Jarník's Notes of the Lecture Course Allgemeine Idealtheorie by B. L. van der Waerden (Göttingen 1927/1928)

Mathematical and Historical Comments

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MATHEMATICAL AND HISTORICAL COMMENTS

Van der Waerden: Allgemeine Idealtheorie – Contents

The text of van der Waerden's lecture course *Allgemeine Idealtheorie* recorded by Jarník in his two notebooks consists of five parts (chapters). The first part does not carry a Roman numeral and has no title. Taking into account the numbering of sections in Chapters III to V (Sections 18 to 31), it is almost certain that the sections in the first chapter should have carried numbers 1 to 8, and the sections in the second chapter numbers 9 to 17. The first notebook ends with Section 19, and the second begins with Section 20. The contents of the whole lecture course are as follows:[1]

[1] Just as a matter of interest, we have included the dates of the individual lectures, which however ended on December 12. The lectures took place twice a week, on Friday and Monday. The course began on Friday, November 4, 1927.

References

Basic information on the literature related to the subject matter is collected in the end the first section and in the beginning of the third section in Chapter I, as well as in the beginnings of Chapters III and V. We have found complete bibliographic references and provide them in the footnotes at the appropriate places of van der Waerden's lectures.

Comments on van der Waerden's lectures

Chapter I.

The content of the first section (*Einleitung*) was delivered in the first lecture on November 4, 1927. The goal was to provide a motivation for the whole lecture course. Among other things, van der Waerden included an example of a "number structure",[2] which does not admit unique factorization into products of irreducible elements:

$$9 = 3 \cdot 3 = \left(2 - \mathrm{i}\sqrt{5}\right) \cdot \left(2 + \mathrm{i}\sqrt{5}\right).$$

A number of concepts that will appear in the subsequent lectures were mentioned – *Ideal*, *Hauptideal*, *Primideal*, *Zahlkörper*, *Körper*, *Erweiterungskörper*, *Norm*, *Polynombereich*, *Modulsystem*, *Basiselemente*, *Ring*, *Teilbarkeit*, *Zerlegbarkeit*, *Unzerlegbarkeit*, *hyperkomplexe Zahlen*, etc.

A key concept of the whole course is the notion of an ideal. An example of a principal ideal, which is the product of two irreducible ideals in the previously mentioned number structure, was given:

$$(3) = \left(3,\ 2 - \mathrm{i}\sqrt{5}\right)\left(3,\ 2 + \mathrm{i}\sqrt{5}\right).$$

The names of mathematicians who significantly contributed to this part of algebra were also mentioned during the introductory lecture: Kummer, Dedekind, Kronecker, Hilbert, König, Max Noether, Lasker, Frobenius, and Emmy Noether.

The second section (*Gruppen*) presents basic facts in group theory. The notion of a group is introduced in an axiomatic way, the defining property being (besides associativity) the solvability of the equations $ax = b$, $ya = b$. The generalized associative law is proved, and followed by the generalized commutative law in the case of a commutative operation. It is shown that the existence of an identity element and the existence of inverse elements follow from the previously mentioned definition of a group. An alternative axiomatic definition of a group is also provided – besides associativity, one requires only the existence of a left identity and left inverses.

Briefly mentioned are abelian groups, the notion of a subgroup, and the structure of cyclic (both finite and infinite) groups. The notion of an isomorphism is introduced, and isomorphic groups are said to be *vom selben Typus*. In several other parts of the course, the same term refers to other isomorphic structures. However, the phrase never appears in the monograph MA.

[2] The example refers to the integral domain $\mathbb{Z}[\mathrm{i}\sqrt{5}]$ obtained from the domain of all integers $\mathbb{Z}$ by adjoining the number $\mathrm{i}\sqrt{5}$. Van der Waerden used the symbol $\sqrt{-5}$.

Van der Waerden restricted the material of this section to what was absolutely necessary for the subsequents parts dealing with commutative rings, ideals, fields, etc. Hence, for example, the notion of a normal subgroup did not appear here.

The third section (*Ringe*) gives an axiomatic definition of rings, demonstrates their basic properties, and discusses special cases of rings: a ring with identity (*Ring mit Einheitselement*), a ring without zero divisors (*Ring ohne Nullteiler*), a noncommutative division ring (skew field, sfield), and a field (*Körper*). A commutative ring without zero divisors is called an integral domain (*Integritätsbereich*), and in general it need not have an identity (the same terminology is used in MA-I, p. 39). Nowadays, it is usually assumed that an integral domain possesses an identity.

In his lectures, van der Waerden employed the term *Einheit* for the identity element of a group, *Einheitselement* for the identity element of a ring, and again *Einheit* for a unit of a ring.[3] He fixed the terminology in the monograph MA: the term *Einselement* was used for the identity elements of both groups and rings (MA-I, pp. 15, 40), while *Einheit* referred to a unit element of a ring (MA-I, p. 63). In MA-I, p. 63, he stated in a footnote: *Das Wort „Einheit“ wird oft als Synonym für „Einselement“ gebraucht. In Untersuchungen über Faktorzerlegung aber sind die beiden Begriffe strong zu trennen, da z. B.* -1 *auch eine Einheit ist.*

The definition of a ring is not quite consistent, since it requires the commutativity of multiplication, as well as both distributivity laws; it is remarked that in the commutative case, one of them is superfluous. The subsequent exposition is restricted to commutative rings and fields. The following important theorem is proved: A finite commutative ring without zero divisors that contains at least one nonzero element is a field.

In this section, van der Waerden used a relatively vague term *Bereich* without providing an exact definition; what he had in mind was a set of certain elements equipped with the operations of addition and multiplication.[4] In place of the term homomorphism (or epimorphism), he employed in his lectures the term *Meromorphismus*.[5] He demonstrated that (in modern terminology) the image of a ring under a homomorphism is another ring. He also introduced the notion of an isomorphism in the usual sense.

The next three sections deal with constructions of rings.

The fourth section (*Quotientenkörper. Ringbildung I.*) describes the classical construction of a fraction field (quotient field, field of quotients) for rings having no zero divisors and at least two elements. It is first explained in the case when the original ring is contained in a certain field, and later in the general case

[3] In Section 31, the term *Einheit* refers to the identity element of a ring.

[4] Van der Waerden used the term *Bereich* also in the beginning of Section 3 of the second chapter. The term does not appear in the subject indices of MA-I and MA-II; it was replaced by another term *System mit doppelter Komposition* having the same sense (see MA-I, p. 37).

[5] The term *Meromorphismus* does not appear in the subject indices of MA-I and MA-II.

without this assumption. In our opinion, this approach to the topic of fraction fields is quite convenient from a pedagogical view. It is also shown that each ring without zero divisors has a unique fraction field (up to isomorphism). A final remark points out the possibility of constructing (in place of a fraction field) a ring of fractions (quotient ring) in the case when the original ring has zero divisors.

The fifth section (*Ringbildung II. Polynomring*) introduces, in the classical way, the ring of polynomials of a single indeterminate, and finitely or infinitely many indeterminates with coefficients from a ring $\mathcal{R}$. The exposition proceeds by showing how the properties of $\mathcal{R}$ translate into the properties $\mathcal{R}[x]$ (existence of an identity, nonexistence of zero divisors), defining the degree of a polynomial, and mentioning substitution of values into a polynomial.

The sixth section (*Ringbildung III. Restklassenring*) describes a one-to-one correspondence between congruence relations on a ring $\mathcal{R}$ and ideals in $\mathcal{R}$, as well as their relation to homomorphic images of $\mathcal{R}$. It introduces the notion of a quotient ring with respect to an ideal or a congruence relation, i.e., the ring of congruence classes (residue class ring). The topic is in fact related to the so-called Fundamental Theorem of Ring Homomorphisms, but this term never appeared throughout the lectures.[6] The term *Meromorphismus* is still being used here.

Recall that the rings under consideration need not have an identity. The next notions to be introduced are the ideal generated by a subset, a basis of an ideal, and a principal ideal. It is noted how the ideals look like in a field, in the integral domain of integers, and what are the corresponding quotient rings. It is shown that each ideal in the integral domain of integers is a principal one.

The language of divisibility and the corresponding notation, as well as the language and notation of congruences, are introduced and used.[7]

The intersection and sum of two ideals (least common multiple, greatest common divisor) are introduced, as well as the notion of a prime ideal (the corresponding quotient ring has no zero divisors).[8] It is shown that maximal ideals in a ring with identity are prime ideals (the corresponding quotient rings are fields).

Section 6a (*Weiteres über Polynomringe*) recalls the division algorithm for polynomials over a field, or over a ring without zero divisors, and the notion of a root of a polynomial. It is shown that a polynomial of degree n (over a field or a ring without zero divisors) has at most n roots, and proved that in the integral domain of polynomials over a field, each ideal is a principal one. The field of rational functions $K(x_1, x_2, \ldots, x_n)$ over a field K is defined as the fraction field of the integral domain $K[x_1, x_2, \ldots, x_n]$.

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[6] See MA-I, p. 57, where the term appears: *Homomorphiesatz für Ringe*.

[7] If an ideal $\mathfrak{a}$ is contained in an ideal $\mathfrak{b}$, then $\mathfrak{a} \equiv 0$ $(\mathfrak{b})$; we say that the ideal $\mathfrak{b}$ is a divisor of the ideal $\mathfrak{a}$, and the ideal $\mathfrak{a}$ is a multiple of the ideal $\mathfrak{b}$.

[8] In the language of congruences, an ideal $\mathfrak{g}$ is a prime ideal if the following holds: $ab \equiv 0$ $(\mathfrak{g})$, $a \not\equiv 0$ $(\mathfrak{g})$ imply $b \equiv 0$ $(\mathfrak{g})$.

The next part of this section returns to the end of the previous section: Once again, it is proved (using the one-to-one correspondence of ideals) that maximal ideals in a ring with identity are prime ideals.

In the seventh section (*Idealtheorie der Euklidischen Ringe*), van der Waerden coined the term Euclidean ring for a principal ideal domain (with identity).[9] All examples that he mentioned (integers, polynomials over a field, Gaussian integers, and three integral domains obtained from the domain of all integers by adjoining a primitive third root of unity, or $\mathrm{i}\sqrt{2}$, or $\sqrt{2}$, respectively) are however Euclidean rings in the present sense (integral domains with a Euclidean norm function and division algorithm).

It was already demonstrated for the integers and polynomials over a field (using the division algorithm) that they are principal ideal domains (see Sections 6 and 6a). For the remaining four of the above-mentioned examples, the validity of the division algorithm is confirmed in Section 7. As a consequence, each ideal is a principal one.

Furthermore, it is shown that in a principal ideal domain, every two elements possess the greatest common divisor, which is their linear combination.

The notion of a unit (*Einheit*) is defined as an invertible element, but the relation of associatedness is omitted. Basic notions of divisibility (now in the language of elements instead of ideals) – divisor and proper divisor – are introduced. An irreducible element is, somewhat unfortunately, referred to as a *Primzahl*.

It is demonstrated that in a principal ideal domain, an irreducible element p generates a prime ideal $\mathfrak{p} = (p)$, and the corresponding quotient ring is a field. At this point, van der Waerden recalled Cauchy's construction of the complex numbers.

Important results of the seventh section are the following theorems (in modern terminology):

A. In a principal ideal domain, each nonzero element has a unique factorization (up to associatedness) into a product of irreducible elements.[10]

B. If $\mathcal{S}$ is a ring with identity and without zero divisors such that the theorem on the existence and uniqueness of factorization into irreducible elements holds, then it holds for $\mathcal{S}[x]$ as well.

   If Σ is the fraction field of this ring $\mathcal{S}$, then a polynomial f in $\mathcal{S}[x]$ is irreducible if and only if it is irreducible in $\Sigma[x]$.

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9 See the definition in the beginning of Section 7: *Ein Euklidischer Ring ist ein Ring ohne Nullteiler und mit dem Einheitselement, wo jedes Ideal Hauptideal ist.* In MA-I, p. 60, he already used the term *Hauptidealring*: *Ein Integritätsbereich mit Einselement, in dem jedes Ideal Hauptideal ist, heißt ein Hauptidealring.*

10 For the integral domain of all integers, this result is referred to as the Fundamental Theorem of Arithmetic.

C. If Σ is a field, then the theorem on the existence and uniqueness of factorization into irreducible elements holds in the integral domains $\Sigma[x_1, \ldots, x_n]$.

Nowadays we know that each Euclidean ring is a principal ideal domain[11] and that each principal ideal domain is an integral domain for which the theorem on the existence and uniqueness of factorization into irreducible elements holds. However, there exist integral domains where the theorem no longer holds. Such an example was provided by van der Waerden already in the introductory motivational section: $\mathbb{Z}[\mathrm{i}\sqrt{5}]$. See also MA-I, p. 66, which mentions the integral domain $\mathbb{Z}[\mathrm{i}\sqrt{3}]$. It is possible to give additional examples of this type ($\mathbb{Z}[2\mathrm{i}]$, $\mathbb{Z}[\mathrm{i}\sqrt{7}]$, etc.).

Van der Waerden provided six examples of Euclidean rings, but no example of a principal ideal domain which is not a Euclidean ring.[12] Neither did he give an example of an integral domain which is not a principal ideal domain, but in which the theorem on the existence and uniqueness of factorization into irreducible elements holds. He could have easily done this. According to the above-mentioned Theorem B, the theorem on the existence and uniqueness of the factorization holds for the integral domain $\mathbb{Z}[x]$ (polynomials over the integral domain $\mathbb{Z}$). On the other hand, the integral domain $\mathbb{Z}[x]$ is not a principal ideal domain, since, for example, the ideal $(2, x)$ is not generated by a single element. Another example is the integral domain $K[x, y]$, where K is a field.

We remark that in the second edition of MA-I published in 1937, van der Waerden already made distinction between Euclidean rings and principal ideal domains (see MA-I, 1937, p. 60), and showed that each Euclidean ring is a principal ideal domain.

Neither in his lectures nor in the 1930 monograph MA-I did van der Waerden distinguish between irreducible elements and prime elements. Let us recall that every prime element is irreducible, but the converse is not true in general. Both concepts coincide in an integral domain where the theorem on the existence and uniqueness of factorization into irreducible elements holds.[13]

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[11] Neither in his lectures nor in his 1930 monograph MA-I did van der Waerden consider integral domains that are now known as the Euclidean domains.

[12] For example, $\mathbb{Z}\left[\frac{1}{2}\left(1 + \mathrm{i}\sqrt{19}\right)\right]$.

[13] MA-I, p. 63, introduces *unzerlegbares Element* or *Primelement*; the corresponding term in the integral domain of all integers is *Primzahl*, and *irreduzibles Polynom* in the integral domain of polynomials. This passage remained unchanged in the second edition of MA-I, as well as in the third edition from 1950.
A prime element is an element p having the following property: if p divides a product ab, then it necessarily divides a or b. In the example from Section 1 of the first chapter, the elements 3, $2 \pm \mathrm{i}\sqrt{5}$ are irreducible, but they are not prime elements.
We remark that the distinction between prime elements and irreducible elements appears e.g. in Jacobson's textbook *Lectures in Abstract Algebra I – Basic Concepts*, D. Van Nostrand Company, Toronto, New York, London, 1951, xii + 217 pages, see pp. 115–116.

Chapter II. *Körpertheorie*

The first section (*Primkörper*) introduces the notion of a prime field. It is shown that every field contains a unique prime field, and that a prime field is isomorphic either to the field of rational numbers, or to one of the fields of residue classes of integers modulo a prime number (quotient ring $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$). The characteristic of a field is also introduced.

The second section (*Einfache Körpererweiterungen*) introduces the ring and field adjunction of a set of elements. It continues by studying the adjunction of a single element (simple extension), defining algebraic and transcendental elements, describing the results of adjoining such elements, and introducing the degree of an algebraic element. The style of exposition is similar to the passage on fraction fields – adjunctions are first considered in a previously given field extension, and later without this assumption. The proof of the uniqueness (up to isomorphism) of a simple algebraic or transcendental extension is followed by an explanation why two roots of an irreducible polynomial give rise to isomorphic (so-called equivalent or conjugate) extensions.

In this context, van der Waerden once again recalled Cauchy's construction of the field of complex numbers.

The third section (*Lineare Abhängigkeit in bezug auf einen Körper*) presents a system of axioms of a vector space over a field, without explicitly mentioning the term vector space (van der Waerden again used the somewhat vague term *Bereich*). The notions of linear dependence and independence are introduced, as well as linear equivalence of two sets (one set generates the other and vice versa). The Steinitz exchange lemma (*Austauschsatz* – with a reference to Steinitz) is proved, and is followed by its corollaries – the notions of a basis (*unabhängige lineare Basis*) and dimension (*Rang*) are introduced.

In MA-I, p. 95, van der Waerden considered a ring containing a field with a common identity element; therefore he needed no axioms; he defined linear dependence and independence of elements of this ring over the given field. However, in his lectures he provided an axiomatic definition of a vector space over a field; thus, it was necessary to add the axiom of a zero element o, e.g. in the form $0 \cdot u = o$ for each u.

In the second edition of MA-I published in 1937, van der Waerden modified the original Section 28 *Lineare Abhängigkeit von Größen in bezug auf einen Körper* (MA-I, 1930, pp. 95–99), which resulted in Section 33 *Lineare Abhängigkeit von Größen über einem Schiefkörper* (MA-I, 1937, pp. 104–109). Here he presented a system of axioms of a module over a (not necessarily commutative) field. He proceeded in a similar way in Section 14, *Vektorräume und hyperkomplexe Systeme* (MA-I, 1937, pp. 46–49), which is missing in the first edition; he provided an axiomatic definition of an *n-gliedriger Linearformenmodul*, or *n-dimensionaler Vektorraum.*

In the fourth section (*Endliche und algebraische Körpererweiterungen*), the results of the previous section are applied to field extensions. Distinction is made between finite and infinite extensions, the degree of a finite extension (*Körpergrad*) is introduced, and the multiplicativity formula for degrees (i.e., transitivity of finite extensions) is proved. Algebraic and transcendental extensions are defined, it is proved that every simple algebraic extension is finite, and every finite extension is algebraic. Another important result of this section is the transitivity of algebraic extensions.

The fifth section (*Galoissche Erweiterungen*) presents the construction of a splitting field of a polynomial (*Zerfällungskörper*, *Wurzelkörper*) and proves that it is unique up to isomorphism. The derivative of a polynomial (*Ableitung*) is introduced, and it is shown that a polynomial has a multiple root if and only if the greatest common divisor of the polynomial and its derivative is nontrivial. Irreducible polynomials are divided into separable and inseparable ones, depending on whether they have only simple roots in their splitting fields. It is demonstrated that irreducible polynomials over a field of characteristic zero are always separable, and a characterization is provided for inseparable polynomials over a field of characteristic p.

A Galois (normal) extension of a field Σ is defined as an extension such that every irreducible polynomial in $\Sigma[x]$ with at least one root in the extension splits in the extension.[14] It is shown that a splitting field of a polynomial in $\Sigma[x]$ is a Galois extension of the field Σ. The notion of a separable element over Σ is introduced (a root of a separable polynomial in $\Sigma[x]$). It is shown that if $\sigma_1, \ldots, \sigma_n$ are algebraic elements over Σ, where each σ_i is separable over $\Sigma(\sigma_1, \ldots, \sigma_{i-1})$ and $\Sigma(\sigma_1, \ldots, \sigma_n)$ is a Galois extension of Σ, then the number of automorphisms of $\Sigma(\sigma_1, \ldots, \sigma_n)$ which leaves the elements of Σ fixed equals the degree of the extension $\Sigma \subset \Sigma(\sigma_1, \ldots, \sigma_n)$. If some of the σ_i are inseparable, then the number of automorphisms is strictly smaller.

Moreover, the field $\Sigma(\sigma_1, \ldots, \sigma_n)$ is separable over Σ, i.e., each of its elements is separable over Σ. It is remarked that an adjunction of finitely many separable elements to Σ gives rise to a separable extension of the field Σ.

The sixth section (*Algebraisch abgeschlossene Körper*) contains the definition of an algebraically closed field. It is shown that if $K \subset \Omega$ is an algebraic extension and every polynomial in $K[x]$ splits into linear factors in $\Omega[x]$, then Ω is algebraically closed. The construction of an algebraically closed extension of a coutable field is presented, and it is shown that the extension is unique up to isomorphism. The end of the section mentions the possibility of a general approach; we remark that in MA-I, p. 199, the theorem is stated without the countability requirement in the following form:

Zu jedem Körper P gibt es einen algebraisch-abgeschlossenen algebraischen Erweiterungskörper Ω. Und zwar ist dieser Körper bis auf äquivalente Erweiterungen eindeutig bestimmt: Je zwei algebraisch-abgeschlossene algebraische Erweiterungen Ω, Ω' von P sind äquivalent.

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[14] In MA-I, p. 103, van der Waerden imposed the additional requirement that the extension has to be an algebraic one.

The proof is based on Zermelo's Well-Ordering Theorem (see MA-I, p. 194), which was published by Ernst Zermelo in 1904.[15]

The seventh section (*Transzendente Erweiterungskörper*) introduces algebraic dependence and independence of elements over a field. By adjoining n algebraically independent elements to a field K, one obtains the so-called purely transcendental extension (*rein transzendente Erweiterung*), which is isomorphic to the field $K(x_1, \ldots, x_n)$. Algebraically equivalent systems are introduced; if they are algebraically independent, they have the same number of elements.

Let $K \subset \Omega$ be an extension and M a subset of Ω. If there exist n algebraically independent elements in M on which all remaining elements of M are algebraically dependent, then we say that the transcendence degree of M over K is n (*Transzendenzgrad*). If a field Ω has transcendence degree of n over K, then it is possible to obtain Ω by adjoining n algebraically independent elements to the field K, followed by adjunction of additional elements which are algebraically dependent on the previous n elements; thus, by performing first a purely transcendental extension, followed by an algebraic extension.

The conclusion of the section mentions the possibility of an infinite transcendental extension by means of Zermelo's Well-Ordering Theorem.

In van der Waerden's lectures, the topic of this short section was mentioned only very briefly. A much more detailed treatment is given in MA-I, pp. 203–208.

We point out that in MA-I, an algebraically independent set is called *irreduzibel*. For the sake of comparison, we include the following more comprehensible formulation from MA-I, p. 205:

Eine Menge $\mathfrak{M}$ heißt irreduzibel (in bezug auf P), wenn kein Element von $\mathfrak{M}$ algebraisch von den übrigen abhängt.

Der Körper $P(\mathfrak{M})$, der durch Adjunktion eines irreduziblen Systems $\mathfrak{M}$ an P entsteht, ist isomorph dem Körper der rationalen Funktionen einer mit $\mathfrak{M}$ gleichmächtigen Menge $\mathfrak{X}$ von Unbestimmten x_i, d. h. dem Quotientenkörper des Polynombereichs $P[\mathfrak{X}]$.

Man nennt jeden Körper $P(\mathfrak{M})$, der durch Adjunktion eines irreduziblen Systems $\mathfrak{M}$ an P entsteht, eine rein transzendente Erweiterung von P.

The eighth section (*Algebraische Funktionen*) introduces algebraic functions of indeterminates (variables) $x_1, \ldots, x_n$ as the elements of an algebraic extension of the field $K(x_1, \ldots, x_n)$, where K is an infinite field. Let $y_1, \ldots, y_m$ be such algebraic functions, and let each y_i be a root of an irreducible polynomial $h_i(z)$ with coefficients from $K(x_1, \ldots, x_n, y_1, \ldots, y_{i-1})$. Then a regular system (*reguläres Argumentwertsystem*) consists of values $x'_1, \ldots, x'_n$ from K or its

[15] E. Zermelo: *Beweis, daß jede Menge wohlgeordnet werden kann*, Mathematische Annalen 59(1904), pp. 514–516. See also E. Zermelo: *Neuer Beweis für die Möglichkeit einer Wohlordnung*, ibid. 65(1908), pp. 107–128.

algebraic extension for which the denominators of the coefficients of the polynomials $h_1(z), \ldots, h_m(z)$ are nonzero. These values gives rise to the so-called corresponding system of values $y'_1, \ldots, y'_m$ (*zugehöriges Funktionswertsystem*), where each y'_i is a root of the polynomial $h_i(z)$, whose coefficients are evaluated at $x'_1, \ldots, x'_n, y'_1, \ldots, y'_{i-1}$.[16]

It is demonstrated that for a polynomial $f \in K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, the following two statements are equivalent:

(1) For each regular system $x'_1, \ldots, x'_n$ and its corresponding system of values $y'_1, \ldots, y'_m$, we have $f(x'_1, \ldots, x'_n, y'_1, \ldots, y'_m) = 0$.

(2) $f(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$.

The ninth section (*Erweiterungen erster und zweiter Art*) begins by recalling some basic facts on the derivative of a polynomial, existence of multiple roots, and the notions of a separable or inseparable polynomial (or its root). All these facts were already presented in the fifth section of the second chapter.

The fifth section of the second chapter was also the starting point for the next theorem: A finite extension $K_r = K(a_1, \ldots, a_r)$ of a field K has in a suitable extension as many isomorphisms which map it into conjugate fields as is the degree of the extension $K \subset K_r$, provided that each a_i is separable over K_{i-1}. In the opposite case, their number is strictly smaller.

Chapter III. *Idealtheorie in Polynombereichen*

In Section 18 (*Der Hilbertsche Basissatz*), it is proved that if each ideal in a ring $\mathcal{R}$ with identity has a finite basis (*in $\mathcal{R}$ gilt der Basissatz*), then the same property is inherited by the ring $\mathcal{R}[x]$, and therefore also by $K[x_1, x_2, \ldots, x_n]$, where K is a field, as well as by $E[x_1, x_2, \ldots, x_n]$, where E is a principal ideal domain.

We remark that commutative rings with identity in which each ideal has a finite basis are sometimes called *Noetherian rings*, or *Noetherian integral domains* if they have no zero divisors.[17] Then Hilbert's theorem may be expressed as follows: If $\mathcal{R}$ is a Noetherian ring, then $\mathcal{R}[x]$ is a Noetherian ring as well.

In Section 19 (*Algebraische Mannigfaltigkeiten*), an algebraic variety (manifold) is defined to be the set of all common roots of a system of polynomials, or of all polynomials belonging to an ideal in the integral domain $K[x_1, x_2, \ldots, x_n]$ (*Nullstelle eines Ideals*); it is a subset of the Cartesian product K^n. If we assign to the given variety all its annihilating polynomials, we get the so-called belonging ideal (*zugehörige Ideal*).[18]

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[16] See MA-II, Engl., 1950, p. 50: ... *allowable system of argument values*, ... *system of function values belonging to these arguments.*

[17] See e.g. the 5th edition of van der Waerden's monograph (MA-II, 1967, p. 120).

[18] If all elements of a variety M are roots of a polynomial f, or of all polynomials in an ideal $\mathfrak{p}$, then it is said that f contains M, or $\mathfrak{p}$ contains (*enthält*) M.

To an intersection of ideals corresponds the union of the appropriate varieties, while a sum of ideals corresponds to the intersection. A variety is called indecomposable or irreducible (*irreduzibel*) if it cannot be obtained as the union of two smaller varieties. It is proved that a variety is irreducible if and only if the belonging ideal is a prime ideal.

It is also possible to study varieties in ${K'}^n$, where K' is an extension of the original field K.

We remark that in MA-II, pp. 51–54, van der Waerden introduced varieties in an algebraic extension of the field K; in his lectures, he mentioned this possibility only in the next-to-last paragraph of Section 19.

Section 20 (*Nullstellentheorie der Primideale*) begins by proving the following theorem. If K is a field and $K(\xi_1, \dots, \xi_n)$ its extension, the system of all polynomials $f \in K[x_1, \dots, x_n]$ satisfying $f(\xi_1, \dots, \xi_n) = 0$ is a prime ideal $\mathfrak{p}$, the quotient ring $K[x_1, \dots, x_n]/\mathfrak{p}$ is isomorphic to the integral domain $K[\xi_1, \dots, \xi_n]$, and thus the corresponding fraction fields are isomorphic. For each proper prime ideal $\mathfrak{p} \subset K[x_1, \dots, x_n]$, there exists a field $K(\xi_1, \dots, \xi_n)$ such that the prime ideal $\mathfrak{p}$ consists of all polynomials f satisfying $f(\xi_1, \dots, \xi_n) = 0$.

The field $K(\xi_1, \dots, \xi_n)$ is a so-called root field (*Nullstellenkörper*) of the prime ideal $\mathfrak{p}$, and $(\xi_1, \dots, \xi_n)$ is a so-called generic root (*allgemeine Nullstelle*) of the prime ideal $\mathfrak{p}$.

The elements $\xi_1, \dots, \xi_n$ might be regarded as algebraic functions.[19] For each regular system, the corresponding system of values gives rise to an element of the variety M corresponding to the ideal $\mathfrak{p}$.[20] The algebraic functions $\xi_1, \dots, \xi_n$ provide a so-called parametric representation (*Parameterdarstellung*) of the variety M.

The dimension (*Dimension*) of a variety M, or a prime ideal $\mathfrak{p}$, is the transcendence degree (*Transzendenzgrad*) of the system $\xi_1, \dots, \xi_n$ over K. It is a pity that in his lectures, van der Waerden gave no illuminating example such as in MA-II.[21]

The following theorems are proved:

A. If $\mathfrak{p}' \subseteq \mathfrak{p}$ are prime ideals having dimensions d', d, then $d' \geq d$, with inequality replaced by equality if and only if $\mathfrak{p}' = \mathfrak{p}$.

B. If a prime ideal $\mathfrak{p}'$ has dimension d', then each of its roots has

[19] Section 89 in MA-II begins with this assumption: *Sind* $\xi_1, \dots, \xi_n$ *algebraische Funktionen von* $t_1, \dots, t_r$, ...

[20] A more detailed and poignant description of the one-to-one correspondence between the ideal $\mathfrak{p}$, variety M and generic root $\xi_1, \dots, \xi_n$ is described in MA-II, pp. 58–61, or in the second edition of MA-II from 1940, pp. 52–59.

[21] For example, in $K[x_1, x_2, x_3]$, we have the prime ideal $(x_1x_3 - x_2^2,\ x_2x_3 - x_1^3,\ x_3^2 - x_1^2x_2)$, whose generic root is $(t^3,\ t^4,\ t^5)$, see MA-II, p. 61.

transcendence degree at most d'. If the dimension of $\mathfrak{p}'$ is zero, then each root is algebraic and generic.

C. To an irreducible nonconstant polynomial p corresponds a prime ideal of dimension $(n-1)$.

Section 21 (*Geometrische Deutung beliebiger Ideale*) is based on the previous results.

To each irreducible factor of a polynomial f corresponds an $(n-1)$-dimensional variety or hypersurface (*Hyperfläche*), which is counted as many times as is the multiplicity of the given factor. To an ideal – as a system of polynomials – corresponds a system of hypersurfaces, and to a prime ideal corresponds a system of hypersurfaces containing the appropriate irreducible variety.

There follow nine examples of ideals in $K[x, y, z]$, where K is the field of all complex numbers that are not prime ideals. Basic properties of the corresponding hypersurfaces are given.

Using the so-called primary properties of a polynomial with respect to the corresponding irreducible variety M, the notion of a primary ideal $\mathfrak{q}$ in the integral domain of polynomials is introduced: An ideal $\mathfrak{q}$ is called primary if there exists an irreducible variety M such that

1. if $fg \equiv 0$ $(\mathfrak{q})$ and g does not contain M, then $f \equiv 0$ $(\mathfrak{q})$,
2. if $f \equiv 0$ $(\mathfrak{q})$, then f contains M,
3. if g contains M, then $g^h \equiv 0$ $(\mathfrak{q})$ for a certain natural number h.

The irreducible variety M can be replaced by the belonging prime ideal $\mathfrak{p}$. In this way, we arrive at the definition of a primary ideal $\mathfrak{q}$ that makes sense in a general ring:

1. if $fg \equiv 0$ $(\mathfrak{q})$, $g \not\equiv 0$ $(\mathfrak{p})$, then $f \equiv 0$ $(\mathfrak{q})$,
2. if $f \equiv 0$ $(\mathfrak{q})$, then $f \equiv 0$ $(\mathfrak{p})$,
3. if $g \equiv 0$ $(\mathfrak{p})$, then $g^h \equiv 0$ $(\mathfrak{q})$ for a certain natural number h.

An equivalent formulation is:

1. if $fg \in \mathfrak{q}$, $g \notin \mathfrak{p}$, then $f \in \mathfrak{q}$,
2. $\mathfrak{q} \subseteq \mathfrak{p}$,
3. if $g \in \mathfrak{p}$, then $g^h \in \mathfrak{q}$ for a certain natural number h.

The relation between a primary ideal and the corresponding prime ideal, which complements property (3), is proved next:

A. The prime ideal corresponding to a primary ideal $\mathfrak{q}$ is the set of all elements f such that $f^h \in \mathfrak{q}$ for a certain natural number h.

Chapter IV. *Allgemeine Idealtheorie*

Section 22 (*Basissatz u. Teilerkettensatz*) shows that the following four statements are equivalent:[22]

(1) *Basissatz*: Every ideal in the ring $\mathcal{R}$ has a finite basis.

(2) *Teilerkettensatz*: The ring $\mathcal{R}$ contains no infinite increasing chain of ideals.

(3) *Maximalsatz*: Every nonempty set of ideals in the ring $\mathcal{R}$ has a maximal element.

(4) *Prinzip der Teilerinduktion*: If an ideal $\mathfrak{a}$ in the ring $\mathcal{R}$ has a property E whenever all ideals containing $\mathfrak{a}$ as a proper subideal (including the ideal $\mathcal{R}$) have this property, then every ideal in $\mathcal{R}$ has property E.

Thus, in a Noetherian ring, all increasing chains of ideals are finite.

Furthermore, it is proved that if a ring $\mathcal{R}$ has these properties, then its homomorphic images have them, too.

In this section of his lecture, when proving the equivalence of statements (1) to (4), van der Waerden included two brief references to the axiom of choice. A more detailed explanation can be found in MA-II, p. 26:

Nämlich: Auf Grund des Auswahlpostulats (§ 58)[23] *denke man sich in jeder nichtleeren Untermenge von* $\mathfrak{o}$ *ein Element ausgezeichnet. Es sei nun* $\mathfrak{a}$ *ein Ideal,* a_1 *das ausgezeichnete Element von* $\mathfrak{a}$*. ...*

In the second edition of MA-II from 1940, the author was already more succinct when proving the implication (2) $\Longrightarrow$ (1):

Es sei nämlich $\mathfrak{a}$ *ein Ideal,* a_1 *irgend ein Element von* $\mathfrak{a}$*. ...* (p. 21)

However, he attached the following footnote: *Beim Beweise wird das Auswahlpostulat benutzt. Vgl. dazu O. Teichmüller, Deutsche Mathematik Bd. 4 (1939) S. 567.*

Section 23 (*Der Zerlegungssatz*) begins by introducing the notion of a primary ideal (*Primärideal*) in a general ring, independently of varieties, hypersurfaces, as well as prime ideals (see § 21). An ideal $\mathfrak{q}$ in a ring $\mathcal{R}$ is called primary, if $ab \equiv 0\ (\mathfrak{q})$, $a \not\equiv 0\ (\mathfrak{q})$ implies $b^h \equiv 0\ (\mathfrak{q})$ for a certain natural number h; the set of all elements $c \in \mathcal{R}$ satisfying $c^h \equiv 0\ (\mathfrak{q})$ for a certain natural number h makes up the corresponding prime ideal $\mathfrak{p}$.

The following theorems, which are linked in a natural way, are proved under the assumption that the ring $\mathcal{R}$ satisfies the condition of finiteness of increasing chains of ideals (*Teilerkettensatz*).

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[22] See MA-II, Engl., 1950, pp. 18–22: *Basis Condition*, *Divisor Chain Condition*, *Maximal Condition*, *Principle of Divisor Induction.*

[23] See § 58. *Das Auswahlpostulat und der Wohlordnungssatz*, MA-I, pp. 194–196.

A. An ideal which is not primary is a nontrivial intersection of two ideals.

B. Every ideal is the intersection of finitely many primary ideals.

C. The intersection of finitely many primary ideals, which correspond to the same prime ideal, is a primary ideal corresponding to the same prime ideal.

D. Every ideal is an irredundant intersection (*unverkürzbare Darstellung als Durchschnitt*) of primary ideals (none of them can be omitted) which correspond to mutually distinct prime ideals. Such an expression is not unique in general, as is demonstrated by an example.

Section 24 (*Idealprodukte und -quotienten*) introduces the product $\mathfrak{a}\mathfrak{b}$ of two ideals $\mathfrak{a}$, $\mathfrak{b}$ by means of the relation $\mathfrak{a}\mathfrak{b} = \big\{\Sigma a_i b_i\,;\ a_i \in \mathfrak{a},\ b_i \in \mathfrak{b}\big\}$, describes the properties of this operation, and its relation to previously defined operations (sum and intersection of ideals).[24] It is shown that if $\mathfrak{q}$ is a primary ideal and if the corresponding prime ideal $\mathfrak{p}$ has a finite basis, then $\mathfrak{p}^h \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ for a certain natural number h.

Next, the quotient $\mathfrak{a} : \mathfrak{b}$ of two ideals $\mathfrak{a}$, $\mathfrak{b}$ is defined by means of the relation $\mathfrak{a} : \mathfrak{b} = \big\{c\,;\ c\mathfrak{b} \equiv 0\ (\mathfrak{a})\big\}$, and its properties are studied.[25] Among other things, it is demonstrated that if $\mathfrak{q}$ is a primary ideal and $\mathfrak{a} \not\equiv 0\ (\mathfrak{q})$, then $\mathfrak{q} : \mathfrak{a}$ is a primary ideal corresponding to the same prime ideal.

Finally, a reverse version of Theorem C from Section 23 is presented:

A. If an irredundant intersection of finitely many primary ideals gives a primary ideal, then all these primary ideals correspond to the same prime ideal.

In Section 25 (*Geometrische Anwendungen des Zerlegungssatzes*), the results of Section 23 are applied to ideals in integral domains of polynomials, and carried over to corresponding varieties (see § 19). The following result is stated only very briefly:

A. Every algebraic variety can be represented as an irredundant union of irreducible varieties. Such an expression is unique.

A more detailed and succinct explanation could have been included in the lectures, possibly as follows: If an ideal $\mathfrak{a}$ is expressed as an irredundant intersection of primary ideals $\mathfrak{q}_i$, to which correspond pairwise distinct prime ideals $\mathfrak{p}_i$, then the variety M corresponding to the ideal $\mathfrak{a}$ is the union of irreducible varieties M_i corresponding to primary ideals $\mathfrak{q}_i$. However, the varieties M_i also correspond to appropriate prime ideals $\mathfrak{p}_i$ (hence, they are irreducible).[26] We distinguish between isolated varieties (M_i is not contained

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[24] The product of ideals appeared already in the first motivational section of the first chapter.

[25] Section 31 introduces the notation $\mathfrak{p}^{-1} = \mathcal{R} : \mathfrak{p}$.

[26] This is not explicitly mentioned in the lectures. See MA-II, p. 65: *Also ist die Mannigfaltigkeit eines Primärideals $\mathfrak{q} \neq \mathfrak{o}$ irreduzibel und gleich der Mannigfaltigkeit des zugehörigen Primideals.*

in any other M_j) and imbedded ones (the remaining ones). Imbedded varieties are not considered in the above-mentioned expression of the variety M.

Furthermore, the following results are proved:

B. *Hilbertsche Nullstellensatz*: If a polynomial f vanishes for all roots of an ideal $\mathfrak{a}$, then $f^h \equiv 0\ (\mathfrak{a})$ for a certain h, which depends only on $\mathfrak{a}$.

C. If polynomials $f_1, \dots, f_r$ vanish for all roots of an ideal $\mathfrak{a}$, then every product of h factors f_i is contained in $\mathfrak{a}$.

Section 26 (*Die Eindeutigkeitssätze*) is a direct continuation of Section 23. Van der Waerden included the following results:

A. Every two representations of an ideal $\mathfrak{a}$ as irredundant intersections of primary ideals, i.e., $\mathfrak{a} = [\mathfrak{q}_1, \dots, \mathfrak{q}_r] = [\mathfrak{q}'_1, \dots, \mathfrak{q}'_{r'}]$, where all $\mathfrak{q}_i$ as well as all $\mathfrak{q}'_j$ correspond to pairwise distinct prime ideals, contain the same number of primary ideals, i.e., $r = r'$, to which correspond (up to ordering) the same prime ideals.

An isolated primary ideal in the decomposition of an ideal $\mathfrak{a}$ is an ideal $\mathfrak{q}_i$ whose corresponding prime ideal $\mathfrak{p}_i$ does not contain any other prime ideal $\mathfrak{p}_j$. An isolated component is an ideal $[\mathfrak{q}_1, \dots, \mathfrak{q}_n]$ such that none of the corresponding prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ contains none of the corresponding prime ideals $\mathfrak{p}_{n+1}, \dots, \mathfrak{p}_s$. The following theorem is proved:

B. Every isolated component $[\mathfrak{q}_1, \dots, \mathfrak{q}_n]$ of the ideal $\mathfrak{a}$ is uniquely determined by the corresponding prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

Section 27 (*Theorie der teilerfremden Ideale*) investigates relatively prime ideals $\mathfrak{a}$, $\mathfrak{b}$ in a ring $\mathcal{R}$ with identity, that is, ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b}) = \mathcal{R}$; this means there exists a representation $1 = a + b$, where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. It is proved that:

A. If $(\mathfrak{a}, \mathfrak{b}) = \mathcal{R}$, $(\mathfrak{a}, \mathfrak{c}) = \mathcal{R}$, then $(\mathfrak{a}, \mathfrak{b}\mathfrak{c}) = \mathcal{R}$ and $(\mathfrak{a}, \mathfrak{b} \cap \mathfrak{c}) = \mathcal{R}$.

B. If $(\mathfrak{a}, \mathfrak{b}) = \mathcal{R}$, then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. If $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are pairwise relatively prime ideals, then $[\mathfrak{a}_1, \dots, \mathfrak{a}_r] = \prod a_i$.[27]

C. If $\mathfrak{q}_1$, $\mathfrak{q}_2$ are primary ideals and $\mathfrak{p}_1$, $\mathfrak{p}_2$ the corresponding prime ideals, then $(\mathfrak{p}_1, \mathfrak{p}_2) = \mathcal{R}$ implies $(\mathfrak{q}_1, \mathfrak{q}_2) = \mathcal{R}$.

D. If $(\mathfrak{a}, \mathfrak{b}) = \mathcal{R}$, then the system of congruences $\xi \equiv \alpha\ (\mathfrak{a})$, $\xi \equiv \beta\ (\mathfrak{b})$ has a solution. If $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are pairwise relatively prime ideals, then the system of congruences $\xi \equiv \alpha_i\ (\mathfrak{a}_i)$ has a solution.[28]

E. If $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ are pairwise relatively prime ideals and if we let

$$\mathfrak{b}_i = [\mathfrak{a}_1, \dots, \mathfrak{a}_{i-1}, \mathfrak{a}_{i+1}, \dots, \mathfrak{a}_r], \quad \mathfrak{c} = [\mathfrak{a}_1, \dots, \mathfrak{a}_r],$$

[27] We recall the notation: $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a} \cap \mathfrak{b}$.

[28] This is the Chinese Remainder Theorem. Van der Waerden did not mention the name, not even in MA-II, p. 46.

then $\mathcal{R} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_r)$, i.e., each element of the ring $\mathcal{R}$ can be expressed in the form $\sum b_i$, where $b_i \in \mathfrak{b}_i$, and this representation is unique modulo $\mathfrak{c}$. If the representation is unique, i.e., if $\mathfrak{c} = 0$, we speak about a direct sum. (One can proceed to a direct sum by replacing $\mathcal{R}$ with the quotient ring $\mathcal{R}/\mathfrak{c}$.)

In Section 28 (*Der Vielfachenkettensatz*), two conditions dealing with chains of ideals in a ring $\mathcal{R}$ are considered.

(1) *Uneingeschränkte Vielfachenkettensatz*: Every descending chain of ideals in the ring $\mathcal{R}$ is finite.

(2) *Eingeschränkte Vielfachenkettensatz*: Every descending chain of ideals in the ring $\mathcal{R}$ that is bounded from below by the nonzero ideal is finite.

The following results are proved:

A. A ring without zero divisors that fulfills condition (1) is a field.

B. Condition (2) holds in $\mathcal{R}$ if and only if condition (1) holds in $\mathcal{R}/\mathfrak{a}$ for each nonzero ideal $\mathfrak{a}$ in the ring $\mathcal{R}$.

C. If condition (2) holds in the ring $\mathcal{R}$, then:
- $\mathcal{R}/\mathfrak{a}$ is a field for each nonzero prime ideal $\mathfrak{a}$ in $\mathcal{R}$.
- Each nonzero prime ideal in the ring $\mathcal{R}$ is maximal.
- Two nonzero prime ideals in the ring $\mathcal{R}$ are relatively prime.

D. Let $\mathcal{R}$ be a ring with identity which contains no infinite increasing chain of ideals (*Teilerkettensatz*). If an ideal $\mathfrak{a}$ is expressed as the intersection of primary ideals in the form $\mathfrak{a} = [\mathfrak{q}_1, \ldots, \mathfrak{q}_r]$ (according to Section 23), then the ideals $\mathfrak{q}_i$ are uniquely determined. Every two primary ideals corresponding to distinct proper prime ideals are relatively prime. Thus, it is possible to write $\mathfrak{a} = \prod \mathfrak{q}_i$.

Chapter V. *Ganze algebraische Grössen*

Section 29 (*Moduln in bezug auf einen Ring*) contains an axiomatic definition of an $\mathcal{R}$-module over a ring $\mathcal{R}$, which need not have an identity. (We recall that in a similar manner, the third section of the second chapter introduced the notion of a vector space over a field, which was however later used only for field extensions.) If N is a submodule of a module M, then M is referred to as a divisor of the module N, and N is a multiple of the module M; this fact is denoted by $N \equiv 0\ (M)$. We remark that the notion of a left $\mathcal{R}$-module is introduced in MA-II, pp. 86–87 (in general, the ring $\mathcal{R}$ need not be commutative).

A finite module (*endlicher $\mathcal{R}$-Modul*) is a module generated by finitely many elements. For modules, the condition of finiteness of increasing chains of submodules (*Teilerkettensatz*) is considered; it is equivalent to the condition of finiteness of all submodules (*Basissatz*).[29] Taking into account that the ring $\mathcal{R}$ is an $\mathcal{R}$-module, the condition of finiteness of increasing chains of submodules in the $\mathcal{R}$-module $\mathcal{R}$ reduces to the condition of finiteness of increasing chains of ideals in the ring $\mathcal{R}$.

A submodule N of a module M is called a divisor of an ideal $\mathfrak{c}$ in a ring $\mathcal{R}$, if $\mathfrak{c}M \equiv 0\ (N)$.

The important results of this section are as follows:[30]

A. If the condition of finiteness of increasing chains of ideals (*Teilerkettensatz*) holds in a ring $\mathcal{R}$ and if M is a finite $\mathcal{R}$-module, then the condition of finiteness of increasing chains of submodules (*Teilerkettensatz*) holds in the $\mathcal{R}$-module M, and therefore every submodule is finite.

B. If the condition of finiteness of bounded descending chains of ideals (*Eingeschränkte Vielfachenkettensatz*) holds in a ring $\mathcal{R}$ and if M is a finite $\mathcal{R}$-module, then the condition of finiteness of bounded descending chains of submodules (*Eingeschränkte Vielfachenkettensatz*) holds in the $\mathcal{R}$-module M.

C. If the condition of finiteness of descending chains of ideals containing a nonzero ideal $\mathfrak{c}$ holds in a ring $\mathcal{R}$ and if M is a finite $\mathcal{R}$-module, then the condition holds for submodules of the $\mathcal{R}$-module M that are divisors of the ideal $\mathfrak{c}$.

We remark that the second edition of MA-I from 1937 included a new section *Vektorräume und hyperkomplexe Systeme* (pp. 46–49) containing an axiomatic definition of an n-dimensional left $\mathcal{R}$-module over a (not necessarily commutative) ring $\mathcal{R}$ with identity (*n-dimensinaler Vektorraum*, *n-gliedriger Linearformenmodul*).

Section 30 (*Theorie der ganzen Grössen*) introduces the notion of an integral quantity or integral algebraic quantity (*ganz*, *ganz algebraisch*) with respect to a ring. Given a pair of rings $\mathcal{R} \subset \mathcal{S}$, an element $a \in \mathcal{S}$ is called an integral quantity over $\mathcal{R}$ if the $\mathcal{R}$-module $(\mathcal{R}, a, a^2, a^3, \dots)$ has a finite basis. This definition generalizes the notion of an algebraic element over a field (see Section 2 in the second chapter).

It is shown that the following statements hold in a ring with identity:

A. Sums and products of integral quantities are integral quantities.

B. A root of a polynomial $a^h + b_1 a^{h-1} + \cdots + b_h$, whose coefficients $b_1, \dots, b_h$ are integral quantities, is an integral quantity.

C. If all elements of the ring $\mathcal{S}$ are integral quantities over $\mathcal{R}$ and an element a is an integral quantity over $\mathcal{S}$, then a is an integral quantity over $\mathcal{R}$.

[29] Lecture participants were referred to the analogous result in Section 22. For a proof see MA-II, p. 87.

[30] The proof of theorem A is carried out for a ring $\mathcal{R}$ with identity, while the proofs of theorems B and C, which are similar, are omitted.

An integral domain $\mathcal{R}$ is called integrally closed (*ganz abgeschlossen*) in its fraction field P if every element $a \in P$ which is an integral quantity over $\mathcal{R}$ is contained in $\mathcal{R}$.

D. If the unique factorization holds in $\mathcal{R}$, then $\mathcal{R}$ is integrally closed in P.

Suppose an integral domain $\mathcal{R}$ has the following properties:

I. *Teilerkettensatz*: each increasing chain of ideals is finite.

II. *Ganz-abgeschlossen im Quotientenkörper*: it is integrally closed in its fraction field P.

If $P(s)$ is a simple algebraic separable extension of the integral domain $\mathcal{R}$ and $\mathcal{S}$ is a ring satisfying $\mathcal{R} \subset \mathcal{S} \subset P(s)$, then the following statements hold:

E. The $\mathcal{R}$-module $\mathcal{S}$ is finite if and only if all elements of $\mathcal{S}$ are integral quantities over $\mathcal{R}$.

F. If the $\mathcal{R}$-module $\mathcal{S}$ is finite, then increasing chains of $\mathcal{R}$-modules in $\mathcal{S}$ are finite (*Teilerkettensatz*), and increasing chains of $\mathcal{S}$-modules (i.e., ideals) are finite as well. See (A), § 29.

G. If all descending chains of ideals in $\mathcal{R}$ containing a nonzero ideal $\mathfrak{c}$ are finite, then all descending chains of submodules in $\mathcal{S}$ that divide the ideal $\mathfrak{c}$ are finite. See (C), § 29.

In Section 31 (*Idealtheorie der ganz-abgeschlossenen Ringe*), integral domains having the following properties are studied:

I. *Teilerkettensatz*: each increasing chain of ideals is finite.

II. *Eingeschränkte Vielfachenkettensatz*: each descending chain of ideals which is bounded from below by a nonzero ideal is finite.[31]

III. *Ganz-abgeschlossen im Quotientenkörper*: it is integrally closed in its fraction field.

Taking into account earlier results (see Sections 23 and 28), each ideal in such an integral domain is the product of relatively prime primary ideals.

The following statements are proved:

A. Every primary ideal is the power of a prime ideal.

B. Every ideal is the product of powers of prime ideals, and this representation is unique.

A reverse statement is proved in the end of this section.

C. If $\mathcal{R}$ is an integral domain with identity where unique factorization into prime ideals holds, then the ring $\mathcal{R}$ is integrally closed in its fraction field.

A final scheme presents several basic types of rings with typical examples and crucial results dealing with ideals.

[31] We remark that at this point, MA-II, p. 97, requires each nonzero prime ideal to be a maximal ideal; this fact is a consequence of the condition II (see § 28).

Final remarks

Transcription. Vojtěch Jarník was quite accurate when copying van der Waerden's lectures. In our transcription of his records, we have corrected only few language errors (e.g., *komutativ*, *Manigfaltigkeit*). More significant changes that we have performed (corrections, insertions) are pointed out in the footnotes (e.g., a substitution of the word *Ideal* by *Integral*). Otherwise, Jarník's lecture notes were kept in their original form – not even minor errors, e.g. in the articles, were corrected. It seems that Vojtěch Jarník never returned to his notes, since he did not correct even obvious and eye-catching errors (e.g., *Ideal – Integral*).

In Jarník's notes, several theorems are highlighted on their left-hand sides, while others are not. In our transcription, the passages highlighted by Jarník are marked by a double line.

One problem we had to face during the transcription process is the division of individual lectures into paragraphs, because the structure in Jarník's records (in small size notebooks) is not always apparent; thus, we have tried to follow the logical structure of the text.

The transcription of notation posed another problem. From Jarník's records, it is not obvious what kind of alphabet did van den Waerden use to denote sets, groups, rings, fields, ideals, etc., and whether he kept the same style throughout the lecture course. In our transcription, we have strived for consistency whenever possible.

Terminology and notation. Some terms used by van der Waerden throughout his lectures never appeared in his monograph *Moderne Algebra* (e.g., *vom selben Typus*), while other were modified (e.g., *Einheit*, *Einheitselement*, *Meromorphismus*).

As far as terminology is concerned, van der Waerden was not always consistent in his lectures. For example, he introduced the term *Integritätsbereich* for a ring without zero divisors, but he kept using the original term *Ring ohne Nullteiler* quite as often. We have already pointed out some terminological curiosities and inaccuracies in the previous text, as well as in the footnotes to the transcription of Jarník's notes. The same remarks apply to notation. However, such small inaccuracies in terminology and notation are inevitably present in every lecture, and especially in its transcription.

To a certain extent, van der Waerden used set-theoretic terminology and notation. Although he used the term *Menge* in the axiomatic definitions of a group and a ring, he mostly avoided it in the subsequent text. A remark in the beginning of Section 4 of the first chapter makes it clear that set-theoretic notation ($\in$, $\subset$, $\subseteq$, etc.) was still not quite common in the year 1927. A further evidence is provided by the fact that van der Waerden did not use the currently common symbol K^n for the n-ary Cartesian power of a field K.

Spirit of the lecture. Van der Waerden's lecture *Allgemeine Idealtheorie* was probably rather demanding for the audience because of its modern scope and high abstractness. Including a larger number of concrete examples would have surely increased its lucidity.

It seems clear that van der Waerden aimed to proceed in an effective way, without unnecessary detours, and omitted notions and facts whenever he was sure they are not needed for the remaining lectures. Thus, the section on groups completely lacks the fundamental notion of a normal subgroup.

In our opinion, some parts of the course would have benefited from a more detailed exposition (e.g., Sections 19, 20, 21, 25).[32] In such cases, we have provided footnotes with references (emphasized more than elsewhere) to the corresponding sections of van der Waerden's monograph. On the other hand, there are certain parts where van der Waerden somewhat redundantly returned to the already covered material (see e.g. the conclusion of Section 6a of the first chapter, or Section 9 of the second chapter).

Comparison of van der Waerden's lectures and his monograph. It is interesting to compare the choice of topics, their ordering and elaboration in van der Waerden's lectures in the year 1927/1928 with his two-volume monograph *Moderne Algebra* from the years 1930 and 1931. At this point, we emphasize that certain parts of the lectures are missing in the monograph.

The following table provides a summary of the relations between the sections of van der Waerden's lectures and the corresponding sections of his two-volume monograph *Moderne Algebra*.

§	**[Chapter I]**	
1.		−−
2.		MA-I, pp. 15–26, 29–30
3.		MA-I, pp. 36–45
4.		MA-I, pp. 46–49
5.		MA-I, pp. 49–53
6.		MA-I, pp. 53–60
6*a*.		MA-I, pp. 52–53, 59–60, 69–70
7.		MA-I, pp. 60–67, 73–76

[32] It is possible that van der Waerden had the feeling of time pressure.

Chapter II

1.	MA-I, pp. 86–87
2.	MA-I, pp. 88–94
3.	MA-I, pp. 95–97
4.	MA-I, pp. 96–99
5.	MA-I, pp. 99–104, 67–68, 113–118
6.	MA-I, pp. 198–203
7.	MA-I, pp. 203–208
8.	MA-II, pp. 54–58
9.	MA-I, pp. 67–69, 113–114

Chapter III

18.	MA-II, pp. 23–25
19.	MA-II, pp. 51–54
20.	MA-II, pp. 58–64
21.	––

Chapter IV

22.	MA-II, pp. 25–27
23.	MA-II, pp. 35–40
24.	MA-II, pp. 27–30, 34–35, 37–38
25.	MA-II, pp. 64–67, 11
26.	MA-II, pp. 40–41
27.	MA-II, pp. 43–48
28.	––

Chapter V

29.	MA-II, pp. 86–88, 109–110, MA-I, 133
30.	MA-II, pp. 88–95
31.	MA-II, pp. 97–103