Chapter VII: Topology of the plane

In: Eduard Čech (author); Miroslav Katětov (author); Aleš Pultr (translator): Point Sets. (English). Praha: Academia, Publishing House of the Czechoslovak Academy of Sciences, 1969. pp. [218]–265.

Persistent URL: `http://dml.cz/dmlcz/402654`

Chapter VII

TOPOLOGY OF THE PLANE

§ 26. Cutting of the plane by a given set

26.1. In the topological study of the plane, a transfer from the plane to the sphere by the so called stereographical projection is often convenient.

The *sphere* is the space $\mathbf{S}_2$ (see 17.10). We put (throughout the whole chapter)

$$\omega = (1, 0, 0) \in \mathbf{S}_2 .$$

If $x + iy \in \mathbf{E}_2$, we put (throughout the whole chapter) $\sigma(x + iy) = (\xi_0, \xi_1, \xi_2) \in$ $\in \mathbf{S}_2 - (\omega)$, where

$$\xi_0 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}, \qquad \xi_1 = \frac{2x}{x^2 + y^2 + 1}, \qquad \xi_2 = \frac{2y}{x^2 + y^2 + 1} .$$

By the proof of theorem 17.10.4 we obtain

26.1.1. *σ is homeomorphic mapping of the plane onto $\mathbf{S}_2 - (\omega)$.*

The mapping σ is termed the *stereographical projection.*

The following theorem is easy to prove:

26.1.2. *Let $M \subset \mathbf{E}_2$. The set M is unbounded if and only if $\omega \in \overline{\sigma(M)}$.*

26.1.3. *Let $M \subset \mathbf{E}_2$. The closure of $\sigma(M)$ in $\mathbf{S}_2$ is:* [1] *$\sigma(\overline{M})$ if M is bounded,* [2] *$\sigma(\overline{M}) \cup (\omega)$ if M is not bounded.*

Proof: By 26.1.1, $\sigma(\overline{M})$ is the closure of $\sigma(M)$ in $\mathbf{S}_2 - (\omega)$, so that 26.1.3 follows from 8.7.1 and 26.1.2.

26.1.4. *Let $M \subset \mathbf{E}_2$, $a \in M$. A continuum $K \subset (M) \cup (\omega)$ containing both $\sigma(a)$ and ω exists if and only if there is a set $C \subset M$ which is closed (in $\mathbf{E}_2$), connected, unbounded, and which contains the point a.*

Proof: I. Let C exist. The set $\sigma(C) \subset \sigma(M)$ is connected by 26.1.1, so that, by 18.1.6, the set $K = \overline{\sigma(C)}$ is also connected. We have $K = \sigma(C) \cup (\omega)$ by 26.1.3, so that $K \subset \sigma(M) \cup (\omega)$, $\sigma(a) \in K$, $\omega \in K$. K is a continuum by 17.2.2 and 17.10.2.

II. Let K exist. By 19.4.1 there exists an irreducible continuum $L \subset K$ between $\sigma(a)$ and ω. Put $Q = L - (\omega)$, $C = \sigma_{-1}(Q)$, so that $a \in C$, $C \subset M$. Evidently

$\bar{Q} = L$, so that, by 26.1.3, C is closed and unbounded. Q is connected by 19.4.2 and hence C is connected by 26.1.1

26.1.5. *Let $M \subset \mathbf{E}_2$ be a bounded set. Let $a \in \mathbf{E}_2 - M$, $b \in \mathbf{E}_2 - M$, $a \neq b$. M cuts the plane between the points a and b if and only if $\sigma(M)$ cuts the sphere between $\sigma(a)$ and $\sigma(b)$.*

Proof: If $\sigma(M)$ cuts $\mathbf{S}_2$ between $\sigma(a)$, $\sigma(b)$, then evidently $\sigma(M)$ cuts $\mathbf{S}_2 - (\omega)$ between $\sigma(a)$, $\sigma(b)$, so that (see 26.1.1) M cuts the plane between a, b.

If $\sigma(M)$ does not cut $\mathbf{S}_2$ between $\sigma(a)$, $\sigma(b)$, there is a continuum $K \subset \mathbf{S}_2 - \sigma(M)$ containing both $\sigma(a)$, $\sigma(b)$. If ω does not belong to K, then (see 26.1.1) $\sigma_{-1}(K) \subset \subset \mathbf{E}_2 - M$ is a continuum containing both points a, b, so that M does not cut the plane between a, b. Thus, let $\omega \in K$. Obviously there is a neighborhood U of ω in $\mathbf{S}_2$ such that $\bar{U} - U$ is a continuum and such that neither $\sigma(a)$ nor $\sigma(b)$ nor any point of $\sigma(M)$ belongs to $\bar{U}$. Let H_1, H_2 be components of $K - U$ such that $\sigma(a) \in H_1$, $\sigma(b) \in H_2$. By 19.1.1 and 19.3.1, H_1, H_2 are continua and we have $H_1 \cap (\bar{U} - U) \neq \emptyset \neq H_2 \cap (\bar{U} - U)$ so that $K_0 = H_1 \cup (\bar{U} - U) \cup H_2$ is also a continuum. We have $\sigma(a) \in K_0$, $\sigma(b) \in K_0$, $\omega \in \mathbf{S}_2 - K_0$, $\sigma(M) \cap K_0 = \emptyset$, so that $\sigma(M)$ does not cut the plane between a, b.

26.2. Let a set $M \subset \mathbf{E}_2$ and a point $a \in \mathbf{E}_2 - M$ be given. Let us associate with every $z \in M$ the point

$$(z) = \frac{z - a}{|z - a|} \in \mathbf{S}_1 .$$

We obtain a mapping f of M into $\mathbf{S}_1$ which plays an important role in following tasks. We denote it by

$$= \pi(M; a).$$

Evidently $\pi(M; a)$ is a continuous mapping of M into $\mathbf{S}_1$ and $\pi(N; a)$ is its partial mapping whenever $N \subset M$.

26.2.1. *Let $M \subset \mathbf{E}_2$, $a \in \mathbf{E}_2 - M$. A necessary and sufficient condition for $\pi(M; a)$ to be inessential is the following: There exists a set $C \subset \mathbf{E}_2$ which is closed (in $\mathbf{E}_2$), connected, unbounded and such that $a \in C$, $C \cap M = \emptyset$.*

Proof of sufficiency: I. Let such a C exist. Let us assume that the mapping $\pi(M; a)$ is essential. We have to reach a contradiction.

II. Since $M \subset \mathbf{E}_2 - C$, $a \in C$, $\pi(\mathbf{E}_2 - C; a)$ is essential by 24.2.6. Thus, by 24.2.18, there is a continuum $K \subset \mathbf{E}_2 - C$ such that the mapping $\pi(K; a)$ is essential.

III. By 17.3.4, $\varrho(K, C) > 0$. Choose an $\varepsilon > 0$, $\varepsilon < \varrho(K, C)$. By 17.2.3 there is a $c > 0$ such that $|x| < c$, $|y| < c$ for $x + iy \in K$. The set C is unbounded so that there is a point $b = b_1 + ib_2 \in C$ such that either $|b_1| > c$ or $|b_2| > c$. By 24.2.7 it follows easily that the mapping $\pi(K; b)$ is inessential.

IV. By 19.1.2 there is a finite sequence $\{a_n\}_0^k$ such that $a_0 = a$, $a_k = b$, $a_n \in C$ $(0 \leqq n \leqq k)$, $\varrho(a_{n-1}, a_n) < \varepsilon$ $(1 \leqq n \leqq k)$. The mapping $\pi(K; a_0)$ is essential by II; the mapping $\pi(K; a_k)$ is inessential by III. Thus, there is an index m $(1 \leqq m \leqq k)$ such that the mapping $\pi(K; a_{m-1})$ isesse ntial and the mapping $\pi(K; a_m)$ is inessential.

V. Put

$$J = \mathop{\mathrm{E}}_{t}[0 \leqq t \leqq 1] .$$

For $t \in J$ we have

$$(1 - t)\, a_m + t a_{m-1} \in \mathbf{E}_2 \tag{1}$$

and we compute easily that

$$\varrho[(1 - t)\, a_m + t a_{m-1}, a_m] = t \,.\, \varrho(a_{m-1}, a_m) < \varepsilon < \varrho(K, C)\,,$$

so that the point (1) does not belong to K.

For $z \in K$, $t \in J$ put

$$\varphi(z, t) = \frac{z - [(1 - t)\, a_m + t a_{m-1}]}{|z - [(1 - t)\, a_m + t a_{m-1}]|} .$$

Then φ is a continuous mapping of $K \times J$ into $\mathbf{S}_1$. The partial mapping $\varphi_{K \times (0)}$ is inessential. The partial mapping $\varphi_{K \times (1)}$ is essential. By 24.3.1, the partial mapping $\varphi_{(z) \times J}$ is inessential for every $z \in K$. Thus, by 24.5.1, the mapping φ is inessential so that by 24.2.6 also $\varphi_{K \times (1)}$ is inessential which is a contradiction.

Proof of necessity: I. Let $\pi(M; a)$ be inessential. Since $\varphi = \pi(\mathbf{E}_2 - (a); a)$ is a continuous mapping of the open set $\mathbf{E}_2 - (a) \supset M$ into $\mathbf{S}_1$ and since $\varphi_M = \pi(M; a)$, by 24.2.16 there exists an open set $G \subset \mathbf{E}_2 - (a)$ such that $M \subset G$ and the mapping $\pi(G; a)$ is inessential. Put $F = \mathbf{E}_2 - G$, so that F is closed and $a \in F$. Let C be the component of F containing the point a. We have $C \cap M = \emptyset$ and C is connected. Moreover, C is closed (see 8.7.4 and 18.2.2). Thus, it suffices to prove that C is not bounded.

II. Let, on the contrary, C be bounded. We have to reach a contradiction. There exists a bounded neighborhood U of C. Obviously C is a component of $F \cap \overline{U}$. $F \cap \overline{U}$ is compact (see 17.2.3) and $F \cap U$ is a neighborhood of C in the space $F \cap \overline{U}$, so that, by 19.1.4 (see also 19.1.5), there exist separated sets A, B such that $F \cap \overline{U} = A \cup B$, $C \subset A \subset U$.

Since A, B are separated, we have first $A \cap B = \emptyset$. Secondly, A, B are closed in $A \cup B = F \cap \overline{U}$ and hence in $\mathbf{E}_2$. Moreover, $A \subset U$ is bounded and hence compact (see 17.2.3). $F - U$ is also a closed set. Since $A \cap B = \emptyset$, $A \subset U$, we have $A \cap [B \cup (F - U)] = \emptyset$. Thus, $\varrho[A, B \cup (F - U)] > 0$ by 17.3.4. Let us choose an $\varepsilon > 0$ with $\varepsilon < \varrho[A, B \cup (F - U)]$. Then,

$$V = \Omega(A, \varepsilon)$$

is an open bounded set (see 8.6). Since $F = A \cup [B \cup (F - U)]$, $\varepsilon < \varrho[A, B \cup \cup (F - U)]$, we have evidently $\bar{V} \cap F = A$, so that $(\bar{V} - V) \cap F = \emptyset$.

III. Put

$$H = \bar{V} - V,$$

so that H is bounded and closed. Moreover, $H \cap F = \emptyset$, i.e. $H \subset G$, so that by I and 24.2.6, $\pi(H; a)$ is an inessential mapping. Hence there exists a continuous mapping φ of H into $\mathbf{E}_1$ such that

$$e^{i\varphi(z)} = \frac{z - a}{|z - a|} \quad \text{for} \quad z \in H.$$

Since $H \subset \mathbf{E}_2$ is closed, by 14.8.3 there exists a continuous mapping ψ of the whole plane into $\mathbf{E}_1$ such that

$$e^{i\psi(z)} = \frac{z - a}{|z - a|} \quad \text{for} \quad z \in H.$$

IV. Since V is bounded, there is a number $c > 0$ such that

$$z \in V \quad \text{implies} \quad |z - a| < c.$$

Denote by Q the set of all $z \in \mathbf{E}_2$ with $|z - a| = c$.

V. Since $a \in F$, $H \subset G$, we have $a \in \mathbf{E}_2 - H$, so that there is a component K of $\mathbf{E}_2 - H$ containing the point a. The set K is connected and by 22.1.4 (see also 22.1.8) it is also open. By 22.1.9 (see also 10.3.2) we have $\bar{K} - K \subset H$.

VI. Define a mapping g of $\mathbf{E}_2$ into $\mathbf{S}_1$ as follows: [1] if $z \in \bar{K}$, then $g(z) = e^{i\psi(z)}$; [2] if $z \in \mathbf{E}_2 - K$, then

$$g(z) = \frac{z - a}{|z - a|}.$$

If simultaneously both $z \in \bar{K}$ and $z \in \mathbf{E}_2 - \bar{K}$, then, by V, $z \in H$ so that both values $g(z)$ are equal by (2). The partial mappings

$$g_{\bar{K}}, \quad g_{\mathbf{E}_2 - K}$$

are evidently continuous so that, by ex. 9.5, g is a continuous mapping of $\mathbf{E}_2$ into $\mathbf{S}_1$. By 24.5.3 g is inessential so that the partial mapping g_Q is also inessential.

VII. As $C \subset A \subset V$, we have $a \in V$. Hence, $K \cap V \neq \emptyset$. If K is not contained in V, we have, by 18.1.8, $K \cap B(V) \neq \emptyset$, i.e. (see 10.3.2) $K \cap H \neq \emptyset$, which is a contradiction. Thus, $K \subset V$, so that by IV, $Q \subset \mathbf{E}_2 - K$, and therefore

$$g_Q = \pi(Q; a).$$

Hence (see VI) $\pi(Q; a)$ is an inesssential mapping of Q into $\mathbf{S}_1$. On the other hand, $\pi(Q; a)$ is evidently a homeomorphic mapping of the simple loop Q onto $\mathbf{S}_1$. Thus, $\pi(Q; a)$ is essential, by 24.3.3 and 24.3.5, which is a contradiction.

26.2.2. *Let $M \subset \mathbf{E}_2$, $a \in \mathbf{E}_2 - M$. The set $\sigma(M)$ cuts the sphere between the points $\sigma(a)$ and ω if and only if the mapping $\pi(M; a)$ of M into $\mathbf{S}_1$ is essential.*

This follows by 26.1.1, 26.1.4 and 26.2.1.

26.2.3. *Let $M \subset \mathbf{E}_2$, $a \in \mathbf{E}_2 - M$, $b \in \mathbf{E}_2 - M$, $a \neq b$. The set $\sigma(M)$ cuts the sphere between the points $\sigma(a)$, $\sigma(b)$ if and only if the mapping*

$$\pi(M; a)/\pi(M; b)$$

of M into $\mathbf{S}_1$ is essential.

Proof: For $z \in \mathbf{E}_2 - (b)$ put

$$h(z) = \frac{z - a}{z - b}.$$

It is easy to prove that h is a homeomorphic mapping of $\mathbf{E}_2 - (b)$ onto $\mathbf{E}_2 - (1)$. Put $N = h(M)$; we have $0 = h(a)$.

Define a mapping k of $\mathbf{S}_2$ into $\mathbf{S}_2$ as follows: First, $k(\omega) = \sigma(1)$; secondly, $k[\sigma(b)] = \omega$; if, thirdly, $\zeta \in \mathbf{S}_2$, $\zeta \neq \omega$, $\zeta \neq \sigma(b)$ there, is exactly one point $z \in \mathbf{E}_2 - (b)$ with $\zeta = \sigma(z)$ and we put $k(\zeta) = \sigma[h(z)]$. It is easy to prove that k is a homeomorphic mapping of $\mathbf{S}_2$ onto $\mathbf{S}_2$ and that $k[\sigma(M)] = \sigma(N)$, $k[\sigma(a)] = \sigma(0)$, $h[\sigma(b)] = \omega$. Thus, $\sigma(M)$ cuts the sphere between the points $\sigma(a)$, $\sigma(b)$ if and only if $\sigma(N)$ cuts the sphere between the points $\sigma(0)$, ω, hence (see 26.2.2) if and only if the mapping $\pi(N; 0)$ of N into $\mathbf{S}_1$ is essential.

Put $f = \pi(M; a)/\pi(M; b)$, $g = \pi(N; 0)$. We have to prove that the mapping f of M into $\mathbf{S}_1$ is inessential if and only if the mapping g of N into $\mathbf{S}_1$ is inessential. This, however, is an easy consequence of the fact that h_M is a homeomorphic mapping of M onto N, since, for every $z \in M$, $f(z) = g[h(z)]$.

26.2.4. *Let $M \subset \mathbf{E}_2$. Let*

$$a_1, a_2, \ldots, a_k$$

be mutually distinct points of the set $\mathbf{E}_2 - M$. Suppose that for $1 \leqq \lambda \leqq k$ there exists no $C \subset \mathbf{E}_2$ closed, connected and unbounded such that $a_\lambda \in C$, $C \cap M = \emptyset$. Suppose that for $1 \leqq \lambda < \mu \leqq k$ there exists no continuum K such that $a_\lambda \in K$, $a_\mu \in K$, $K \cap M = \emptyset$. Let $n_1, n_2, \ldots, n_k$ be integers. Let the mapping

$$\prod_{\lambda=1}^{k} [\pi(M; a_\lambda)]^{n_\lambda}$$

of M into $\mathbf{S}_1$ be inessential. Then all the numbers $n_1, n_2, \ldots, n_k$ are equal to zero.

Proof: I. Let, on the contrary, some of the numbers $n_1, n_2, \ldots, n_k$ not be zero. Since our assumption concerning the points $a_1, a_2, \ldots, a_k$ remains preserved if we omit some of them, we may assume that none of the numbers $n_1, n_2, \ldots, n_k$ is equal to zero.

II. By 24.2.16 we conclude easily that there is an open set $G \supset M$ such that G contains none of the points $a_1, a_2, \ldots, a_k$ and the mapping

$$\prod_{\lambda=1}^{k} [\pi(G; a_\lambda)]^{n_\lambda}$$

is inessential. Put

$$F = \mathbf{E}_2 - G .$$

Let C_λ $(1 \leqq \lambda \leqq k)$ be the component of F containing the point a_λ. The set F is closed so that (see 8.7.4 and 18.2.2) also the sets C_λ $(1 \leqq \lambda \leqq k)$ are closed. Moreover, C_λ are connected and we have $a_\lambda \in C_\lambda$, $C_\lambda \cap M = \emptyset$, so that the sets C_λ are bounded and hence (see 17.2.3) compact. Thus, for every λ $(1 \leqq \lambda \leqq k)$ either $C_\lambda = (a_\lambda)$ or C_λ is a continuum such that $a_\lambda \in C_\lambda$, $C_\lambda \cap M = \emptyset$. It follows easily that C_λ $(1 \leqq \lambda \leqq k)$ are mutually distinct, and hence disjoint, components of F.

III. If $\mu = 0$, it is easy to construct a closed, connected and unbounded set $T_\mu \subset \mathbf{E}_2$ such that $T_\mu \cap C_\lambda \neq \emptyset$ for exactly μ of the k sets $C_1, C_2, \ldots, C_k$. Let such a T_μ exist for some μ $(0 \leqq \mu \leqq k - 1)$. We are going to show that also $T_{\mu+1}$ exists.
Choose an index λ $(1 \leqq \lambda \leqq k)$ with $T_\mu \cap C_\lambda = \emptyset$. Choose a point $b \in T_\mu$. Choose a simple arc $A \subset \mathbf{E}_2$ with end points a_λ, b, oriented in such a way that a_λ is the initial point (see 20.2.5). Denote by P the union of the C_ν $(1 \leqq \nu \leqq k)$ with $C_\nu \cap T_\mu = \emptyset$. Then P is a closed set and $a_\lambda \in A \cap P$, hence $A \cap P \neq \emptyset$. Hence (see 20.2.7) there is a last point c of the ordered set $A \cap P \subset A$. Evidently $c \neq b$ so that (see 20.1.8) there exists a simple arc $B \subset A$ with end points b, c. Obviously we may put $T_{\mu+1} = T_\mu \cup B$.

IV. Thus, there exists a set $T_{k-1} = T$ which is closed, connected and not bounded and such that $T \cap C_\lambda = \emptyset$ for exactly one of the indices λ $(1 \leqq \lambda \leqq k)$. For certainty let

$$T \cap C_1 = \emptyset, \quad T \cap C_\lambda \neq \emptyset \quad (2 \leqq \lambda \leqq k) .$$

Put $S = T \cup \bigcup_{\lambda=2}^{k} C_\lambda$. The set S is closed, connected and not bounded and we have

$$S \cap C_1 = \emptyset , \quad C_\lambda \subset S \quad (2 \leqq \lambda \leqq k) ,$$

hence

$$a_\lambda \in S \quad (2 \leqq \lambda \leqq k) .$$

V. The set C_1 is bounded and $\mathbf{E}_2 - S$ is its neighborhood. Thus, there exists a bounded neighborhood U of the set C_1 such that $U \cap S = \emptyset$. By 10.1.2 we may assume that $\overline{U} \cap S = \emptyset$. Since $C_1 \subset U$ is a component of the set F, C_1 is evidently a component of $F \cap \overline{U}$. $F \cap \overline{U}$ is compact (see 17.2.3) and $F \cap U$ is a neighborhood of C_1 in the space $F \cap \overline{U}$. Thus, by 19.1.4 (see also 19.1.5), there exist separated A, B such that $F \cap \overline{U} = A \cup B$, $C_1 \subset A \subset U$. Since A, B are separated, we have $A \cap \cap B = \emptyset$ and A, B are closed in $A \cup B = F \cap \overline{U}$, and consequently in $\mathbf{E}_2$. Moreover, $A \subset U$ is bounded and hence compact (see 17.2.3). $S \cup (F - U)$ is also a closed set. Since $A \cap B = \emptyset$, $A \subset U$, we have $A \cap [B \cup S \cup (F - U)] = \emptyset$. Thus, $\varrho[A, B \cup S \cup (F - U)] > 0$ by 17.3.4. Choose an $\varepsilon > 0$ with $\varepsilon < \varrho[A, B \cup S \cup \cup (F - U)]$. Then

$$V = \Omega(A, \varepsilon)$$

is an open bounded set. As $S \cup F = A \cup [B \cup S \cup (F - U)]$, $\varepsilon < \varrho[A, B \cup S \cup \cup (F - U)]$, we have $\bar{V} \cap (S \cup F) = A$, so that $(\bar{V} - V) \cap (S \cup F) = \emptyset$.

VI. Put $H = \bar{V} - V = B(V)$ (see 10.3.2). Then we have $S \cap H = \emptyset$ and, moreover, $H \cap F = \emptyset$, i.e. $H \subset G$, so that the mapping

$$\prod_{\lambda=1}^{k} [\pi(H; a_\lambda)]^{n_\lambda}$$

is inessential. The set S is closed, connected and not bounded. Moreover, $S \cap H = \emptyset$ and, for $2 \leqq \lambda \leqq k$, $a_\lambda \in C_\lambda \subset S$, so that, by 26.2.1, the mapping $\pi(H; a_\lambda)$ is inessential for $2 \leqq \lambda \leqq k$. Thus, by 24.2.4 and 24.2.5, also the mapping

$$\prod_{\lambda=2}^{k} [\pi(H; a_\lambda)]^{-n_\lambda}$$

is inessential so that, by 24.2.4, also $[\pi(H; a_1)]^{n_1}$ is inessential. As $n_1 \neq 0$, the mapping $\pi(H; a_1)$ is, by 24.2.10, also inessential.

VII. Thus, by 26.2.1 there exists a set Q which is closed, connected and not bounded, such that $a_1 \in Q$, $Q \cap H = \emptyset$.

As $a_1 \in V$, we have $Q \cap V \neq \emptyset$. Since Q is not bounded and V is bounded, V does not contain Q. Thus, by 18.1.8, $Q \cap B(V) \neq \emptyset$, i.e. $Q \cap H \neq \emptyset$, which is a contradiction.

26.2.5. *Let $M \subset \mathbf{E}_2$. Let*

$$a_1, a_2, \ldots, a_k \quad (k \geqq 1)$$

be mutually distinct points of the set $\mathbf{E}_2 - M$. Let the set $\sigma(M)$ cut the sphere between every two of the points

$$\omega, \sigma(a_1), \ldots, \sigma(a_k).$$

Let $n_1, n_2, \ldots, n_k$ be integers. Let

$$\prod_{\lambda=1}^{k} [\pi(M; a_\lambda)]^{n_\lambda}$$

be an inessential mapping of M into $\mathbf{S}_1$. Then all the numbers $n_1, n_2, \ldots, n_k$ are equal to zero.

This follows easily by 26.1.1, 26.1.4 and 26.2.4.

26.3. 26.3.1. *Let $A \subset \mathbf{S}_2$, $B \subset \mathbf{S}_2$. Let the sets A, B be either both closed in $A \cup B$ or both open in $A \cup B$. Let*

$$a_0, a_1, \ldots, a_k \quad (k \geqq 1) \tag{1}$$

be mutually distinct points of $\mathbf{S}_2 - (A \cup B)$. Let neither A nor B cut the sphere between some two points from (1). Let the set $A \cap B$ have at most k components. Then there are indices λ, μ $(0 \leqq \lambda < \mu \leqq k)$ such that $A \cup B$ does not cut the sphere between the points a_λ, a_μ.

Proof: By 17.10.3 we may assume that $a_0 = \omega$. Evidently there are sets $C \subset \mathbf{E}_2$, $D \subset \mathbf{E}_2$ and mutually different points $\alpha_\lambda \in \mathbf{E}_2 - (C \cup D)$ $(1 \leqq \lambda \leqq k)$ such that $\sigma(C) = A$, $\sigma(D) = B$, $\sigma(a_\lambda) = a_\lambda$ $(1 \leqq \lambda \leqq k)$. We conclude easily by 26.1.1 that the sets C, D are either both open in $C \cup D$ or both closed in $C \cup D$ and that $C \cap D$ has at most k components.

Since neither $A = \sigma(C)$ nor $B = \sigma(D)$ cuts the sphere between some two of the points $a_0 = \omega$, $a_\lambda = \sigma(\alpha_\lambda)$ $(1 \leqq \lambda \leqq k)$, we conclude by 26.2.2 that the mappings $\pi(C; \alpha_\lambda)$ $(1 \leqq \lambda \leqq k)$ of C into $\mathbf{S}_1$ and the mappings $\pi(D; \alpha_\lambda)$ $(1 \leqq \lambda \leqq k)$ of D into $\mathbf{S}_1$ are inessential.

By 24.2.12 there are integers $n_1, n_2, \ldots, n_k$ such that not all of them are equal to zero and the mapping

$$\prod_{\lambda=1}^{k} [\pi(C \cup D; \alpha_\lambda)]^{n_\lambda}$$

of $C \cup D$ into $\mathbf{S}_1$ is inessential. Thus, by 26.2.5, there are two distinct points amongst $\omega = a_0$, $\sigma(\alpha_\lambda) = a_\lambda$ $(1 \leqq \lambda \leqq k)$ such that $C \cup D$ does not cut the sphere between them.

26.3.2. *Let $A \subset \mathbf{S}_2$, $B \subset \mathbf{S}_2$. Let the sets A, B be either both closed or both open. Let $k = 1, 2, 3, \ldots$. Let both sets A, B be connected; let $A \cap B$, however, have more than k components. Then $\mathbf{S}_2 - (A \cup B)$ has more than k components.*

Proof: The sets A, B, $A \cap B$ are either closed or open. In the first case they are compact by 17.2.2 and 17.10.2. In the second case they are locally connected by 22.1.3 and 22.1.14 and topologically complete by 15.5.2, 17.2.1 and 17.10.2. Thus, in both cases (see 19.5.9 and 22.3.2) the constituants of any of the sets A, B, $A \cap B$ coincide with its components.

Since A, B are connected and since $A \cap B$ has more than k components, we see that A, B are semicontinua and further, that there exist points $a_\lambda \in A \cap B$ $(0 \leqq \lambda \leqq k)$ such that distinct ones of them belong to distinct constituants of $A \cap B$.

Put $C = \mathbf{S}_2 - A$, $D = \mathbf{S}_2 - B$, so that the sets C, D are either both open (in $\mathbf{S}_2$, hence also in $C \cup D$), or both closed (in $\mathbf{S}_2$, hence also in $C \cup D$). As $a_\lambda \in A$, $A = \mathbf{S}_2 - C$ and A is a semicontinuum, C cuts the sphere between no two of the points a_λ $(0 \leqq \lambda \leqq k)$. The same holds certainly for the set D. If the set

$C \cap D$ has at most k components, there are, by 26.3.1, indices λ, μ such that $0 \leqq \lambda < \mu \leqq k$ and the set $C \cup D$ does not cut the sphere between a_λ, a_μ. If follows that (see 19.5.10) both points a_λ, a_μ belong to the same constituant of $\mathbf{S}_2 - (C \cup D) = A \cap B$, which is a contradiction. Thus, the set $C \cap D = \mathbf{S}_2 - (A \cup B)$ has more than k components.

26.4. 26.4.1. *Let $M \subset \mathbf{S}_2$ be a closed set. Let M cut the sphere between points a, b. Then there exists a component of M which cuts the sphere between the points a, b.*

Proof: By 17.10.3 we may assume that $b = \omega$. Then there exists a set $N \subset \mathbf{E}_2$ and a point $\alpha \in \mathbf{E}_2 - N$ such that $\sigma(N) = M$, $\sigma(\alpha) = a$. If no component K of M cuts the sphere between the points a, ω, the mapping $\pi(H; \alpha)$, where $H = \sigma_{-1}(K)$, is inessential by 26.2.2. All the components of N have by 26.1.1 the form $H = \sigma_{-1}(K)$ where K are all the components of M. Thus, the mapping $\pi(N; \alpha)$ is inessential by 24.2.17, since M is compact by 17.2.2 and 17.10.2, so that N is compact by 26.1.1. Then, by 26.2.2, $M = \sigma(N)$ does not cut the sphere between a, ω. This is a contra- diction.

26.4.2. *Let $M \subset \mathbf{S}_2$ be a locally connected set. Let M cut the sphere between points a, b. Then there is a component of M, cutting the sphere between the points a, b.*

The *proof* is similar to the proof of theorem 26.4.1.

26.4.3. *Let $M \subset \mathbf{S}_2$, $N \subset \mathbf{E}_1$ be homeomorphic sets. Then $\mathbf{S}_2 - M$ is a semicontinuum.*

Proof: It is easy to show (even in different ways) that $M \neq \mathbf{S}_2$. Thus, if the statement does not hold, there are points $a \in \mathbf{S}_2 - M$, $b \in \mathbf{S}_2 - M$ such that M cuts the sphere between them. By 17.10.3 we may proceed under the assumption of $b = \omega$. By 26.2.2 the mapping $\pi[\sigma_{-1}(M); \sigma_{-1}(a)]$ would be essential. On the other hand, $\sigma_{-1}(M)$ is homeomorphic with $N \subset \mathbf{E}_1$, so that we see easily by 24.3.7 that every continuous mapping of $\sigma_{-1}(M)$ into $\mathbf{S}_1$ is inessential.

26.4.4. *Let $M \subset \mathbf{E}_2$ be a bounded set. Let M be homeomorphic with a set $N \subset \mathbf{E}_1$. Then $\mathbf{E}_2 - M$ is a semicontinuum.*

This follows easily by 26.1.1, 26.1.5 and 26.4.3.

26.4.5. *Let $M_1 \subset \mathbf{S}_2$, $M_2 \subset \mathbf{S}_2$. Let h be a homeomorphic mapping of M_1 onto M_2. Let $a_1 \in M_1$, $a_2 \in h(a_1)$. Let a_1 be an interior point of M_1 (in $\mathbf{S}_2$). Then a_2 is an interior point of M_2 (in $\mathbf{S}_2$).*

Proof: Assume the contrary. As a_1 is an interior point of M_1 in $\mathbf{S}_2$, it is easy to find a neighborhood U_1 of a_1 in the space M_1 such that there exists a homeomorphic mapping k of $\mathbf{E}_2$ onto U_1. Evidently there is a neighborhood $V_1 \subset U_1$ of a_1 in M_1 such that $k_{-1}(V_1)$ is bounded.

Evidently $U_2 = h(U_1)$, $V_2 = h(V_1)$ are neighborhoods of the point a_2 in M_2. Choose a $b \in U_2$, $b \neq a_2$. There is a number $\delta > 0$ such that $x \in M_2$, $\varrho(a_2, x) < \delta$ imply $x \in V_2$. Since a_2 is not an interior point of M_2 in $\mathbf{S}_2$, there is a point $c \in \mathbf{S}_2 - M_2$ such that $\varrho(a_2, c) = r < \delta$. Let Q be the set of all $x \in \mathbf{S}_2$ with $\varrho(a_2, x) = r$. It is easy to prove that Q is homeomorphic with $\mathbf{S}_1$ and that Q cuts $\mathbf{S}_2$ between the points a_2, b. Thus, the set $Q \cap M_2 = Q \cap U_2$ cuts U_2 between the points a_2, b. We have $Q \cap M_2 \subset Q - (c)$, so that evidently there is an $N \subset \mathbf{E}_1$ homeomorphic with $Q \cap M_2$.

For $z \in \mathbf{E}_2$ put $\varphi(z) = h[k(z)]$, so that φ is a homeomorphic mapping of the plane onto U_2. There exist points $\alpha \in \mathbf{E}_2$, $\beta \in \mathbf{E}_2$ and a set $R \subset \mathbf{E}_2$ such that $\varphi(\alpha) = a_2$, $\varphi(\beta) = b$, $\varphi(R) = Q \cap M_2$. As $Q \cap M_2$ cuts U_2 between the points a_2, b, R cuts the plane between the points α, β. We have $\varphi_{-1}(V_2) = k_{-1}(V_1)$, $Q \cap M_2 \subset V_2$ so that the set R is bounded. Evidently, R is homeomorphic with N. This is a contradiction by 26.4.4.

26.4.6. *Let $M \subset \mathbf{S}_2$, $a \in \mathbf{S}_2 - M$, $b \in \mathbf{S}_2 - M$, $a \neq b$. Let M not cut the sphere between the points a, b. Let M be connected. Let $M \subset N \subset \overline{M}$. Let N cut the sphere between the points a, b. Then there is at least one point $c \in N - M$ such that the set $M \cup (c)$ cuts the sphere between a, b. If C is the set of all such points c, then C is closed in N.*

Proof: By 17.10.3 we may assume that $b = \omega$. There exist sets $M_0 \subset \mathbf{E}_2$, $N_0 \subset \mathbf{E}_2$ and a point $\alpha \in \mathbf{E}_2$ such that $\sigma(M_0) = M$, $\sigma(N_0) = N$, $\sigma(\alpha) = a$. By 26.1.1, M_0 is connected and $M_0 \subset N_0 \subset \overline{M}_0$, so that M_0 is dense in N_0. We have $\alpha \in \mathbf{E}_2 - N_0$, so that $\pi(N_0; \alpha)$ is a continuous mapping of N_0 into $\mathbf{S}_1$ and $\pi(M_0; \alpha)$ is its partial mapping. This partial mapping is inessential by 26.2.2.

By 24.2.19 there exists a set $C_0 \subset N_0 - M_0$ such that C_0 is closed in N_0 and, for $Z_0 \subset N_0 - M_0$, the mapping $\pi(M_0 \cup Z_0; \alpha)$ is essential if and only if $Z_0 \cap C_0 \neq \emptyset$. Put $C = \sigma(C_0)$. The set C is closed in N by 26.1.1. By 26.2.2, for $Z \subset N - M$, the set $M \cup Z$ cuts the sphere between the points a, ω if and only if $Z \cap C \neq \emptyset$. Since N cuts the sphere between the points a, ω, we have $(N - M) \cap \cap C \neq \emptyset$ and hence $C \neq \emptyset$.

26.5. 26.5.1. *Let either $P = \mathbf{S}_2$ or $P = \mathbf{E}_2$. Let F be a closed set in P. Then the constituants of $P - F$ coincide with its components. They are open.*

Proof: P is complete by 15.1.3, 17.2.1 and 17.10.2. P is locally connected by 22.1.8 and 22.1.14. $P - F$ is open in P. Thus, the space $P - F$ is topologically complete by 15.5.2 and locally connected by 22.1.3 so that our theorem follows by 22.1.4 and 22.3.2.

It is easy to prove the following theorem

26.5.2. *Let $M \subset \mathbf{E}_2$ be a bounded set. Then $\mathbf{E}_2 - M$ has exactly one unbounded component — denote it by H. The set $\mathbf{S}_2 - \sigma(M)$ has the following components: first, $\sigma(H) \cup (\omega)$, secondly, all the sets $\sigma(K)$ where K are bounded components of $\mathbf{E}_2 - M$.*

26.5.3. *Let either $P = \mathbf{S}_2$ or $P = \mathbf{E}_2$. Let $C \subset P$ be a simple arc. Then $P - C$ is connected and $B(P - C) = C$.*

Proof: I. The set $P - C$ is connected, since, by 26.4.3 and 26.4.4, it is a semi-continuum.

II. By 10.3.2 we have $B(P - C) \subset C$. If there is a point $a \in C - B(P - C)$, it is evidently an interior point of C in P. By 26.4.5 this is impossible for $P = \mathbf{S}_2$. By means of the stereographical projection it follows easily that this is also impossible in the case of $P = \mathbf{E}_2$.

26.5.4. (Jordan theorem.) *Let either $P = \mathbf{S}_2$ or $P = \mathbf{E}_2$. Let $C \subset P$ be a simple loop. Then $P - C$ has exactly two components; denote them by G_1, G_2. We have $B(G_1) = B(G_2) = C$.*

Proof: I. Choose $a \in C$, $b \in C$, $a \neq b$. By 21.1.2 there are simple arcs C_1, C_2 such that

$$C_1 \cup C_2 = C\,, \quad C_1 \cap C_2 = (a) \cup (b)\,.$$

The sets C_1, C_2 are closed and connected. $C_1 \cap C_2$ has two components. Thus, by 26.3.2 (see also 26.5.2), $P - C$ has at least two components.

II. If the set $P - C$ had more than two components, there would be, by 26.5.1, points α, β, γ in $P - C$ such that C would cut P between any two of them. If $P = \mathbf{S}_2$, we obtain a contradiction with theorem 26.3.1, since C_1, C_2 are closed in $C = C_1 \cup C_2$, $C_1 \cap C_2$ has two components and (by 26.4.3) neither C_1 nor C_2 cuts $\mathbf{S}_2$ between some two of the points α, β, γ. By 26.1.5 it follows easily that we may obtain an analogous contradiction also in the case of $P = \mathbf{E}_2$.

III. Thus, $P - C$ has exactly two components G_1, G_2. We have to prove that $B(G_1) = B(G_2) = C$. Choose $a_1 \in G_1$, $a_2 \in G_2$. Then C separates a_1 from a_2 in P. If $D \subset C \neq D$, then it follows by 26.4.3 and 26.4.4 that D does not separate a_1 from a_2 in P. Thus, C is an irreducible cut of the locally connected space P between the points a_1, a_2, so that by 22.1.10 there are connected sets Γ_1, Γ_2 such that $a_1 \in \Gamma_1$, $a_2 \in \Gamma_2$, $\Gamma_1 \cup \Gamma_2 \subset P - C$, $B(\Gamma_1) = B(\Gamma_2) = C$.

By 22.1.9, Γ_1, Γ_2 are components of $P - C$. Thus, $\Gamma_1 = G_1$, $\Gamma_2 = G_2$ and hence $B(G_1) = B(G_2) = C$.

Let $C \subset \mathbf{E}_2$ be a simple loop. By 26.5.2 and 26.5.4, $\mathbf{E}_2 - C$ has exactly one bounded and exactly one unbounded component. The bounded component of $\mathbf{E}_2 - C$ is called the *interior* of the loop C; denote it by

$$V(C)\,.$$

The other component of $\mathbf{E}_2 - C$ is called the *exterior* of C; denote it by

$$W(C).$$

By 26.5.1 the sets $V(C)$ and $W(C)$ are open. By 26.5.4,

$$B[V(C)] = B[W(C)] = C.$$

26.6. 26.6.1. *Let $Q \subset \mathbf{S}_2$. Define the set $L(Q) \subset \mathbf{S}_2$ as in* 22.2 (*putting $P = \mathbf{S}_2$*). *Let $a \in \mathbf{S}_2 - Q$, $b \in \mathbf{S}_2 - Q$, $a \neq b$. If Q does not cut the sphere between the points a, b, then neither does the set*

$$M = Q \cup L(Q) - [(a) \cup (b)].$$

Proof: We may assume that $b = \omega$ (see 17.10.3), so that $Q \subset \mathbf{S}_2 - (\omega)$. There exists a set $Q_0 \subset \mathbf{E}_2$ and a point $\alpha \in \mathbf{E}_2 - Q_0$ such that $\sigma(\alpha) = a$, $\sigma(Q_0) = Q$. Define $L(Q_0) \subset \mathbf{E}_2$ as in 22.2. By 26.1.1 it follows easily that $\sigma[L(Q_0)] = L(Q) - (\omega)$. By 26.2.2 the mapping $\pi(Q_0; \alpha)$ is inessential, so that by 24.4.1 the mapping $\pi[Q_0 \cup L(Q) - (\alpha); \alpha]$ is also inessential. On the other hand $\sigma[Q_0 \cup L(Q_0) - - (\alpha)] = M$, so that, by 26.2.2, M does not cut the sphere between the points a, ω.

26.6.2. *Let $Q \subset \mathbf{S}_2$, $a \in \mathbf{S}_2 - Q$, $b \in \mathbf{S}_2 - Q$, $a \neq b$. Let Q be locally connected. If Q does not cut the sphere between the points a, b, then there exists a set $M \subset \subset \mathbf{S}_2 - [(a) \cup (b)]$ such that* [1] *$Q \subset M \subset \bar{Q}$,* [2] *$M$ is $\mathbf{G}_\delta(\mathbf{S}_2)$,* [3] *$M$ is locally connected,* [4] *M does not cut the sphere between the points a, b.*

Proof: Put

$$M = L(Q) - [(a) \cup (b)].$$

By 22.2.2, $Q \subset M$. By the definition of $L(Q)$ we have $M \subset \bar{Q}$. By 22.2.3 (see also 13.1.2) the set M is $\mathbf{G}_\delta(\mathbf{S}_2)$. By 22.2.4, M is locally connected. As $Q \subset M$, by 26.6.1, M does not cut the sphere between the points a, b.

26.6.3. *Let either $P = \mathbf{S}_2$ or $P = \mathbf{E}_2$. Let $Q \subset P$, $a \in P - Q$, $b \in P - Q$, $a \neq b$. If $P = \mathbf{E}_2$, let Q not be bounded. Let Q be $\mathbf{G}_\delta(P)$. Let Q be locally connected. Let Q cut P between a and b. Then there is a simple loop $C \subset Q$ cutting P between a and b.*

Proof will be done e.g. for $P = \mathbf{S}_2$ (the case $P = \mathbf{E}_2$ may be transferred to $P = \mathbf{S}_2$ by means of theorem 26.1.5). By 17.10.3 we may assume that $b = \omega$. There exists a set $Q_0 \subset \mathbf{E}_2$ and a point $\alpha \in \mathbf{E}_2 - Q$ such that $\sigma(\alpha) = a$, $\sigma(Q_0) = Q$. By 26.1.1 it follows easily that Q_0 is locally connected and that it is $\mathbf{G}_\delta(\mathbf{E}_2)$, so that Q_0 is a topologically complete space (see 15.1.3 and 15.5.2). By 26.2.2, the mapping $\pi(Q_0; \alpha)$ is essential. Hence, by 24.4.2, there exists a simple loop $C_0 \subset Q_0$ such that the mapping $\pi(C_0; \alpha)$ is essential. Then $C = \sigma(C_0)$ is a simple loop, we have $C \subset Q$ and, by 26.2.2, C cuts $\mathbf{S}_2$ between the points a, ω.

26.6.4. *Let $Q \subset \mathbf{S}_2$, $a \in \mathbf{S}_2 - Q$, $b \in \mathbf{S}_2 - Q$, $a \neq b$. Let Q be locally connected. Let Q cut the sphere between the points a, b. Let no set $X \subset Q \neq X$ closed in Q cut the sphere between the points a, b. Then Q is a simple loop.*

Proof: Put $M = L(Q) - [(a) \cup (b)]$. By the definition of $L(Q)$ (see 22.2) we obtain $M \subset \bar{Q}$. The set M is $\mathbf{G}_\delta(\mathbf{S}_2)$ by 22.2.3. By 22.2.4 M is locally connected. By 22.2.2, $Q \subset M$, so that M cuts the sphere between the points a, b. Thus, by 26.6.3, there is a simple loop $C \subset M$ which cuts the sphere between the points a, b.

It suffices to prove that $Q = C$. Let, on the contrary, $Q \neq C$. If $Q \subset C$, there exists a set $N \subset \mathbf{E}_1$ homeomorphic with Q. This is, however, impossible by 26.4.4, as Q cuts the sphere between the points a, b. Thus, C does not contain Q, so that C is not equal to M. As $C \subset M$, there is a point $c \in M - C$. The set C is compact, hence (see 17.2.2) it is closed in $\mathbf{S}_2$. Thus (see 10.1.2), there is a neighborhood U of the set C such that $c \in \mathbf{S}_2 - \bar{U}$. If we had $Q \subset \bar{U}$, then

$$c \in M \subset L(Q) \subset \bar{Q} \subset \bar{U},$$

which is impossible. Thus $Q \cap \bar{U} \neq Q$. On the other hand, $X = Q \cap \bar{U}$ is closed in Q. Hence, $Q \cap \bar{U}$ does not cut the sphere between the points a, b. 26.6.1 yields that the set

$$M_0 = (Q \cap \bar{U}) \cup L(Q \cap \bar{U}) - [(a) \cup (b)]$$

does not cut the sphere between the points a, b either. As $C \subset M \subset L(Q)$ and as $\bar{U} \supset C$ is closed, we obtain easily by the definition of $L(Q)$, $L(Q \cap \bar{U})$ that $C \subset L(Q \cap \bar{U})$. Thus, $C \subset M_0$, so that C does not cut the sphere between the points a, b. This is a contradiction.

26.6.5. *Let $Q \subset \mathbf{S}_2$. Let the set Q be $\mathbf{G}_\delta(\mathbf{S}_2)$. Let Q be locally connected. Then the constituants of $\mathbf{S}_2 - Q$ coincide with its components.*

Proof: Let a, b belong to distinct constituants of $\mathbf{S}_2 - Q$ so that Q cuts the sphere between them; let, however, both points a, b belong to the same component K of $\mathbf{S}_2 - Q$. We have to reach a contradiction. By 26.6.3 there exists a simple loop $C \subset Q$ which cuts the sphere between points a, b. Hence (see 26.5.1), a, b are not in the same component of $\mathbf{S}_2 - C$. This is a contradiction, as both a, b belong to the connected set

$$K \subset \mathbf{S}_2 - Q \subset \mathbf{S}_2 - C.$$

26.6.6. *Let $Q \subset \mathbf{S}_2$. Let Q be locally connected. Let M be a constituant of $\mathbf{S}_2 - Q$. Let*

$$a \in \bar{M} - M, \quad b \in M.$$

Then there exists a continuum K such that

$$a \in K, \quad b \in K, \quad K - (a) \subset M.$$

Proof: Define $L(Q)$ as in 22.2. By 8.2.1 there exists a sequence $\{c_n\}_1^\infty$ such that $c_n \in M$ for every n and $c_n \to a$. By 22.2.2 we have $Q \subset L(Q)$. Put

$$Q_0 = L(Q) - [(a) \cup (b) \cup \bigcup_{n=1}^{\infty} (c_n)] .$$

Q_0 is $\mathbf{G}_\delta(\mathbf{S}_2)$ by 22.2.3 (see also ex. 13.11). $Q_0 \cup (a)$ is locally connected by 22.2.4, so that Q_0 is locally connected by 22.1.3. Let M_0 be the component of $\mathbf{S}_2 - Q_0$ containing the point b. By 18.2.2 (see also 8.7.1) we have

$$\overline{M}_0 - M_0 \subset Q_0 .$$

By 26.6.5, M_0 is a semicontinuum, so that M is a constituant of $\mathbf{S}_2 - Q_0$ (see 19.5.8). For $n = 1, 2, 3, \ldots$ we have $b \in M$, $c_n \in M$, so that Q does not cut the sphere between the points b, c_n. By 26.6.1, the set

$$L(Q) - [(b) \cup (c_n)] = Q \cup L(Q) - [(b) \cup (c_n)]$$

does not cut the sphere between the points b, c_n either. On the other hand $Q_0 \subset L(Q) - [(b) \cup (c_n)]$. Thus, Q_0 does not cut the sphere between the points b, c_n, so that both the points b, c_n belong to the same constituant of the set $\mathbf{S}_2 - Q_0$, i.e. $c_n \in M_0$. As $c_n \to a$, we have $a \in \overline{M}_0$. As $a \in \mathbf{S}_2 - Q_0$, $\overline{M}_0 - M_0 \subset Q_0$, we have $a \in M_0$. As $a \in M_0$, $b \in M_0$, $a \neq b$ and as M_0 is a semicontinuum, there exists a continuum $K_0 \subset M_0$ containing both a and b. Thus, by 19.4.1, there is an irreducible continuum K between the points a, b such that $K \subset M_0$.

It remains to be proved that $K - (a) \subset M$. The set $K - (a)$ is connected by 19.4.2. Moreover, $K \subset M_0 \subset \mathbf{S}_2 - Q_0$, so that $K - (a) \subset \mathbf{S}_2 - Q_1$, where

$$Q_1 = L(Q) - [(b) \cup \bigcup_{n=1}^{\infty} (c_n)] .$$

The set Q_1 is, similar to Q_0, a locally connected $\mathbf{G}_\delta(\mathbf{S}_2)$-set. Since $K - (a)$ is a connected subset of $\mathbf{S}_2 - Q_1$ and since $b \in K - (a)$, we have $K - (a) \subset N$, where N is the component of $\mathbf{S}_2 - Q_1$ containing the point b. By 26.6.5, N is a semicontinuum. On the other hand,

$$Q_1 \supset L(Q) - M \supset Q .$$

Thus, N is a subset of the constituant of $\mathbf{S}_2 - Q$ containing the point b, i.e. $N \subset M$, so that really $K - (a) \subset M$.

26.6.7. *Let $Q \subset \mathbf{S}_2$ be a locally connected set. Then the constituants of $\mathbf{S}_2 - Q$ are closed in $\mathbf{S}_2 - Q$.*

Proof: Let M be a constituant of $\mathbf{S}_2 - Q$. We have to prove (see 8.7.1), that $\overline{M} - M \subset Q$. On the other hand, let there be a point

$$a \in \overline{M} - (M \cup Q) .$$

By 26.6.6 there exists a continuum K such that $a \in K$, $K - (a) \subset M$. As $a \in \mathbf{S}_2 - Q$, $M \subset \mathbf{S}_2 - Q$, we have $K \subset \mathbf{S}_2 - Q$. As K is a continuum, K is a subset of one constituant of $\mathbf{S}_2 - Q$. Since $\emptyset \neq K - (a) \subset M$ and since M is a constituant of $\mathbf{S}_2 - Q$, we have $K \subset M$ and hence $a \in M$ which is a contradiction.

26.7. 26.7.1. *Let $M \subset \mathbf{S}_2$ be a closed set. Let M be locally connected. Let $\varepsilon > 0$. Then the set $\mathbf{S}_2 - M$ has only a finite number of components with diameter greater than ε.*

Proof: I. Otherwise there exists (see 17.10.2) a sequence $\{a_n\}_1^\infty$ and a point b such that $\lim a_n = b$, all the a_n belong to $\mathbf{S}_2 - M$ and, if G_n is the component of $\mathbf{S}_2 - M$ containing a_n, the sets G_n are mutually distinct and every one of them is more than ε in diameter.

II. We have $b \in M$. In fact, otherwise (see 22.1.4 and 22.1.14) there exists an open connected $G \subset \mathbf{S}_2 - M$ such that $b \in G$. Since $\lim a_n = b$, there exists an index n such that $a_n \in G$, $a_{n+1} \in G$. Since a_n and a_{n+1} do not belong to the same component of $\mathbf{S}_2 - M$ and since $G \subset \mathbf{S}_2 - M$ is connected, this is impossible.

III. By 23.1.6 (see also 17.2.2) there exists a finite system $\mathfrak{A}$ of point sets such that [1] the sets $A \in \mathfrak{A}$ are closed and connected, [2] the union of all the sets $A \in \mathfrak{A}$ is equal to M, [3] every set $A \in \mathfrak{A}$ is less than $\frac{1}{6}\varepsilon$ in diameter.

Divide $\mathfrak{A}$ into three parts $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ as follows: [1] $A \in \mathfrak{A}$ belongs to $\mathfrak{A}_1$, if and only if $b \in A$, [2] if $A \in \mathfrak{A} - \mathfrak{A}_1$ then $A \in \mathfrak{A}_2$ if there is, and $A \in \mathfrak{A}_3$ if there is not a set $B \in \mathfrak{A}_1$ with $A \cap B \neq \emptyset$.

Denote by C_i $(i = 1, 2, 3)$ the union of all the sets $A \in \mathfrak{A}_i$. The sets C_1, C_2, C_3 are then closed and every one of them has a finite number of components; we have $C_1 \cup C_2 \cup C_3 = M$, $b \in C_1 - (C_2 \cup C_3)$, $C_1 \cap C_3 = \emptyset$ and finally,

$$\varrho(b, x) < \frac{1}{6}\varepsilon \quad \text{for} \quad x \in C_1,$$

$$\varrho(b, x) < \frac{1}{3}\varepsilon \quad \text{for} \quad x \in C_2.$$

Since b does not belong to the closed set $C_2 \cup C_3$, there is a $\delta > 0$ such that $\delta < \frac{1}{8}\varepsilon$ and such that

$$\varrho(b, x) > \delta \quad \text{for} \quad x \in C_2 \cup C_3 .$$

IV. Denote by T_1 the set of all $x \in \mathbf{S}_2$ with $\varrho(b, x) \leqq \delta$, so that $T_1 \cap (C_2 \cup C_3) = \emptyset$. Denote by T_2 the set of all $x \in \mathbf{S}_2$ with $\varrho(b, x) \geqq \frac{1}{2}\varepsilon$, so that $T_2 \cap (C_1 \cup C_2) = \emptyset$. Evidently, T_1, T_2 are continua.

V. Since $\lim a_n = b$, $\delta > 0$, there is an index p such that

$$n \geqq p \quad \text{implies} \quad \varrho(a_n, b) < \delta .$$

VI. If $p \leqq m < n$, then $C_2 \cup C_3$ does not cut the sphere between the points a_m, a_n as $T_1 \subset \mathbf{S}_2 - (C_2 \cup C_3)$ is a continuum containing both the points a_m, a_n.

VII. If $p \leqq m < n$, then $C_1 \cup C_2$ does not cut the sphere between the points a_m, a_n. In fact, if G_m, G_n are more than ε in diameter, there exist points $\alpha_m \in G_m$, $\alpha_n \in G_n$ such that $\varrho(a_m, \alpha_m) > \frac{1}{2}\varepsilon$, $\varrho(a_n, \alpha_n) > \frac{1}{2}\varepsilon$. We have

$$\varrho(b, \alpha_m) \geqq \varrho(a_m, \alpha_m) - \varrho(b, a_m) > \frac{1}{2}\varepsilon - \delta > \frac{1}{3}\varepsilon,$$

hence $\alpha_m \in T_2$ and similarly $\alpha_n \in T_2$.

By 26.5.1 there exists a continuum $K_m \subset G_m$ containing both points a_m, α_m and a continuum $K_n \subset G_n$ containing both points a_n, α_n. The set $K = K_m \cup T_2 \cup K_n$ is a continuum contained in $\mathbf{S}_2 - (C_1 \cup C_2)$ and containing both points a_m, a_n, so that $C_1 \cup C_2$ does not cut the sphere between these points.

VIII. If $m < n$, then (see 26.5.1) M cuts the sphere between the points a_m, a_n. Since

$$M = (C_1 \cup C_2) \cup (C_2 \cup C_3)$$

with closed summands, we obtain by VI and VII and 26.3.1 that the set

$$(C_1 \cup C_2) \cap (C_2 \cup C_3) = C_2$$

has infinitely many components, which is a contradiction (see III).

26.7.2. *Let $M \subset \mathbf{S}_2$ be a closed set. Let M be locally connected. Let G be a component of $\mathbf{S}_2 - M$. Let $G \subset N \subset \bar{G}$. Then N is locally connected.*

Proof: I. G is open in $\mathbf{S}_2$ (see 22.1.4 and 22.1.14), so that, by 22.1.2, N is locally connected in every point $x \in G$.

II. Let $a \in N - G$. We have to prove that N is locally connected at a. Choose an $\varepsilon > 0$. By 22.1.1 it suffices to prove that there is a $\delta > 0$ such that for every $x \in N$ with $\varrho(a, x) < \delta$ there is a connected $S \subset N$ with $a \in S$, $x \in S$, $d(S) \leqq 2\varepsilon$.

Obviously it suffices to prove this for ε such that there exists a point $b \in G$ with $\varrho(a, b) > \varepsilon$.

III. Denote by T the set of all $x \in \mathbf{S}_2$ with $\varrho(a, x) \geqq \varepsilon$, so that $b \in T$. It is easy to prove that $M \cup T$ is closed and locally connected.

If K is a component of $\mathbf{S}_2 - (M \cup T)$, then K is a connected subset of $\mathbf{S}_2 - M$, so that (see 18.2.5) we have either $K \subset G$ or $K \cap G = \emptyset$.

IV. Let K be a component of $\mathbf{S}_2 - (M \cup T)$ such that $K \subset G$. Then $\bar{K} \cap T \neq \emptyset$. Assume the contrary. By 18.2.2 (see also 8.7.1), $\bar{K} - K \subset M \cup T$, and hence (see 10.3.2 and 22.1.4) $B(K) = \bar{K} - K \subset M$. Since K is a connected subset of the open $\mathbf{S}_2 - M$, we see by 22.1.9 that K is a component of $\mathbf{S}_2 - M$. As $K \subset G$, we have $K = G$ which is a contradiction, since $K \cap T = \emptyset$, $b \in G \cap T$.

V. Denote by $\mathfrak{A}$ the system of all components K of $\mathbf{S}_2 - (M \cup T)$ such that $K \subset G$. By III, $G - (M \cup T) = G - T$ is the union of all sets $K \in \mathfrak{A}$. Divide $\mathfrak{A}$ into three parts $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ as follows: $K \in \mathfrak{A}_1$ if $a \in \overline{K}$. If $K \in \mathfrak{A} - \mathfrak{A}_1$, then $K \in \mathfrak{A}_2$ ($K \in \mathfrak{A}_3$, respectively) if the diameter of K is greater than (less than or equal to) $\frac{1}{2}\varepsilon$.

Denote by C_i ($i = 1, 2, 3$) the union of all $K \in \mathfrak{A}_i$, so that

$$G - T = C_1 \cup C_2 \cup C_3.$$

VI. For every $K \in \mathfrak{A}_1$, $K \cup (a)$ is connected by 18.1.7. Thus, $C_1 \cup (a)$ is connected by 18.1.5.

VII. $\mathfrak{A}_2$ is a finite system by 26.7.1, so that $\overline{C}_2$ is the union of all $\overline{K}$ with $K \in \mathfrak{A}_2$. Thus, a does not belong to $\overline{C}_2$. It follows easily by IV that a does not belong to $\overline{C}_3$ either. Hence, there is a number $\delta > 0$ such that $\delta < \varepsilon$ and

$$x \in \overline{C_2 \cup C_3} \quad \text{implies} \quad \varrho(a, x) \geqq \delta.$$

VIII. Let $x \in N$, $\varrho(a, x) < \delta$. Since $\delta < \varepsilon$, x does not belong to T. We have

$$x \in N \subset \overline{G} = \overline{G - T} \cup \overline{T} = \overline{G - T} \cup T,$$

and hence

$$x \in \overline{G - T} = \overline{C}_1 \cup \overline{C_2 \cup C_3},$$

so that, by VII, $x \in \overline{C}_1$.

IX. Thus, the set $S = C_1 \cup (a) \cup (x)$ is connected by VI and 18.1.7. Evidently $S \subset N$, $a \in S$, $x \in S$. Moreover, $S \subset \mathbf{S}_2 - T$ so that $d(S) \leqq 2\varepsilon$.

26.7.3. *Let $M \subset \mathbf{S}_2$ be a closed set. Let M be locally connected. Let G be a component of $S_2 - M$. Let $a \in \overline{G} - G$. Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for every $b \in G$ with $\varrho(a, b) < \delta$ there is a simple arc C with end points a, b such that $C - (a) \subset G$ and the diameter of C is less than ε.*

Proof: G is open by 22.1.4 and 22.1.14, so that $G \cup (a)$ is $\mathbf{G}_\delta(\mathbf{S}_2)$ by 13.1.3 and 13.2. $G \cup (a)$ is locally connected by 26.7.2. $[G \cup (a)] \cap \Omega(a, \frac{1}{3}\varepsilon) = H$ is open in $G \cup (a)$ and hence locally connected (see 22.1.3). Let Γ be the component of H containing a. Γ is locally connected by 22.1.6 and open in $G \cup (a)$ by 22.1.4. Thus, Γ is $\mathbf{G}_\delta(\mathbf{S}_2)$ by 8.7.5, 13.1.1 and 13.1.2. Consequently, Γ is a topologically complete space by 15.5.2, 17.2.1 and 17.10.2. Certainly, Γ is connected and we know that it is locally connected.

Since Γ is open in $G \cup (a)$, there is a $\delta > 0$ such that

$$x \in G, \quad \varrho(a, x) < \delta \quad \text{imply} \quad x \in \Gamma.$$

Let $b \in G$, $\varrho(a, b) < \delta$. Then $b \neq a$, $b \in \Gamma$. By 22.3.1 there is a simple arc $C \subset \Gamma$ with end points a, b. We have $C - (a) \subset \Gamma - (a) \subset G$. As $\Gamma \subset \Omega(a, \frac{1}{3}\varepsilon)$, we have $d(C) \leqq \frac{2}{3}\varepsilon < \varepsilon$.

26.7.4. *Let $M \subset \mathbf{S}_2$ be a closed set. Let G be a component of $\mathbf{S}_2 - M$. Put $H = B(G)$, so that* (see 22.1.9 and 22.1.14) *$H \subset M$. Let $a \in H$. If both the sets*

$$G \cup (a),\ M$$

are locally connected at the point a, then H is also locally connected at the point a.

Proof: Assume the contrary. By 22.1.1 there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is a $b \in H$ with $\varrho(a, b) < \delta$ and such that every connected subset of H containing both a and b is greater than or equal to ε in diameter.

Put $\Omega = \Omega(a, \frac{1}{3}\varepsilon)$, so that Ω is a neighborhood of a in $\mathbf{S}_2$. Evidently Ω is homeomorphic to $\mathbf{E}_2$, therefore locally connected by 22.1.8 and unicoherent by 25.2.3.

Let K be the component of $[G \cup (a)] \cap \Omega$ containing the point a. Let L be the component of $M \cap \Omega$ containing the point a. Since $G \cup (a)$, M are locally connected at a, there is a number $\delta > 0$ such that

$$x \in G,\quad \varrho(a, x) < \delta \quad \text{imply} \quad x \in K,$$
$$x \in M,\quad \varrho(a, x) < \delta \quad \text{imply} \quad x \in L.$$

The definition of ε yields easily the existence of a point $b \in H \cap \Omega$ such that $\varrho(a, b) < \delta$ and the fact that the points a, b belong to distinct components of $H \cap \bar{\Omega}$. On the other hand, $H \cap \bar{\Omega}$ is compact (see 10.3.1, 17.2.2 and 17.10.2), so that (see 19.1.5) a and b belong to distinct quasicomponents of $H \cap \bar{\Omega}$ and hence also in distinct quasicomponents of $H \cap \Omega \subset H \cap \bar{\Omega}$. Thus, the set $\Omega - (H \cap \Omega) = \Omega - H$ separates the point a from the point b in Ω. Thus (see 22.1.12), there exists a $C \subset \Omega - H$ which is an irreducible cut of Ω between the points a, b. C is connected by 25.1.2.

Since $b \in H \subset M$, $\varrho(a, b) < \delta$, we have $b \in L$ and, of course, also $a \in L$. Thus L is a connected subset of Ω containing both a and b. Since C separates a from b in Ω, we have $L \cap C \neq \emptyset$. On the other hand, $L \subset M \subset \mathbf{S}_2 - G$, so that

$$C - G \neq \emptyset.$$

If $x \in G - K$, we have $\varrho(a, x) \leqq \delta$. Hence also

$$x \in \overline{G - K} \quad \text{implies} \quad \varrho(a, x) \leqq \delta.$$

On the other hand, $b \in H \subset \bar{G} \subset \overline{G - K} \cup \bar{K}$, $\varrho(a, b) < \delta$, so that $b \in \bar{K}$. Thus, $K \cup (b)$ is a connected (see 18.1.7) subset of Ω containing both points a, b. Since C separates a from b in Ω, we have $[K \cup (b)] \cap C \neq \emptyset$ and hence $K \cap C \neq \emptyset$. On the other hand, $K \subset G \cup (a)$, so that $G \cap C \neq \emptyset$.

As $C - G \neq \emptyset \neq G \cap C$ and as C is connected, $H \cap C \neq \emptyset$ by 18.1.8. This is a contradiction.

26.7.5. *Let $M \subset \mathbf{S}_2$ be a closed set. Let M be locally connected. Let G be a component of $\mathbf{S}_2 - M$. Then $B(G)$ is locally connected.*

This follows by 26.7.2 and 26.7.4.

26.8. 26.8.1. *Let $M \subset \mathbf{E}_2$ be a compact set. Let f be a continuous mapping of M into $\mathbf{S}_1$. Then there exists a finite number of points $a_\lambda \in \mathbf{E}_2 - M$ $(1 \leqq \lambda \leqq k)$ and integers n_λ $(1 \leqq \lambda \leqq k)$ such that the mapping*

$$f \cdot \prod_{\lambda=1}^{k} [\pi(M; a_\lambda)]^{n_\lambda}$$

is inessential.

Proof: I. If $a \in \mathbf{E}_2$, $b \in \mathbf{E}_2$, $a \neq b$, define a segment $S(a, b)$ similarly as in exercises to § 19; $S(a, b)$ is, of course, a simple arc with end points a, b.

II. If $a = a_1 + ia_2 \in \mathbf{E}_2$ and if s is a positive number, denote by $\Delta(a, 2s)$ the set of all $x + iy \in \mathbf{E}_2$ with $|x - a_1| \leqq s$, $|y - a_2| \leqq s$. The set $\Delta(s, 2s)$ will be called a *square* and the point a is said to be its *centre*. *Edges* of the square $\Delta(a, 2s)$ are the segments

$$S(a - s - si,\ a - s + si), \qquad S(a + s - si,\ a + s + si),$$
$$S(a - s - si,\ a + s - si), \qquad S(a - s + si,\ a + s + si).$$

The union of all four edges of a square $\Delta(a, 2s)$ is said to be its *perimeter* and is denoted by $\mathbf{D}(a, 2s)$. Evidently $\mathbf{D}(a, 2s)$ is a simple loop, and $\Delta(a, 2s) - \mathbf{D}(a, 2s)$ is its interior. The points

$$a - s - si, \quad a - s + si, \quad a + s - si, \quad a + s + si$$

are termed the *vertices* of the square.

III. By 24.2.15 there exists an open set $G \subset \mathbf{E}_2$ such that $M \subset G$ and there exists a continuous mapping g of G into $\mathbf{S}_1$ with $|f(z) - g(z)| < 2$ for every $z \in M$.

IV. 17.2.3 and 17.3.4 yield the existence of an integer $m > 1$ such that: [1] for every $x + iy \in M$, $|x| < m$, $|y| < m$, [2] if $\varrho(x + iy, M) < 2m^{-1}$, then $x + iy \in G$.

Order the points

$$\frac{\mu + \nu i}{m} + \frac{1 + i}{2m} \qquad (-m^2 \leqq \mu < m^2,\ -m^2 \leqq \nu < m^2)$$

into a one-to-one sequence $\{c_\lambda\}_1^{4m^4}$. Put $\Delta_\lambda = \Delta(c_\lambda, m^{-1})$, $D_\lambda = \mathbf{D}(c_\lambda, m^{-1})$, $K = \bigcup_{\lambda=1}^{4m^4} \Delta_\lambda$, so that $K = \Delta(0, 2m)$.

Denote by K_0 the union of all Δ_λ $(1 \leqq \lambda \leqq 4m^4)$ with $\Delta_\lambda \cap M \neq \emptyset$. Evidently

$$M \subset K_0 \subset G \cap K.$$

Denote by K_1 the set obtained from K_0 by adjoining of all the vertices of all the squares Δ_λ $(1 \leqq \lambda \leqq 4m^4)$. Put $K_2 = K_0 \cup \bigcup_{\lambda=1}^{4m^4} D_\lambda = K_1 \cup \bigcup_{\lambda=1}^{4m_4} D_\lambda$.

V. Evidently there exists a continuous mapping g_1 of K_1 into $\mathbf{S}_1$ such that $g_1(z) = g(z)$ for every $z \in K_0$. If $S(a, b)$ is an edge of some of the squares Δ_λ ($1 \leqq \lambda \leqq 4m^4$), there is evidently a continuous mapping h of $S(a, b)$ into $\mathbf{S}_1$ such that $h(a) = g_1(a)$, $h(b) = g_1(b)$. Consequently there exists a continuous mapping g_2 of K_2 into $\mathbf{S}_1$ such that the partial mapping $(g_2)_{K_1}$ coincides with g_1.

VI. The square Δ_λ ($1 \leqq \lambda \leqq 4m^4$) is said to be *free*, if Δ_λ is not a subset of K_2. (Evidently, $\Delta_\lambda \subset K_2$ if and only if $\Delta_\lambda \subset K_0$, i.e. if and only if $\Delta_\lambda \cap M \neq \emptyset$.) Denote by Λ the set of $\lambda (1 \leqq \lambda \leqq 4m^4)$ for which the square Δ_λ is free.

Let $\lambda \in \Lambda$, so that $\Delta_\lambda \cap K_2 = D_\lambda$. D_λ is a simple loop. It is easy to prove that D_λ may be oriented in such a way that the mapping $\pi(D_\lambda; c_\lambda)$ has degree (see 24.3.2) equal to one. Denote by n_λ the degree of $(g_2)_{D_\lambda}$, so that n_λ is an integer.

VII. Put

$$k = g_2 \cdot \prod_{\lambda \in \Lambda} [\pi(K_2; c_\lambda)]^{-n_\lambda}.$$

If $\lambda \in \Lambda$, then the degree of the mapping $(g_2)_{D_\lambda}$ is n_λ, the degree of the mapping $\pi(D_\lambda; c_\lambda)$ is $+1$, and it is easy to prove that for $\mu \in \Lambda$, $\mu \neq \lambda$, the degree of the mapping $\pi(D_\mu; c_\lambda)$ is zero. Thus (see 24.3.4) the degree of k_{D_λ} is zero so that it is inessential by 24.3.3. Thus, for every $\lambda \in \Lambda$ there is a continuous mapping φ_λ of D_λ into $\mathbf{E}_1$ such that $e^{i\varphi_\lambda(z)} = k(z)$ for every $z \in D_\lambda$. By 14.8.3 there exists a continuous mapping ψ_λ of Δ_λ into $\mathbf{E}_1$ such that $\psi_\lambda(z) = \varphi_\lambda(z)$ for every $z \in D_\lambda$.

VIII. Since $K = K_2 \cup \bigcup_{\lambda \in \Lambda} \Delta_\lambda$, $\Delta_\lambda \cap K_2 = D_\lambda$ for $\lambda \in \Lambda$, there exists evidently a continuous mapping v of K into $\mathbf{S}_1$ such that [1] $v(z) = k(z)$ for every $z \in K_2$, [2] $v(z) = e^{i\psi_\lambda(z)}$ for $\lambda \in \Lambda$, $z \in \Delta_\lambda$.

K is obviously a cartesian product of two simple arcs, so that the mapping v is inessential by 24.3.1 and 24.5.1. Put

$$u = f \cdot \prod_{\lambda \in \Lambda} [\pi(M; c_\lambda)]^{-n_\lambda},$$

so that u is a continuous mapping of M into $\mathbf{S}_1$. For $z \in M$ we have $g_2(z) = g(z)$; therefore $z \in M$ implies $|u(z) - k(z)| = |f(z) - g(z)| < 2$, and hence u is inessential by 24.2.6 and 24.2.8.

26.8.2. *Let $P = \mathbf{S}_2$ or $P = \mathbf{E}_2$. Let $M \subset P$ be a compact set. Let $k = 1, 2, 3, \ldots$. The set $P - M$ has more than k components if and only if there exist k continuous mappings f_λ ($1 \leqq \lambda \leqq k$) of M into $\mathbf{S}_1$ such that the mapping*

$$\prod_{\lambda=1}^{k} f_\lambda^{n_\lambda}$$

cannot be inessential, if the integers n_λ ($1 \leqq \lambda \leqq k$) are not all equal to zero.

Proof: I. By 17.2.3, 17.10.3, 24.5.4, 26.1.1 and 26.5.2 it suffices to prove the theorem under the assumption of $P = \mathbf{E}_2$.

II. Let $\mathbf{E}_2 - M$ have more than k components. By 26.5.2 there exist mutually distinct bounded components K_λ $(1 \leqq \lambda \leqq k)$ of $\mathbf{E}_2 - M$. Choose an $a_\lambda \in K_\lambda$ and put $f_\lambda = \pi(M; a_\lambda)$. By 26.2.4 (see also 26.5.1) we see easily that the mapping $\prod_{\lambda=1}^{k} f_\lambda^{n_\lambda}$ is inessential only if $n_1 = \ldots = n_k = 0$.

III. Let there exist continuous mappings f_λ $(1 \leqq \lambda \leqq k)$ of M into $\mathbf{S}_1$ such that $\prod_{\lambda=1}^{k} f_\lambda^{n_\lambda}$ is essential whenever at least one of n_λ is not zero. We have to prove that the set $\mathbf{E}_2 - M$ has more than k components. Assume the contrary.

By 26.5.2 (see also 17.2.3), $\mathbf{E}_2 - M$ has exactly one unbounded component. Let us denote this by K_0 and the remaining components of $\mathbf{E}_2 - M$ by K_μ $(1 \leqq \mu \leqq h)$, so that, by the assumption,

$$0 \leqq h < k.$$

For every λ $(1 \leqq \lambda \leqq k)$ there is, by 26.8.1, a finite number of points and integers $c_\nu^{(\lambda)} \in \mathbf{E}_2 - M$ $(1 \leqq \nu \leqq r_\lambda)$ and $m_\nu^{(\lambda)}$ $(1 \leqq \nu \leqq r_\lambda)$ such that the mapping

$$f_\lambda \cdot \prod_{\nu=1}^{r_\lambda} [\pi(M; c_\nu^{(\lambda)})]^{m_\nu^{(\lambda)}}$$

is inessential.

For every μ $(0 \leqq \mu \leqq h)$ choose a point $a_\mu \in K_\mu$. As M is bounded, we may choose a_0 such that $x \in M$, $a_0 - x = y_1 + iy_2$ imply $y_1 > 0$. By 26.2.3 (see also 26.1.5 and 26.5.1) we may associate with every pair of indices λ, ν $(1 \leqq \lambda \leqq k,\ 1 \leqq \nu \leqq r_\lambda)$ an index μ $(0 \leqq \mu \leqq h)$ such that the mapping $\pi(M; c_\nu^{(\lambda)})/\pi(M; a_\mu)$ is inessential. Moreover, we see easily by 24.2.7 that the mapping $\pi(M; a_0)$ is inessential. Thus (see 24.2.4 and 24.2.5) there are integers $n_{\lambda\mu}$ $(1 \leqq \lambda \leqq k,\ 1 \leqq \mu \leqq h)$ such that the mappings

$$f_\lambda \cdot \prod_{\mu=1}^{h} [\pi(M; a_\mu)]^{n_{\lambda\mu}}$$

$(1 \leqq \lambda \leqq k)$ are inessential so that, for every choice of integers x_λ $(1 \leqq \lambda \leqq k)$, the mapping

$$\prod_{\lambda=1}^{k} f_\lambda^{x_\lambda} \cdot \prod_{\mu=1}^{h} [\pi(M; a_\mu)]^{\sum_{\lambda=1}^{k} n_{\lambda\mu} x_\lambda}$$

is inessential. Thus, the mapping $\prod_{\lambda=1}^{k} f_\lambda^{x_\lambda}$ is inessential if the integers x_λ $(1 \leqq \lambda \leqq k)$ are such that $\sum_{\lambda=1}^{k} n_{\lambda\mu} x_\lambda = 0$ for $1 \leqq \mu \leqq h$.

As $h < k$, we may choose such integers without putting $x_1 = \ldots = x_k = 0$. This is a contradiction.

26.8.3. *Let $C \subset \mathbf{E}_2$ be an oriented simple loop. For every $a \in \mathbf{E}_2 - C$, $\pi(C; a)$ is a continuous mapping of C into $\mathbf{S}_1$. The degree (see 24.3.2) of this mapping is equal to zero if and only if $a \in W(C)$.*

Proof: By 24.3.3, the degree of $\pi(C; a)$ is equal to zero if and only if this mapping is inessential. This holds by 26.2.2, 26.5.1 and 26.5.2 if and only if $a \in W(C)$.

26.8.4. Let $C \subset \mathbf{E}_2$ be a simple loop. With respect to one of both its orientations (see 21.2.3), the degree of $\pi(C; a)$ is equal to $+1$ for every $a \in V(C)$.

This orientation is called *positive* and the second one is called *negative*. By 24.3.2, with respect to the negative orientation, the degree of the mapping $\pi(C; a)$ is equal to -1 for every $a \in V(C)$.

Proof: I. Let $a \in V(C)$, $b \in V(C)$. By 26.5.1 and 26.5.2, $\sigma(C)$ does not cut the sphere between the points $\sigma(a)$, $\sigma(b)$ so that the mapping $\pi(C; a)/\pi(C; b)$ is inessential by 26.2.3. Thus, both mappings $\pi(C; a)$, $\pi(C; b)$ have the same degree by 24.3.3 and 24.3.4.

II. It remains to be proved that, if an orientation of a simple loop C and a point $a \in V(C)$ are chosen, the degree of $\pi(C; a)$ is equal to ± 1. By 24.3.6 there is a continuous mapping f of C into $\mathbf{S}_1$ such that its degree is equal to $+1$. By 26.8.1 there are points $a_\lambda \in \mathbf{E}_2 - C\,(1 \leqq \lambda \leqq k)$ and integers $n_\lambda\,(1 \leqq \lambda \leqq k)$ such that the mapping

$$f \cdot \prod_{\lambda=1}^{k} [\pi(C; a_\lambda)]^{n_\lambda}$$

is inessential, so that (see 24.3.3) its degree is equal to zero. Thus (see 24.3.4)

$$1 + \sum_{\lambda=1}^{k} n_\lambda r_\lambda = 0,$$

where r_λ is the degree of $\pi(C; a_\lambda)$. As $\mathbf{E}_2 - C = V(C) \cup W(C)$, we have for every index λ $(1 \leqq \lambda \leqq k)$ either $a_\lambda \in W(C)$ or $a_\lambda \in V(C)$. In the first case $r_\lambda = 0$ by 26.8.3; in the second case $r_\lambda = s$ by I. Thus, there is an integer n with $ns = 1$. Thus, $s = \pm 1$.

Exercises

26.1. Let $M \subset \mathbf{E}_2$ be a closed unbounded connected set. Then there is at least one point $a \in M$ such that every component of $M - (a)$ is unbounded. If there is exactly one such point, then M is homeomorphic with the set of all $x \in \mathbf{E}_1$ with $x \geqq 0$.

26.2. Let $M \subset \mathbf{E}_2$ be an irreducible continuum between points $a \in M$, $b \in M$. Then M has no interior points.

26.3. Let $\mathfrak{K}$ be a disjoint system of continua $K \subset \mathbf{E}_2$. Let the union of all $K \in \mathfrak{K}$ be the whole plane. Then there is a continuum $K \in \mathfrak{K}$ such that $\mathbf{E}_2 - K$ is connected.

26.4. For $1 \leqq i \leqq n$ let $C_i \subset \mathbf{E}_2$ be a simple arc with end points a, b. For $1 \leqq i < j \leqq n$, let $C_i \cap C_j = (a) \cup (b)$. Then $\mathbf{E}_2 - \bigcup_{i=1}^{n} C_i$ has exactly n components.

26.5. Let a locally connected $M \subset \mathbf{E}_2$ be an irreducible cut of the plane between points a, b. Then M is a simple loop.

26.6. Let a bounded $C \subset \mathbf{E}_2$ be an irreducible cut of the plane between points a, b. Let $K \subset C$ be a continuum. Then $C - K$ is either void or connected.

26.7. Let $C \subset \mathbf{E}_2$ be a simple loop. Let K_1, K_2 be continua. Let $K_1 \cup K_2 \subset \overline{V(C)}$, $K_1 \cap K_2 = \emptyset$. Let $a_1 \in C \cap K_1$, $b_1 \in C \cap K_1$, $a_2 \in C \cap K_2$, $b_2 \in C \cap K_2$, $a_1 \neq b_1$. Then there exists a simple arc $C_1 \subset C$ with end points a_1, b_1 such that $(a_2) \cup (b_2) \subset C_1$.

26.8. We cannot replace the word "continua" in ex. 26.7 by the words "connected sets".

26.9. For $1 \leqq i \leqq n$ let $C_i \subset \mathbf{E}_2$ be a simple loop. Let $\bigcup_{i=1}^{n} V(C_i)$ be connected. Then there exists a simple loop $C \subset \bigcup_{i=1}^{n} C_i$ such that $\bigcup_{i=1}^{n} V(C_i) \subset V(C)$.

26.10. Let $M \subset \mathbf{E}_2$ be a locally connected set. Let M be $\mathbf{G}_\delta(\mathbf{E}_2)$. Let $a \in M$. Let there be no continuum K with $K \cap M = (a)$. Let $\varepsilon > 0$. Then there exists a simple loop $C \subset M$ of less than ε in diameter such that $a \in V(C)$.

26.11. Let $K \subset \mathbf{E}_2$ be a continuum. $\mathbf{E}_2 - K$ is connected if and only if for every $\varepsilon > 0$ there is a simple loop C with

$$K \subset V(C) \subset \Omega(K, \varepsilon)\,.$$

26.12. Let $C \subset \mathbf{E}_2$ be a simple arc. There exists a simple loop $C_0 \subset \mathbf{E}_2$ such that $C \subset C_0$.

In exercises 26.13 and 26.14, P_5 is the space from the exercises to § 19.

26.13. Let $a = (0, \frac{1}{2})$. For every $x \in \mathbf{E}_2 - P_5$ there is a continuum $K \subset \mathbf{E}_2$ such that $x \in K$, $K \cap P_5 = (a)$. If there is an $\varepsilon > 0$ given, we may choose the point x with $\varrho(a, x) < \varepsilon$ and such that every continuum K is more than $> \frac{1}{2}$ in diameter.

26.14. Let P be the set of all $x + iy \in \mathbf{E}_2$ with either $x + iy \in P_5$ or $-x + iy \in P_5$. Let $a = (0, \frac{1}{2})$. Then there is no continuum K with $K \cap P = (a)$.

26.15. B. Knaster constructed in 1921 a continuum $K \subset \mathbf{E}_2$ such that there is no simple arc $C \subset K$. Assuming this result, prove that there exists a set $M \subset \mathbf{E}_2$ and a point $a \in M$ such that: [1] M is $\mathbf{G}_\delta(\mathbf{E}_2)$, [2] M is $\mathbf{F}_\sigma(\mathbf{E}_2)$, [3] there is a continuum $H \subset \mathbf{E}_2$ with $H \cap M = (a)$, [4] there is no simple arc $C \subset \mathbf{E}_2$ such that $C \cap M = (a)$.

26.16. Let $M \subset \mathbf{E}_2$ be a closed set. Let $a \in M$. Let $K \subset \mathbf{E}_2$ be a continuum such that $K \cap M = (a)$. Let $\varepsilon > 0$. Then there is a simple arc $C \subset \mathbf{E}_2$ with $C \subset \Omega(K, \varepsilon)$ and $C \cap M = (a)$.

In ex. 26.17—26.19, $C_i \subset \mathbf{E}_2$ $(1 \leqq i \leqq n;\ n = 2, 3, 4, \ldots)$ are simple loops such that $C_i \cap C_j = \emptyset$ for $1 \leqq i < j \leqq n$.

26.17. Let $C_i \subset W(C_j)$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \neq j$. Then $P - \bigcup_{i=1}^{n} C_i$ has exactly $n + 1$ components. These are the sets $V(C_i)$ $(1 \leqq i \leqq n)$ and the set $\bigcup_{i=1}^{n} W(C_i) = K$; we have $B(K) = \bigcup_{i=1}^{n} C_i$.

26.18. Let there exist an index λ $(1 \leqq \lambda \leqq n)$ such that $C_i \subset V(C_\lambda)$ for $1 \leqq i \leqq n$, $i \neq \lambda$, and that $C_i \subset W(C_j)$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i \neq \lambda \neq j \neq i$. Then $P - \bigcup_{i=1}^{n} C_i$ has exactly $n + 1$ components. These are: the set $W(C_\lambda)$, the sets $V(C_i)$ $(1 \leqq i \leqq n,\ i \neq \lambda)$, and the set $V(C_\lambda) - \bigcup_{\substack{i=1 \\ i \neq \lambda}}^{n} V(C_i) = K$. We have $B(K) = \bigcup_{i=1}^{n} C_i$.

26.19. If neither the assumptions of ex. 26.17 nor the assumptions of ex. 26.18 are satisfied, the set $P - \bigcup_{i=1}^{n} C_i$ has also $n + 1$ components. However, the boundary of none of them is equal to $\bigcup_{i=1}^{n} C_i$.

26.20. Exercises 26.17—26.19 may be generalized as follows: The assumption that $C_i \cap C_j$ are void will be replaced by the assumption that every set

$$C_i \cap \bigcup_{j=1}^{i-1} C_j \quad (2 \leqq i \leqq n)$$

is either void or connected (and $C_i \neq C_j$ for $1 \leqq i < j \leqq n$).

In exercises 26.21 and 26.22, $C_1 \subset \mathbf{E}_2$, $C_2 \subset \mathbf{E}_2$ are simple loops and $\mathbf{C}_1, \mathbf{C}_2$ denote their positive orientation, a_1, b_1, c_1 are three distinct points of C_1, a_2, b_2, c_2 are three distinct points of C_2.

26.21. Let $C_1 \subset V(C_2)$. Let $(a_1, b_1, c_1) \in \mathbf{C}_1$. We have $(a_2, b_2, c_2) \in \mathbf{C}_2$ if and only if there exist three disjoint simple arcs A_1, A_2, A_3 such that

$$\begin{aligned} C_1 \cap A_1 &= (a_1), \quad C_1 \cap A_2 = (b_1), \quad C_1 \cap A_3 = (c_1), \\ C_2 \cap A_1 &= (a_2), \quad C_2 \cap A_2 = (b_2), \quad C_2 \cap A_3 = (c_3). \end{aligned} \tag{1}$$

26.22. Let $C_1 \subset W(C_2)$, $C_2 \subset W(C_1)$. Let $(a_1, b_1, c_1) \in \mathbf{C}_1$. We have $(a_2, b_2, c_2) \in \mathbf{C}_2$ if and only if there are no simple arcs A_1, A_2, A_3 such that (1) holds.

26.23. Let $C \subset \mathbf{E}_2$ be a simple arc with end points a, b. Let $\mathfrak{S}$ be the system of all simple arcs $K \subset \mathbf{E}_2$ such that $C \cap K$ contains exactly one point which is an end point of K, and such that $(a) \neq C \cap K \neq (b)$. The system $\mathfrak{S}$ may be divided into two disjoint subsystems $\mathfrak{S}_1$, $\mathfrak{S}_2$ which have the following property: If $K_1 \in \mathfrak{S}$, $K_2 \in \mathfrak{S}$, $K_1 \cap K_2 \subset C$, there is a simple arc Γ and a simple loop Δ such that [1] $\Gamma \subset \mathbf{E}_2 - C$, [2] one of the end points of Γ belongs to K_1, the other to K_2, [3] $\Gamma \subset \Delta \subset \Gamma \cup K_1 \cup K_2 \cup C$. For every such Δ it holds that: [1] if $K_1 \in \mathfrak{S}_1$, $K_2 \in \mathfrak{S}_1$ or $K_1 \in \mathfrak{S}_2$, $K_2 \in \mathfrak{S}_2$, then either $(a) \cup (b) \subset V(\Delta)$ or $(a) \cup (b) \subset W(\Delta)$, [2] if $K_1 \in \mathfrak{S}_1$, $K_2 \in \mathfrak{S}_2$, then either $a \in V(\Delta)$, $b \in W(\Delta)$, or $a \in W(\Delta)$ $b \in V(\Delta)$.

In exercise 26.24, S designates the segment as in exercises to § 19.

26.24. Let $a \in \mathbf{E}_2$, $b \in \mathbf{E}_2$, $a \neq b$. Let $C \subset \mathbf{E}_2$ be a simple loop such that $S(a, b) \subset C$. Let $S(a, b,)$ be oriented in such a way that a is the initial point, and let C be oriented coherently. Let $c \in \mathbf{E}_2$, $d \in S(a, b)$, $a \neq d \neq b$, $S(c, d) - (d) \subset V(C)$. Let

$$\frac{c - d}{b - d} = x + iy .$$

The given orientation of C is positive if and only if y is positive.

26.25. Let $Q_1 \subset \mathbf{E}_2$ be the set consisting of the points $+i$, $-i$, the points $x + i \sin(x^{-1})$, $0 < x \leqq 1$, and, finally, the points $(1 + i)x + i \sin(x^{-1})$, $0 < x \leqq 1$. Then $\mathbf{E}_2 - Q_1$ is a semicontinuum.

26.26. Let $Q_2 \subset \mathbf{E}_2$ be the set consisting of the points $+i$, $-i$, the points $\sqrt{(\varrho^2 - y^2)} + iy$ with $1 < \varrho \leqq 2$, $y = \sin 1/(\varrho - 1)$, and, finally, of the points $-\sqrt{(\varrho^2 - y^2)} + iy$ with $1 < \varrho \leqq 2$, $y = \sin 1/(\varrho - 1)$. Let $a = 0$, $b = 2i$. Then Q_2 cuts the plane between the points a, b.

26.27. Let $M_1 \subset \mathbf{E}_2$, $M_2 \subset \mathbf{E}_2$ be closed sets. Let there exist a homeomorphic mapping of M_1 onto M_2. If $\mathbf{E}_2 - M_1$ has a finite number k of constituants, then $\mathbf{E}_2 - M_2$ has also k constituants. If $\mathbf{E}_2 - M_1$ has an infinite number of constituants, then also $\mathbf{E}_2 - M_2$ has an infinite number of constituants.

26.28. One cannot omit in exercise 26.27 the word "closed". This may be seen, e.g., by replacing M_1, M_2 with some of $A_1 \cup B_1$, $A_2 \cup B_1$, $A_2 \cup B_2$ (where A_1 is the set of all real x, A_2 is the set of all x with $-1 < x < 1$, B_1 is the set of all ix with real x, B_2 is the set of all ix with $-1 < x < 1$). One cannot replace the word "closed" by the word "bounded", either. This follows by the example with $M_1 = Q_1$ (see ex. 26.25), $M_2 = Q_2$ (see ex. 26.26).

§ 27. Topological characterization of the sphere

27.1. Let P be a metric space. We say that P is a *spherical space*, if it has the following properties:

(α) *P is a locally connected continuum,*

(β) *$P - (a)$ is connected for every $a \in P$,*

(γ) *if $A \subset P$, $B \subset P$ are closed sets such that $A \cap B$ is either void or connected, and if a, b are two distinct points of $P - (A \cup B)$ such that neither A nor B separates the point a from the point b in P, then $A \cup B$ does not separate a from b either.*

The term "spherical space" is motivated by the following fact:

27.1.1. *A metric space is spherical if and only if it is homeomorphic with $\mathbf{S}_2$.*

Proof of this theorem is the principal aim of this section. $\mathbf{S}_2$ is evidently a spherical space. In fact, $\mathbf{S}_2$ has property (α) by 17.10.2, 19.2.5 and 22.1.14; $\mathbf{S}_2$ has property (β) by 17.10.4 and 19.2.4; $\mathbf{S}_2$ has property (γ) by 26.3.1 (see also 19.5.11 and 22.3.3). Since (α), (β), (γ) are topological properties, every metric space homeomorphic with $\mathbf{S}_2$ is spherical. To finish the proof of theorem 27.1.1, we have to prove the theorem:

27.1.2. *Let P, Q be spherical spaces. Then P and Q are homeomorphic.*

Proof of this theorem will be done in section 27.3. First, we have to prove some simple theorems concerning spherical spaces.*)

27.2. 27.2.1. *A spherical space P is unicoherent.*

Proof: Let us assume the contrary. By 25.1.2 [see also property (α)] there are points $a \in P$, $b \in P$, $a \neq b$ and an irreducible cut $M \subset P - [(a) \cup (b)]$ of P between the points a, b such that the set M is not connected. Since P is connected, we have certainly $M \neq \emptyset$. M is closed by 18.5.4. Hence, there exist disjoint closed sets A, B such that $A \neq \emptyset \neq B$, $M = A \cup B$. Since M is an irreducible cut between points a, b, neither A nor B separates the point a from the point b. As $A \cap B = \emptyset$, property (γ) yields that $A \cup B = M$ does not separate a from b either, which is a contradiction.

27.2.2. *Let P be a spherical space. Let $A \subset P$, $B \subset P$ be open sets such that $A \cap B$ is either void or connected. Let a, b be two distinct points of $P - (A \cup B)$ such that neither A nor B separates a from b in P. Then $A \cup B$ does not separate a from b in P either.*

*) By theorems of § 26 we see easily that all the theorems of section 27.2 are true for $P = \mathbf{S}_2$. Thus, it follows by 27.1.2 that these theorems are true for every spherical space P. Theorem 27.1.2, however, is not proved yet. Thus, these theorems must be deduced directly from the properties (α), (β), (γ).

Proof: Assume the contrary. P is locally connected [see property (α)]. Thus (see 22.1.12), there is an irreducible cut $M \subset A \cup B$ of P between the points a, b. M is closed by 18.5.4, so that $M - A$, $M - B$ are also closed. As $M \subset A \cup B$, we have $(M - A) \cap (M - B) = \emptyset$, so that $M - A$, $M - B$ are separated (see 10.2.1). Hence (see 10.2.7), there exist disjoint open U, V with $U \cap V = \emptyset$, $U \supset M - A$, $V \supset M - B$. Put $A_0 = M - U$, $B_0 = M - V$. Then A_0, B_0 are closed sets and we see easily that $A_0 \subset A$, $B_0 \subset B$, $A_0 \cup B_0 = M$. $A_0 \cap B_0$ is compact by 17.2.2; $A \cap B$ is open and either connected or void. Thus (see 23.2.5), there exists a closed set C such that $A_0 \cap B_0 \subset C \subset A \cap B$ and that C is either connected or void.

As $A_0 \cup C \subset A$, $A_0 \cup C$ does not separate a from b in P. This holds also for $B_0 \cup C$. On the other hand, the set $(A_0 \cup C) \cup (B_0 \cup C) \supset A_0 \cup B_0 = M$ separates a from b. Thus, property (γ) yields that $(A_0 \cup C) \cap (B_0 \cup C) = (A_0 \cap B_0) \cup C = C$ is neither connected nor void, which is a contradiction.

27.2.3. *Let P be a spherical space. Let $C \subset P$ be a simple arc. Then $P - C$ is connected.*

Proof: Let us assume the contrary. We obtain, by property (β) and 20.1.2, that $C \neq P$. Thus, there are points $a \in P - C$, $b \in P - C$ such that C separates a from b in P. By 22.1.12 [see also property (α)] there exists an irreducible cut $D \subset C$ of P between a, b. D is closed by 18.5.4 and connected by 25.1.2 and 27.2.1 [see also property (α)]. Moreover, D is not a one-point set. Thus (see 17.2.2), D is a continuum. As $D \subset C$, D is a simple arc (see 20.1.13). Hence (see 20.1.9) there exist simple arcs $D_1 \subset D$, $D_2 \subset D$ such that $D_1 \cup D_2 = D$ and $D_1 \cap D_2$ is a one-point set. As D is an irreducible cut of P between points a, b, neither D_1 nor D_2 separates a from b. By (γ), $D_1 \cup D_2 = D$ does not separate a from b either. This is a contradiction.

27.2.4. *Let P be a spherical space. Let $C \subset P$ be a simple arc. Then C has no interior points.*

Proof: Let there be, on the contrary, a non-void open $G \subset C$. C is closed (see 17.2.2), so that $\bar{G} \subset C$. By 27.2.3, $P - C \neq \emptyset$. Choose $a \in G$, $b \in P - C$. By 18.5.3, $B(G) = \bar{G} - G \subset C$ separates a from b. By 22.1.12 there exists an irreducible cut $D \subset B(G) \subset C$ of P between a and b. We obtain a contradiction similarly as in the previous proof.

27.2.5. *Let P be a spherical space. Let $A \subset P$, $B \subset P$ be disjoint closed sets. Let $C \subset P$ be a simple arc with end points $a \in A$, $b \in B$. Let $A \cap C = (a)$, $B \cap C = (b)$. Then $C - [(a) \cup (b)]$ is a subset of a component G of $P - (A \cup B)$ and $G - C$ is a component of $P - (A \cup B \cup C)$.*

Proof: $C - [(a) \cup (b)]$ is a subset of a component G of $P - (A \cup B)$ by 18.2.5 and 20.1.5. G is open by 22.1.4 [see property (α)], so that $G - C \neq \emptyset$ by 27.2.4. Choose a $c \in G - C$.

We have to prove that $x \in P - (A \cup B \cup C)$ is in the same component of $P - (A \cup B \cup C)$ with c, if and only if $x \in G$. First, let Δ be a component of $P - (A \cup B \cup C)$ and let $c \in \Delta$, $x \in \Delta$. Then Δ is a connected subset of $P - (A \cup B)$ and $c \in \Delta \cap G$, so that, by 18.2.5, $\Delta \subset G$ and hence $x \in G$.

Secondly, let c and x belong to distinct components of $P - (A \cup B \cup C)$. We have to reach a contradiction with the assumption of $x \in G$. $P - (A \cup B \cup C)$ is locally connected by 22.1.3, so that, by 22.1.5, $A \cup B \cup C$ separates c from x. Since $c \in G$, $x \in G$ and $G \subset P - (A \cup B)$ is connected, $A \cup B$ does not separate c from x, so that A does not separate c from x either. By 27.2.3 C does not separate c from x either. As $A \cap C = (a)$, $A \cup C$ does not separate c from x by property (γ). Similarly we may prove that $B \cup C$ does not separate c from x. Since $(A \cup C) \cap (B \cup C) = C$, by property ($\gamma$) $(A \cup C) \cup (B \cup C) = A \cup B \cup C$ does not separate c from x. This is a contradiction.

27.2.6. *Let P be a spherical space. Let $K \subset P$ be a continuum. Let $C \subset P$ be a simple arc with end points $a \in K$, $b \in K$. Let $C - [(a) \cup (b)] \subset P - K$. Then $C - [(a) \cup (b)]$ is a subset of a component G of $P - K$. $G - C$ has exactly two components G_1, G_2. We have*

$$C \subset B(G_i) \subset C \cup B(G) \cup K \qquad (i = 1, 2).$$

Proof: I. $C - [(a) \cup (b)]$ is a subset of a component G of $P - K$ by 18.2.5 and 20.1.5. $P - G$ is connected (see 22.1.13), so that G does not separate a from b. By 27.2.3, $P - C$ does not separate a from b either. On the other hand, $G \cup (P - C) = P - [(a) \cup (b)]$ separates a from b. Moreover, G and $P - C$ are open sets (see 17.2.2 and 22.1.4), so that, by 27.2.2, $G \cap (P - C) = G - C$ is neither void nor connected. Thus, $G - C$ has at least two components.

II. Let G_0 be a component of $G - C$. G and G_0 are open by 22.1.4, so that $B(G) = \overline{G} - G$, $B(G_0) = \overline{G}_0 - G_0$ by 10.3.2. We have $B(G_0) \subset \overline{G}_0 \subset \overline{G} = G \cup B(G)$. By 22.1.9, $B(G_0) \subset P - (G - C)$. Thus, $B(G_0) \subset C \cup B(G)$. By 22.1.9, $B(G) \subset K$. To prove that

$$C \subset B(G_0) \subset C \cup B(G) \subset C \cup K,$$

we have to prove that $C \subset B(G_0)$.

Let us assume the contrary. Since $B(G_0)$ is closed and since C is a simple arc, we prove easily that there exist simple arcs C_1, C_2 such that

$$a \in C_1, \qquad b \in C_2, \qquad C_1 \cap C_2 \neq \emptyset, \qquad C_1 \cup C_2 \subset C,$$
$$C - (C_1 \cup C_2) \neq \emptyset, \qquad B(G_0) \subset C_1 \cup C_2 \cup K.$$

Choose a point $c \in C - (C_1 \cup C_2)$ and a point $d \in G_0$. Evidently $c \in P - \overline{G}_0$, so that (see 18.5.2) $B(G_0)$ separates c from d. Since $B(G_0) \subset C_1 \cup C_2 \cup K$, $C_1 \cup C_2 \cup K$ also separates c from d. Evidently $(c) \cup (d) \subset K$, so that K does not separate c from d. By 27.2.3 neither C_1 nor C_2 separates c from d. Since $C_1 \cap K =$

$= (a)$, $C_2 \cap K = (b)$, by property (γ) neither $C_1 \cup K$ nor $C_2 \cup K$ separates c from d. On the other hand, $(C_1 \cup K) \cap (C_2 \cup K) = K$ is connected, so that, by property (γ), $(C_1 \cup K) \cup (C_2 \cup K) = C_1 \cup C_2 \cup K$ does not separate c from d, which is a contradiction.

III. Choose a $c \in C$, $a \neq c \neq b$. Choose (see 18.3.1 and 20.1.2) points r, s in $C - (c)$ such that this set is not connected between r and s. We may evidently assume $a \neq r \neq b$, $a \neq s \neq b$.

$G - (c)$ is open and non-void. It is also connected. Otherwise (see 18.3.1) it would contain points h, k such that $P - [G - (c)] = (P - G) \cup (c)$ would separate h from k. As G is connected, $P - G$ does not separate h from k. By property (β), (c) does not separate h from k either. As $(P - G) \cap (c) = \emptyset$, by property ($\gamma$) $(P - G) \cup \cup (c)$ does not separate h from k.

Thus, $G - (c)$ is connected. It is also locally connected by 22.1.3. Moreover, $G - (c)$ is a topologically complete space by 15.5.2 [see also (α) and 17.2.1]. Hence (see 22.3.1), there is a simple arc $D \subset G - (c)$ with end points r, s. As $C - (c)$ is not connected between r and s, D is not contained in C. Thus, there is a point $t \in D - C$. By 20.1.9 there is a simple arc D_1 with end points r, t and a simple arc D_2 with end points t, s such that $D_1 \cup D_2 = D$, $D_1 \cap D_2 = (t)$. By 20.2.7 we see easily that there is a simple arc $E_1 \subset D_1$ with end points u, t and a simple arc $E_2 \subset D_2$ with end points t, v such that $C \cap E_1 = (u)$, $C \cap E_2 = (v)$. Put $E_0 = = E_1 \cup E_2$. Then E_0 is (see 20.1.10) a simple arc with end points u, v and we have $C \cap E_0 = (u) \cup (v)$.

$E_0 - [(u) \cup (v)]$ is a connected (see 20.1.5) subset of $G - C$. Hence, there is a component G_0 of $G - C$ such that

$$E_0 - [(u) \cup (v)] \subset G_0 .$$

IV. We have to prove that $G - C$ has at most two components. Let us assume the contrary. Then there exist, besides the described component G_0, two other components G_1, G_2 of $G - C$. Choose $g_1 \in G_1$, $g_2 \in G_2$. Then (see 22.1.3, 22.1.5 and property (α)] $P - (G - C) = (P - G) \cup C$ separates g_1 from g_2. Consequently, $(P - G) \cup C \cup E_0$ separates g_1 from g_2.

It is easy to prove that there exist simple arcs C', C'', C''' such that $C = C' \cup \cup C'' \cup C'''$, $C' \cap C'' = \emptyset$, $a \in C'$, $b \in C''$ and either $C' \cap C''' = (u)$, $C'' \cap C''' = (v)$ or $C' \cap C''' = (v)$, $C'' \cap C''' = (u)$. Let, e.g., $C' \cap C''' = (u)$, $C'' \cap C''' = (v)$.

Choose $y \in C'''$, $u \neq y \neq v$. By II we have $y \in B(G_1) \cap B(G_2)$. Thus, $G_1 \cup (y)$, $G_2 \cup (y)$ are connected (see 18.1.7), so that $G_1 \cup (y) \cup G_2$ is connected (see 18.1.4). Evidently $[G_1 \cup (y) \cup G_2] \cap [(P - G) \cup C' \cup C'' \cup E_0] = \emptyset$, so that $(P - G) \cup \cup C' \cup C'' \cup E_0$ does not separate g_1 from g_2.

Choose a $z \in C'$, $a \neq z \neq u$. By II, $z \in B(G_1) \cap B(G_2)$, so that (again by 18.1.7 and 18.1.4) $G_1 \cup (z) \cup G_2$ is connected. Evidently $[G_1 \cup (z) \cup G_2] \cap (C''' \cup E_0) = \emptyset$, so that $C''' \cup E_0$ does not separate g_1 from g_2.

$[(P - G) \cup C' \cup C'' \cup E_0] \cap (C''' \cup E_0) = E_0$ is connected, so that, by property (γ), $[(P - G) \cup C' \cup C'' \cup E_0] \cup (C''' \cup E_0) = (P - G) \cup C \cup E_0$ does not separate g_1 from g_2, which is a contradiction.

27.2.7. *Let P be a spherical space. Let $C \subset P$ be a simple loop. $P - C$ has exactly two components and C is the boundary of both of them.*

Proof: Choose $u \in C$, $v \in C$, $u \neq v$. By 21.1.2 there exist simple arcs C_1, C_2 with end points u, v such that $C_1 \cup C_2 = C$, $C_1 \cap C_2 = (u) \cup (v)$.

Put $K = C_1$, $G = P - K$. Then K is a continuum (see 20.1.1), C_2 is a simple arc with end points $u \in K$, $v \in K$, G is a component of $P - K$ (see 27.2.3) and $C_2 - [(u) \cup (v)] \subset G$. Thus, by 27.2.6, $G - C_2 = P - C$ has exactly two components G_1, G_2 and we have

$$C_2 \subset B(G_i) \subset C_1 \cup C_2 \qquad (i = 1, 2).$$

After an analogous reasoning in which we interchange C_1 and C_2 we obtain

$$C_1 \subset B(G_i) \subset C_1 \cup C_2 \qquad (i = 1, 2).$$

Thus, $B(G_1) = B(G_2) = C_1 \cup C_2 = C$.

27.2.8. *Let P be a spherical space. Let $M_1 \subset P$ be a one-point set or a continuum. Let $M_2 \subset P$ be a one-point set or a continuum. Let $M_1 \cap M_2 = \emptyset$. For $i = 1, 2, 3$ let $C_i \subset P$ be a simple arc with end points $a_i \in M_1$, $b_i \in M_2$. Put $C_i^* = C_i - [(a_i) \cup \cup b_i)]$ $(i = 1, 2, 3)$. Let*

$$C_1^* \cap C_2^* = C_1^* \cap C_3^* = C_2^* \cap C_3^* = \emptyset,$$
$$(C_1^* \cup C_2^* \cup C_3^*) \cap (M_1 \cup M_2) = \emptyset.$$

$P - (M_1 \cup M_2)$ has a component G such that

$$C_1^* \cup C_2^* \cup C_3^* \subset G.$$

$G - (C_1 \cup C_2 \cup C_3)$ has exactly three components G_1, G_2, G_3. We have

$$C_2 \cup C_3 \subset B(G_1), \qquad C_3 \cup C_1 \subset B(G_2), \qquad C_1 \cup C_2 \subset B(G_3),$$
$$C_1^* \cap B(G_1) = C_2^* \cap B(G_2) = C_3^* \cap B(G_3) = \emptyset,$$
$$B(G_1) \cup B(G_2) \cup B(G_3) \subset C_1 \cup C_2 \cup C_3 \cup M_1 \cup M_2.$$

Proof: I. If $C_1^* \cup C_2^* \cup C_3^*$ is not a subset of a component of $P - (M_1 \cup M_2)$, then [see 22.1.3, 22.1.5 and property (α)] there exist points $c_1 \in C_i^*$, $c_2 \in C_j^*$ $(i, j = 1, 2, 3)$ such that $M_1 \cup M_2$ separates c_1 from c_2. Sets $C_i^* \cup (b_i)$, $C_j^* \cup (b_j)$ are connected (see, e.g., 20.1.2). Thus (see 18.1.4) $C_i^* \cup C_j^* \cup M_2$ is connected. On the other hand, $(c_1) \cup (c_2) \subset C_i^* \cup C_j^* \cup M_2 \subset P - M_1$, so that M_1 does not separate c_1 from c_2. Similarly we may prove that M_2 does not separate c_1 from c_2. As $M_1 \cap M_2 = \emptyset$, we obtain by property (γ) that $M_1 \cup M_2$ does not separate c_1 from c_2 which is a contradiction.

Thus, there is a component G of $P - (M_1 \cup M_2)$ with

$$C_1^* \cup C_2^* \cup C_3^* \subset G.$$

G is open (see 22.1.4).

II. By 27.2.5, $G - C_1$ is a component of $P - (C_1 \cup M_1 \cup M_2)$. $C_1 \cup M_1 \cup M_2$ is a continuum (see 17.2.2 and 18.1.4). By 27.2.6, $(G - C_1) - C_2 = G - (C_1 \cup C_2)$ has exactly two components G_3, G_3' and we have

$$C_2 \subset B(G_3) \cap B(G_3'),$$
$$B(G_3) \cup B(G_3') \subset C_1 \cup C_2 \cup M_1 \cup M_2 \subset P - C_3^*.$$

III. We may repeat the argument in II with interchanged C_2, C_1. We obtain

$$C_1 \subset B(G_3) \cap B(G_3').$$

IV. By 18.2.5 (see also 20.1.5) we have either $C_3^* \subset G_3$ or $C_3^* \subset G_3'$. Let, e.g., $C_3^* \subset G_3'$ and hence $C_3^* \cap G_3 = \emptyset$. Evidently $G_3 \subset G - (C_1 \cup C_2 \cup C_3)$. We obtain, by 22.1.9,

$$B(G_3) \subset P - [G - (C_1 \cup C_2)] \subset P - [G - (C_1 \cup C_2 \cup C_3)].$$

Thus, by 22.1.9, G_3 is a component of $G - (C_1 \cup C_2 \cup C_3)$.

V. Hence, the set $G - (C_1 \cup C_2 \cup C_3)$ has a component G_3 such that

$$C_1 \cup C_2 \subset B(G_3), \qquad C_3^* \cap B(G_3) = \emptyset,$$
$$B(G_3) \subset G_1 \cup C_2 \cup C_3 \cup M_1 \cup M_2.$$

We may repeat the whole part of the proof done till now with an arbitrary permutation of C_1, C_2, C_3. Thus, $G - (C_1 \cup C_2 \cup C_3)$ has a component G_1 such that

$$C_2 \cup C_3 \subset B(G_1), \qquad C_1^* \cap B(G_1) = \emptyset,$$
$$B(G_1) \subset C_1 \cup C_2 \cup C_3 \cup M_1 \cup M_2,$$

and a component G_2 such that

$$C_3 \cup C_1 \subset B(G_2), \qquad C_2^* \cap B(G_2) = \emptyset,$$
$$B(G_2) \subset C_1 \cup C_2 \cup C_3 \cup M_1 \cup M_2.$$

The components G_1, G_2, G_3 are distinct, since their boundaries are distinct.

VI. It remains to prove that $G - (C_1 \cup C_2 \cup C_3)$ has at most three components. We have (see II and IV)

$$G - (C_1 \cup C_2 \cup C_3) = G_3 \cup (G_3' - C_3).$$

Thus, it suffices to prove that $G_3' - C_3$ has at most two components. In fact, every component of $G_3' - C_3$ is a connected subset of $G - (C_1 \cup C_2 \cup C_3)$ and hence (see 18.2.5) it is a subset of a component of $G - (C_1 \cup C_2 \cup C_3)$.

VII. $C_1 \cup C_2 \cup M_1 \cup M_2$ is a continuum (see 17.2.2 and 18.1.4). As $G_3' \subset P - (C_1 \cup C_2 \cup M_1 \cup M_2)$, $B(G_3') \subset C_1 \cup C_2 \cup M_1 \cup M_2$, G_3' is, by 22.1.9, a component of $P - (C_1 \cup C_2 \cup M_1 \cup M_2)$. On the other hand, C_3 is a simple arc, the end points of which, a_3 and b_3, belong to $C_1 \cup C_2 \cup M_1 \cup M_2$, and we have $C_3 - [(a_3) \cup (b_3)] \subset G_3'$. Thus, $G_3' - C_3$ has exactly two components by 27.2.6.

27.2.9. *Let P be a spherical space. Let $C \subset P$ be a simple loop. Let $a \in C$, $b \in C$, $a \neq b$, so that* (*see* 21.1.2) *there exist simple arcs $C_1 \subset P$, $C_2 \subset P$ with end points a, b such that*

$$C_1 \cup C_2 = C, \qquad C_1 \cap C_2 = (a) \cup (b).$$

Let $C_3 \subset P$ be a simple arc with end points a, b. Let $C \cap C_3 = (a) \cup (b)$. There exists a component Q of $P - C$ such that

$$C_3 - [(a) \cup (b)] \subset Q.$$

$Q - C_3$ has exactly two components G_1, G_2 and we have

$$B(G_1) = C_2 \cup C_3, \qquad B(G_2) = C_1 \cup C_3.$$

Proof: I. Obviously $P - [(a) \cup (b)] \neq \emptyset$. If $P - [(a) \cup (b)]$ were not connected, then it would contain (see 18.3.1) points c, d such that $(a) \cup (b)$ would separate c from d. On the other hand, by property (β), neither (a) nor (b) separates c from d, so that, by property (γ), $(a) \cup (b)$ does not separate c from d. Thus, $P - [(a) \cup (b)]$ is connected.

II. Put $M_1 = (a)$, $M_2 = (b)$, $a_1 = a_2 = a_3 = a$, $b_1 = b_2 = b_3 = b$. Then the assumptions of theorem 27.2.8 are satisfied. By I, the set G from the quoted theorem is equal to $P - [(a) \cup (b)]$. Thus, $P - (C_1 \cup C_2 \cup C_3)$ has exactly three components G_1, G_2, G_3 and we have

$$B(G_1) = C_2 \cup C_3, \qquad B(G_2) = C_3 \cup C_1, \qquad B(G_3) = C_1 \cup C_2 = C.$$

We have

$$P - (C \cup C_3) = G_1 \cup G_2 \cup G_3$$

with disjoint summands. G_3 is, by 22.1.9, a component of $P - C$. This last set has, by 27.2.7, one more component Q and we have

$$P - C = Q \cup G_3$$

with disjoint summands. Thus,

$$P - (C \cup C_3) = (Q - C_3) \cup G_3$$

with disjoint summands, so that

$$Q - C_3 = G_1 \cup G_2$$

with disjoint summands.

It remains to prove that G_1 and G_2 are components of $Q - C_3$. Let this statement be not true, e.g., concerning G_1. Then there exists a connected subset G_1' of $Q - C_3$ such that $G_1 \subset G_1' \neq G_1$. Then, G_1' is a connected subset of $P - (C_1 \cup \cup C_2 \cup C_3)$ and we have $G_1 \subset G_1' \neq G_1$. This is a contradiction, since G_1 is a component of $P - (C_1 \cup C_2 \cup C_3)$.

27.2.10. *Let P be a spherical space. Let $K \subset P$ be a locally connected continuum. Let G be a component of $P - K$. Put $H = B(G)$. Then H is a locally connected continuum and we have $H \subset K$.*

Proof: I. By 22.1.9 we have $H \subset K$.

II. By 25.1.1 and 27.2.1, either H is a continuum, or $H = (c)$ is a one-point set. Let $H = (c)$. By 22.1.9, G is a component of $P - H = P - (c)$. $P - (c)$ is connected [see property (β)]. Thus, $G = P - (c)$, so that $K \subset (c)$, which is a contradiction. Thus, H is a continuum.

III. It remains to be proved that the continuum H is locally connected. Assume the contrary. By 22.2.5 (see also 22.2.2) there exist distinct points $a \in H$, $b \in H$ and a disjoint sequence of continua $\{H_n\}_1^\infty$ such that $H_n \subset H$, $\lim \varrho(a, H_n) = = \lim \varrho(b, H_n) = 0$.

IV. Choose an $\varepsilon > 0$ such that $\varrho(a, b) > 6\varepsilon$. By 23.1.2 there is an $\alpha > 0$ such that, whenever $x \in K$, $y \in K$, $\varrho(x, y) < \alpha$, there is a connected $S \subset K$ such that $x \in S$, $y \in S$, $d(S) < \varepsilon$. Evidently $\alpha \leqq \varepsilon$.

As $\lim \varrho(a, H_n) = \lim \varrho(b, H_n) = 0$, there is an index p such that for every $n > p$ there are points $a_n \in H_n$, $b_n \in H_n$ with $\varrho(a, a_n) < \alpha$, $\varrho(b, b_n) < \alpha$.

Since $H_{p+1}, H_{p+2}, H_{p+3}$ are disjoint continua, there exists (see 17.3.4) an $\eta > 0$ such that $\eta < \varepsilon$, $2\eta < \varrho(H_{p+1}, H_{p+2})$, $2\eta < \varrho(H_{p+1}, H_{p+3})$, $2\eta < \varrho(H_{p+2}, H_{p+3})$. By 23.1.2 there is a $\beta > 0$ such that, whenever $x \in K$, $y \in K$, $\varrho(x, y) < \beta$, there is a connected $S \subset K$ such that $x \in S$, $y \in S$, $d(S) < \eta$. We may assume that $2\beta \leqq \eta$.

V. Let $n > p$. Since $H \subset K$, we have $a \in K$, $a_n \in K$. Moreover, $\varrho(a, a_n) < \alpha$, so that there is a connected $S_n \subset K$ such that $a \in S_n$, $a_n \in S_n$, $d(S_n) < \varepsilon$. Put

$$M_1 = \overline{S_{p+1} \cup S_{p+2} \cup S_{p+3}}.$$

M_1 is a continuum (see 17.2.2, 18.1.5 and 18.1.6). We have $M_1 \subset K$, $d(M_1) \leqq 2\varepsilon$, $a \in M_1$, $a_{p+1} \in M_1$, $a_{p+2} \in M_1$, $a_{p+3} \in M_1$.

Similarly we may prove that there exists a continuum $M_2 \subset K$ such that $d(M_2) \leqq \leqq 2\varepsilon$, $b \in M_2$, $b_{p+1} \in M_2$, $b_{p+2} \in M_2$, $b_{p+3} \in M_2$.

We have $M_1 \cap M_2 = \emptyset$; we have even

$$\Omega(M_1, \varepsilon) \cap \Omega(M_2, \varepsilon) = \emptyset.$$

In fact, let there exist a point x with $\varrho(x, M_1) < \varepsilon$, $\varrho(x, M_2) < \varepsilon$. Then there are $x_1 \in M_1$, $x_2 \in M_2$ such that $\varrho(x, x_1) < \varepsilon$, $\varrho(x, x_2) < \varepsilon$. Since $a \in M_1$, $d(M_1) \leqq 2\varepsilon$, we have $\varrho(a, x_1) \leqq 2\varepsilon$. Similarly $\varrho(b, x_2) \leqq 2\varepsilon$. Thus, $\varrho(a, b) \leqq \varrho(a, x_1) + \varrho(x_1, x) + \varrho(x, x_2) + \varrho(x_2, b) < 2\varepsilon + \varepsilon + \varepsilon + 2\varepsilon$, i.e. $\varrho(a, b) < 6\varepsilon$, which is a contradiction.

VI. H_{p+1} is a connected subset of $K \cap \Omega(H_{p+1}, \beta)$. Hence (see 18.2.5) there exists a component Q of $K \cap \Omega(H_{p+1}, \beta)$ such that $H_{p+1} \subset Q$. Since K is a locally connected space and since $K \cap \Omega(H_{p+1}, \beta)$ is open in K, Q is open in K by 22.1.4. Thus, Q is locally connected by 22.1.3. As Q is open in the compact space K, Q is, by 15.5.2 and 17.2.1, a topologically complete space. Moreover, Q is, of course, connected. We have $a_{p+1} \in H_{p+1} \subset Q$, $b_{p+1} \in Q$, $a_{p+1} \neq b_{p+1}$. Thus, by 22.3.1, there exists a simple arc $T \subset Q$ with end points a_{p+1}, b_{p+1}. We have $a_{p+1} \in M_1$, $b_{p+1} \in M_2 \subset P - M_1$. Thus, by 20.2.7 (see also 20.1.8), there exists a simple arc $E \subset D$ with end points u_1, b_{p+1} such that $E \cap M_1 = (u_1)$. We have $u_1 \in M_1 \subset P - M_2$, $b_{p+1} \in M_2$. Thus, by 20.2.7, there is a simple arc $C_1 \subset E$ with end points u_1, v_1 such that $C_1 \cap M_2 = (v_1)$.

We have proved that there exists a simple arc C_1 with end points u_1, v_1 such that

$$C_1 \subset K \cap \Omega(H_{p+1}, \beta)\,, \quad C_1 \cap M_1 = (u_1)\,, \quad C_1 \cap M_2 = (v_1)\,.$$

Similarly we may prove the existence of a simple arc C_2 with end points u_2, v_2 such that

$$C_2 \subset K \cap \Omega(H_{p+2}, \beta)\,, \quad C_2 \cap M_1 = (u_2)\,, \quad C_2 \cap M_2 = (v_2)$$

and the existence of a simple arc C_3 with end points u_3, v_3 such that

$$C_3 \subset K \cap \Omega(H_{p+3}, \beta), \qquad C_3 \cap M_1 = (u_3), \qquad C_3 \cap M_2 = (v_3).$$

We have

$$C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \emptyset\,.$$

In fact, if there were, e.g., a point $x \in C_1 \cap C_2$, we would have $\varrho(x, H_{p+1}) < \beta$, $\varrho(x, H_{p+2}) < \beta$ and consequently $\varrho(H_{p+1}, H_{p+2}) < 2\beta$. This is a contradiction, since $\varrho(H_{p+1}, H_{p+2}) > \eta \geqq 2\beta$.

VII. If there were $C_1 \subset \Omega(M_1, \varepsilon) \cup \Omega(M_2, \varepsilon)$, we would have

$$C_1 = [C_1 \cap \Omega(M_1, \varepsilon)] \cup [C_1 \cap \Omega(M_2, \varepsilon)]\,.$$

The sets on the right hand side are open in C_1 and, by V, disjoint; thus, they are separated. Moreover, they are non-void, since $u_1 \in C_1 \cap M_1$, $v_1 \in C_1 \cap M_2$. This is impossible, as C_1 is a connected set.

Thus, there is a point $x_1 \in C_1$ such that $\varrho(x_1, M_1) \geqq \varepsilon$, $\varrho(x_1, M_2) \geqq \varepsilon$. As $C_1 \subset \Omega(H_{p+1}, \beta)$, there is a point $y_1 \in H_{p+1}$ such that $\varrho(x_1, y_1) < \beta$. On the other hand, $x_1 \in C_1 \subset K$, $y_1 \in H_{p+1} \subset K$. Thus (see IV), there exists a connected $S_1 \subset K$ such that $x_1 \in S_1$, $y_1 \in S_1$, $d(S_1) < \eta$.

We have $x_1 \in S_1$, $d(S_1) < \eta$, $\varrho(x_1, M_1) \geqq \varepsilon > \eta$. Thus, $S_1 \cap M_1 = \emptyset$. Similarly we may prove that $S_1 \cap M_2 = \emptyset$.

We have $S_1 \cap C_2 = \emptyset$. Let there be, on the contrary, a point $z \in S_1 \cap C_2$. As $C_2 \subset \Omega(H_{p+2}, \beta)$, there is a point $t \in H_{p+2}$, $\varrho(z, t) < \beta$. As $d(S_1) < \eta$, we have $\varrho(y_1, z) < \eta$. Thus, $\varrho(y_1, t) \leqq \varrho(y_1, z) + \varrho(z, t) < \eta + \beta < 2\eta$. This is a contradiction since $y_1 \in H_{p+1}$, $t \in H_{p+2}$, $\varrho(H_{p+1}, H_{p+2}) > 2\eta$. Similarly we may prove that $S_1 \cap C_3 = \emptyset$.

Thus, we have proved that there exist points $x_1 \in C_1$, $y_1 \in H_{p+1}$ and a connected set S_1 such that

$$x_1 \in S_1 , \quad y_1 \in S_1 , \quad S_1 \subset P - (C_2 \cup C_3 \cup M_1 \cup M_2) .$$

Similarly we may prove that there exist points $x_2 \in C_2$, $y_2 \in H_{p+2}$ and a connected set S_2 such that

$$x_2 \in S_2 , \quad y_2 \in S_2 , \quad S_2 \subset P - (C_1 \cup C_3 \cup M_1 \cap M_2) ,$$

further, that there exist points $x_3 \in C_3$, $y_3 \in H_{p+3}$ and a connected set S_3 such that

$$x_3 \in S_3 , \quad y_3 \in S_3 , \quad S_3 \subset P - (C_1 \cup C_2 \cup M_1 \cup M_2) .$$

VIII. The set G is a component of $P - K$. Since $M_1 \cup M_2 \subset K$, there exists (see 18.2.5) a component Γ of $P - (M_1 \cup M_2)$ such that $G \subset \Gamma$.

We have $y_1 \in H_{p+1} \subset H \subset B(G) \subset \bar{G}$, so that $G \cup (y_1)$ is connected by 18.1.7. Since $y_1 \in S_1$ and since S_1 is also connected, $[G \cup (y_1)] \cup S_1 = G \cup S_1$ is connected by 18.1.4. Since $G \subset P - (M_1 \cup M_2)$, $S_1 \subset P - (M_1 \cup M_2)$, $G \cup S_1$ is a subset of a component of $P - (M_1 \cup M_2)$. Since $\emptyset \neq G \subset \Gamma$, we have $G \cup S_1 \subset \Gamma$.

Thus, we proved that $S_1 \subset \Gamma$. Similarly we may prove that $S_2 \subset \Gamma$, $S_3 \subset \Gamma$.

IX. M_1 and M_2 are disjoint continua. For $i = 1, 2, 3$, C_i is a simple arc with end points $u_i \in M_1$, $v_i \in M_2$. Moreover, $C_1^* \cap (M_1 \cup M_2) = \emptyset$, where $C_1^* = C_i - [(u_i) \cup (v_i)]$ and, further, $C_1 \cap C_2 = C_1 \cap C_3 = C_3 \cap C_2 = \emptyset$. Thus, by 27.2.8, $C_1^* \cup C_2^* \cup C_3^*$ is a subset of a component of $P - (M_1 \cup M_2)$. On the other hand, $x_1 \in S_1 \cap C_1$, $S_1 \subset P - (M_1 \cup M_2)$ and hence $x_1 \in C_1^*$. As $S_1 \subset \Gamma$, we have

$$C_1^* \cup C_2^* \cup C_3^* \subset \Gamma .$$

By 27.2.8, $\Gamma - (C_1 \cup C_2 \cup C_3)$ has exactly three components G_1, G_2, G_3 and we have

$$C_2 \cup C_3 \subset B(G_1) , \quad C_3 \cup C_1 \subset B(G_2) , \quad C_1 \cup C_2 \subset B(G_3) ,$$
$$C_1^* \cap B(G_1) = C_2^* \cap B(G_2) = C_3^* \cap B(G_3) = \emptyset ,$$
$$B(G_1) \cup B(G_2) \cup B(G_3) \subset C_1 \cup C_2 \cup M_1 \cup M_2 .$$

X. G is connected. We have $C_1 \cup C_2 \cup C_3 \subset K$, $G \cap K = \emptyset$. Moreover, $G \subset \Gamma$. Thus, G is contained in a component of $\Gamma - (C_1 \cup C_2 \cup C_3)$, i.e. we have either $G \subset G_1$ or $G \subset G_2$ or $G \subset G_3$. If suffices to finish the proof under the assumption of $G \subset G_1$.

We have

$$B(G_1) \subset C_2 \cup C_3 \cup M_1 \cup M_2 \subset K.$$

We have $G \cap K = \emptyset$, $S_1 \cap (C_2 \cup C_3 \cup M_1 \cup M_2) = \emptyset$. Hence,

$$(G \cup S_1) \cap B(G_1) = \emptyset.$$

On the other hand, $G \cup S_1$ is connected (see VIII) and we have

$$\emptyset \neq G \subset (G \cup S_1) \cap G_1.$$

Thus (18.1.8), $G \cup S_1 \subset G_1$. This is a contradiction, since $x_1 \in C_1 \cap S_1$, $G_1 \cap \cap C_1 = \emptyset$.

27.2.11. *Let P be a spherical space. Let $a \in P$, $b \in P$, $a \neq b$. Let $M \subset P$ be a closed locally connected set. Let M separate the point a from the point b in P. Then there exists a simple loop $C \subset M$ separating a from b in P.*

Proof: I. By 22.1.12, 25.1.2 and 27.2.1 [see also property (α)], there exists a component K of M separating a from b. K is not a one-point set [see property (β)]. Thus (see 17.2.2, 18.2.2 and 22.1.6), K is a locally connected continuum.

II. Let G be the component of $P - K$ containing a. Put $H = B(G)$. By 27.2.10, $H \subset K$ and H is a locally connected continuum. Denote by Γ the component of $P - H$ containing b. By 27.2.10, $C \subset H \subset K$ and C is a locally connected continuum. By 22.1.11, C is an irreducible cut of P between the points a, b. It remains to be shown that C is a simple loop.

III. Choose $u \in C$, $v \in C$, $u \neq v$. By 21.4, it suffices to show that $C - [(u) \cup (v)]$ is not connected. Let us assume the contrary. Since u, v are distinct points of a locally connected set C, there is a connected neighborhood U of u in C and a connected neighborhood V of v in C such that $\overline{U} \cap \overline{V} = \emptyset$. The sets $\overline{U}$, $\overline{V}$ are continua (see 17.2.2 and 18.1.6). There is an $\varepsilon > 0$ such that

$$x \in C, \ \varrho(x, u) < 2\varepsilon \Rightarrow x \in U,$$
$$x \in C, \ \varrho(x, v) < 2\varepsilon \Rightarrow x \in V.$$

Denote by S the set of all $x \in S$ with $\varrho(x, u) \geqq \varepsilon$, $\varrho(x, v) \geqq \varepsilon$. Then $S \subset C - - [(u) \cup (v)]$ and S is closed in C. $C - [(u) \cup (v)]$ is connected and open in C. C is a locally connected continuum. Thus, by 23.2.5 there exists a continuum T such that $S \subset T \subset C$.

IV. As $B(\Gamma) = C$, $B(G) \supset C$, the sets $\Gamma \cup (u)$, $G \cup (u)$ are connected (see 18.1.7), so that (see 18.1.4), $\Gamma \cup G \cup (u)$ is also connected. On the other hand, $\Gamma \cup G \cup (u)$ contains both a and b and we have $[\Gamma \cup G \cup (u)] \cap (T \cup \overline{V}) = \emptyset$. Thus, $T \cup \overline{V}$ does not separate a from b in P. Similarly we may prove that $T \cup \overline{U}$ does not separate a from b. Since $(T \cup \bar{U}) \cap (T \cup \bar{V})$ is a continuum, we obtain by property (γ) that $T \cup \bar{U} \cup \bar{V}$ does not separate a from b. This is a contradiction, since evidently $T \cup \bar{U} \cup \bar{V} = C$.

27.2.12. *Let P be a spherical space. Let $a \in P$, $b \in P$, $a \neq b$. Let $\varepsilon > 0$. Then there exists a simple loop $C \subset \Omega(a, \varepsilon)$ separating a from b in P.*

Proof: Property (α) yields, by 23.2.4, that $P = \bigcup_{i=1}^{m} P_i$ where P_i are locally connected continua, $2d(P_i) < \varepsilon$, $3d(P_i) < \varrho(a, b)$. If $1 \leqq i \leqq m$, let: [1] $i \in A_1$ if $a \in P_i$, [2] $i \in A_2$ if on the one hand i does not belong to A_1 and on the other hand there is an index j $(1 \leqq j \leqq m)$ such that $j \in A_1$, $P_i \cap P_j \neq \emptyset$, [3] $i \in A_3$ if neither $i \in A_1$ nor $i \in A_2$. For $k = 1, 2, 3$ denote by Q_k the union of all the P_i $(1 \leqq i \leqq m)$ with $i \in A_k$. Evidently, Q_1, Q_2, Q_3 are closed and locally connected (see 23.1.11) and we have $Q_1 \cup Q_2 \cup Q_3 = P$, $a \in Q_1 - Q_2$, $b \in Q_3 - Q_2$, $Q_1 \cap Q_3 = \emptyset$, $Q_1 \cup Q_2 \subset \Omega(a, \varepsilon)$. Thus, Q_2 separates a from b and, by 27.2.6, there exists a simple loop $C \subset Q_2$ separating a from b.

27.3. Let P be a spherical space. The word "*map*" (more precisely, map of the space P) will signify a system $\mathfrak{M}$ of a finite number (greater than or equal to 2) of simple arcs, satisfying certain assumptions stated below. The simple arcs $S \in \mathfrak{M}$ are said to be the *edges* of the map $\mathfrak{M}$. The union of all the edges of a map $\mathfrak{M}$ will be denoted by $|\mathfrak{M}|$. Every end point of an edge of a map is termed a *vertex* of the map. The components of $P - |\mathfrak{M}|$ are said to be the *faces* of the map $\mathfrak{M}$. We assume that: [1] if $a \in P$ belongs to two distinct edges of the map $\mathfrak{M}$, then a is an end point of every edge in which it is contained; [2] if G is a face of $\mathfrak{M}$, then $B(G)$ is a simple loop and $B(G)$ is a union of some edges of $\mathfrak{M}$.

Let $\mathfrak{M}$ be a map, let S be its edge, let $a \in S$ not be a vertex of $\mathfrak{M}$. Then (see 20.1.9) $S = S_1 \cup S_2$, where S_1, S_2 are simple loops with $S_1 \cap S_2 = (a)$. Denote by $\mathfrak{M}_1$ the system of simple arcs obtained from $\mathfrak{M}$ omitting S and adding S_1, S_2. We see easily that $\mathfrak{M}_1$ is a map. We say that $\mathfrak{M}_1$ is an *elementary refinement of the first kind* of the map $\mathfrak{M}$.

Let $\mathfrak{M}$ be a map, let a, b be two of its vertices, let G be its face, let $a \in B(G)$, $b \in B(G)$ and let S be a simple arc with the end points a, b such that $S - [(a) \cup \cup (b)] \subset G$. Denote by $\mathfrak{M}_2$ the system of all the simple arcs obtained from $\mathfrak{M}$ adding S. We see easily (see 22.1.9 and 27.2.9), that $\mathfrak{M}_2$ is a map. We say that $\mathfrak{M}_2$ is an *elementary refinement of the second kind* of the map $\mathfrak{M}$.

Let $\mathfrak{M}'$, $\mathfrak{M}''$ be maps. We say that $\mathfrak{M}''$ is a *refinement* of the map $\mathfrak{M}'$ if either $\mathfrak{M}' = \mathfrak{M}''$ or there is a finite sequence $\{\mathfrak{M}_i\}_0^m$ of maps such that: [1] $\mathfrak{M}_0 = \mathfrak{M}'$, $\mathfrak{M}_m = \mathfrak{M}''$, [2] if $1 \leqq i \leqq m$, then $\mathfrak{M}_i$ is an elementary refinement of the first or second kind of $\mathfrak{M}_{i-1}$.

If $\mathfrak{M}''$ is a refinement of $\mathfrak{M}'$ and $\mathfrak{M}'''$ a refinement of $\mathfrak{M}''$, then evidently $\mathfrak{M}'''$ is a refinement of $\mathfrak{M}'$.

27.3.1. *Let $\mathfrak{M}$ be a map of a spherical space P. Let G be its face. Let S be a simple arc with end points $a \in B(G)$, $b \in B(G)$. Let $S - [(a) \cup (b)] \subset G$. Then there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that S is an edge of $\mathfrak{M}'$ and $|\mathfrak{M}'| = |\mathfrak{M}| \cup S$.*

Proof: There are evidently maps $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that: [1] either $\mathfrak{M}_1 = \mathfrak{M}$ or $\mathfrak{M}_1$ is an elementary refinement of the first kind of $\mathfrak{M}$, [2] either $\mathfrak{M}_2 = \mathfrak{M}_1$ or $\mathfrak{M}_2$ is an elementary refinement of the first kind of $\mathfrak{M}_1$, [3] the points a, b are vertices of $\mathfrak{M}_2$. Evidently there exists a map $\mathfrak{M}'$ such that $S \in \mathfrak{M}'$ and $\mathfrak{M}'$ is an elementary refinement of the second kind of $\mathfrak{M}_2$. Evidently $\mathfrak{M}'$ has the required properties.

27.3.2. *Let $\mathfrak{M}$ be a map of a spherical space P. Let G be its face. Let C be a simple loop. Let $C \subset \bar{G}$; let $C - G$ contain at most one point. There exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that $C = |\mathfrak{M}'|$.*

Proof: I. Let $C - G = (a)$ and hence $a \in B(G)$. Choose points $b_1 \in C$, $b_2 \in P - G$ such that $b_1 \neq a \neq b_2$. By properties (α), (β) and by 22.3.1 (see also 15.5.2, 17.2.1 and 22.1.3) there exists a simple loop $B \subset P - (a)$ with end points b_1, b_2. We have $b_1 \in B \cap G$, $b_2 \in B - G$, so that, by 18.1.8, $B \cap B(G) \neq \emptyset$. We see easily by 20.2.7 that there exist points $a_1 \in C$, $a_2 \in B(G)$ and a simple arc $A \subset B$ with end points a_1, a_2 such that $A \cap C = (a_1)$, $A - (a_2) \subset G$. As $A \subset B$, we have $a_1 \neq a \neq a_2$. As C is a simple loop, there exist (see 21.1.2) simple loops C_1, C_2 with end points a, a_1 such that $C = C_1 \cup C_2$, $C_1 \cap C_2 = (a) \cup (a_1)$.

Evidently $A \cup C_1$ is a simple loop with end points $a \in B(G)$, $a_2 \in B(G)$, and $A \cup C_1 - [(a) \cup (a_2)] \subset G$. Thus (see 27.3.1), there exists a refinement $\mathfrak{M}_1$ of $\mathfrak{M}$ such that $|\mathfrak{M}_1| = |\mathfrak{M}| \cup A \cup C_1$. The set $C_2 - [(a) \cup (a_1)]$ is a connected (see 20.1.5) subset of $P - |\mathfrak{M}_1|$ so that there exists a face Γ of $\mathfrak{M}_1$ with $C_2 - ((a) \cup (a_1)) \subset \Gamma$. Obviously $a \in B(\Gamma)$, $a_1 \in B(\Gamma)$, so that (see 27.3.1) there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}_1$ such that $|\mathfrak{M}'| = |\mathfrak{M}_1| \cup C_2$, and hence $C \subset |\mathfrak{M}'|$.

II. Let $C \subset G$. Choose points $r_1 \in C$, $r_2 \in P - G$. There exists (see 22.3.1) a simple arc $R \subset P$ with end points r_1, r_2. By 18.1.8, $R \cap B(G) \neq \emptyset$. We see easily by 20.2.7 that there exists a simple arc $A \subset R$ with end points $a_1 \in C$, $a_2 \in B(G)$ such that $A \cap C = (a_1)$, $A - (a_2) \subset G$. $P - A$ is open and connected (see 27.2.3). Hence, $P - A$ is a connected, locally connected and topologically complete space, so that (see 22.3.1) there exists a simple arc $S \subset P - A$ with end points $s_1 \in C$, $s_2 \in P - G$. We have $S \cap B(G) \neq \emptyset$, so that there is a simple arc $B \subset S$ with end points $b_1 \in C$, $b_2 \in B(G)$ such that $B \cap C = (b_1)$, $B - (b_2) \subset G$. As $B \subset S$, we have $a_1 \neq b_1$. Thus, there are simple arcs C_1, C_2 with $C_1 \cup C_2 = C$, $C_1 \cap C_2 = (a_1) \cup (b_1)$.

Evidently $A \cup C_1 \cup B$ is a simple arc with end points $a_2 \in B(G)$, $b_2 \in B(G)$ and

$$A \cup C_1 \cup B - ((a_2) \cup (b_2)) \subset G.$$

Thus (see 27.3.1), there exists a refinement $\mathfrak{M}_1$ of $\mathfrak{M}$ such that $|\mathfrak{M}_1| = |\mathfrak{M}| \cup \cup A \cup C_1 \cup B$. $C_2 - [(a_1) \cup (b_1)]$ is a connected subset of $P - |\mathfrak{M}_1|$, so that there exists a face Γ of $\mathfrak{M}_1$ such that $C_2 - [(a_1) \cup (b_1)] \subset \Gamma$. Evidently $a_1 \in B(\Gamma)$. $b_1 \in B(\Gamma)$, so that (see 27.3.1) there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}_1$ such that $|\mathfrak{M}'| = = |\mathfrak{M}_1| \cup C_2$ and hence $C \subset |\mathfrak{M}'|$.

27.3.3. *Let $\mathfrak{M}$ be a map of a spherical space P. Let $a \in P$, $b \in P$, $a \neq b$. Let $C \subset P$ be a simple loop. Let C separate a from b in P. Let $\varepsilon > 0$. There exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ and a simple loop C_0 such that:* [1] $C_0 \subset |\mathfrak{M}'|$, [2] $\varrho(x, C) < \varepsilon$ *for* $x \in C_0$, [3] C_0 *separates a from b.*

Proof: I. If $C \subset |\mathfrak{M}|$, we may put $\mathfrak{M}' = \mathfrak{M}$, $C_0 = C$. If $C \cap |\mathfrak{M}|$ contains at most one point then $C - |\mathfrak{M}|$ is connected, so that there exists a face G of $\mathfrak{M}$ such that $C - |\mathfrak{M}| \subset G$. By 27.3.2 there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that $C \subset |\mathfrak{M}'|$ and we may again put $C_0 = C$.

Thus, let $C - |\mathfrak{M}| \neq \emptyset$ and let $C \cap |\mathfrak{M}|$ contain at least two distinct points.

II. By 20.1.12 we may put $|\mathfrak{M}| = \bigcup_{i=1}^{m} S_i$, where S_i are simple loops of less than

$$\min\,[\varepsilon, \varrho(a, C), \varrho(b, C)]$$

in diameter. Denote by A the union of all S_i ($1 \leqq i \leqq m$) with $C \cap S_i \neq \emptyset$. We have $A \neq \emptyset$, $a \in P - A$, $b \in P - A$ and A is compact and locally connected. We have $C - |\mathfrak{M}| = C - A$.

III. Let us assume that there exists a component T of $C - A$ such that $T \cup A = \overline{T} \cup A$ (see 8.7.1 and 18.2.2) separates a from b in P. T is a connected subset of $P - |\mathfrak{M}|$, so that there exists a face G of $\mathfrak{M}$ such that $T \subset G$. $\overline{T}$ is (see 21.1.6) a simple arc, the end points of which, t_1, t_2, belong to $B(G)$ and $\overline{T} - [(t_1) \cup (t_2)] = = T \subset G$. Thus (see 27.3.1) there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that $|\mathfrak{M}'| = = |\mathfrak{M}| \cup \overline{T} = |\mathfrak{M}| \cup T$. Since A is compact and locally connected, this holds also for $\overline{T} \cup A$. Thus (see 27.2.11), there exists a simple loop $C_0 \subset \overline{T} \cup A \subset \subset C \cup A$ separating a from b. Evidently C_0 has the required properties.

IV. It remains to prove the statement under the assumption that for no component T of $C - A$, $T \cup A = \overline{T} \cup A$ separates a from b in P.

Since C separates a from b in P, evidently $C - A$ separates a from b in $P - A$. On the other hand, $P - A$ is locally connected (see 22.1.3), so that, by 22.1.12, there is an irreducible cut $R \subset C - A$ of $P - A$ between a and b.

Evidently $R \cup A = \overline{R} \cup A$ (see 18.5.4) separates a from b in P.

V. Let T be *a* component of $C - A$ with $T \cap R \neq \emptyset$. Let us prove that the two end points of the simple arc $\overline{T}$ (see 21.1.6) belong to distinct components of A.

Let, on the contrary, there be a component A_0 of A such that $\overline{T} - T \subset A_0$. Put $A_1 = A - A_0$, $R_1 = R - T$. Since R is an irreducible cut of $P - A$ between the points a, b, R_1 does not separate a from b in $P - A$,*) so that $R_1 \cup A$ does not separate a from b in P.

Since $T \cup A = \overline{T} \cup A$ does not separate a from b in P, $T \cup A_0$ does not separate a from b in P either. $R_1 \cup A = \overline{R}_1 \cup A$ and $T \cup A_0 = \overline{T} \cup A_0$ are closed and $(R_1 \cup A) \cap (T \cup A_0) = A_0$ is connected. Thus, by property (γ), $(R_1 \cup A) \cup \cup (T \cup A_0) \supset R \cup A$ does not separate a from b in P, which is a contradiction.

VI. A is compact and locally compact and hence (see 23.1.4) it has a finite number of components and there is (see 17.3.4) a $\delta > 0$ such that the distances of distinct components of A are greater than δ.

Denote by $\mathfrak{S}$ the system of all components T of $C - A$ with $T \cap R \neq \emptyset$. For every $T \in \mathfrak{S}$, $\overline{T}$ is (see V) a simple arc, the end points of which belong to distinct components of A, so that $\overline{T}$ is more than δ in diameter. On the other hand, $\overline{T}$ are simple arcs contained in the simple loop C, and two distinct ones of them have at most their end points in common. Thus (see 21.1.7), $\mathfrak{S}$ is finite. Let T_j ($1 \leqq j \leqq n$) be all the elements of the system $\mathfrak{S}$. By 27.3.1 there exist maps $\mathfrak{M}_0 = \mathfrak{M}, \mathfrak{M}_1, \ldots, \mathfrak{M}_n$ such that $\mathfrak{M}_j$ ($1 \leqq j \leqq n$) is a refinement of the map $\mathfrak{M}_{j-1}$ and that $|\mathfrak{M}_j| = = |\mathfrak{M}_{j-1}| \cup T_j = |\mathfrak{M}_{j-1}| \cup \overline{T}_j$. Put $\mathfrak{M}' = \mathfrak{M}_n$, so that $\mathfrak{M}'$ is a refinement of $\mathfrak{M}$ and $|\mathfrak{M}'| = |\mathfrak{M}| \cup \bigcup_{j=1}^{n} \overline{T}_j$.

Since $R \cup A \subset A \cup \bigcup_{j=1}^{n} T_j$, $A \cup \bigcup_{j=1}^{n} T_j$ separates a from b in P. Thus (see 27.2.11), there exists a simple loop $C_0 \subset A \cup \bigcup_{j=1}^{n} T_j$ separating a from b in P. Obviously C_0 has the required properties.

27.3.4. *Let $\mathfrak{M}$ be a map of a spherical space P. Let $\varepsilon > 0$. There exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that every face of $\mathfrak{M}'$ is less than or equal to ε in diameter.*

Proof: I. $P \times P$ is compact (see ex. 17.2). Let Q be the set of all $(x, y) \in P \times P$ with $\varrho(x, y) \geqq \varepsilon$. Q is closed in $P \times P$ and hence it is compact.

II. For every couple $(x, y) \in Q$ we may, by 27.2.12, choose a simple loop $C(x, y) \subset P$ separating x from y in P. Then we may choose a connected neighborhood $U(x, y)$ of x in P and a connected neighborhood $V(x, y)$ of y in P such that

$$\overline{U(x,y)} \cap C(x, y) = \overline{V(x, y)} \cap C(x, y) = \emptyset .$$

III. If $(x, y) \in Q$, then $Q \cap [U(x, y) \times V(x, y)]$ is (see ex. 8.13) a neighborhood of (x, y) in Q. Thus (see 17.5.4), we may find a finite sequence $\{(x_i, y_i)\}_1^m$ such that $(x_i, y_i) \in Q$ for $1 \leqq i \leqq m$ and that

$$Q \subset \bigcup_{i=1}^{m} (U(x_i, y_i) \times V(x_i, y_i)) .$$

*) As $T \cap R \neq \emptyset$, it cannot be $R_1 = R$.

IV. There exists a $\delta > 0$ such that

$$\varrho[U(x_i, y_i), \quad C(x_i, y_i)] > \delta, \quad \varrho[V(x_i, y_i), \quad C(x_i, y_i)] > \delta$$

for $1 \leqq i \leqq m$. Applying theorem 27.3.3 m times, we obtain a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ and simple loops $C_i \subset P$ $(1 \leqq i \leqq m)$ such that: [1] $C_i \subset |\mathfrak{M}'|$ $(1 \leqq i \leqq m)$, [2] $z \in C_i \Rightarrow \varrho[z, C(x_i, y_i)] < \delta$ $(1 \leqq i \leqq m)$, [3] C_i separates x_i from y_i in P $(1 \leqq i \leqq m)$.

V. Let G be a face of the map $\mathfrak{M}'$. We have to prove that G is less than or equal to ε in diameter. Let there exist, on the contrary, points $x \in G$, $y \in G$ with $\varrho(x, y) > \varepsilon$. Then $(x, y) \in Q$, so that there is an index i $(1 \leqq i \leqq m)$ such that $(x, y) \in U(x_i, y_i) \times V(x_i, y_i)$. The distances of $U(x_i, y_i)$, $V(x_i, y_i)$ from $C(x_i, y_i)$ are greater than δ and the distancee of every point $z \in C_i$ from $C(x_i, y_i)$ is less than δ. Thus, $U(x_i, y_i)$, $V(x_i, y_i)$ are connected subsets of $P - C_i$. On the other hand, $x_i \in U(x_i, y_i)$, $x \in U(x_i, y_i)$, $y_i \in V(x_i, y_i)$, $y \in V(x_i, y_i)$ and C_i separates x_i from y_i in P. Hence, C_i separates x from y in P. As $C_i \subset |\mathfrak{M}'|$, $|\mathfrak{M}'|$ separates x from y in P. This is a contradiction, since $x \in G$, $y \in G$ and G is a connected subset of $P - |\mathfrak{M}'|$.

27.3.5. *Let $\mathfrak{M}$ be a map of a spherical space P. Then $\mathfrak{M}$ has a finite number of faces and P is the union of the closures of these faces.*

Proof: The boundary of every face of the map $\mathfrak{M}$ is a union of some edges of $\mathfrak{M}$. Hence, there is only a finite number of sets H which are a boundaries of faces of $\mathfrak{M}$. If $B(G) = H$, then G is (see 22.1.9) a component of $P - H$. H is a simple loop. Thus (see 27.2.7), each H is a boundary of at most two faces. Thus, $\mathfrak{M}$ has only a finite number of faces. Consequently (see section 8.1), the union of the closures of all the faces of $\mathfrak{M}$ is equal to $\overline{P - |\mathfrak{M}|}$. The set $|\mathfrak{M}|$ is closed and, by 12.2.4 and 27.2.4, nowhere dense, so that $\overline{P - |\mathfrak{M}|} = P$.

27.3.6. *Let P be a spherical space. Let $C \subset P$ be a simple loop. Let K be a component of $P - C$. Let $a \in C$, $b \in C$, $a \neq b$. Then there is a simple arc S with end points a, b such that $S - [(a) \cup (b)] \subset K$.*

Proof: I. K is open (see 22.1.4), so that $M = K \cup (a) \cup (b)$ is $\mathbf{G}_\delta(P)$. Thus (see 15.5.2 and 17.2.1), M is a topologically complete space. The points a, b belong to $\bar{K}$ by 27.2.7, so that M is connected by 18.1.7. By 22.3.1 it suffices to prove that M is locally connected. At every $x \in K$, K is locally connected and, hence, also M is locally connected at every $x \in K$ (see 22.1.2). It remains to prove that M is locally connected also at the points a, b. Certainly is suffices to prove this for the point a.

II. Choose an $\varepsilon > 0$. We have to prove that there exists a $\delta > 0$ and a connected $S \subset (a) \cup K$ such that $a \in S$, $K \cap \Omega(a, \varepsilon) \subset S \subset \Omega(a, \varepsilon)$.

By 27.2.7 there exists a map $\mathfrak{M}$ of P with $|\mathfrak{M}| = C$. By 27.3.4 there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}$ such that every face of $\mathfrak{M}'$ is less than ε in diameter. The map $\mathfrak{M}'$ has (see 27.3.5) a finite number of faces; denote them by G_i $(1 \leqq i \leqq m)$. By 27.3.5, $P = \bigcup_{i=1}^{m} \overline{G}_i$.

If $1 \leqq i \leqq m$, let: [1] $i \in A_1$ if $G_i \subset K$, $a \in \overline{G}_i$, [2] $i \in A_2$ if $a \in P - \overline{G}_i$. There is a $\delta > 0$ such that $i \in A_2$, $x \in \overline{G}_i$ imply $\varrho(a, x) > \delta$.

If $i \in A_1$, we have $a \in \overline{G}_i$, $G_i \subset K \cap \overline{G}_i$, so that $(a) \cup (K \cap \overline{G}_i)$ is a connected set (see 18.1.7). Hence (see 18.1.5), also $(a) \cup \bigcup_{i \in A_1} (K \cup \overline{G}_i) = S$ is connected. Evidently $S \subset (a) \cup K$, $a \in S$, $S \subset \Omega(a, \varepsilon)$.

We have to prove that $K \cap \Omega(a, \delta) \subset S$. Hence, let $x \in K$, $\varrho(a, x) < \delta$. As $P = \bigcup_{i=1}^{m} \overline{G}_i$, there is an index i $(1 \leqq i \leqq m)$ with $x \in \overline{G}_i$. As $\varrho(a, x) < \delta$, i does not belong to A_2, so that $a \in \overline{G}_i$. As $x \in K \cap \overline{G}_i$ and as K, G_i are open, we have $K \cap G_i \neq \emptyset$ by 10.2.6. On the other hand, G_i is connected and we have $G_i \subset P - |\mathfrak{M}'|$, $B(K) = C = |\mathfrak{M}| \subset |\mathfrak{M}'|$. Thus, $G_i \cap B(K) = \emptyset \neq K \cap G_i$, so that, by 18.1.8, $G_i \subset K$. As $a \in \overline{G}_i$, we have $i \in A_1$. As $i \in A_1$, $x \in K \cap \overline{G}_i$, we have $x \in S$.

Let P, Q be spherical spaces. Let $\mathfrak{M}$ be a map of P. Let $\mathfrak{N}$ be a map of Q. We say that the maps P, Q are *isological* (and we speak, more precisely, about an *isology* (f, g, h) between $\mathfrak{M}$ and $\mathfrak{N}$), if: [1] there exists a one-to-one mapping f of the system of all vertices of $\mathfrak{M}$ onto the system of all vertices of $\mathfrak{N}$, [2] there exists a one-to-one mapping g of the system of all edges of $\mathfrak{M}$ onto the system of all edges of $\mathfrak{N}$, [3] there exists a one-to-one mapping h of the system of all faces of $\mathfrak{M}$ onto the system of all faces of $\mathfrak{N}$, [4] if a, b are the end points of an edge S of $\mathfrak{M}$, then $f(a)$, $f(b)$ are the end points of the edge $g(S)$ of $\mathfrak{N}$, [5] if G is a face of a map $\mathfrak{M}$ and if $B(G) = \bigcup_{i=1}^{m} S_i$, where S_i are edges of $\mathfrak{M}$, then $B(h(G)) = \bigcup_{i=1}^{m} g(S_i)$.

If $\mathfrak{M}'$ is a refinement of $\mathfrak{M}$, if $\mathfrak{N}'$ is a refinement of $\mathfrak{N}$, and if (f', g', h') is an isology between $\mathfrak{M}'$ and $\mathfrak{N}'$, we say that (f', g', h') is an *extension* of the isology (f, g, h), if [1] for every vertex a of $\mathfrak{M}$, $f'(a) = f(a)$; [2] if S is an edge of $\mathfrak{M}$, if S' is an edge of $\mathfrak{M}'$ and if $S' \subset S$, then $g'(S') \subset g(S)$; [3] if G is a face of $\mathfrak{M}$, if G' is a face of $\mathfrak{M}'$ and if $G' \subset G$, then $h'(G') \subset h(G)$.

If $\mathfrak{M}''$ is a refinement of $\mathfrak{M}'$, if $\mathfrak{N}''$ is a refinement of $\mathfrak{N}'$, and if an isology (f'', g'', h'') between the map $\mathfrak{M}''$ and the map $\mathfrak{N}''$ is an extension of an isology (f', g', h') between the map $\mathfrak{M}'$ and the map $\mathfrak{N}'$, which itself is an extension of an isology (f, g, h) then evidently (f'', g'', h'') is an extension of the isology (f, g, h).

27.3.7. *Let P, Q be spherical spaces. Let $\mathfrak{M}$ be a map of P. Let $\mathfrak{N}$ be a map of Q. Let (f, g, h) be an isology between $\mathfrak{M}$ and $\mathfrak{N}$. Let $\mathfrak{M}'$ be a refinement of $\mathfrak{M}$. Then there* *exists a refinement $\mathfrak{N}'$ of $\mathfrak{N}$ and an isology (f', g', h') between $\mathfrak{M}'$ and $\mathfrak{N}'$ such that (f', g', h') is an extension of the isology (f, g, h).*

Proof may be done, of course, under the assumption that $\mathfrak{M}'$ is an elementary refinement of the first or of the second kind of $\mathfrak{M}$. (Then, $\mathfrak{N}'$ will be the same refinement of $\mathfrak{N}$.) This is quite evident for elementary refinements of the first kind and may be easily proved for elementary refinements of the second kind considering theorems 27.2.9 and 27.3.6.

Now, we are able to prove theorem 27.1.2.

Proof: I. Choose (see 27.2.12) a simple loop $C \subset P$ and a simple loop $D \subset Q$. It is easy to construct (see 27.2.7) a map $\mathfrak{M}_0$ of P and a map $\mathfrak{N}_0$ of Q such that $|\mathfrak{M}_0| = C$, $|\mathfrak{N}_0| = D$ and such that there exists an isology (f_0, g_0, h_0) between $\mathfrak{M}_0$ and $\mathfrak{N}_0$.

II. We shall construct recursively a sequence $\{\mathfrak{M}_n\}_0^\infty$ of maps of P and a sequence $\{\mathfrak{N}_n\}_0^\infty$ of maps of Q such that $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are the maps just constructed and such that: [1] for $n = 1, 2, 3, \ldots$, $\mathfrak{M}_n$ is a refinement of $\mathfrak{M}_{n-1}$ and $\mathfrak{N}_n$ is a refinement of $\mathfrak{N}_{n-1}$, [2] for $n = 0, 1, 2, \ldots$ there exists an isology (f_n, g_n, h_n) between $\mathfrak{M}_n$ and $\mathfrak{N}_n$ (already constructed for $n = 0$), [3] for $n = 1, 2, 3, \ldots$, (f_n, g_n, h_n) is an extension of the isology $(f_{n-1}, g_{n-1}, h_{n-1})$, [4] for $n = 1, 2, 3, \ldots$, every face of the map $\mathfrak{M}_n$ and every face of the map $\mathfrak{N}_n$ is less than n^{-1} in diameter.

III. Let us assume to be determined, for some $n = 1, 2, 3, \ldots$, the map $\mathfrak{M}_{n-1}$ of P, the map $\mathfrak{N}_{n-1}$ of Q and an isology $(f_{n-1}, g_{n-1}, h_{n-1})$ between $\mathfrak{M}_{n-1}$ and $\mathfrak{N}_{n-1}$. We have to determine the maps $\mathfrak{M}_n$, $\mathfrak{N}_n$ and the isologies (f_n, g_n, h_n). By 27.3.4, there exists a refinement $\mathfrak{M}'$ of $\mathfrak{M}_{n-1}$ such that every face of $\mathfrak{M}'$ is less than n^{-1} in diameter. By 27.3.7 we may determine a refinement $\mathfrak{N}'$ of $\mathfrak{N}_{n-1}$ such that there exists an isology (f', g', h') between $\mathfrak{M}'$ and $\mathfrak{N}'$, which is a refinement of $(f_{n-1}, g_{n-1}, h_{n-1})$. By 27.3.4 there exists a refinement $\mathfrak{N}_n$ of $\mathfrak{N}'$ such that every face of $\mathfrak{N}_n$ is less than n^{-1} in diameter. By 27.3.7 there exists a refinement $\mathfrak{M}_n$ of $\mathfrak{M}'$ such that there exists an isology (f_n, g_n, h_n) between $\mathfrak{M}_n$ and $\mathfrak{N}_n$, which is an extension of the isology (f', g', h'). The maps $\mathfrak{M}_n$, $\mathfrak{N}_n$ and the isology (f_n, g_n, h_n) evidently satisfy the assumptions pronounced in II.

IV. Let $x \in P$. By 27.3.5, for every n $(= 0, 1, 2, \ldots)$ there exists at least one face of $\mathfrak{M}_n$ such that its closure contains the point x. Let $G_i^{(n)}$ $(1 \leqq i \leqq k_n)$ be all such faces of $\mathfrak{M}_n$. Put

$$A_n(x) = \bigcup_{i=1}^{k_n} \overline{G_i^{(n)}}, \quad B_n(x) = \bigcup_{i=1}^{k_n} \overline{h_n(G_i^{(n)})}.$$

If $1 \leqq j \leqq k_{n+1}$, then $G_j^{(n+1)}$ is a connected subset of $P - |\mathfrak{M}_{n+1}| \subset P - |\mathfrak{M}_n|$, so that there is a face G of $\mathfrak{M}_n$ such that $G_j^{(n+1)} \subset G$. As $x \in \overline{G_j^{(n+1)}}$, we have $x \in \overline{G}$. Hence, there exists an index i such that $1 \leqq i \leqq k_n$, $G = G_i^{(n)}$. Thus, for every j $(1 \leqq j \leqq k_{n+1})$ there is an i $(1 \leqq i \leqq k_n)$ such that $G_j^{(n+1)} \subset G_i^{(n)}$. Since the isology $(f_{n+1}, g_{n+1}, h_{n+1})$ is an extension of (f_n, g_n, h_n), we also have $h_{n+1}(G_j^{(n+1)}) \subset h_n(G_i^{(n)})$. Thus, $A_{n+1}(x) \subset A_n(x)$, $B_{n+1}(x) \subset B_n(x)$. Evidently $B_n(x) \neq \emptyset$, $B_n(x) = \overline{B_n(x)}$. Thus, by 17.5.1, $\bigcap_{n=1}^{\infty} B_n(x) \neq \emptyset$.

If $1 \leqq i \leqq k_n$, we have $x \in \overline{G_i^{(n)}}$. Since (f_n, g_n, h_n) is an isology between the map $\mathfrak{M}_n$ and the map $\mathfrak{N}_n$, we may prove easily that there exists a point $y_n \in Q$ such that $y_n \in \overline{h_n(G_i^{(n)})}$ for $1 \leqq i \leqq k_n$. The sets $h_n(G_i^{(n)})$ are less than n^{-1} in diameter. Thus, $B_n(x)$ are less than $2 \,.\, n^{-1}$ in diameter. Thus, $\bigcap_{n=1}^{\infty} B_n(x)$ is a one-point set, i.e. there is a point $y \in Q$ with $(y) = \bigcap_{n=1}^{\infty} B_n(x)$. Put $y = \varphi(x)$.

V. Thus, we have defined a mapping φ of P into Q. In the same way we define a mapping ψ of Q into P. It is easy to prove that, for $x \in P$, $y \in Q$,

$$y = \varphi(x) \quad \text{if and only if} \quad x = \psi(y) .$$

Thus, φ is a one-to-one mapping of P onto Q and we have $\psi = \varphi_{-1}$.

VI. Choose a point $a \in P$ and a number $\varepsilon > 0$. There is an index n with $4 \,.\, n^{-1} < \varepsilon$. By 27.3.5 there is a $\delta > 0$ such that $\varrho(a, \overline{G}) < \delta$ for every face G of $\mathfrak{M}_n$ such that $a \in P - \overline{G}$. If $x \in P$, $\varrho(a, x) < \delta$, by 27.3.5 there is a face G of $\mathfrak{M}_n$ such that $x \in \overline{G}$. Since $\varrho(a, x) < \delta$, we obtain $a \in \overline{G}$ by the choice of δ.

Thus, $\overline{G} \subset A_n(a) \cap A_n(x)$ (see IV), so that $h_n(\overline{G}) \subset B_n(a) \cap B_n(x)$. Thus, $B_n(a) \cap B_n(x) \neq \emptyset$. In IV, we took notice of the fact that the sets $B_n(x)$ are less than $2 \,.\, n^{-1}$ in diameter. Evidently $\varphi(a) \in B_n(a)$, $\varphi(x) \in B_n(x)$, so that $\varrho(\varphi(x), \varphi(a)) < 4 \,.\, n^{-1} < \varepsilon$. Thus (for a given $a \in P$), for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in P$, $\varrho(a, x) < \delta$ imply $\varrho(\varphi(a), \varphi(x)) < \varepsilon$. Thus, φ is continuous. By the same argument (or, by 17.4.6), ψ is also continuous, i.e., φ is homeomorphic.

27.4. 27.4.1. *Let P be a locally connected unicoherent space. Let $G \subset P$ be an open connected set. Let $n = 0, 1, 2, \ldots$. $P - G$ has exactly n components if and only if $B(G)$ has exactly n components.*

Remark: By 22.1.14 and 25.2.4, or by 27.1.1 and 27.2.1 we may put, in theorem 27.4.1, $P = \mathbf{S}_2$.

Proof: I. Obviously it suffices to prove that the number of components of $P - G$ is less than or equal to n, if and only if the number of components of $B(G)$ is less than or equal to n.

II. Let the number of components of $B(G)$ be less than or equal to n. We have to prove that the number of components of $P - G$ is less than or equal to n. Let us assume the contrary. Then there exist mutually distinct components A_i $(0 \leqq i \leqq n)$ of $P - G$. Since P is unicoherent, it is connected. Evidently $\emptyset \neq A_i \neq P$. Thus (see 18.1.8), $B(A_i) \neq \emptyset$. As P is locally connected, we have (see 22.1.9)

$B(A_i) \subset B(P - G) = B(G)$. The sets A_i are closed (see 18.2.2 and 8.7.4), so that $B(A_i) \subset A_i$. Hence, there is a point $a_i \in A_i \cap B(G)$ $(0 \leqq i \leqq n)$. Since there are less than or equal to n components, we have $n > 0$ and there is a component K of $B(G)$ and indices j, k $(0 \leqq j < k \leqq n)$ such that $a_j \in K$, $a_k \in K$. Since G is open, we have $B(G) \subset P - G$. Thus, a_j, a_k belong to a connected subset K of $P - G$ so that they belong to the same component of $P - G$. This is a contradiction.

III. Let the number of components of $P - G$ be less than or equal to n. We have to prove that the number of components of $B(G)$ is less than or equal to n. Let us assume the contrary.

First, if $n = 0$, we have $P - G = \emptyset$, hence $G = P$, hence $B(G) = 0$ and hence the number of components of $B(G)$ is zero. Thus, let $n \geqq 1$.

By 18.3 and 18.3.11 there are mutually distinct quasicomponents A_i $(0 \leqq i \leqq n)$ of $B(G)$. Choose an $a_i \in A_i$ $(0 \leqq i \leqq n)$. As G is open, we have $B(G) \subset P - G$. Since the number of components of $P - G$ is less than or equal to n, there exists a component K of $P - G$ and indices j, k $(0 \leqq j < k \leqq n)$ with $a_j \in K$, $a_k \in K$.

Since a_j, a_k belong to distinct quasicomponents of $B(G) = \overline{G} - G$, the set $P - B(G) = G \cup (P - G)$ separates a_j from a_k in P. By 22.1.12 there exists an irreducible cut $C \subset G \cup (P - \overline{G})$ of P between the points a_j, a_k. C is connected by 25.1.2. As $C \subset G \cup (P - \overline{G})$, we have, by 18.1.2 (see also 10.2.2), either $C \subset G$ or $C \subset P - \overline{G}$.

If $C \subset G$ then G separates a_j from a_k in P. This is a contradiction, since $a_j \in K$, $a_k \in K$ and K is a connected subset of $P - G$. If $C \subset P - \overline{G}$, then $P - \overline{G}$ separates a_j from a_k in P. This is a contradiction, since $a_j \in \overline{G}$, $a_k \in \overline{G}$ and $\overline{G}$ is connected (see 18.1.6).

27.4.2. *Let P be a compact, locally connected and unicoherent space. Let $G \subset P$, $\Gamma \subset P$ be connected open sets. Let there exist a homeomorphic mapping f of G onto Γ. Let $n = 0, 1, 2, \ldots$. $B(\Gamma)$ has exactly n components if and only if $B(G)$ has exactly n components.*

Remark: By 17.10.2, 22.1.14 ane 25.2.4 we may put, in theorem 27.4.2, $P = \mathbf{S}_2$.

Proof: I. Evidently it suffices to prove that if the number of components of $B(G)$ is greater than n, then the number of components of $B(\Gamma)$ is also greater than n.

II. Let $B(G)$ have more than n components. Then (see ex. 10.3 and 18.11) we may put $B(G) = \bigcup_{i=0}^{n} A_i$ where $A_i \neq \emptyset$ and the sets A_i are disjoint and closed in $B(G)$ and hence (see 10.3.1) closed in P. A_i are compact (see 17.2.2), so that (see 17.3.4) there is an $\varepsilon > 0$ such that $0 \leqq j < k \leqq n$ implies $\varrho(A_j, A_k) > 2\varepsilon$. (For $n = 0$ choose the $\varepsilon > 0$ arbitrarily.)

III. For $0 \leqq i \leqq n$ put $U_i = \Omega(A_i, \varepsilon)$. Evidently U_i are disjoint and open (see 8.6). Moreover, $B(G) = \overline{G} - G \subset \bigcup_{i=0}^{n} U_i$.

Put $C = G - \bigcup_{i=0}^{n} U_i$, so that $C \subset G$. Obviously $C = \overline{G} - \bigcup_{i=0}^{n} U_i$, so that C is closed. Thus (see 17.2.2), C is compact, so that $f(C) \subset \Gamma$ is also compact and hence (see 17.4.2) closed in P.

IV. Choose an $a_i \in A_i$ $(0 \leqq i \leqq n)$. By 8.2.1 there exist sequences $\{b_{i\lambda}\}_{\lambda=1}^{\infty}$ $(0 \leqq i \leqq n)$ such that $b_{i\lambda} \in G$, $\lim_{\lambda\to\infty} b_{i\lambda} = a_i$. We may assume that, for every i, λ, $\varrho(b_{i\lambda}, a_i) < \varepsilon$, and hence $b_{i\lambda} \in G \cap U_i$.

As $b_{i\lambda} \in G$, there exist points $f(b_{i\lambda}) \in \Gamma$. As P is compact, we may, for $0 \leqq i \leqq n$, find a subsequence $\{c_{i\lambda}\}_{\lambda=1}^{\infty}$ of $\{b_{i\lambda}\}_{\lambda=1}^{\infty}$ such that $\lim_{\lambda\to\infty} f(c_{i\lambda}) = \alpha_i$ exists. We have $f(c_{i\lambda}) \in \Gamma$ and hence $\alpha_i \in \overline{\Gamma}$ $(0 \leqq i \leqq n)$.

If, for some i $(0 \leqq i \leqq n)$, there were $\alpha_i \in \Gamma$, we would have (since f is homeomorphic) $\lim_{\lambda\to\infty} c_{i\lambda} = f_{-1}(\alpha_i) \in G$, which is a contradiction, since (see 7.1.2) $\lim_{\lambda\to\infty} c_{i\lambda} = a_i \in \overline{G} - G$. Thus, $\alpha_i \in \overline{\Gamma} - \Gamma = B(\Gamma)$ for $0 \leqq i \leqq n$. Thus, $B(\Gamma) \neq \emptyset$ and the proof for $n = 0$ is finished.

V. For $n > 0$, it remains to prove that the points α_i $(0 \leqq i \leqq n)$ belong to distinct components of $B(\Gamma)$. Let us assume the contrary. Then there are indices j, k $(0 \leqq j < k \leqq n)$ and a component K of $B(G)$ such that $\alpha_j \in K$, $\alpha_k \in K$.

Since P is locally connected and as the set $f(C) \subset \Gamma \subset P - B(\Gamma) \subset P - ((\alpha_j) \cup (\alpha_k))$ is closed, there exists a connected neighborhood V_1 of α_j and a connected neighborhood V_2 of α_k such that $V_1 \cup V_2 \subset P - f(C)$. $V_1 \cup V_2 \cup K$ is connected (see 18.1.4). Since $\lim_{\lambda\to\infty} f(c_{j\lambda}) = \alpha_j$, $\lim_{\lambda\to\infty} f(c_{k\lambda}) = \alpha_k$, there exists an index μ with $f(c_{j\mu}) \in V_1$, $f(c_{k\mu}) \in V_2$.

VI. Since $c_{j\mu} \in U_j$, $c_{k\mu} \in U_k$, since the sets U_i $(0 \leqq i \leqq n)$ are disjoint and open, and since $C = G - \bigcup_{i=0}^{n} U_i$, C evidently separates $c_{j\mu}$ from $c_{k\mu}$ in G. Since f is a homeomorphic mapping, $f(C)$ separates $f(c_{j\mu})$ from $f(c_{k\mu})$ in Γ. Thus, $f(C) \cup (P - \Gamma)$ separates $f(c_{j\mu})$ from $f(c_{k\mu})$ in P.

Hence (see 21.1.12), there exists an irreducible cut $S \subset f(C) \cup (P - \Gamma)$ of P between $f(c_{j\mu})$ and $f(c_{k\mu})$. By 25.1.2, S is connected. Since $f(C)$, $P - \Gamma$ are separated (see 10.2.1), we have, by 18.1.2, either $S \subset f(C)$ or $S \subset P - \Gamma$.

If $S \subset f(C)$, then $f(C)$ separates $f(c_{j\mu})$ from $f(c_{k\mu})$ in P. This is a contradiction, since $f(c_{j\mu})$, $f(c_{k\mu})$ belong to the connected set $V_1 \cup V_2 \cup K \subset P - f(C)$. If $S \subset P - \Gamma$, then $P - \Gamma$ separates $f(c_{j\mu})$ from $f(c_{k\mu})$ in P. This is a contradiction, since Γ is connected and contains both $f(c_{j\mu})$, $f(c_{k\mu})$.

27.4.3. *A set $G \subset \mathbf{S}_2$ is homeomorphic to $\mathbf{E}_2$ if and only if:* [1] *G is open and connected,* [2] *$\mathbf{S}_2 - G$ is connected.*

Proof: I. Let G be homeomorphic to $\mathbf{E}_2$. Then (see 26.1.1) G is homeomorphic to $\mathbf{S}_2 - (\omega)$. Thus, G is open by 26.4.5. G is connected by 19.2.4. $P - G$ is connected by 27.4.2 (see also 27.4.1).

II. Let G be open and connected, let $\mathbf{S}_2 - G$ be connected. By 17.9.1, G is a separable and locally compact space. Hence (see 17.9.2), there exists a compact space P and a point $a \in P$ such that there exists a homeomorphic mapping φ of G onto $P - (a)$.

It suffices to prove that P is a spherical space. In fact, P is then homeomorphic to $\mathbf{S}_2$ by 27.1.1, so that G is homeomorphic with $\mathbf{E}_2$ by 17.10.4.

Thus, we have to prove that P has properties (α), (β), (γ) stated in 27.1.

III. Define a mapping f of $\mathbf{S}_2$ onto P as follows. If $x \in G$, let $f(x) = \varphi(x)$; if $x \in \in \mathbf{S}_2 - G$, put $f(x) = a$. (We have $\mathbf{S}_2 - G \neq \emptyset$, since $\mathbf{S}_2 - G$ is connected.)

We shall prove that f is continuous. Let $x_n \in \mathbf{S}_2$, $x \in \mathbf{S}_2$, $x_n \to x$; we have to prove that $f(x_n) \to f(x)$. First, if $x \in G$, then (as G is open) there is an index p such that, for $n > p$, $x_n \in G$ and hence $f(x_n) = \varphi(x_n)$, so that $\lim f(x_n) = \lim \varphi(x_n) = = \varphi(x) = f(x)$. Secondly, let $x \in \mathbf{S}_2 - G$, hence, $f(x) = a$. We have to prove that $\lim f(x_n) = a$. Assume the contrary. Then there is an $\varepsilon > 0$ and a subsequence $\{y_n\}_1^\infty$ of $\{x_n\}$ such that $\varrho[a, f(y_n)] \geqq \varepsilon$ for every n. Thus, for every n, $f(y_n) \neq a$, i.e. $y_n \in G$, $f(y_n) = \varphi(y_n)$. Denote by M the set of all $z \in P$ with $\varrho(a, z) \geqq \varepsilon$. M is closed in P and hence (see 17.2.2) compact. Thus, there is a subsequence $\{y'_n\}_1^\infty$ of $\{y_n\}$ and a point $z \in M$ with $\lim \varphi(y'_n) = z$. As $M \subset P - (a)$ and φ is homeomorphic, we have $\lim y'_n = \varphi_{-1}(z) \in \varphi_{-1}[P - (a)] = G$. This is a contradiction, since $\lim y'_n = x \in \mathbf{S}_2 - G$ by 7.1.2.

IV. Since f is continuous, P is a continuum by 18.1.10 and 19.2.5. G is locally connected by 22.1.3 and 22.1.14. Since φ is a homeomorphic mapping, $\varphi(G) = = P - (a)$ is also locally connected. Hence, P is locally connected at every $x \in P - - (a)$, so that $P - L(P) \subset (a)$ by 22.2.1. As P is a continuum, we have $L(P) = P$ by 22.2.5, so that P is locally connected by 22.2.2. Thus, P has property (α).

V. Property (β) requires $P - (y)$ connected for every $y \in P$. This is evident for $y = a$, since $P - (a)$ is homeomorphic with the connected G. Thus, let $y \neq a$, so that there is an $x \in G$ with $P - (y) = f[\mathbf{S}_2 - (x)]$. $\mathbf{S}_2 - (x)$ is connected, so that $P - (y)$ is connected by 18.1.10.

VI. It remains to prove that P has property (γ). Let A, B be sets closed in P and such that $A \cap B$ is either void or connected, and let u, v be two distinct points of $P - (A \cup B)$ such that neither A nor B separates u from v in P. We have to prove that $A \cup B$ does not separate u from v in P.

$\mathbf{S}_2$ is compact. Moreover, f is a continuous mapping of $\mathbf{S}_2$ onto P such that, for $y \in P$, $f_{-1}(y)$ is either a one-point set, or $f_{-1}(y) = \mathbf{S}_2 - G$, so that $f_{-1}(y)$ is connected for every $y \in P$. Thus (see 19.1.8), $f_{-1}(S)$ is connected whenever $S \subset P$ is connected.

Choose points $r \in \mathbf{S}_2$, $s \in \mathbf{S}_2$ with $f(r) = u$, $f(s) = v$. Put $A_0 = f_{-1}(A)$, $B_0 = = f_{-1}(B)$. Evidently, r, s are distinct points of $\mathbf{S}_2 - (A_0 \cup B_0)$.

A_0 and B_0 are closed in $\mathbf{S}_2$ (see 9.2). Evidently $A_0 \cap B_0 = f_{-1}(A \cap B)$. Since $A \cap B$ is void or connected, $A_0 \cap B_0$ is also void or connected.

Since A does not separate u from v in P, u, v belong to the same quasicomponent of $P - A$. On the other hand, P is locally connected and $P - A$ is open. Thus (see 22.1.3 and 22.1.5), u, v belong to the same component K of $P - A$. Since K is connected, $f_{-1}(K)$ is also connected. Moreover, $f_{-1}(K) \subset \mathbf{S}_2 - A_0$, $r \in f_{-1}(K)$, $s \in f_{-1}(K)$. Thus, A_0 does not separate r from s in $\mathbf{S}_2$. Similarly we may prove that B_0 does not separate r from s in $\mathbf{S}_2$. $\mathbf{S}_2$ has property (γ). Thus, $A_0 \cup B_0$ does not separate r from s in $\mathbf{S}_2$.

If $A \cup B$ separates u from v in P, we have $P - (A \cup B) = U \cup V$, $u \in U$, $v \in V$, $U \cap V = \emptyset$ and U, V are open in $P - (A \cup B)$ and hence in P. Then, however, $\mathbf{S}_2 - (A_0 - B_0) = f_{-1}(U) \cup f_{-1}(V)$, $r \in f_{-1}(U)$, $s \in f_{-1}(V)$, $f_{-1}(U) \cap f_{-1}(V) = \emptyset$, and $f_{-1}(U), f_{-1}(V)$ are (see 9.2) open, i.e. $A_0 \cup B_0$ separates r from s in $\mathbf{S}_2$, which is a contradiction.

27.4.4. *Let $G \subset \mathbf{S}_2$, $\Gamma \subset \mathbf{S}_2$. Let G be open and connected; let $\mathbf{S}_2 - G$ have a finite number n $(= 0, 1, 2, \ldots)$ of components. Γ is homeomorphic with G if and only if: [1] Γ is open and connected, [2] $\mathbf{S}_2 - \Gamma$ has n components.*

Remark: Theorem 27.4.3 is a consequence (see 17.10.4) of the case with $n = 1$ of theorem 27.4.4. Of course, the proof of theorem 27.4.3 was not superfluous; we shall need theorem 27.4.3 in the proof of theorem 27.4.4.

Proof: I. Let Γ be homeomorphic with G, so that Γ is connected. Γ is open by 26.4.5 and $\mathbf{S}_2 - \Gamma$ has n components by 27.4.1 and 27.4.2.

II. Choose mutually distinct points $s_\lambda \in \mathbf{S}_2$ $(\lambda = 1, 2, 3, \ldots)$ and put $M_n = \mathbf{S}_2 - \bigcup_{\lambda=1}^{n} (s_\lambda)$; hence, $M_0 = \mathbf{S}_2$, $M_{n+1} = M_n - (s_{n+1})$. Since two sets, each of which is homeomorphic with a third one, are homeomorphic, it suffices to prove, for every n $(= 0, 1, 2, \ldots)$, the following theorem $\mathbf{V}_n$: Let $G \subset \mathbf{S}_2$ be open and connected, let $\mathbf{S}_2 - G$ have n components; then G is homeomorphic with M_n.

Theorem $\mathbf{V}_0$ is evident: if $\mathbf{S}_2 - G$ has no component, we have $\mathbf{S}_2 - G = \emptyset$ and hence $G = \mathbf{S}_2 = M_0$. Theorem $\mathbf{V}_1$ follows by 27.4.3, as we remarked above. Thus, it suffices to prove theorem $\mathbf{V}_{n+1}$ assuming the validity of theorem $\mathbf{V}_n$ (for a given $n \geqq 1$).

III. Thus, let theorem $\mathbf{V}_n$ be valid for a given $n = 1, 2, 3, \ldots$, and let $G \subset \mathbf{S}_2$ be open and connected; let $\mathbf{S}_2 - G$ have $n + 1$ components. We have to prove that G is homeomorphic with M_{n+1}.

Denote by K_i $(0 \leqq i \leqq n)$ the components of $\mathbf{S}_2 - G$, so that $K_i \subset \mathbf{S}_2$ are connected closed sets.

Put $G_0 = G \cup K_0$. Then G_0 is an open set. By 19.3.1 we have $\overline{G} \cap K_0 = B(G) \cap K_0 = B(\mathbf{S}_2 - G) \cap K_0 \neq \emptyset$, so that $G_0 = [G \cup (\overline{G} \cap K_0)] \cup K_0$ is connected by 18.1.5 and 18.1.7. We have $\mathbf{S}_2 - G_0 = \bigcup_{i=1}^{n} K_i$, so that $\mathbf{S}_2 - G_0$ has n components. Thus, by $\mathbf{V}_n$ there exists a homeomorphic mapping f of G_0 onto M_n.

Put $f(G) = \Gamma$, $f(K_0) = \mathscr{L}$. K_0 is connected, so that $\mathscr{L}$ is also connected; K_0 is closed in $\mathbf{S}_2$ and hence compact, so that $\mathscr{L}$ is also compact and hence closed in $\mathbf{S}_2$. G is connected, so that Γ is also connected. We have $\mathbf{S}_2 = M_n \cup (\mathbf{S}_2 - M_n) = = \Gamma \cup \mathscr{L} \cup \bigcup_{\lambda=1}^{n} (s_\lambda)$ with disjoint summands. Since $\mathscr{L}$ is closed, we have evidently $\bigcup_{\lambda=1}^{n} (s_\lambda) \subset \bar{\Gamma}$ and hence $\Gamma \subset \mathbf{S}_2 - \mathscr{L} \subset \bar{\Gamma}$, so that $\mathbf{S}_2 - \mathscr{L}$ is connected. Moreover. certainly $\mathbf{S}_2 - \mathscr{L}$ is open and $\mathbf{S}_2 - (\mathbf{S}_2 - \mathscr{L}) = \mathscr{L}$ is connected. Thus, by 27.4.3, there exists a homeomorphic mapping φ of $\mathbf{S}_2 - \mathscr{L}$ onto $\mathbf{E}_2$. Put $\varphi(s_\lambda) = u_\lambda$ $(1 \leqq \lambda \leqq n)$.

For $x \in G$ put $g(x) = \varphi[f(x)]$. Obviously g is a homeomorphic mapping of G onto $\mathbf{E}_2 - \bigcup_{\lambda=1}^{n} (u_\lambda)$. Hence, it suffices to prove that the sets $M_{n+1} = \mathbf{S}_2 - \bigcup_{\lambda=1}^{n+1} (s_\lambda)$, $\mathbf{E}_2 - \bigcup_{\lambda=1}^{n} (u_\lambda)$ are homeomorphic, which is easy (see 17.10.4).

Exercises

27.1. Deduce theorem 27.2.5 from 27.1.1 and from theorems of § 26.

27.2. Similarly deduce theorem 27.2.6.

27.3. Similarly deduce theorem 27.2.8.

27.4. Deduce theorem 27.2.9 from 27.2.6 and 27.2.7 without use of theorem 27.2.8.

27.5. Generalize theorem 27.8.8 (and its proof) in such a way that one may speak about n simple arcs instead of C_1, C_2, C_3.

27.6. Proving theorem 27.1.2, we chosed a simple loop $C \subset P$ and a simple loop $D \subset Q$. We constructed there a homeomorphic mapping φ of P onto Q such that $\varphi(C) = D$. From this we may prove easily the following theorem: Let $C_1 \subset \mathbf{S}_2$ and $C_2 \subset \mathbf{S}_2$ be simple loops and let G_i be a component of $\mathbf{S}_2 - C_i$ $(i = 1, 2)$. There exists a homeomorphic mapping φ of $\bar{G}_1$ onto $\bar{G}_2$. We have $\varphi(C_1) = C_2$. [In addition theorem 26.4.5 yields that under every homeomorphic mapping φ of $\bar{G}_1$ onto $\bar{G}_2$ we have $\varphi(C_1) = C_2$.]