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In Nekvasilová, Jana (ed.), Circuli: 1933–2003. Prague: National Technical Museum, Department of the History of Technology; Prague: Society for the History of Science and Technology (ISBN 80-7037-125-0/pbk). Acta Historiae Rerum Naturalium necnon Technicarum (New Series) 7, 73-80 (2003)

Persistent URL: http://dml.cz/dmlcz/501199

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Abstract. In the years 1930–1938 Čech turned his attention to topology, publishing 30 papers on different topics in that area. The paper presents Čech's topological contributions from that period, among them two major ones: Čech homology theory and Čech-Stone compactification.

The two decades between two World Wars were a period of an intensive development of topological ideas towards greater abstraction and generality. Aiming at the concept of a general topological space, there were recognized several steps of generality like Hausdorff, regular, completely regular, or normal spaces. There were also recognized several types of general spaces like metric, separable, complete, or compact ones and initiated the theory of dimension[1]. On the other hand, there also was a strong tendency to keep together two main branches of topology, that is, combinatorial topology (as it was then called, later it became algebraic topology), patterned after polyhedra and possessing a strong geometric flavor, and general topology which was basically influenced by analytical considerations[2].

Eduard Čech (1893–1960), undoubtedly the greatest Czech mathematician in the XX$^{\text{th}}$ century and one of the great names in topology, was then already known for his achievements in differential geometry and continued to work in that area, but in the second decade of that inter-war period he fell tempted to try also topology. And he was largely successful also there, contributing substantial innovations to that new field. All his topological papers were later reedited (the papers which appeared originally in Czech have been translated into French or English) in a separate volume [3].

[1] For a review of these concepts see any book on general topology, e.g., J. L. Kelley, *General Topology*, Princeton, N.J.: Van Nostrand, 1955; R. Engelking, *General Topology*, Warszawa: PWN – Polish Scientific Publishers, 1977.

[2] An early history of topology is covered by, e.g., J. H. Manheim, *The Genesis of Point Set Topology*, Oxford: Pergamon Press, 1964; Guy Hirsch, *Topologie*, in the book: J. Dieudonné, *Geschichte der Mathematik 1700–1900*, Berlin: VEB – Deutscher Verlag der Wissenschaften, 1985, 639–697; J. Dieudonné, *Une brève histoire de la topologie*, in the book: J.-P. Pier (editor), *Development of Mathematics 1900–1950*, Basel·Boston·Berlin: Birkhäuser Verlag, 1994 (the two latter articles are dealing mainly with the development of algebraic topology).

[3] *Topological papers of Eduard Čech*, Prague: Academia, 1968.

After some minor contributions like another proof of the theorem of Jordan or some technical lemmas in homology modulo 2[4], he turned his attention to dimension. Since 1874 there was a problem whether distinct Euclidean spaces are topologically different[5]. The problem has been answered in the positive by Brouwer only in 1913[6], thus confirming the value of topological ideas and reviving old hopes for a theory of dimension. And in fact, the theory has been initiated independently (for metric separable spaces) by Menger and Urysohn in the early twenties[7]. Their fundamental concept is now called the small inductive dimension and denoted ind. Its definition, for a given topological space X, runs as follows:

(i) if $X \neq \emptyset$, then $\mathrm{ind}\, X = -1$,

(ii) if for any point x of X there is an arbitrarily small open neighborhood U of x such that its boundary $\overline{U} \backslash U$ is of dimension $\mathrm{ind}\,(\overline{U} \backslash U) \leq n-1$, then $\mathrm{ind}\, X \leq n$.

One can slightly change that definition by replacing the words "point x" in (ii) by the words "closed subset F". This is a new concept of dimension, called the great inductive dimension and denoted Ind. Essentially it is equivalent to Brouwer's concept from 1913 but it was Čech who first gave to it a formal definition (for the class of normal spaces) and used it to prove some important results like the additive theorem (dimension of a countable union of closed sets is equal to the limit superior of their dimensions), the theorem on monotony of dimension, and others — for wider classes of spaces like normal ones.

[4] E. Čech, *Une démonstration du théorème de Jordan*, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat., (6) **12** (1930), 386-388; E. Čech, *Trois théorèmes sur l'homologie,* Spisy Přírod. Fak. Univ. Brno **144** (1931), 21 pp.

[5] For the meaning of that problem to topology and the history of its solution cf. J. W. Dauben, *The invariance of dimension: problems in the early development of set theory and dimension*, Hist. Math. **2** (1975), 273–288; R. Duda, *The origins of the concept of dimension*, Coll. Math. **42** (1979), 95–110; D. M. Johnson, *The problem of the invariance of dimension in the growth of modern topology*, Arch. Hist. Ex. Sci. **20** (1979), 97–188 and **25** (1981), 85–226.

[6] L. E. J. Brouwer, *Über den natürlichen Dimensionsbegriff*, J. reine angew. Math. **142** (1913), 146–152.

[7] K. Menger, *Über die Dimension von Punktmengen*, Monatsh. für Math. u. Phys. **33** (1923), 148–160 and **34**(1924), 137–161; P. S. Urysohn, *Les multiplicités cantoriennes*, C. R. Acad. Sci. Paris **175** (1922), 440–442; P. S. Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fund. Math. **7** (1925), 30–137 and **8** (1926), 225–331; K. Menger, *Dimensionstheorie*, Leipzig-Berlin: Springer, 1928.

Strictly speaking, he formulated the definition and announced results[8] but two papers providing details and proofs appeared in Czech[9].

His last paper on dimension (from 1933) is worth mentioning also for extracting the exact meaning of an old concept of dimension due to Lebesgue[10]. Čech gave that concept a formal definition and in that way started to use, as it was later called[11], the covering dimension, denoted dim. It seems worth to notice that it is sometimes called the Čech-Lebesgue dimension[12].

Eduard Čech was a great teacher (he wrote 7 textbooks for secondary schools) and a patriotic man. Complaining in 1932 that topology has not yet achieved a status in university teaching which it deserves, he wrote an extensive paper "en langue tchèque. Pour [...] je trouve bon de faire précéder l'exposé propre du sujet [...] par un aperçu sommaire de notions bien connues [...]"[13]. A didactical tendency and an inclination to write "en langue tchèque" not only books and surveys but also some original contributions were a characteristic feature of his activity throughout the years. Such an attitude obviously did him harm by restraining dissemination of his results and thus diminishing his influence in the world. In spite of that his influence has soon become great.

Besides exposition, that extensive paper revealed also Čech's early interest in problems concerning connectedness, in particular irreducible connectedness between several points. This led him to the concept of a "tree" which for him was a sort of a dendrite for general topological spaces. Related to it there is a short paper[14] on continua which can be mapped into a segment in such a way that the inverse images of points are

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8 E. Čech, *Sur la théorie de la dimension*, C. R. Acad. Sci. Paris **193** (1931), 976–977.

9 E. Čech, *Dimense dokonale normálních prostorů* [Dimension of perfectly normal spaces], Rozpr. Čes. Akad. Věd (13) **42** (1932), 22 pp. (Bull. Int. Acad. Sci. Boh. (1932), 18pp.); – ,*Příspěvek k theorii dimense* [Contribution to the theory of dimension], Časopis Pěst. Mat. **62** (1933), 277–291.

10 H. Lebesgue, *Sur la non applicabilité de deux domaines appartenant à des espaces de n et n+p dimensions*, Math. Ann. **70** (1911), 166–168.

11 Cf. W. Hurewicz, H. Wallman, *Dimension Theory*, Princeton 1940.

12 Cf. R. Engelking, *Dimension Theory*, Warszawa: PWN – Polish Scientific Publishers & Amsterdam·Oxford·New York: North-Holland, 1978.

13 E. Čech, *Množství irreducibilně souvislá mezi n body* [On sets which are irreducible between n points], Časopis pro pěstování matematiky a fysiky **61** (1932), 109–129.

14 E. Čech, *Une nouvelle classe de continus*, Fund. Math. **18** (1931), 85–87.

finite sets. And in another related paper[15] he offers some simplifications to the Menger's proof of *n-Bogensatz.*

An influential advocate for restoring the unity of topology was in those years P. Aleksandrov, whose best paper on the subject has then newly appeared[16]. One of his most important notions serving that aim was that of the nerve of a covering which offers a sort of a polyhedral approximation to the whole covered space. Formally, for a given family of sets $\{U_s\}$ one can consider an abstract simplicial complex whose vertices are U_s and simplexes are all finite families $U_{s_1}, U_{s_2}, \ldots, U_{s_k}$ such that $U_{s_1} \cap \ldots U_{s_k} \neq \emptyset$. Such a complex is called the nerve of the family $\{U_s\}$ and this notion has turned to be of great importance for topological considerations. To give one example, take a compact Hausdorff space X and consider coverings of X with disjoint interiors. The set of all such coverings can be partially ordered by the relation of inclusion of its elements ($\{U_s\} \leq \{V_t\}$ iff each V_t is contained in some U_s) and for two coverings in that order one can define a simplicial mapping σ between their nerves which transforms V_t into that unique U_s for which $V_t \subset U_s$. The family of all such nerves together with the simplicial mappings between them is called the spectrum of X. It is an inverse system of polyhedra and mappings whose limit, as Aleksandrov proved, is X itself. This is the fundamental way in which an arbitrary (compact) topological space can be approximated by polyhedra.

Using these concepts of nerve and spectrum, and probably motivated also by the earlier homology theory of Vietoris[17], Čech developed an original and quite general homology theory of his own[18]. While Vietoris has been using infinite complexes where vertices are the points of the space, Čech's idea was to use the spectrum of Aleksandrov in order to define a homology group of a given (compact) space as the inverse limit of classic homology groups of the nerves of its suitable coverings. Since each nerve is a polyhedron, it was a successful bridge between general topological spaces and techniques developed for polyhedra, apparently

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[15] E. Čech, *Sur les arcs indépendants dans un continu localement connexe*, Spisy Přírod. Fak. Univ. Brno, **193** (1934), 10 pp.

[16] P. Alexandroff, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Ann. of Math. (2) **30** (1929), 101–187.

[17] L. Vietoris, *Über den höheren Zusammenhang kompakten Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. **97** (1927), 454–472.

[18] E. Čech, *Théorie générale de l'homologie dans un espace quelconque,* Fund. Math. **19** (1932), 149–183.

the best bridge discovered so far. However, the meaning of the new theory has been far greater. The theory is still called Čech homology (and cohomology) theory[19] and it remains one of the most general in the field.

One should add that in dealing with combinatorial topology (the original name for algebraic topology) Čech has tended, as he himself admitted[20], to a unification of its methods and ways of reasoning with those in general topology or, somewhat more precisely, to a discovery of the general substance of the homology theory, of the theory of manifolds etc., with the aim to incorporate it into the theory of general topological spaces. Čech's homology theory was an important contribution to that program but it was also a strong impulse for a further development of algebraic topology.

In 1932 there was a Congress of Mathematicians in Zurich during which Čech delivered two communications, one envisaging a general theory of topological manifolds[21] (not necessarily combinatorial) and second, proposing a definition of higher homotopy groups[22]. In that time the only considered manifolds were those with a combinatorial structure due to which one could apply to them simplicial homology theory and in that way discover, e.g., duality theorems. But Čech already possessed then his more general homology theory and so he was able to offer a more general concept of a manifold in which duality theorems, exposed *via* his homology theory, still hold[23]. A similar idea has been contemplated by S. Lefschetz[24] and these were the first two instances of a theory of topological manifolds, now an important chapter of topology.

The magnificent idea expressed in the second communication of considering continuous mappings $(S^n, p) \to (X, x_0)$ of the n-dimensional sphere S^n into a topological space X, both with fixed points, as a way to

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[19] For a full treatment the reader is advised to consult the book: S. Eilenberg, N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[20] E. Čech, *Les théorèmes de dualité en topologie*, C. R. 2-e Congr. Math. Slav. Praha, (1934), 17–25.

[21] E. Čech, *La notion de variété et les théorèmes de dualité*, Verh. des intern. Mathematikerkongresses Zurich **2** (1932), 194.

[22] E. Čech, *Höherdimensionale Homotopiegruppen*, Verh. des intern. Mathematikerkongresses **2** (1932), 203.

[23] E. Čech, *Théorie générale des variétés et de leur théorèmes de dualité*, Ann. of Math. (2) **34** (1933), 621–730.

[24] S. Lefschetz, *On generalized manifolds*, Amer. J. Math. **55** (1933), 469–504.

define n-dimensional homotopy group $\pi_n(X, x_0)$, has not been realized by Čech. One of the reasons why he did not pursue the idea could be a criticism that the groups arising out of his definition did not extend the following property of the fundamental group: abelianized fundamental group $\pi_1(X, x_0)$ is equal to 1-dimensional singular homology group $H_1(X)$. It was expected that abelianized n-dimensional homotopy group should be equal to n-dimensional singular homology group which was not the case with the Čech's definition. It is regrettable because higher homotopy groups have later played a major role. Thus all the fame of discovering all those higher homotopy groups went to Hurewicz who defined them (equivalently but in a different way) and developed their theory[25].

Having studied two papers by Alexandroff and Urysohn[26] and by Tychonoff[27], the first of which has initiated a systematic study of compact spaces and the second envisaged a compactification of completely regular spaces, Eduard Čech has defined, for every completely regular space X, its specific compactification βX[28]. His idea was to consider the family of all continuous functions $f\colon X \to [0,1]$ such that $f(p) = 0$ and $f(x) = 1$ for all $x \in F$, where p is a point of X and F is a closed subset of X not containing p. Taking now the cube C with so many edges I_f as there are functions f, one may transform X into C by defining $f(x)$ to be the coordinate of x with respect to I_f. The transformation is an embedding $X \to C$ and the closure of $f(X)$ in C is, by the definition, the required compactification βX. As one can easily see, this compactification is the greatest one and it possesses also other interesting properties which are studied to this day [29]. In the literature it is called Čech compactification or Čech-Stone compactification because American mathematician M. H. Stone has simultaneously come to an equivalent

[25] W. Hurewicz, *Beiträge zur Theorie der Deformationen*, Proc. Acad. Amsterdam **38** (1935), 112–119 and **38** (1935), 521–528 and **39** (1936), 117–126 and **39** (1936), 215–224.

[26] P. Alexandroff, P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verhandlungen der Kon. Akad. Amsterdam, 1929.

[27] A. Tychonoff, *Über die topologische Erweiterungen von Räumen*, Math. Ann. **102** (1930)

[28] E. Čech, *On bicompact spaces*, Ann. of Math. **38** (1937), 823–844.

[29] Some account of a further work on β-compactification can be found in the book: K. Kuratowski, *Topologie*, vol. II, New York ·London: Academic Press & Warszawa: PWN – Polish Scientific Publishers, 1968, 18–20.

concept[30]. Stone considered the set M of maximal ideals in the ring $C(X)$ of all continuous real functions on X. Defining basis in M to consist of the sets $U_f = \{\Delta\colon f \notin \Delta\}$, he was able to show that M is a compact space and that the mapping $x \to \Delta(x)$, where $\Delta(x)$ is the maximal ideal consisting of all functions vanishing in x, defines an embedding of X into M. Thus M is a compactification of X. Being the maximal one, it is equivalent to βX. Contexts of the two definitions, however, were different and it seems that Čech better recognized generality of the construction.

In that paper on β-compactification Čech has also introduced some new topological concepts which later turned to be of some value. If a completely regular space X is of the G_δ type in βX, that is, if X is the common part of a family of countably many open subsets of βX, then it is called topologically complete (complete in the sense of Čech). Čech proved that for such a topologically complete space X holds the Baire category theorem. Since it was then the most general type of spaces X enjoying that important property, the concept has proved important and soon became the object of study for itself. Another new and valuable concept introduced in that paper was that of a perfectly normal space.

Problems concerning Čech-Stone compactification are still among more interesting ones and they are important not only for general topology[31]. Many of them are important for foundations of mathematics, especially those concerning compactification of the set of natural numbers[32].

The construction of β-compactification of completely regular spaces has been accompanied by his book point sets[33] and a paper on general topological spaces[34]. Both were fairly original and modern (they are admirably precise and didactically outstanding) but, being published in Czech, they could influence only Czech mathematical community. This they did and with a good success but a little later Bourbaki offered

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[30] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), 375–481.

[31] Cf. N. Neil, D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, Berlin·New York: W. de Gruyter, 1998.

[32] A survey of open problems in that area is provided by the article: K. P. Hart, J. van Mill, *Open problems on $\beta\omega$*, in the book: J. van Mill, G. M. Reed (editors), *Open problems in topology*, Amsterdam etc.: Elsevier Science Publishers (North-Holland), 1990, 97–125.

[33] E. Čech, *Bodové množiny I* [Point-sets I], Praha 1936.

[34] E. Čech, *Topologické prostory* [Topological spaces], Časopis Pěst. Mat. **66** (1937), D225–D264.

a different approach to general topology and it was Bourbaki's book[35] that has become a canonical model for the theory, since then commonly accepted. Among Čech's unpublished papers was found a manuscript of *Bodové množiny II.*

In the period 1930–1938 Čech published altogether 30 papers and 1 book (in Czech) on topology, of which we have described a part. The selection was personal but everybody should agree that he made two major contributions, namely Čech homology theory and Čech-Stone compactification, and was fairly close to invent higher homotopy groups. His extraordinary topological intuition is also well reflected by the push he gave to dimension theory by advancing two nearly then forgotten dimension concepts Ind and dim or by the initiative to start the general theory of manifolds. All his contributions were extremely original and have exercised great influence up to the present day but the impact of those two which bear his name was by far the greatest one. They secured for Čech not only wide recognition but also a place in the history of mathematics. Later on Čech returned to differential geometry although after War World II there appeared one more paper (joint with J. Novák)[36] and two more books on topology, *Topologické prostory* in 1959 and *Bodové množiny* in 1966 (the latter was a posthumous edition consisting of the first three chapters of *Bodové množiny I* and of the manuscript of *Bodové množiny II*), the two in Czech. Although the books were revised (by his students and friends) and later translated into English[37], their time has already passed: having been written about thirty years before their publication, they could hardly compete with later ones.

Topological decade was a sort of a break in Čech's continuing interest in differential geometry, to which he devoted more time and more energy than to topology, publishing nearly twice as many papers. Nevertheless, if the Čech's name remains vivid in mathematics, it is due rather to that topological break than to anything else.

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[35] N. Bourbaki, *Eléments de mathématique, Topologie générale*, Paris: Hermann: 1940.

[36] E. Čech, J. Novák, *On regular and combinatorial imbedding*, Časopis Pěst. Mat. **72** (1947), 7–16.

[37] E. Čech, *Topological spaces*, Revised edition by Z. Frolík and M. Katětov, Praha: Academia, 1966; – , *Point sets*, Praha 1968.