Otakar Borůvka

Geometric elements in the theory of transformations of ordinary second-order linear differential equations [working paper]

Symposium of Differential Equations and Dynamical Systems. Mathematics Institute, University of Warwick, 1968-9, 19-22

Persistent URL: http://dml.cz/dmlcz/500133

Geometric elements in the theory of transformations of ordinary second-order linear differential equations.

1. In the following lecture I should like to give a short survey of the present state of the theory of transformations of ordinary 2nd order linear differential (d.) equ tions as well as of the geometric elements occurring in this theory. It is a theory in the real domain, of qualitative and global character. The geometric ele ents I shall speak about conc rn the centro - affin differential global geometry of plane-curves. To be brief, I hall just speak about the theory of transf rmation ithout always stating that it is a question of tran formations of linear 2nd order dif. equations.

2. The basic problem of the theory of transformations was formulated by the German mathematician, E. E. KUMMER, in 1834, as follows:

Suppose two given 2nd order dif. equation

$$(q) \qquad y'' = q(t)y \, , \qquad \ddot{Y} = Q(T)Y \qquad (Q) \, ,$$

wh re the coefficients q , Q , which are sometimes briefly called carriers of the dif. equati ns (q) or (Q) , are continuous functions in certain open, bounded or unbounded intervals $j = (a,b)$ or $J = (A,B)$.

We are to determine the functions $w(t)$, $X(t)$ such that, for every integral $Y(T)$ of the dif. equation (Q), the function

$$y(t) = w(t).Y\left[X(t)\right]$$

is a solution of the d. equation (q) . We a ume that $w(t) \neq 0$, $X'(t) \neq 0$.

The analysis of this problem directly leads to the following non-linear dif. equation of the 3rd order, called Kummer's dif. equation:

$$(Qq) \qquad -\{X,t\} + Q(X)X'^2 = q(t)$$

where t stands for the independent variable, X for the unknown function and the symbol $\{\ \}$ for Schwarz's derivative of the function X at the point t :

$$\{X,t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\frac{X''^2(t)}{X'^2(t)} \quad .$$

We can show that every function X satisfying Kummer's problem is a solution of the dif. equation (Qq) , where $w(t) = c : \sqrt{|X'(t)|}$ (c = const.). Vice versa, in the same way one finds, from every solution of the dif. equation (Qq), solution of Kummer's problem. We can say that the entio ed theory of transformations is, in fact, an analysis of Kummer's dif. equation (Qq) .

3. The main notions of the theory of transformations are the notions of the first and the second phases of the linear dif. equation (q) .

Let us first consider the notion of the first phases.

Let u, v be an arbitrary basis of the dif. equation (q), i. e. a sequence of two linearly independent integrals of the dif. equation (q) .

By the first phase of the basis u, v we understand every function $\alpha(t)$ continuous in the interval j , in which it satisfies - except the zeros of the function v - the relation $\operatorname{tg} \alpha(t) = u(t):v(t)$. We see that there exists just one countable system of the first phases of the basis u, v , the differences between the single phases of this system being integer multiples of the number π .

By the first phase of the dif. equation (q) we understand a first phase of some basis of the dif. equation (q) .

It is important to note that the first phasesof the dif. equ tion (q) are unbounded, both above and below, exactly when the dif. equation (q) is oscillatory. The dif. equation (q) is called oscillatory when its integrals have an infinite number of zeros in both directions towards the ends a, b of the interval j . For example, the dif. equation $y'' = -y$ in the interval $j = (-\infty, \infty)$ is oscillatory.

Besides the first phases of the basis u, v or the dif. equation (q) we can, analogously, define the second phases of the basis u, v or the dif. equation (q) by means of the formula $\operatorname{tg} \beta(t) = u'(t):v'(t)$.

Let us note that the function $\vartheta(t) = \beta(t) - \alpha(t)$ formed by means of an arbitrary second phase β and first phase α , belonging to the same basis u, v of the dif. equation (q) , is called polar function of the basis u, v .

4. An important section of the theory of transformations is formed by the theory of the so-called central dispersions. Central dispersions are certain functions of one variable which, in a certain sense, describe the dispersion, that is to say, distribution of the zeros of the integrals of the dif. equations (q) and the derivatives of these integrals.

Let us now consider some arbitrary dif. equation (q) which is oscillatory. In some cases it is convenient to assume that the function q is always < 0 . Note that, from the geometric point of view, this assumption ensures that the integral curves of the dif. equation (q) are regular, i. e. locally convex and without points of inflection.

In what follows we shall therefore assume $q(t) < 0$ for $t \in j$.

ell, let $t \in j$ be an arbitrary nu ber and u r v an rbitrary integral of the dif. equation (q) , which has or whose derivative has, at the point t the value 0 : $u(t) = 0$, $v'(t) = 0$. Let, furthermore, $n = 1, 2, \dots$ and d note by:

$\varphi_n(t)$ or $\varphi_{-n}(t)$, respectively, the n-th zero of the integral u that follows or precedes the zero t ;

$\psi_n(t)$ or $\psi_{-n}(t)$, respectively, the n-th zero of the function v' that follows or precedes the zero t ;

$\chi_n(t)$ or $\chi_{-n}(t)$, respectively, the n-th zero of the function u' that follows or precedes t ;

$\omega_n(t)$ or $\omega_{-n}(t)$, respectively, the n-th zero of the integral v that follows or precedes t .

Let now $\nu = \pm 1, \pm 2, \dots$. The functions φ_ν or ψ_ν , χ_ν , ω_ν are called central dispersions of the first, second, third, fourth kind with the index ν , respectively, and - in particular - the functions φ_1 , ψ_1 , χ_1 , ω_1 are called basic central dispersions of the respective kind. These definitions evidently do not depend upon the choice of the inte rals u and v , so that the mentioned central dispersio s represent elements given by the dif. equation (q) .

The situation is illustrated by the following figures:

5. The analytic apparatus of the theory of central dispersions

The preceding notions are connected by numerous mutual relations which can be expressed by simple and often elegant formulas representing an effective analytic apparatus for solving various problems from the theory of transformations and, more generally, from the theory of 2nd order linear dif. equations.

As an example, let me introduce the formula

$$\varphi(t) = t_0 + \frac{1}{\alpha_0'} \int_{\alpha_0}^{\alpha+\varepsilon\pi} (\exp 2 \int_{\alpha_0}^{\sigma} \operatorname{cotg} h(\rho) d\rho)\, d\sigma ,$$

expressing the basic central dispersion φ by means of an arbitrary first phase $\alpha = \alpha(t)$ of the dif. equation (q) and some polar function $\mathcal{h}(t) = \beta(t) - \alpha(t)$ corresponding to the first phase α : $h(\alpha) = \mathcal{h}(t)$; α_0 , α_0' ($\neq 0$) denote the values of the functions α and α', respectively, at the number t and ε = sgn α' .

This formula yields

$$\varphi'(t) = \exp 2 \int_{\alpha}^{\alpha+\varepsilon\pi} \operatorname{cotg} h(\rho)\, d\rho$$

and furthermore

$$\frac{\varphi''(t)}{\varphi'(t)} = 2\alpha_0'\left[\operatorname{cotg} h(\alpha + \varepsilon\pi) - \operatorname{cotg} h(\alpha)\right] . \exp\left(-2\int_{\alpha_0}^{\alpha} \operatorname{cotg} h(\rho) d\rho\right)$$

We see that the basic central dispersion $\varphi(t)$ of the dif. equation (q) is linear of the form

(1) $$\varphi(t) = t + k \quad (k = \text{const.}, \ k > 0)$$

if and only if, the polar function h is periodic with period π and satisfies the condition

$$\int_0^{\pi} \operatorname{cotg} h(\rho)\, d\rho = 0 .$$

This result we shall meet again later, when considering the geometric applications of the theory of central dispersions. Note that it is exactly the mentioned conditions upon which the validity of formula (1) depends that are characteristic of the dif.

equations (q) , the zeros of whose integrals are ranged equidistantly, that is to say, so that the distance k between any two neighbouring zeros is the same .

6. A brief survey of the present state of the theory of transformations.

The theory of transformations consists, fundamentally, of two parts. One part is the so-called theory of dispersions, whereas the other part is the general theory of transformations. Each of these theories is further divided into two parts according to the following scheme:

Theory of transformations			
Theory of dispersions		General theory of transformations	
Central dispersions	General dispersions	General transformations	Complete transformations

The theory of dispersions deals with oscillatory dif. equations (q) .

The notion of central dispersions has been introduced already. From the standpoint of the theory of transformations, the central dispersions are characterized by the fact that they transform the integrals u of the dif. equation (q) and their derivatives u' into themselves or that they transform the integrals u into the derivatives u' or the derivatives u' into the integrals u . For example, every central dispersion of the first kind φ_ν transforms every integral u of the dif. equation (q) into itself according to the formula

$$u(t) = (-1)^\nu \frac{u[\varphi_\nu(t)]}{\sqrt{\varphi_\nu'(t)}} \qquad \left(w(t) = (-1)^\nu : \sqrt{\varphi_\nu'(t)}\right)$$

and in the case of central dispersions of higher kinds the situation is similar. In the theory of central dispersions a number of problems have been studied. Let us, for instance, mention the determination of all the dif. equations (q) having the same basic central φ_1 , or the determination of all the dif. equations (q) in which the basic central dispersions φ_1 and ψ_1 or χ_1 and ω_1 coincide, and others. Some of these problems are closely related to the centro-affine geometry of plane -curves, as we shall see later.

The second part of the theory of dispersions is formed, as I have already said, by the theory of general dispersions. This theory studies the mutual transformations of two oscillatory dif. equations (q) , (Q) . It starts with the constructive definition of certain functions of one variable based on the dif. equations (q) , (Q) and are called general dispersions of the dif. equations (q) , (Q) . Studying the properties of these functions one finds that the general dispersions of dif. equations (q) , (Q) are exactly all the integrals of Kummer's dif. equation (Qq) . We may say that the theory of general dispersions is, essentially, a constructive integration theory of Kummer's dif. equation in the oscillatory case. It should be noted that the theory of general dispersions leads to an extensive algebraic theory of transformations of oscillatory dif. linear equations, as we shall hear later.

Let us now proceed to a brief survey of the theory of general transformations. This theory concerns arbitrary dif. equations (q) , (Q) , whether they are oscillatory or not.

The central point in this theory is given by the theorem about the existence and uniqueness of the integrals of Kummer's dif. equation (Qq) . This theorem ensures the existence of just one integral X of the dif. equation (Qq) , satisfying the given Cauchy initial conditions of the 2nd order. It is the question of the widest integral in the sense that every solution of the dif. equation (Qq) , satisfying the same initial conditions, is a part of this widest integral. We can show that this widest integral X is given by the formula

$$X(t) = A^{-1}\alpha(t) \quad ,$$

where α , A are convenient first phases of the dif. equation (q) or (Q) , respectively. It may further be shown that the curve determined by the integral X passes from one end of the right-angled domain $j \times J$ to the other ; j and J are of course the definition intervals of the dif. equations (q) , (Q), respectively. In case of increasing integrals X , for example, the situation is illustrated by the following figure:

Further problems studied in the general theory of transformations concern the properties of the solutions X of Kummer's dif. equation (Qq) as well as the relations of these solutions to the integrals of the dif. equations (q) , (Q) . The fundamental relation in this respect is, of course, the formula

$$y(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}}$$

expressing the transformation of an arbitrary integral Y of the dif. equation (Q) into a certain integral y of the dif. eq. (q).

An important section of the general theory of transformations is formed by the so-called complete transformations. The starting-point of this theory is the mentioned theorem about the existence and uniqueness of the integrals of Kummer's dif. equation (Qq). In fact, the definition interval i of the widest solution $X(t)$ of the dif. equation (Qq) does not coincide, generally, with the definition interval j of the dif. equation (q), neither do the values of the function X form the entire interval J of the dif. equation (Q). The solution $X(t)$ is called complete if $i = j$ and if the values of the function X form the interval J. The object of the theory of complete transformations is to study the necessary and sufficient conditions for the existence of complete solutions X of the dif. equation (Qq), to determine the generality of such solutions and to describe the structure of the set of all complete solutions of the dif. equation (Qq). Even from this brief outline of the object of the theory of complete transformations one may realize that it is a vast theory, deeply affecting the notions of the theory of 2nd order linear dif. equations. And we may add that it is, on the whole, a most satisfactory theory, since it leads to results of a definite kind.

7. The algebraic theory of general dispersions.

Let me now speak about the algebraic theory of general dispersions which belong, in my opinion, to the smartest parts of the theory of transformations. I shall again consider the linear dif. equations (q), (Q), etc., which are oscillatory and, moreover, assume that their interval of definition is always the interval $j = (-\infty, \infty)$. To the algebraic theory in question both these assumptions are essential.

Let us use the term phase function for every function which is defined in the interval $j = (-\infty, \infty)$, which is unbounded on both sides, which belongs to class C_3 (i. e. always has a continuous derivative of the 3rd order) and whose first derivative α' is always different from 0 : $\alpha'(t) \neq 0$ for $t \in j$. The importance of the phase functions for the theory of transformations consists in the fact that every first phase of the dif. equation (q) is a phase function but that there also holds, vice versa: Every phase function α is the first phase of the dif. equation (q) whose carrier is given by the formula $q(t) = -\{\alpha, t\} - \alpha'^2(t)$. For this reason we simply speak about phases instead of phase functions.

Let us consider the set $\mathcal{G}$ composed of all the phases and define, in $\mathcal{G}$, a binary operation - i. e. multiplication - by composing the functions, so that the product $\alpha\beta$ of an arbitrary sequence of two elements α, β in $\mathcal{G}$ is the composed function $\alpha[\beta(t)]$. One can easily see that the set together with this multiplication is a group. This group is called group of phases. Its unit is, of course, the function $\varphi_0(t) = t$. We can say that the algebr ic theory of general dispersions is a study of the structure properties of the group $\mathcal{G}$ and their relations to the general dispersions of linear dif. equations of the 2nd order.

When studying the structure of group $\mathcal{G}$ we meet, first of all, its sub-group $\mathcal{P} \subset \mathcal{G}$, which is composed of all the increasing phases. This subgroup $\mathcal{P}$ has the index 2 and is invariant in $\mathcal{G}$, so that the factor-group $\mathcal{G}/\mathcal{P}$ is composed of two classes one of which is $\mathcal{P}$, whereas the other is formed by all the decreasing phases.

In the algebraic theory we are dealing with, an important place is held by a certain subgroup $\mathfrak{f}$ of the group $\mathcal{G}$, namely the so-called fundamental subgroup: $\mathfrak{f} \subset \mathcal{G}$. This fundamental subgroup is composed exactly of all the phases of the dif. equation $y'' = -y$ in the interval $j = (-\infty, \infty)$. Its importance lies in the following property: If we form the decomposition $\bar{E}$ of the group $\mathcal{G}$ into right classes with regard to the subgroup $\mathfrak{f}$, so that $\bar{E} = \mathcal{G}/_r \mathfrak{f}$, then every class $\mathfrak{f}\alpha \in \bar{E}$ is composed of phases of exactly one dif. equation (q), namely of phases of the dif. equation with the carrier $q(t) = -\{\gamma, t\} - \gamma'^2(t)$, where γ stands for an arbitrary phase of that class. The correspondence between the classes of the decomposition $\bar{E}$ and the dif. equations (q) is a one-to-one correspondence. A further important result is given by the theorem:

The set $I(Q,q)$ of all general dispersions of the dif. equations (q), (Q) or all the integrals of Kummer's equation (Qq), defined in the interval j, is given by the formula:

$$I(Q,q) = A^{-1}\mathfrak{f}\alpha$$

where A, α stand for an arbitrary first phase of the dif. equation (Q) or (q), respectively.

From this result it follows in particular ($Q = q$, $A = \alpha$) that the set $I(q,q)$ of general dispersions of the dif. equation (q), in other words, the set of all integrals of Kummer's equation (qq) in the interval j, is a subgroup in the group $\mathcal{G}$, conjugated with the fundamental subgroup $\mathfrak{f}$: $I(q,q) = \alpha^{-1}\mathfrak{f}\alpha$.

For lack of time I cannot deal with the study of the algebraic structure of the group $\mathcal{G}$ any longer. I only wish to add that considerations based on this study have made it possible to solve several problems which would otherwise be very difficult to approach. By this method it was found, for example, that the set of all the dif. equations (q) in the interval $j = (-\infty, \infty$ having the same basic central dispersion φ_1, always has the cardinal number of the continuum, regardless of the choice of the function φ_1. Furthermore, one has characterized the general dispersions of the dif. equations (q), (Q) by means of thei relations to the groups $I(Q,Q)$, $I(q,q)$, one has found common solutions of two Kummer's equations, etc.

8. Geometric elements in the transformation theory.

Let us now proceed to study the geometric elements occurring in the theory of transformations. As I have already said, the theory of transformations represents an effective means of investigating centroaffine differential properties of plane-curves of global character.

Consider, in a centroaffine plane, an arbitrary curve C, defined by its parametric coordinates U, V, in some open interval J. Suppose these coordinates are related to a certain fixed coordinate system given by two vectors x_1, x_2 with the origin O. With regard to the method applied in what follows, we shall first assume that the functions U, V belong to class C_2, i. e., have, in the interval J, continuous derivatives of the 2nd order and, furthermore, that their Wronskian $W = UV' - U'V$ is always different from 0. Upon these conditions there exists a dif. linear equation of the 2nd order with continuous coefficients , B.

$$\text{(A)} \qquad \ddot{Y} + A\dot{Y} + BY = 0$$

characterized by the property that the functions U, V are its integrals and that these integrals are independent of each other.

Let us now study the differential centroaffine properties of the curve C, i. e. the properties independent with regard to the centroaffine transformations as well as to the transformations of the parameter. The dif. equation (A) evidently represents a definition-equation of the curve C in the sense that the integral curves of the dif. equation (a) define the curve C up to the centroaffine transformations. As to the transformations of the parameter, we can choose the parameter of the curve C so as to transform the dif. equation (A) into Jacobi's form (q) $y'' = q(t)y$ with a continuous coefficient q in some interval $j = (a,b)$ (bounded or unbounded). We see that, for a convenient choice of the parameter $t \in j$, the curve C may be defined, up to the centroaffine transformations, by the lin dif. equation of the form (q).

Let us, next, assume the curve C to be regular, i. e. locally convex and without points of inflection. This assumption can be expressed by the fact that the carrier q of the corresponding dif. equation (q) is always different from 0: $q(t) \neq 0$ for $t \in j$.

On the whole, we then suppose that the definition-equation of the curve C is of the form (q) whilst the function q is always different from 0.

Well, first there is the question of expressing the fundamental centroaffine invariants of the curve C, namely the centro affine oriented arc $s(t|t_0)$ and the centroaffine curvature $k(t$ of the curve C by means of the function q. These invariants are given by the following formulas:

$$(1) \quad s(t|t_0) = \operatorname{sgn} w \cdot \int_{t_0}^{t} \sqrt{|q(\sigma)|}\, d\sigma ,$$

$$(2) \quad k(t) = \operatorname{sgn} w \cdot \left(\frac{1}{\sqrt{|q(t)|}}\right)' , \quad (t, t_0 \in j) ,$$

where w stands for the Wronskian formed by the coordinates $u(t)$, $v(t)$ of the curve C.

Note that if the function q is periodic with period p, then, according to (2), for $t, t+p \in j$:

$$\int_t^{t+p} k(\sigma)\,d\sigma = 0 .$$

9. Geometric significance of the central dispersions.

Let us now consider the geometric significance of the centra dispersions. To that purpose we shall denote by $P(t)$, $t \in j$, the point of the curve C given by the parameter t and, furthe by $\tau(t)$ the tangent of the curve C at the point $P(t)$.

Then the following theorems apply:

1° Two points $P(t_1)$, $P(t_2)$ of the curve C, determined by the values t_1, t_2 of the parameter t, different from each other, lie in the same straight line passing through the origin O, if and only if t_1, t_2 are zeros of the same integral of the dif. equation (q).

2° Two tangents $\tau(t_1)$, $\tau(t_2)$ of the curve C, determined by the values t_1, t_2 of the parameter t, different from each other, are parallel, if and only if t_1, t_2 are zeros of the derivative of the same integral of the dif. equation (q).

3° The tangent $\tau(t_2)$ and the straight line $OP(t_1)$ are parallel, if and only if there exists an integral v of the dif. equation (q) vanishing at t_2 and whose derivative vanishes at t_1: $v'(t_1) = v(t_2) = 0$.

From this as well as from the definition of central dispersions we realize the geometric significance of central dispersion

For every value $t \in j$ and every index $\nu = \pm 1, \pm 2, \ldots$ there holds: The points $P(t)$, $P[\varphi_\nu(t)]$ lie in the same strai line passing through the origin O;
the tangents $\tau(t)$, $\tau[\psi_\nu(t)]$ are parallel;
the straight line $OP(t)$ and the tangent $\tau[\chi_\nu(t)]$ are paralle
the tangent $\tau(t)$ and the straight line $OP[\omega_\nu(t)]$ are paralle

In all these cases the mentioned values of the parameter are not only a sufficient but also a necessary condition of the described position of the points and tangents of the curve C.

The theory of central dispersions has been successfully applied in the study of a certain class of plane-curves, the so-called curves (F). By a curv (F) we understand a plane-curve which has with regard to a given pencil of straight lines through a point O, F, the following position: Every line p of the pencil F intersects the curve, at least, at two differen points, and in such a way that the tangents of the curve are, at all points of intersection with the straight line p, parallel with each other.

Examples of curves (F) are the ellipses as well as the logarithmic spirals.

Well, considering the geometric significance of central dispersions I have spoken about, we easily realize that the curves (F) are defined by linear dif. equations (q) characterized by the fact that their basic central dispersions of the 1st and 2nd kind coincide: $\varphi_1(t) = \psi_1(t)$.

Dif. equations (q) with this property of the basic central dispersions have been thoroughly studied. Every dif. equation (q) of this kind has, in particular, the property that its basic central dispersion of the 1st (and thus even of the 2nd) kind $\varphi_1(t)$ ($= \psi_1(t)$) is linear of the form $\varphi_1(t) = ct + k$ (c and k are constants, $c > 0$). These dif. equations (q) also include the dif. equations $(\overline{q})$ whose basic central dispersions of the 3rd and the 4th kind coincide: $\chi_1(t) = \omega_1(t)$. The curves defined by the dif. equations $(\overline{q})$ are the so-called J. Radon curves, often dealt with in the literature. They are characterized by the property that not only are the tangents of the curve at the points of intersection parallel with each other but, at the same time, the straight line p' of the pencil F, parallel with these tangents, intersects the curve at the points at which the tangents are parallel with the straight line p.

For lack of time I cannot give a more detailed description of the results concerning the curves (F). Let me just point out that one has found, in particular, the finite equation in polar coordinates of all curves (F):

$$r = C^{\alpha} \cdot F(\alpha) \quad ;$$

C (> 0) is a constant and F ($\in C_2$) a positive periodic functi[on] with period π, which satisfies a certain differential inequali[ty] expressing that the corresponding curve (F) is regular.

10. Centrosymmetric curves.

A further section of the theory of transformations including geometric elements concerns the dif. equations (q) in the interval $j = (-\infty, \infty)$, which are oscillatory and such that the distance between every two zeros od their integrals is constant and always the same. It is therefore a question of oscillatory dif. equations (q) which have the basic central dispersion of the 1st kind of the form $\varphi_1(t) = t + k$, where $k > 0$ is a constant. We shall briefly write φ instead of φ_1. These dif. equations (q) have been studied by many authors.

First, let us note that the problem of determining all the dif. equations (q) defined in the interval $j = (-\infty, \infty)$ and having the mentioned basic central dispersion $\varphi(t)$ is included in the following, more general problem: Determine all the oscillatory dif. equations (q) in a certain interval J, which have a given basic central dispersion $\varphi(t)$. It can be shown that, for every choice of the function $\varphi(t)$ (having certain properties belonging to every basic central dispersion of the 1st kind) there exist dif. equations (q) whose basic central dispersion is $\varphi(t)$ and that the set of all such dif. equations (q) has, for every choice of the function $\varphi(t)$, the same potency, namely that of the continuum, $\aleph$. We may even show that all the dif. equations (q) having a given basic central dispersion $\varphi(t)$ are determined by the formula:

$$(1) \qquad q(t) = q_0(t) + \frac{p''(t)}{p(t)} + 2\,\frac{y_0'(t)}{p(t)} \cdot \frac{p'(t)}{y_0(t)} ,$$

where $q_0(t)$ is the carrier of an arbitrary dif. equation with the basic central dispersion $\varphi(t)$, $y_0(t)$ is an arbitrary integral ot the dif. equation (q_0) and $p(t)$ an arbitrary function with certain properties which I shall, for lack of time, not introduce.

Well, let us now return to the special case $\varphi(t) = t + k$ and choose, for example, $k = \pi$. In this case the formula (1) yields, for the notation $p(t) = p(0) \cdot \exp f(t)$,

(2) $\bar{q}(t) = -1 + f''(t) + f'^2(t) + 2 . f'(t) . \operatorname{cotg} t$,

where f is an arbitrary function in the interval $j = (-\infty, \infty$ with the following properties:

$f(t + \pi) = f(t)$ for $t \in j$; $f \in C_2$; $f(0) = f'(0) = 0$;

$$\int_0^{\pi} \frac{\exp[-2f(\sigma)] - 1}{\sin^2 \sigma} \, d\sigma = 0 .$$

Let us now consider, for a while, the oscillatory dif. equations (q) in the interval $j = (-\infty, \infty)$, with the property $\varphi(t) = t + \pi$. The carriers q of these dif. equations are therefore given by the formula (2). Let us denote by the symbol Q_{π} the set of all the carriers that are always ≤ 0. From the formula (2) Mr. F. Neumann has recently deduced an elegant result which may be expressed by the formula

(3) $$\max_{q \in Q_{\pi}} \int_0^{\pi} |q(\sigma)|^p \, d\sigma = \pi ,$$

where p is an arbitrary number satisfying the inequality $0 < p \leq 1$ and the maximum is reached exactly for the function $q(t) = -1$.

Well, what are the geometric elements occurring in connectio with the theory in question ?

Let C be an arbitrary regular curve defined by the parametric coordinates $U(T)$, $V(T)$ which are the integrals of a certain dif. oscillatory equation (Q) in the interval $j = (-\infty$

Suppose the curve C is symmetric with regard to the origin O of the considered coordinate system. This symmetry of the curve C is expressed by the fact that the values of the amplitude $R(T) = \sqrt{U^2(T) + V^2(T)}$ for every two values of the parameter, T, $\Phi(T)$, are the same:

$$(4) \qquad R\left[\Phi(T)\right] = R(T) \quad ;$$

Φ evidently stands for the basic central dispersion of the 1st kind of the dif. equation (Q). Mention now that in the analytic apparatus of the theory of transformations the following formulas are occurring

$$A\left[\Phi(T)\right] = A(T) + \pi \cdot \operatorname{sgn} \dot{A}(T) \; ; \qquad R(T) = \sqrt{\frac{-W}{\dot{A}(T)}} \quad ,$$

where A stands for an arbitrary (first) phase of the dif. equation (Q) and W for the Wronskian of the basis U, V.

These formulas, together with (4) yield $\dot{\Phi}(T) = 1$, so that we have $\Phi(T) = T + K$ $(K > 0)$. Vice versa, it can be shown that this linear form of the basic central dispersion Φ is sufficient for the curve C to be symmetric with regard to the origin O.

Let us now transform the parameter of the curve C by substituting $T = \frac{K}{\pi} t$. This transformation keeps Jacobi's form of the dif. equation (Q) and we can show that the basic central dispersion $\varphi(t)$ of the newly formed dif. equation (q) is $\varphi(t) = t + \pi$.

Thus we arrive, first of all, at the result that every regular plane-curve C which is symmetric with regard to the origin O of the coordinate system may be defined by the dif. equat (q) with a carrier of the form (2), f denoting a convenient function.

The curve C is evidently closed and its centroaffine length s is given, according to (1) , by the formula

$$S = \int_0^{2\pi} \sqrt{|q(\sigma)|}\, d\sigma .$$

Employing the formula (3) for $p = \frac{1}{2}$, we get the inequality

$$S \leq 2\pi ,$$

the equality being reached just for the ellipse $u(t) = \sin t$, $v(t) = \cos t$.

Thus we have arrived at the following result:

The centroaffine length of every regular centro-symmetric plane-curve defined by an oscillatory dif. equation (q) in the interval $j - (-\infty, \infty)$ is always at most equal to 2π , the equality occuring only in the case of $q(t) = -1$.

Now let me close my lecture with a few remarks.

Recently a number of further geometric applications of the theory of transformations have been arrived at by H. Guggenheimer (Brooklyn) and will be published in an article which will appear in our periodical Archivum Mathematicum in Brno. I believe the theory of transformations yields, for the differential global geometry of plane-curves, many further possibilities, for example in Minkowski's geometry. Its value in this respect consists, in m opinion, in the fact that it has been systematically developed and forms a complete theory rich in notions and methods, so that it presents a wide theoretical basis with a powerful analytic app ratus for solving various problems of dif. geometry. If you are interested in more detailed information about the theory of trans formation you will find it in my book "Lineare Differentialtrans- formationen 2. Ordnung", published in Berlin (DDR) in 1967. At present it is being translated into English and will be published the University Press.