$$

The algebraic polynomial in the bracket has the following property. If $-\alpha_k$ is some root of this polynomial then $-\dfrac{1}{\bar{\alpha}_k}$ is also a root, but these two roots are different only if $|\alpha_k| \neq 1$. Therefore

$$\{A_n x^{2n} + \ldots + A_n\} = A_n \prod_1^{2\nu} (x + e^{i\varphi_k}) \,.\, \prod_1^{\mu} (x + \alpha_k)\left(x + \frac{1}{\bar{\alpha}_k}\right), \qquad (1,2)$$

where

$$\nu + \mu = n \text{ and } \alpha_k = r_k e^{i\psi_k}, \; 0 < r_k < 1.$$

The real angels φ_k and ψ_k are not independent, because

$$\sum \varphi_k + \sum 2\psi_k \equiv 0 \;(\mathrm{mod}\, 2\pi) \qquad (1,3)$$

must be satisfied. It is easily seen that also an inverse statement holds true. Every such product represents an algebraic polynomial of the form above. Therefore the polynomial $P(\varphi)$ can be written as follows:

$$2P(\varphi) = A_n e^{-ni\varphi+\nu i\varphi} \prod_1^{2\nu} (e^{\frac{1}{2}i\varphi} + e^{i\varphi_k - \frac{1}{2}i\varphi}) \cdot \prod_1^{\mu} (e^{i\varphi} + r_k e^{i\psi_k})(e^{i\varphi} + \frac{1}{r_k} e^{i\psi_k}).$$

We have

$$\nu - n = -\mu,\; e^{\frac{1}{2}i\varphi} + e^{i\varphi k - \frac{1}{2}i\varphi} = 2e^{\frac{1}{2}i\varphi k} \cos\tfrac{1}{2}(\varphi - \varphi_k),$$

$$(1 + r_k e^{i\psi_k - i\varphi})(e^{i\varphi} + \frac{1}{r_k} e^{i\psi_k}) = e^{i\psi_k}(r_k + \frac{1}{r_k} + 2\cos(\varphi - \psi_k)).$$

It is convenient to put

$$r_k + \frac{1}{r_k} = 2c_k, \text{ where } c_k > 1, \text{ if } 0 < r_k < 1. \text{ Hence}$$

$$2P(\varphi) = \varepsilon 2^{n+\nu} A_n \prod_1^{2\nu} \cos\tfrac{1}{2}(\varphi - \varphi_k) \cdot \prod_1^{\mu} (c_k + \cos(\varphi - \psi_k)) \qquad 1{,}4$$

where as a consequence of (1,3)

$$\varepsilon = \exp\left(i \sum_1^{2\nu} \tfrac{1}{2}\varphi_k + i \sum_1^{\mu} \psi_k\right) = \pm 1.$$

Hence every polynomial (1,1) is expressible uniquely in the form (1,4). The analogy with algebraic polynomials is obvious. Though this result is almost trivial it enables us to find parametric representations for coefficients of special classes of polynomials.

2. *Non negative polynomials.* A non negative polynomial (1,1) is a polynomial which cannot assume negative values for any real φ. It follows immediately from (1,4) that in this case the first product must have the form

$$\varepsilon \prod_1^{\nu} \cos^2 \tfrac{1}{2}(\varphi - \varphi_k) = 2^{-\nu} \prod_1^{\nu} (1 + \cos(\varphi - \varphi_k))$$

because both the second product $\prod_1^{\mu}$ and A_n are positive for every value of φ. The only condition for the angles is therefore

$$\sum_1^{\nu} \varphi_k + \sum_1^{\mu} \psi_k \equiv 0 \pmod{2\pi}. \qquad (2{,}1)$$

Hence the necessary and sufficient condition for (1,1) to be a non negative polynomial is that

$$P(\varphi) = 2^{n-1} A_n \prod_1^{n} (c_k + \cos(\varphi - \psi_k)), \qquad (2{,}2)$$

where

$$A_n > 0,\; c_k \geqq 1,\; \sum_1^{n} \psi_k \equiv 0 \pmod{2\pi}.$$

The coefficients of the polynomial are expressible by the integrals of Fourier. The number a_0 is positive, as $P(\varphi) \geqq 0$.

Now this is a transcendental operation. Making use of (1,2), which in this case has the form

$$x^{2n} + \frac{A_{n-1}}{A_n} x^{2n-1} + \ldots + \frac{A_1}{A_n} x^{n+1} + \frac{2a_0}{A_n} x^n + \frac{\bar{A}_1}{A_n} x^{n-1} + \ldots$$
$$\ldots + \frac{\bar{A}_{n-1}}{A_n} x + 1 = \prod_1^n (x + r_k e^{i\varphi_k}) \left(x + \frac{1}{r_k} e^{i\varphi_k}\right), \ 0 < r_k \leqq 1$$

elementary evaluation of the coefficients becomes obvious. Thus e. g.

$$\frac{A_{n-1}}{A_n} = \sum_1^n \left(r_k + \frac{1}{r_k}\right) e^{i\varphi_k} = 2 \sum_1^n c_k e^{i\varphi_k}$$

and so on.

In case that one of the two positive numbers a_0, $A_n = a_n$ is fixed, the value of second is given by

$$\frac{2a_0}{A_n} = \text{the } n^{-th}$$

elementary symmetr. function of the $2n$ given parameters

$$r_k e^{i\varphi_k}, \frac{1}{r_k} e^{i\varphi_k}.$$

Thus for $n = 2$, $a_0 = 1$ we obtain

$$A_2 = \frac{1}{2c_1c_2 + \cos 2\psi_1}, \quad A_1 = \frac{2c_1 e^{i\psi_1} + 2c_2 e^{-i\psi_1}}{2c_1c_2 + \cos 2\psi_1}.$$

For $n = 3$, $a_0 = 1$, $\psi_1 + \psi_2 + \psi_3 \equiv 0 \pmod{2\pi}$:

$$A_3 = 1 / 4c_1c_2c_3 + 2c_1 \cos(\psi_2 - \psi_3) + 2c_2 \cos(\psi_3 - \psi_1) + 2c_3 \cos(\psi_1 - \psi_2),$$
$$A_2 = 2A_3(c_1 e^{i\psi_1} + c_2 e^{i\psi_2} + c_3 e^{i\psi_3}),$$
$$A_1 = A_3(4c_1c_2 e^{i(\psi_1+\psi_2)} + 4c_2c_3 e^{i(\psi_2+\psi_3)} + 4c_3c_1 e^{i(\psi_3+\psi_1)} + e^{2i\psi_1} + e^{2i\psi_2} + e^{2i\psi_3}).$$

For $\psi_k = 0$, $a_0 = 1$, $r_k = 1$, $k = 1, 2, \ldots, n$, the polynomial of n^{-th} degree is

$$P(\varphi) = 1 + \frac{2}{(2n)_n} \sum_1^n (2n)_{n-k} \cos k\varphi = \frac{2^n}{(2n)_n} (1 + \cos\varphi)^n.$$

The real roots of this polynomial are $\varphi \equiv \pi \pmod{2\pi}$ each of them having multiplicity n.

3. *Algebraic polynomials with positive real part.* Is

$$P(\varphi) = 1 + \sum_1^{n-1} (a_k \cos k\varphi - b_k \sin k\varphi) + a_n \cos n\varphi, \ a_n > 0$$

a non negative trigon. polynomial, then the real part of the algebraic polynomial

$$P(z) = 1 + \sum_{1}^{n-1} A_k z^k + a_n z^n,\ A_k = a_k + ib_k \tag{3,1}$$

is non negative within the circle $|z| < 1$ and vice versa. This follows from the well known fact that the real part of an analytic function cannot reach its absolute minimum within the circle. Thus the polynomials

$$P(z) = 1 + 2\frac{c_1 e^{i\psi_1} + c_2 e^{-i\psi_1}}{2c_1c_2 + \cos 2\psi_1} z + \frac{1}{2c_1c_2 + \cos 2\psi_1} z^2,$$

$$P(z) = 1 + 2/(2n)_n \sum_{1}^{n} (2n)_{n-k}\, z^k$$

have positive r. part within $|z| < 1$.

It is perhaps not without interest to compare our results with the general theory of power series due to C. Caratheodory[1]) and O. Toeplitz[2]). The necessary and sufficient conditions for the power series

$$f(z) = 1 + \sum_{1}^{\infty} A_k z^k$$

to be convergent within the circle $|z| < 1$ and to have positive r. part are that the determinants

$$\delta_1 = 2,\ \delta_2 = \begin{vmatrix} 2, & A_1 \\ \bar{A}_1 & 2 \end{vmatrix},\quad \delta_3 = \begin{vmatrix} 2 & A_1 & A_2 \\ \bar{A}_1 & 2 & A_1 \\ \bar{A}_2 & \bar{A}_1 & 2 \end{vmatrix},\ \ldots \tag{3,2}$$

are either all positive (> 0), or

$$\delta_1 > 0,\ \delta_2 > 0,\ \ldots,\ \delta_n > 0,\ \delta_{n+1} = \delta_{n+2} = \ldots = 0.$$

For polynomials only the first case can occur. Now if we apply this theorem, an infinite number of conditions $\delta_k > 0$, $(k = 1, 2, 3, \ldots)$ for only n coefficients must be satisfied and a finite subset of these inequalities does not give sufficient conditions. Thus for the linear polynomial $1 + A_1 z$ the infinite set of inequalities is

$$|A_1| < 2,\ |A_1| < \sqrt{2},\ |A_1| < \sqrt{5} - 1,\ \ldots$$

and no finite subset leads to the trivial necessary and s. condition $|A_1| \leqq 1$. To deduce n. and s. conditions for a polynomial in a satisfactory form is of course possible but it is not an easy task. For polynomials of higher degree the determinant criterion has no practical value at all.

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1) Math. Ann B. 64 (1907), p. 95 and Rend. di Palermo, B. 32 (1911), p. 193.

2) Gött. Nachr. 1910, p. 489 and Rend. di Palermo, B. 32, p. 191.

4. *Similar problems.* The elementary method described in the first two paragraphs is by no means limited to the question treated there. To illustrate this assertion let

$$P(z) = \sum_{0}^{n} A_k z^n \tag{4,1}$$

be a bounded polynomial $|P(z)| \leqq 1$ within the circle $|z| \leqq 1$. The n. and s. conditions for coefficients of such polynomials are as follows

$$1 - P(e^{i\varphi}) \,.\, \bar{P}(e^{-i\varphi}) \geqq 0, \;\; 0 \leqq \varphi < 2\pi. \tag{4,2}$$

But the left side of this inequality is easily seen to be a non negative trigon. polynomial of n^{-th} degree. The first coefficient of this polyn. is

$$a_0 = 1 - \sum_{0}^{n} |A_k|^2 \text{ and, by (2,2), } a_0 > 0.$$

Therefore each $|A_k| \leqq 1$ and if some $|A_k|$ equals 1 all the others must vanish. This is a well known theorem of the theory of bounded power series, but its proof is based on Cauchy integral or some other general property of analytic functions. From (4,2) a finite number of parametric conditions for A_k can be deduced by means of § 1 and 2. But if we apply the general theory of bounded power series due to I. SCHUR[3]) to a polynomial, the same difficulty as in (3,2) arises. The number of conditions given by that theory is again infinite.

Some other problems that can be solved by the method used in this paper are: 1. $A_0 \neq 0$ being a given number, to find all polynomials $P(z) = A_0 + \sum_{1}^{n} A_k z^k$, which do not vanish within the circle $|z| < 1$. 2. $P_1(z)$ being a given polynomial not vanishing for $|z| \leqq 1$, to find all rational functions $\dfrac{P_2(z)}{P_1(z)}$ having positive real part within the circle or bounded within the circle etc.

As a final remark, I cannot suppress the suggestion that a more profound study of properties of polynomials would be very useful to the general theory of power series. The theory of polynomials is entirely independent of integral theorems (e. g. Cauchy integral) and it is a purely algebraic question.

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[3]) I. SCHUR: Jour. für die r. und a. Math. B. 147 (1917), p. 205 and B. 148 (1918), p. 122.

Několik vlastností trigonometrických a algebraických mnohočlenů.

Souhrn.

V této práci jest proveden rozklad trigonometrického mnohočlenu (1,1) v součin tvaru (1,4). Z toho plynou pro mnohočleny nikoliv záporné nebo ohraničené nutné a postačující podmínky pro jejich koeficienty. Těchto podmínek jest konečný počet, kdežto obecná theorie mocninných řad vede i v případě polynomů k nekonečnému počtu podmínek a jest tedy pro mnohočleny nevhodná.