Chapter IV

CONNECTEDNESS

§ 18. General theorems concerning connectedness

18.1. A metric space P is said to be *connected* if [1] $P \neq \emptyset$, [2] if $P = A \cup B$ with separated (see 10.2) summands, we have either $A = \emptyset$ or $B = \emptyset$. Since every point set Q embedded into a metric space P is (see 6.3) a metric space, we need not define the notion of connected point set. Connectedness is a topological notion (see 9.3).

The following theorem is evident:

18.1.1. *Every one-point space is connected.*

18.1.2. *Let $P = A \cup B$ with separated summands. Let $S \subset P$ be connected. Then either $S \subset A$ (and hence $S \cap B = \emptyset$), or $S \subset B$ (and hence $S \cap A = \emptyset$).*

Proof: We have $S = (S \cap A) \cup (S \cap B)$ with separated (see 10.2.4) summands. Hence, either $S \cap A = \emptyset$ and then $S \subset B$ or $S \cap B = \emptyset$ and then $S \subset A$.

18.1.3. *Let $P \neq \emptyset$. For every couple a, b of distinct points $a \in P$, $b \in P$, let there be a connected $S(a, b) \subset P$ with $a \in S(a, b)$, $b \in S(a, b)$. Then P is connected.*

Proof: Let $P = A \cup B$ with separated A, B. We have to prove that either $A = \emptyset$ or $B = \emptyset$. Let, on the contrary, $a \in A$, $b \in B$. Then $A \cap S(a, b) \neq \emptyset \neq B \cap S(a, b)$, so that, by 18.1.2, $S(a, b)$ is not connected. This is a contradiction.

18.1.4. *Let $S_1 \subset P$, $S_2 \subset P$ be connected sets and let $S_1 \cap S_2 \neq \emptyset$. Then $S_1 \cup S_2$ is a connected set.*

Proof: We have $\emptyset \neq S_1 \subset S_1 \cup S_2$ and hence $S_1 \cup S_2 \neq \emptyset$. Let $S_1 \cup S_2 = A \cup B$ with separated A, B. We have to prove that either $A = \emptyset$ or $B = \emptyset$. Let, on the contrary, $A \neq \emptyset \neq B$. By 18.1.2, either $S_1 \cap A = \emptyset$ or $S_1 \cap B = \emptyset$; similarly either $S_2 \cap A = \emptyset$ or $S_2 \cap B = \emptyset$. Hence, one of the following cases occurs: [1] $S_1 \cap A = S_2 \cap A = \emptyset$, [2] $S_1 \cap B = S_2 \cap B = \emptyset$, [3] $S_1 \cap A = S_2 \cap B = \emptyset$, [4] $S_1 \cap B = S_2 \cap A = \emptyset$. In case [1] we have $A = (S_1 \cup S_2) \cap A = \emptyset$, which is a contradiction. Similarly we obtain $B = \emptyset$ in case [2]. In cases [3] and [4] we have $S_1 \cap S_2 = S_1 \cap S_2 \cap (A \cup B) = \emptyset$. This is also a contradiction.

18.1.5. *Let $a \in P$. Let $\mathfrak{S} \neq \emptyset$ be a system of connected subsets of P such that every $S \in \mathfrak{S}$ containts the point a. Then the set*

$$T = \bigcup_{S \in \mathfrak{S}} S$$

is connected.

Proof: Evidently $T \neq \emptyset$. Let $\alpha \in T$, $\beta \in T$. Then there are sets $S_1 \in \mathfrak{S}$, $S_2 \in \mathfrak{S}$ such that $\alpha \in S_1$, $\beta \in S_2$. Put $S(\alpha, \beta) = S_1 \cup S_2$. We have $S(\alpha, \beta) \subset T$, $\alpha \in S(\alpha, \beta)$, $\beta \in S(\alpha, \beta)$. Moreover, the set $S(\alpha, \beta)$ is connected by 18.1.4, as $S_1 \cap S_2 \neq \emptyset$, since $a \in S_1 \cap S_2$. Thus, the set T is connected by 18.1.3.

18.1.6. *Let a set $S \subset P$ be connected. Then $\bar{S}$ is also connected.*

Proof: We have $S \subset \bar{S}$, hence $\bar{S} \neq \emptyset$. Let $\bar{S} = A \cup B$ with separated summands. We have to prove that either $A = \emptyset$ or $B = \emptyset$. The set A is closed in $A \cup B = \bar{S}$, and $\bar{S}$ is closed in P; hence (see 8.7.4), A is closed in P, i.e. $A = \bar{A}$. Similarly $B = \bar{B}$. Since $S \subset \bar{S}$, we have, by 18.1.2, either $S \subset A$ or $S \subset B$, hence, either $\bar{S} \subset \bar{A} = A$ or $\bar{S} \subset \bar{B} = B$. We have $\bar{S} = A \cup B$ and A, B are (separated, hence) disjoint. Thus, either $B = \emptyset$ or $A = \emptyset$.

18.1.7. *Let a set $S \subset P$ be connected. Let $S \subset T \subset \bar{S}$. Then T is also connected.*

Proof: This follows from 18.1.6, as $T = T \cap \bar{S}$, i.e. (see 8.7.1) it is the closure of the set S in T.

18.1.8. *Let $A \subset P$, $S \subset P$. Let S be connected. Let $A \cap S \neq \emptyset \neq S - A$. Then $S \cap B(A) \neq \emptyset$.*

Proof: We have $P = A \cup (P - A) = \bar{A} \cup \overline{P - A}$ and hence $S = (S \cap \bar{A}) \cup (S \cap \overline{P - A})$. The sets $S \cap \bar{A}$, $S \cap \overline{P - A}$ are nonvoid. Since S is connected, they are not separated, hence (see 10.2.1) they are not disjoint, hence $\emptyset \neq S \cap \bar{A} \cap \overline{P - A} = S \cap B(A)$.

18.1.9. *Let a connected space P contain two distinct points. Then P is uncountable.*

Proof: Let $a \in P$, $b \in P$, $a \neq b$, hence $\varrho(a, b) > 0$. It suffices to prove that for every $\varepsilon > 0$, $\varepsilon < \varrho(a, b)$ there is an $x \in P$ with $\varrho(a, x) = \varepsilon$. Let $0 < \varepsilon < \varrho(a, b)$, $A = \underset{x}{\mathrm{E}}[\varrho(a, x) < \varepsilon]$. Then $A \cap P \neq \emptyset \neq P - A$ and hence $B(A) \neq \emptyset$ by 18.1.8. Evidently $B(A) \subset \underset{x}{\mathrm{E}}[\varrho(a, x) = \varepsilon]$.

18.1.10. *Let f be a continuous mapping of a connected space P onto a metric space Q. Then Q is a connected space.*

Proof: Since $P \neq \emptyset$ we also have $Q \neq \emptyset$. Let $Q = A \cup B$ with separated A, B. We have to prove that either $A = \emptyset$ or $B = \emptyset$. We have $P = f_{-1}(Q) = f_{-1}(A) \cup f_{-1}(B)$.

We have $A \cap B = \emptyset$, so that $f_{-1}(A) \cap f_{-1}(B) = \emptyset$. Moreover, A, B are closed in $A \cup B = Q$, so that (see 9.2) $f_{-1}(A)$, $f_{-1}(B)$ are closed in P. Thus, $f_{-1}(A)$, $f_{-1}(B)$ are separated. Since $P = f_{-1}(A) \cup f_{-1}(B)$ is connected, we have either $f_{-1}(A) = \emptyset$ and hence $A = \emptyset$ or $f_{-1}(B) = \emptyset$ and hence $B = \emptyset$.

18.1.11. *Let P be a connected space. Let a set $Q \subset P$ be either connected or void. Let $P - Q = A \cup B$ with separated A, B. Then each of the two sets $Q \cup A$, $Q \cup B$ is either connected or void.*

Proof will be done e.g. for $Q \cup A$. Let $Q \cup A = H \cup K$ with separated H, K. We have to prove that either $H = \emptyset$ or $K = \emptyset$. Since $Q \subset H \subset K$ we have, by 18.1.2, either $Q \subset H$ or $Q \subset K$. E.g. let $Q \subset H$ so that $K \subset A$. Thus, K and B are separated by 10.2.4, so that K and $H \cup B$ are separated by 10.2.5. We have $K \cup (H \cup B) = (H \cup K) \cup B = (Q \cup A) \cup B = Q \cup (A \cup B) = Q \cup (P - Q) = P$ and P is connected, so that either $K = \emptyset$ or $H \cup B = \emptyset$, which implies $H = \emptyset$.

18.1.12. *Let $A \subset P$, $B \subset P$ be non-void closed sets. Let $A \cup B$ be connected; let $A \cap B$ be connected. Then A, B are also connected.*

Proof: We have $(A - B) \cap (B - A) = \emptyset$. Moreover, $\overline{A - B} \subset \bar{A} = A$, and hence $\overline{A - B} \cap (B - A) = \emptyset$ and similarly $\overline{B - A} \cap (A - B) = \emptyset$. Hence, the sets $A - B$, $B - A$ are (see 10.2.3) separated. Thus, all the assumptions of theorem 18.1.11, where we put $A \cup B$, $A \cap B$, $A - B$, $B - A$ instead of P, Q, A, B respectively, are satisfied. Hence, $A = (A \cap B) \cup (A - B)$, $B = (A \cap B) \cup (B - A)$ are connected (since they are non-void).

18.1.13. *Let P and Q be connected spaces. Then $P \times Q$ is a connected space.*

Proof: We have $P \neq \emptyset \neq Q$ and hence $P \times Q \neq \emptyset$. Choose an $a = (a_1, a_2) \in \in P \times Q$ and $b = (b_1, b_2) \in P \times Q$. By 18.1.3 it suffices to show that there exists a connected $S \subset P \times Q$ such that $a \in S$, $b \in S$. The set $S_1 = P \times (a_2)$ is homeomorphic with P. Since connectedness is a topological property, S_1 is connected. Similarly, $S_2 = (b_1) \times Q$ is homeomorphic with Q and hence connected. We have $a \in S_1$, $b \in S_2$ and hence $(a) \cup (b) \subset S = S_1 \cup S_2 \subset P \times Q$. As $(b_1, a_2) \in S_1 \cap S_2 \neq \emptyset$, S is connected by 18.1.4.

18.2. Let P be a metric space. A set $K \subset P$ is said to be a *component* of the space P if it is a maximal connected subset of P, i.e., if [1] K is connected, [2] $A \subset P$ connected and $A \supset K$ implies $A = K$. Since every set $Q \subset P$ is a metric space, we need not define the notion of component of a point set. The notion of component is a topological notion (see 9.3).

Obviously, $\emptyset$ has no components.

18.2.1. *Every point $a \in P$ belongs to exactly one component of the space P.*

Proof: Denote by $\mathfrak{S}$ the system of all connected parts of the space P containing the point a. By 18.1.1 $(a) \in \mathfrak{S}$, hence $\mathfrak{S} \neq \emptyset$. The union T of all sets of the system $\mathfrak{S}$ is conntected by 18.1.5. Evidently T is a component of the space P containing the point a. Let K also be a component of P containing a. We have $K \in \mathfrak{S}$, hence, $K \subset T$. As K is a component and T is connected, we obtain $K = T$.

18.2.2. *Components of any space P are closed sets.*

Proof: Let K be a component of a space P. Then $\bar{K}$ is connected (by 18.1.5) and $K \subset \bar{K}$. Since K is a component, we have $\bar{K} = K$.

The following theorem is evident:

18.2.3. *A space P is connected if and only if it has exactly one component.*

18.2.4. *Let K_1, K_2 be two distinct components of a space P. Then K_1, K_2 are separated sets.*

Proof: The sets K_1, K_2 are closed in P by 18.2.2 and hence they are closed in $K_1 \cup K_2$. Moreover, $K_1 \cap K_2 = \emptyset$ by 18.2.1.

18.2.5. *Let $S \subset P$ be a connected set. Then there is exactly one component K of P such that $S \subset K$.*

Proof: By 18.2.1 there is at most one such component, as $S \neq \emptyset$. Choose $a \in S$. By 18.2.1 P has a component K containing the point a. The set $S \cup K$ is connected (by 18.1.4) and contains the component K. Hence, $S \cup K = K$, i.e. $S \subset K$.

18.3. Let P be a metric space. Let $a \in P$, $b \in P$. We say that P *is connected between the points a and b*, if in every decomposition $P = A \cup B$ with separated summands both a and b belong to the same summand A or B. This always holds if $a = b$.

18.3.1. *A space $P \neq \emptyset$ is connected if and only if it is connected between a and b for every choice of the points $a \in P$, $b \in P$.*

Proof: I. Let P be connected and let $a \in P$, $b \in P$. If $P = A \cup B$ with separated summands, then one of the summands is void, so that the other contains both the points a, b.

II. Let P be not connected. Then $P = A \cup B$ with separated non-void A, B. Choose $a \in A$, $b \in B$. Obviously P is not connected between a and b.

18.3.2. *Let $a \in P$, $b \in P$, $c \in P$. Let P be connected between a and b. Let P be connected between b and c. Then P is connected between a and c.*

Proof: Let $P = A \cup B$ with separated summands and let, e.g., $a \in A$. We have to prove that also $c \in A$. Since $a \in A$ and since P is connected between a and b we have $b \in A$. Since $b \in A$ and since P is connected between b and c, we also have $c \in A$.

18.3.3. *Let $S \subset P$ be a connected set. Let $a \in S$, $b \in S$. Then P is connected between a and b.*

Proof: Let $P = A \cup B$ with disjoint summands and let, e.g., $a \in A$. We have to prove that also $b \in A$.

By 18.1.2 we have either $S \subset A$ or $S \subset B$. Since $a \in S$ belongs to A, we have $S \subset A$ and hence also $b \in A$.

A set $Q \subset P$ is termed a *quasicomponent* of the space P, if [1] $Q \neq \emptyset$, [2] P is connected between points a and b whenever $a \in Q$, $b \in Q$, [3] P is not connected between a and b whenever $a \in Q$, $b \in P - Q$. Evidently, $\emptyset$ has no quasicomponents.

18.3.4. *Every point $a \in P$ is contained in exactly one quasicomponent of the space P.*

Proof: I. Denote by Q the set of all $x \in P$ such that P is connected between a and x. We have $a \in Q$ and hence $Q \neq \emptyset$. If $x \in Q$, $y \in Q$, then P is connected between x and a and also between a and y. Thus, P is connected between x and y by 18.3.2. If $x \in Q$, $y \in P - Q$ then P is not connected between x and y. But P is connected between a and x; if it were connected between x and y, it would be connected between a and y by 18.3.2, i.e. y would belong to Q, which is a contradiction. Thus, Q is a quasicomponent of P.

II. Let Q_1 and Q_2 be quasicomponents of P, containing the point a. If $x \in Q_1$, then P is connected between a and x, so that $x \in Q_2$. Hence, $Q_1 \subset Q_2$ and similarly $Q_2 \subset Q_1$. Hence $Q_1 = Q_2$.

18.3.5. *Points $a \in P$, $b \in P$ belong to the same quasicomponent of P if and only if P is connected between a and b.*

Proof: Let Q be the quasicomponent of P (see 18.3.4) containing the point a. By part I of the previous proof, $b \in Q$ if and only if P is connected between a and b.

18.3.6. *Quasicomponents of any space P are closed sets.*

Proof: Let, on the contrary, a quasicomponent Q of a space P be not closed. Then we may choose a point $a \in Q$ and a point $b \in \bar{Q} - Q$. Then P is not connected between a and b so that $P = A \cup B$ with disjoint summands, $a \in A$, $b \in B$. As A, B are separated, they are closed in $A \cup B = P$ and hence $A = \bar{A}$. If $x \in Q$, then P is connected between a and x. Since $P = A \cup B$ with separated A, B and since

$a \in A$ we have $x \in A$. Thus, $x \in Q$ implies $x \in A$, i.e. $Q \subset A$, so that $\bar{Q} \subset \bar{A} = A$ and hence $b \in A$. This is a contradiction.

From 18.3.1 and 18.3.5 we obtain

18.3.7. *A space P is connected if and only if it has exactly one quasicomponent.*

18.3.8. *Let Q_1, Q_2 be two distinct quasicomponents of a space P. Then Q_1, Q_2 are separated sets.*

Proof: The sets Q_1, Q_2 are closed in P by 18.3.6; hence, they are also closed in $Q_1 \cup Q_2$. Moreover, $Q_1 \cap Q_2 = \emptyset$ by 18.3.4.

18.3.3 and 18.3.5 yield

18.3.9. *Every component of any space P is a subset of some quasicomponent of P.*

18.3.10. *A quasicomponent Q of a space P is a component of P if and only if it is connected.*

Proof: I. If Q is not connected, it is not a component, since every component is connected.

II. If Q is connected, then (see 18.2.5) Q is a subset of some component K, and K is (see 18.3.9) a subset of a quasicomponent Q'. Then $\emptyset \neq Q \subset K \subset Q'$ so that $Q = Q'$ by 18.3.4. Thus, $Q = K$.

18.3.11. *Let P have a finite number of quasicomponents. Then every quasicomponent is connected.*

Proof: Let, on the contrary, a quasicomponent Q_1 not be connected, so tha) $Q_1 = A \cup B$ with separated non-void A, B. Choose $a \in A$, $b \in B$. Let Q_i ($2 \leqq i \leqq n$t be all the other quasicomponents of P. (We have $n \geqq 2$, i.e. $Q_1 \neq P$, since Q_1 is not connected between a and b and P is connected between a and b.) The sets A and B are separated. A and Q_i ($2 \leqq i \leqq n$) also are separated (see 18.3.8 and 10.2.4). Hence (see 10.2.5), A and $B \cup \bigcup_{i=2}^{n} Q_i$ are separated. We have $A \cup (B \cup \bigcup_{i=2}^{n} Q_i) =$

$$= (A \cup B) \cup \bigcup_{i=2}^{n} Q_i = \bigcup_{i=1}^{n} Q_i = P, \ a \in A, \ b \in B \cup \bigcup_{i=2}^{n} Q_i.$$

Thus, P is not connected between $a \in Q_1$ and $b \in Q_1$ which is a contradiction.

18.4. Let $a \in P$, $b \in P$. *A set chain from a to b* is a finite sebuence $\{M_i\}_{i=1}^{m}$ of point sets such that [1] $a \in M_1$, [2] $b \in M_m$, [3] $M_i \cap M_{i+1} \neq \emptyset$ for $1 \leqq i \leqq m - 1$.

18.4.1. *Let $a \in P$, $b \in P$. For every finite system $\mathfrak{A}$ of open sets with $\bigcup_{X \in \mathfrak{A}} X = P$ assume* *that we may choose a set chain from a to b out of $\mathfrak{A}$. Then P is connected between a and b.*

Proof: Let, on the contrary, P not be connected between a and b. Then $P = A \cup B$ with separated summands, $a \in A$, $b \in B$. Denote by $\mathfrak{A}$ the system consisting of the sets A and B. Then $\mathfrak{A}$ is a finite system of open sets with $\bigcup_{X \in \mathfrak{A}} X = P$. Hence, there is a set chain $\{M_i\}_{i=1}^m$ from a to b such that $M_i \in \mathfrak{A}$. Thus, for every i either $M_i = A$ or $M_i = B$. As $A \cap B = \emptyset$, $a \in A$, $a \in M_1$, we have $M_1 = A$. As $M_i \cap M_{i+1} \neq \emptyset$ $(1 \leqq i \leqq m - 1)$ we obtain by induction that $M_i = A$ for every i $(1 \leqq i \leqq m)$. Hence, $M_m = A$, so that $b \in A$, which is a contradiction.

18.4.2. *Let P be connected between $a \in P$ and $b \in P$. Let $\mathfrak{A}$ be a system of open sets with $\bigcup_{X \in \mathfrak{A}} X = P$. Then we may choose a set chain from a to b out of $\mathfrak{A}$.*

Proof: Choose an $A \in \mathfrak{A}$ with $a \in A$. Denote by $\mathfrak{A}_1$ the system of all the $X \in \mathfrak{A}$ such that there is a finite sequence $\{A_i\}_{i=1}^m$ with [1] $A_i \in \mathfrak{A}$ $(1 \leqq i \leqq m)$, [2] $A_1 = A$, [3] $A_m = X$, [4] $A_i \cap A_{i+1} \neq \emptyset$ $(1 \leqq i \leqq m - 1)$.

Put $\mathfrak{A}_2 = \mathfrak{A} - \mathfrak{A}_1$. If $X \in \mathfrak{A}_1$, $Y \in \mathfrak{A}$, $X \cap Y \neq \emptyset$, we have, obviously, $Y \in \mathfrak{A}_1$. Hence,

$$X \in \mathfrak{A}_1,\ Y \in \mathfrak{A}_2 \Rightarrow X \cap Y = \emptyset . \tag{1}$$

Choose a $B \in \mathfrak{A}$ with $b \in B$. If $B \in \mathfrak{A}_1$, we obviously may choose a set chain from a to b out of $\mathfrak{A}$. Thus, let $B \in \mathfrak{A}_2$. Put $S_1 = \bigcup_{X \in \mathfrak{A}_1} X$, $S_2 = \bigcup_{X \in \mathfrak{A}_2} X$. Then the sets S_1 and S_2 are open and $S_1 \cup S_2 = P$. Moreover, $S_1 \cap S_2 = \emptyset$ by (1). Hence, S_1 and S_2 are separated. Evidently $a \in S_1$, $b \in S_2$ so that P is not connected between a and b.

18.5. *Let $Q \subset P$, $a \in P$, $b \in P$.* We say that *the set Q separates the point a from the point b in P* if $a \in P - Q$, $b \in P - Q$ and if the set $P - Q$ is not connected between a and b, hence, if $P - Q = A \cup B$ with separated summands such that $a \in A$, $b \in B$. If $Q = (q)$ is a one-point set separating a from b in P, we also say that *the point q separates the point a from the point b in P.*

18.5.1. *Let a set $Q \subset P$ separate a point a from a point b in the space P. Then there is a set $F \subset Q$ closed in P which also separates a from b in P.*

Proof: We have $P - Q = A \cup B$ with separated summands, $a \in A$, $b \in B$. By 10.2.7 there are open U, V such that $U \cap V = \emptyset$, $U \supset A$, $V \supset B$. Put $F = P - (U \cup V)$. Then F is a closed set, $F \subset Q$ and $P - F = U \cup V$ with separated summands, $a \in U$, $b \in V$.

18.5.2. *Let $Q \subset P$, $a \in Q - B(Q)$, $b \in P - \bar{Q}$. Then the set $B(Q)$ separates the point a from the point b in the space P.*

Proof: We have $B(Q) = \bar{Q} \cap \overline{P - Q}$, hence $P - B(Q) = (P - \bar{Q}) \cup (P - \overline{P - Q})$ with separated summands. As $b \in P - \bar{Q}$ while neither $a \in B(Q)$ nor $a \in P - \bar{Q}$, we have $a \in P - (\overline{P - Q})$.

18.5.3. *Let $Q \subset P$ be an open set and let $a \in Q$, $b \in P - \bar{Q}$. Then the set $B(Q)$ separates the point a from the point b in P.*

This is a particular case of theorem 18.5.2, since $Q = Q - B(Q)$ (see 10.3.2).

We say that *a set $Q \subset P$ separates the space P*, if P is connected and $P - Q$ is not connected. If a one-point set (q) separates P, we also say that *the point q separates the space P.*

We say that a set $Q \subset P$ is an *irreducible cut* of the space P between points a, b, if [1] Q separates a from b in P, [2] if $M \subset Q$ separates the point a from the point b in P, then $M = Q$.

18.5.4. *If $Q \subset P$ is an irreducible cut of P between points a, b then Q is a closed set.*

This follows from 18.5.1.

Exercises

18.1. Connected spaces are either one-point or dense-in-itself.

18.2. Let $P = A \cup B$ with connected summands. P is connected if and only if A, B are not separated.

18.3. Let $\mathfrak{S} \neq \emptyset$ be a system of connected parts of a space P. Let there be no separated sets $A \in \mathfrak{S}$, $B \in \mathfrak{S}$. Then the set $\bigcup_{S \in \mathfrak{S}} S$ is connected.

18.4. A space P is connected if and only if for every $X \subset P$

$$\emptyset \neq X \neq P \quad \text{implies} \quad B(X) \neq \emptyset.$$

18.5. Let P be a connected space. Let $Q \subset P$. Let $P - Q = A \cup B$ with separated non-void summands. Let for $X \subset Q \neq X$ the set $P - X$ be connected. Then the sets $A \cup Q$, $B \cup Q$ are connected.

18.6. In theorem 18.1.12 we may replace the word "closed" by the word "open".

18.7. If $P \times Q$ is a connected space, then both spaces P, Q are connected.

18.8. Let P, Q be infinite connected spaces. Let $a \in P \times Q$. Then the set $P \times Q - (a)$ is connected.

18.9. Let a set M be dense in a space P. Let M have a finite number of components. Then P has at most as many components as M.

18.10. The components of a space $P \times Q$ are identical with the sets $M \times N$ where M, N varies over the components of P, Q respectively.

18.11.* We may write $P = \bigcup_{i=1}^{n} A_i$ with separated non-void summands if and only if P has at least n components.

18.12. Let P be a connected space. Let a set $Q \subset P$ have a finite number of components. Let $P - Q = A \cup B$ with separated A, B. Then $Q \cup A$ has at most as many components as Q.

18.13. Let $A \subset P$, $B \subset P$ be closed sets. Let $A \cup B$ be connected; let $A \cap B$ have a finite number of components. Then A has at most as many components as $A \cap B$.
18.14. In exercise 18.13 we may replace the word "closed" by the word "open".
18.15. Let a, b, c be three distinct points of a connected space P. Let the point a separate the point b from the point c. Then the point b does not separate the point a from the point c.
18.16. Let a, b, c be three distinct points of a connected space P. Let no $x \in P$ separate a either from b or from c. Then no $x \in P — (a)$ separates the point b from the point c.
18.17. Let a, b be two distinct points of a connected space P. Let M be the set of all $x \in P$ separating the point a from the point b. Then we may define an ordering of M as follows: If $x \in M$, $y \in M$ then "x precedes y" means that the point x separates the point a from the point y.
18.18. The ordering defined in exercise 18.17 turns to its inverse, if we interchange the points a, b.
18.19. Let P be a connected separable space. Let $M \subset P$ be an uncountable set; let every $x \in M$ separate the space P. Then there exist two points a, b and an uncountable $N \subset M$ such that every $x \in N$ separates the point a from the point b.

In exercises 18.20—18.23, the proposition "P is $\sigma(a, b)$" means that a, b are two distinct points of a connected space P and that no connected closed $M \neq P$ contains both the points a, b.

18.20. Let P be $\sigma(a, b)$. Let a set $M \subset P — [(a) \cup (b)]$ separate P. Then the points a, b belong to distinct components of $P — M$.
18.21. Let P be $\sigma(a, b)$. Let a set $M \neq P$ be closed and connected. Then $P — M$ has at most two components.
18.22. If in ex. 18.21 $a \in M$, then $P — M$ is connected.
18.23. Let $a \neq b \neq c \neq a$. Let P be simultaneously $\sigma(a, b)$, $\sigma(a, c)$, $\sigma(b, c)$. Let $P = A \cup B$ with closed connected summands. Then either $A = P$ or $B = P$.

§ 19. Connectedness of compact spaces

19.1. A *continuum* is a connected compact space containing more than one point. The notion of continuum is a topological notion (see 9.3). Some authors use the term continuum also for one-point sets.

19.1.1. *Let Q be a component of a compact space P. Then Q is either a one-point set or a continuum.*

Proof: Q is connected. By 18.2.2 and 17.2.2 Q is compact.

Let $\varepsilon > 0$. Let $a \in P$, $b \in P$. An *ε-chain from the point a to the point b in the space P* is a finite point sequence $\{a_i\}_{i=1}^{m}$ such that [1] $a_1 = a$, [2] $a_m = b$, [3] $\varrho(a_i, a_{i+1}) < \varepsilon$ for $1 \leqq i \leqq m - 1$.

19.1.2. *Let a metric space P be connected between $a \in P$ and $b \in P$. Let $\varepsilon > 0$. Then there is an ε-chain from a to b in P.*

Proof: Let $\mathfrak{A}$ be the system of all sets $\Omega(x, \frac{1}{2}\varepsilon)$ where x varies over P. The sets $\Omega(x, \frac{1}{2}\varepsilon)$ are (see 8.6) open and $\bigcup_{x \in P} \Omega(x, \frac{1}{2}\varepsilon) = P$, so that, by 18.4.2, there may be chosen a set chain $\{A_i\}_{i=1}^{m}$ from a to b out of $\mathfrak{A}$. Put $A_i = \Omega(x_i, \frac{1}{2}\varepsilon)$ $(1 \leqq i \leqq m)$.

Let $x_0 = a$, $x_{m+1} = b$. Then $\{x_i\}_{i=0}^{m+1}$ is an ε-chain from a to b. We have, first, $x_0 = a \in A_1 = \Omega(x_1, \frac{1}{2}\varepsilon)$, hence $\varrho(x_0, x_1) < \frac{1}{2}\varepsilon < \varepsilon$. Secondy $x_{m+1} = b \in A = \Omega(x_m, \frac{1}{2}\varepsilon)$, hence $\varrho(x_m, x_{m+1}) < \frac{1}{2}\varepsilon < \varepsilon$. Finally, let $1 \leqq i \leqq m - 1$; then there is a point $z \in A_i \cap A_{i+1} = \Omega(x_i, \frac{1}{2}\varepsilon) \cap \Omega(x_{i+1}, \frac{1}{2}\varepsilon)$, hence $\varrho(x_i, z) < \frac{1}{2}\varepsilon$, $\varrho(z, x_{i+1}) < \frac{1}{2}\varepsilon$ and hence $\varrho(x_i, x_{i+1}) < \varepsilon$.

19.1.3. *Let P be a compact space. Let $a \in P$, $b \in P$. For every $\varepsilon > 0$, let there be an ε-chain from a to b in P. Then P is connected between a and b.*

Proof: Let, on the contrary, P not be connected between a and b. Then $P = A \cup B$ with separated A, B such that $a \in A$, $b \in B$. The sets A and B are closed in P, hence (see 17.2.2), they are compact. As $A \cap B = \emptyset$, we have, by 17.3.4, $\varrho(A, B) > 0$. Let $0 < \varepsilon < \varrho(A, B)$. Let $\{a_i\}_{i=1}^{m}$ be an ε-chain from a to b. We have $a_1 = a \in A$. Since $\varrho(a_i, a_{i+1}) < \varepsilon < \varrho(A, B)$ $(1 \leqq i \leqq m - 1)$ we can prove by induction that $a_i \in A$ $(1 \leqq i \leqq m)$. Thus, $b = a_m \in A$, which is a contradiction.

19.1.4. *Let Q be a quasicomponent of a compact space P. Let U be a neighborhood of the set Q. Then $P = A \cup B$ with separated summands such that $Q \subset A \subset U$.*

Proof: The case $U = P$ is trivial ($A = P$, $B = \emptyset$). Hence, let $P - U \neq \emptyset$. Choose an $a \in Q$. If $x \in P - U$, then (see 18.3.5), P is not connected between a and x so that there exist separated sets $H(x)$ and $K(x)$ such that $a \in H(x)$, $x \in K(x)$, $P = H(x) \cup K(x)$. If $y \in Q$, then P is connected between a and y, so that $y \in H(x)$. Thus, $Q \subset H(x)$. The sets $H(x)$ and $K(x)$ are open in P so that the sets $K(x) - U$, $H(x) - U$ are open in $P - U$. Since $P - U$ is compact (see 17.2.2) and $\bigcup_{x \in P - U} (K(x) - U) = P - U$, we may, by 17.5.4, find a finite number of points $x_i \in P - U$ $(1 \leqq i \leqq m)$ such that $\bigcup_{i=1}^{m} (K(x_i) - U) = P - U$, i.e. $\bigcup_{i=1}^{m} K(x_i) \supset P - U$. Since $H(x_i) = P - K(x_i)$, we have $\bigcap_{i=1}^{m} H(x_i) = P - \bigcup_{i=1}^{m} K(x_i)$, hence $\bigcap_{i=1}^{m} H(x_i) \subset U$. Put $A = \bigcap_{i=1}^{m} H(x_i)$, $B = \bigcup_{i=1}^{m} K(x_i)$, so that $Q \subset A \subset U$, $A \cup B = P$. Since $H(x_i)$ and $K(x_i)$ are separated, A and $K(x_i)$ are also separated, by 10.2.4, so that A and B are separated by 10.2.5.

19.1.5. *In compact spaces the quasicomponents are identical with the components.*

Proof: By 18.2.1, 18.3.4 and 18.3.10 it suffices to show that every quasicomponent Q of a compact space P is connected. Let, on the contrary, Q not be connected. As $Q \neq \emptyset$, we have $Q = A \cup B$ with non-void separated A, B. The sets A, B are closed by 8.7.4 and 18.3.6; hence, they are compact by 17.2.2. Moreover, $A \cap B = \emptyset$, so that $\varrho(A, B) > 0$ by 17.3.4. Let $0 < 2\varepsilon < \varrho(A, B)$, so that $\Omega(A, \varepsilon) \cap \Omega(B, \varepsilon) = \emptyset$. The set $\Omega(A, \varepsilon) \cup \Omega(B, \varepsilon)$ is a neighborhood of the set $Q = A \cup B$. Hence, by 19.1.4, $P = H \cup K$ with separated summands such that $Q \subset H \subset \Omega(A, \varepsilon) \cup \Omega(B, \varepsilon)$;

hence, $H = H_1 \cup H_2$, where $H_1 = H \cap \Omega(A, \varepsilon)$, $H_2 = H \cap \Omega(B, \varepsilon)$. Evidently $A \subset H_1$, $B = H_2$. The sets H_1 and K are separated by 10.2.4; the sets H_1 and H_2 are separated by 10.2.7; hence, H_1 and $H_2 \cup K$ are separated by 10.2.5. Moreover, $P = H_1 \cup (H_2 \cup K)$, $A \subset H_1$, $B \subset H_2 \cup K$. Thus, P is not connected between a and b whenever we choose $a \in A$, $b \in B$. This is a contradiction (see 18.3.5), as $a \in Q$, $b \in Q$.

19.1.6. *Let $\{A_n\}_1^\infty$ be a sequence of continua. Let $A_n \supset A_{n+1}$ for $n = 1, 2, 3, \ldots$ Then $\bigcap_{n=1}^{\infty} A_n$ is either a one-point set or a continuum.*

Proof: We may assume that $A_1 = P$, so that P is compact. Put $C = \bigcap_{n=1}^{\infty} A_n$. By 17.5.1, $C \neq \emptyset$. Moreover, the set C is closed and hence compact (see 17.2.2). It remains to prove that C is connected. Let, on the contrary, $C = C_1 \cup C_2$ with non-void separated C_1, C_2. By 10.2.7 there exist open sets U_1, U_2 such that $U_1 \cap U_2 = \emptyset$, $U_1 \supset C_1$, $U_2 \supset C_2$. If $A_n \subset U_1 \cup U_2$, then $A_n = (A_n \cap U_1) \cup \cup (A_n \cap U_2)$ with separated summands and $A_n \cap U_1 \supset C_1 \neq \emptyset$, $A_n \cap U_2 \supset C_2 \neq \emptyset$ which is impossible, since A_n is connected. Hence, $A_n - (U_1 \cup U_2) \neq \emptyset$ for every n. Since $A_n - (U_1 \cup U_2) \supset A_{n+1} - (U_1 \cup U_2)$, we have by 17.5.1

$$\emptyset \neq \bigcap_{n=1}^{\infty} [A_n - (U_1 \cup U_2)] = C - (U_1 \cup U_2),$$

which is a contradiction.

19.1.7. *Let P be a compact space. Let $\{A_n\}_1^\infty$ be a sequence of connected sets such that $\underline{\mathrm{Lim}}\, A_n \neq \emptyset$. Then $\overline{\mathrm{Lim}}\, A_n$ is either a one-point set or a continuum.*

Proof: Let $a \in \underline{\mathrm{Lim}}\, A_n$. Then [see 8.8.(1)] $a \in \overline{\mathrm{Lim}}\, A_n$, so that $\overline{\mathrm{Lim}}\, A_n \neq \emptyset$. We see easily (ex. 8.18), that the set $\overline{\mathrm{Lim}}\, A_n$ is closed. Hence, $\overline{\mathrm{Lim}}\, A_n$ is compact by 17.2.2. It remains to show that $\overline{\mathrm{Lim}}\, A_n$ is connected. Let us assume the contrary. Then there are separated sets H, K such that $\overline{\mathrm{Lim}}\, A_n = H \cup K$, $a \in H$, $K \neq \emptyset$. By 10.2.7 there exist open U, V with $U \cap V = \emptyset$, $U \supset H$, $V \supset K$. As $a \in \underline{\mathrm{Lim}}\, A_n$, there is a sequence $\{a_n\}$ such that $a_n \to a$, $a_n \in A_n$ for every n. Choose a $b \in K \subset \subset \overline{\mathrm{Lim}}\, A_n$. There exist indices $i_1 < i_2 < i_3 < \ldots$ and a sequence $\{b_{i_n}\}_{n=1}^{\infty}$ such that $b_{i_n} \to b$, $b_{i_n} \in A_{i_n}$ for every n. Since U is a neighborhood of the point $a = = \lim a_n$ and V is a neighborhood of the point $b = \lim b_{i_n}$ there is an index p such that $n > p$ implies $a_{i_n} \in U$, $b_{i_n} \in V$, which implies $A_{i_n} \cap U \neq \emptyset \neq A_{i_n} \cap V$. Since the sets $A_{i_n} \cap U$, $A_{i_n} \cap V$ are separated and since A_{i_n} is connected, there exists a $c_{i_n} \in A_{i_n} \cup (U \cup V)$ for $n > p$. As P is compact, there is a subsequence $\{j_n\}$ of $\{i_n\}$ such that $\lim c_{j_n} = c$ exists. As $c_{j_n} \in P - (U \cup V)$ and as $P - (U \cup V)$ is closed, we have $c \in P - (U \cup V)$. This is a contradiction, as $c \in \overline{\mathrm{Lim}}\, A_n = H \cup K \subset \subset U \cup V$.

19.1.8. *Let f be a continuous mapping of a compact space P onto a metric space Q. Let $f_{-1}(y) \subset P$ be connected for every $y \in Q$. If $S \subset Q$ is connected, then $f_{-1}(S) \subset P$ is also connected.*

Proof: Let $f_{-1}(S)$ not be connected. Obviously $f_{-1}(S) \neq \emptyset$, so that there exist non-void separated $A \subset P$, $B \subset P$ with $f_{-1}(S) = A \cup B$. Evidently $S = f(A) \cup \cup f(B)$, $f(A) \neq \emptyset \neq f(B)$. Since S is connected, $f(A)$ and $f(B)$ are not separated, so that, by 10.2.3, we have either $f(A) \cap \overline{f(B)} \neq \emptyset$ or $\overline{f(A)} \cap f(B) \neq \emptyset$. Let, e.g., $f(A) \cap \overline{f(B)} \neq \emptyset$. $\overline{B}$ is compact by 17.2.2. Hence, $f(\overline{B})$ is compact by 17.4.2. Thus, $f(\overline{B})$ is closed in Q by 15.2.1 and 17.2.1. Hence, $\overline{f(B)} \subset f(\overline{B})$ by 8.4, so that $f(A) \cap f(\overline{B}) \neq \emptyset$. Hence there are points $x_1 \in A$, $x_2 \in \overline{B}$, $y \in f(A) \subset S$ with $f(x_1) = = f(x_2) = y$. We have $(x_1) \cup (x_2) \subset f_{-1}(y) \subset f_{-1}(S) = A \cup B$. As $A \cap \overline{B} = \emptyset$ by 10.2.3, x_2 does not belong to A; thus, $x_2 \in B$. Thus, $f_{-1}(y) = (f_{-1}(y) \cap A) \cup \cup (f_{-1}(y) \cap B)$ with non-void separated (see 10.2.4) summands, i.e. $f_{-1}(y)$ is not connected, which is a contradiction.

19.2. 19.2.1. *Let $a \in \mathbf{E}_1$, $b \in \mathbf{E}_1$, $a < b$. Then $J = \underset{x}{\mathrm{E}}[a \leqq x \leqq b]$ is a continuum.*

Proof: The set J is compact (see 17.2.3) and it contains more than one point. It remains to prove that it is connected. This follows easily from 18.3.1 and 19.1.3.

We call an *interval* every part of $\mathbf{E}_1$ containing more than one point and such that it contains every z with $x < z < y$ whenever it contains x and y.

19.2.2. *Let $M \subset \mathbf{E}_1$. M is connected if and only if it is either a one-point set or an interval.*

Proof: I. Every one-point set is connected by 18.1.1. The connectedness of intervals may be easily proved by 18.1.5 and 19.2.1.

II. Let M be neither a one-point set nor an interval. If $M \neq \emptyset$, there are numbers x, y, z with $x \in M$, $y \in M$, $z \in \mathbf{E}_1 - M$. Evidently $M = (M \cap \underset{t}{\mathrm{E}}[t < z]) \cup \cup (M \cap \underset{t}{\mathrm{E}}[t > z])$ with non-void separated summands so that M is not connected.

19.2.3. *Let P be a connected space. Let f be a finite continuous function on P. Then either f is a constant function or $f(P)$ is an interval.*

This follows easily from 18.1.10 and 19.2.2.

19.2.4. *The euclidean space $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ is connected.*

Proof: $\mathbf{E}_1$ is connected by 19.2.2. Since $\mathbf{E}_{m+1} = \mathbf{E}_m \times \mathbf{E}_1$, the connectedness of every $\mathbf{E}_m$ follows by induction from 18.1.13.

19.2.5. *The spherical space* (see Chapter III, 17.10) $\mathbf{S}_m$ ($m = 1, 2, 3, \ldots$) *is connected.*

Proof: Choose an $a \in \mathbf{S}_m$. Evidently $\overline{\mathbf{S}_m - (a)} = \mathbf{S}_m$. $\mathbf{S}_m - (a)$ is connected by 17.10.4 and 19.2.4. Hence, $\mathbf{S}_m$ is connected by 18.1.6.

19.3. 19.3.1. *Let P be a continuum. Let a set $F \subset P$ be closed and let $\emptyset \neq F \neq P$. Let K be a component of F. Then $K \cap B(F) \neq \emptyset$.*

Proof: Let, on the contrary, $K \subset G$, where $G = F - B(F) = F - \overline{P - F}$. Then G is a neighborhood of K in F. As K is (see 19.1.5) a quasicomponent of a compact (see 17.2.2) space F, by 19.1.4 we obtain that $F = A \cup B$ with separated summands such that $K \subset A \subset G$. We have $A \cap B = \emptyset$ and the sets A, B are closed in P by 8.7.4. As $F = A \cup B$ we have

$$P = A \cup (B \cup \overline{P - F}) \tag{1}$$

with closed summands. We have $A \cap B = \emptyset$ and also $A \cap \overline{P - F} = \emptyset$, as $A \subset G = F - \overline{P - F}$. Thus, the summands in (1) are separated; as P is a continuum, one of them is void. We have $\emptyset \neq K \subset A$. Hence, $B \cup \overline{P - F} = \emptyset$, hence $P - F = \emptyset$ i.e. $F = P$, which is a contradiction.

19.3.2. *Let P be a continuum. Let $G \subset P$ be an open set such that $\emptyset \neq G \neq P$. Let K be a component of the set G. Then $K \cap B(G) \neq \emptyset$.*

Proof: Suppose that $\overline{K} \cap B(G) = \overline{K} \cap (\overline{G} - G) = \emptyset$. We have $K \subset G$, hence $\overline{K} \subset \overline{G}$. As $\overline{K} \cap (\overline{G} - G) = \emptyset$, we have $\overline{K} \subset G$. Hence, $\overline{K} \cap (P - G) = \emptyset$ so that, by 17.3.4 (see also 17.2.2) we have $\varepsilon = \varrho(\overline{K}, P - G) > 0$. Let $F = \mathop{\mathrm{E}}_x[\varrho(x, P - G) \geqq \frac{1}{2}\varepsilon]$. Then F is a closed set. We have $F \subset G$, hence $F \neq P$ and $\overline{K} \subset F$, hence $F \neq \emptyset$. The set $\overline{K}$ is connected (see 18.1.6), hence (see 18.2.5), $\overline{K} \subset L$ where L is a component of the set F. Since $F \subset G$, we have, by 18.2.5, $L \subset M$, where M is a component of G. Thus, $\emptyset \neq K \subset \overline{K} \subset L \subset M$, where K and M are components of the set G, so that (see 18.2.1) $K = M$. Hence, $\overline{K} = L$ is a component of the set F. Hence, by 19.3.1, $\overline{K} \cap B(F) \neq \emptyset$. Evidently $B(F) \subset \mathop{\mathrm{E}}_x[\varrho(x, P - G) = \frac{1}{2}\varepsilon]$. Hence, for every $a \in \overline{K} \cap B(F)$, $\varrho(a, P - G) = \frac{1}{2}\varepsilon < \varepsilon = \varrho(\overline{K}, P - G)$. This is a contradiction.

19.3.3. *Let P be a continuum. Let $a \in P$. Let $\varepsilon > 0$. Then there is a continuum $K \subset P$ such that $a \in K \subset \Omega(a, \varepsilon)$.*

Proof: Let $F = \overline{\Omega}(a, \frac{1}{2}\varepsilon)$. We have $a \in F \subset \Omega(a, \varepsilon)$, hence $F \neq \emptyset$ and F is a closed set. If $F = P$, we may choose $K = F$. Thus, let $F \neq P$ and let K be the component of F containing the point a. We have $a \in K \subset \Omega(a, \varepsilon)$ and the set K is connected. Moreover, K is closed by 8.7.4 and 18.2.2, hence (see 17.2.2), K is compact. Thus, K is a continuum, if $K \neq (a)$. By 19.3.1 $\emptyset \neq K \cap B(F) \subset K - (a)$, so that $K \neq (a)$.

19.4. We say that P is an *irreducible continuum between points a and b*, if: [1] P is a continuum, [2] $a \in P$, $b \in P$, [3] if $K \subset P$ is a continuum and if $a \in K$, $b \in K$, then $K = P$. By 19.3.3 necessarily $a \neq b$.

19.4.1. *Let P be a continuum. Let $a \in P$, $b \in P$, $a \neq b$. Then P contains at least one irreducible continuum between the points a and b.*

Proof: Let $\mathfrak{A}$ be the system of all continua $A \subset P$ such that $a \in A$, $b \in A$. We have $P \in \mathfrak{A}$ and hence $\mathfrak{A} \neq \emptyset$. If $A_n \in \mathfrak{A}$, $A_n \supset A_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$ by 19.1.6. Hence, by 16.4 (see also 17.2.6) there is at least one minimal set K in $\mathfrak{A}$. K is obviously an irreducible continuum between the points a and b.

19.4.2. *Let P be an irreducible continuum between $a \in P$ and $b \in P$. Then the sets $P - [(a) \cup (b)]$, $P - (a)$, $P - (b)$ are connected.*

Proof: I. Put $Q = P - [(a) \cup (b)]$. Then $\bar{Q} = P$, i.e. both a and b belong to $\bar{Q}$. Indeed, $Q \neq \emptyset$ by 18.1.9, and $P = \bar{Q} \cup (P - \bar{Q})$ with closed (see 8.3.4), hence separated, summands, so that $P - \bar{Q} = \emptyset$.

II. By I and 18.1.7 it suffices to show that the set Q is connected.

Let us assume that Q is not connected. As $Q \neq \emptyset$, $Q = A \cup B$ with non-void separated summands. By I, $P = \bar{A} \cup \bar{B}$. By 10.2.3 $A \cap \bar{B} = \emptyset = \bar{A} \cap B$ and hence $\bar{A} \neq P \neq \bar{B}$. As P is connected and $P = \bar{A} \cup \bar{B}$, $\bar{A} \neq \emptyset \neq \bar{B}$, we have $\bar{A} \cup \bar{B} \neq \emptyset$. We have $\bar{A} \cap \bar{B} \cap Q = (A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$. Thus $\emptyset \neq \bar{A} \cap \bar{B} \subset (a) \cup (b)$, so that we may assume that $a \in \bar{A} \cap \bar{B}$. Moreover, $b \in P = \bar{A} \cup \bar{B}$, so that we may assume $b \in \bar{A}$. First, let $\bar{A}$ be connected between the points a and b. Then a and b belong to the same quasicomponent K of the space $\bar{A}$ (see 18.3.5). $\bar{A}$ is compact (see 17.2.2), so that (see 19.1.5) K is a component of $\bar{A}$. As $a \in K$, $b \in K$, $a \neq b$, K is a continuum (see 19.1.1). P is an irreducible continuum between the points a, b, so that $K = P$. This is a contradiction, since $K \subset \bar{A} \neq P$.

It remains to investigate the case that $\bar{A}$ is not connected between a and b. Then $\bar{A} = C_1 \cup D_1$ with separated summands such that $a \in C_1$, $b \in D_1$. If b does not belong to $\bar{B}$, we have $P = \bar{A} \cup \bar{B} = D_1 \cup (C_1 \cup \bar{B})$ with separated non-void D_1, $C_1 \cup \bar{B}$ which is a contradiction. Thus, both the points a, b belong to $\bar{B}$. If $\bar{B}$ is connected between a and b, we obtain a similar contradiction as we did above. If $\bar{B}$ is not connected between a and b, then $\bar{B} = C_2 \cup D_2$ with disjoint summands such that $a \in C_2$, $b \in D_2$. Then $P = \bar{A} \cup \bar{B} = (C_1 \cup C_2) \cup (D_1 \cup D_2)$ with separated disjoint $C_1 \cup C_2$, $D_1 \cup D_2$, which is a contradiction.

19.5. *A semicontinuum* is a non-void metric space P such that for every $a \in P$, $b \in P$, $a \neq b$, there is a continuum $K \subset P$ such that $a \in K$, $b \in K$. The notion of semicontinuum is a topological notion (see 9.3).

19.5.1 *Every one-point space is a semicontinuum.* This is evident.

19.5.2. *Every continuum is a semicontinuum.* This is also evident.

19.5.3. *Every semicontinuum is connected.* This follows from 18.1.3.

19.5.4. *A compact semicontinuum is either a one-point space or a continuum.* This follows from 19.5.3.

Let P be a metric space. A set $S \subset P$ is said to be a *constituant* of the space P if it is a maximal semicontinuum in P, i.e. if: [1] S is a semicontinuum, [2] $A \subset P$, A semicontinuum, $A \supset S$ imply $A = S$. The notion of constituant is a topological notion (see 9.3).

Obviously, 0 has no constituants.

19.5.5. *Every point $a \in P$ belongs to exactly one constituant of P.*

Proof: Denote by $\mathfrak{S}$ the system containing the one-point set (a) and all the continua $K \subset P$ such that $a \in K$. If $K_1 \in \mathfrak{S}$, $K_2 \in \mathfrak{S}$, then $a \in K_1 \cup K_2$ and the set $K_1 \cup K_2$ is connected by 18.1.4. Moreover, it is easy to prove (see ex. 17.4) that $K_1 \cup K_2$ is compact. Thus, $K_1 \in \mathfrak{S}$, $K_2 \in \mathfrak{S}$ imply $K_1 \cup K_2 \in \mathfrak{S}$. Denote by S the union of all the sets of $\mathfrak{S}$. Then $a \in S \subset P$. If $x \in S$, $y \in S$ there are $K_1 \in \mathfrak{S}$, $K_2 \in \mathfrak{S}$ with $x \in K_1$, $y \in K_2$. We have $(x) \cup (y) \subset K_1 \cup K_2 \in \mathfrak{S}$. Hence, S is a semicontinuum. Let $T \subset P$ also be a semicontinuum and let $S \subset T$ and hence $a \in T$. If $x \in T$, $x \neq a$, there is a continuum $K \subset T$ with $(a) \cup (x) \subset T$. We have $K \in \mathfrak{S}$, hence, $K \subset S$, $x \in S$. Thus, $T \subset S$, so that $T = S$. Hence, S is a constituant. Let S^* be another constituant with $a \in S^*$. If $x \in S^*$, then either $x = a$ or there is a continuum $K \subset S^*$ with $(a) \cup (x) \subset K$, so that $x \in S$. Hence, $S^* \subset S$ and similarly $S \subset S^*$. Thus, $S^* = S$.

The following theorem is evident.

19.5.6. *A space P is a semicontinuum if and only if it has exactly one constituant.*

18.2.5 and 19.5.3 yield

19.5.7. *Every constituant of P is contained in a component of P.*

The following theorem is evident.

19.5.8. *A component K of P is a constituant of P if and only if K is a semicontinuum.*

19.5.9. *In compact spaces, the constituants are identical with the components.*

Proof: Let K be a component of a compact space P. K is connected. By 18.2.2, K is closed in P, so that K is compact by 17.2.2. Hence, K is a semicontinuum by 19.5.1 and 19.5.2. Thus, K is a constituant of P by 19.5.8.

Let P be a metric space. Let $Q \subset P$. We say that *Q cuts P between points a and b*, if $a \in P - Q$, $b \in P - Q$, $a \neq b$ and if, for every continuum $K \subset P$ such that $a \in K$, $b \in K$ we have $K \cap Q \neq \emptyset$. The following theorem is evident.

19.5.10. *Let $Q \subset P$, $a \in P$, $b \in P$. Q cuts P between the points a, b if and only if the points a, b belong to distinct constituants of $P - Q$.*

19.5.11. *Let $Q \subset P$ separate a point a from a point b in P. Then Q cuts P between the points a, b.*

Proof: We have $P - Q = A \cup B$ with separated summands, $a \in A$, $b \in B$. Let K be the component of $P - Q$ containing the point a. By 18.1.2, $K \subset A$. Let H be the constituant of P containing the point a. By 19.5.7, $H \subset K$. Hence, $H \subset A$, so that $b \in (P - Q) - H$. Thus, Q cuts P between a, b by 19.5.10.

19.5.12. *Let a compact set $Q \subset P$ cut P between points a, b. Then there is a compact set $M \subset Q$ such that* [1] *M cuts P between a, b,* [2] *if $H \subset M$ is compact and if H cuts P between a, b then $H = M$.*

Proof: Let us denote by $\mathfrak{A}$ the system of all compact $A \subset Q$ cutting P between a, b. We have $Q \in \mathfrak{A}$ and hence $\mathfrak{A} \neq \emptyset$. Let $A_n \in \mathfrak{A}$, $A_n \supset A_{n+1}$ $(n = 1, 2, 3, \ldots)$; put $A = \bigcap_{n=1}^{\infty} A_n$. The sets A_n are closed in Q by 17.2.2, so that A is also closed in Q and consequently A is compact by 17.2.2. Let us assume that A does not cut P between a, b. Since $Q \supset A$ cuts P between a, b, we have $a \in P - A$, $b \in P - A$. Hence, there is a continuum $K \subset P$ such that $a \in K$, $b \in K$, $K \cap A = \emptyset$. As $A_n \in \mathfrak{A}$, we have $K \cap A_n \neq \emptyset$ for every n. The sets $K \cap A_n$ are closed in K and $K \cap A_n \supset$ $\supset K \cap A_{n+1}$. Hence, by 17.5.1, $\bigcap_{n=1}^{\infty} K \cap A_n = K \cap A \neq \emptyset$, which is a contradiction. Hence the required set M exists by 16.4 (see also 17.2.6).

Exercises

First, we describe twelve examples of metric spaces $P_1, P_2, \ldots, P_{12}$; all of them are subspaces of $\mathbf{E}_2$. We shall use the following abbreviation in order to simplify the description. If $a = (a_1, a_2) \in \mathbf{E}_2$, $b = (b_1, b_2) \in \mathbf{E}_2$, then $\mathbf{S}(a, b)$ denotes the set of all $x = (x_1, x_2) \in \mathbf{E}_2$ such that

$$\begin{aligned} x_1 &= a_1(1 - t) + b_1 t \\ x_2 &= a_2(1 - t) + b_2 t. \end{aligned} \qquad (0 \leqq t \leqq 1)$$

($\mathbf{S}$ is, of course, the initial of the word "segment".)

Put $a_0 = (0, 0)$, $b_0 = (0, 1)$ and, for $n = 1, 2, 3, \ldots$ put $a_n = (n^{-1}, 0)$, $b_n = (n^{-1}, 1)$. Put

$$P_1 = \bigcup_{n=0}^{\infty} [\mathbf{S}(a_n, b_n) \cup \mathbf{S}(a_0, a_1) \cup \mathbf{S}(b_0, b_1)].$$

Preserve the meaning of the symbols a_n ($n = 0, 1, 2, \ldots$) and b_0. Put

$$P_2 = \bigcup_{n=0}^{\infty} S(a_n, b_0).$$

Denote by D the Cantor discontinuum (see 17.8). Put $b_0 = (0, 1)$. Put

$$P_3 = \bigcup_{\xi} S(\xi, b), \qquad \xi = (x, 0), \qquad x \in D.$$

Put $a_1 = (0, 0)$, $a_2 = (1, 0)$. For $n = 1, 2, 3, \ldots, 1 \leqq i \leqq 2^n - 1$ put $b_{ni} = (i \,.\, 2^{-n}, 0)$, $c_{ni} = (i \,.\, 2^{-n}, 2^{-n})$. Put

$$P_4 = S(a_1, a_2) \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n - 1} S(b_{ni}, c_{ni}).$$

Put $a_1 = (0, 0)$, $a_2 = (0, 1)$ and, for $n = 1, 2, 3, \ldots$ put $b_n = (n^{-1}, 0)$, $c_n = (n^{-1}, 1)$. Put

$$P_5 = S(a_1, a_2) \cup \bigcup_{n=1}^{\infty} [S(b_n, c_n) \cup S(c_n, b_{n+1})].$$

Put $a_1 = (0, 0)$, $a_2 = (1, 0)$ and, for $n = 1, 2, 3, \ldots, 1 \leqq i \leqq 2^{n-1}$ put $b_{ni} = [(2i - 1) \,.\, 2^{-n}, 2^{-n})]$. Put

$$P_6 = S(a_1, a_2) \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} [S(b_{ni}, b_{n+1,2i-1}) \cup S(b_{ni}, b_{n+1,2i})].$$

Put $a = (0, 0)$; for $n = 1, 2, 3, \ldots$ put $b_n = (n^{-1}, 0)$; for $n = 1, 2, 3, \ldots, i = 1, 2, 3, \ldots$ put $c_{ni} = [(n + 1)^{-1}, n^{-1} \,.\, i^{-1}]$. Put

$$P_7 = S(a, b_1) \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} S(b_n, c_{ni}).$$

Denote by A_1 the set of all (x, y) with $x = -1$, $y \geqq 0$; denote by A_2 the set of all (x, y) with $x = 1, y \geqq 0$. For $n = 1, 2, 3, \ldots$ put

$$a_{4n-3} = (1 - 2^{-2n+1}, 0), \qquad a_{2n} = (0, n), \qquad a_{4n-1} = (-1 + 2^{-2n}, 0).$$

Put

$$P_8 = A_1 \cup A_2 \cup \bigcup_{n=1}^{\infty} S(a_n, a_{n+1}).$$

For $n = 1, 2, 3, \ldots$ denote by A_n the set of all (x, y) with $x = n^{-1}$, $y \geqq 0$; B_n is the set of all (x, y) such that $n^2(x^2 + y^2) = 1$ and either $x \leqq 0$ or $y \leqq 0$. Put

$$P_9 = \bigcup_{n=1}^{\infty} (A_n \cup B_n).$$

For $n = 0, 1, 2, \ldots$ denote by A_n the set of all (x, y) with $x = n$, y arbitrary. Denote by B_n the set of all (x, y) with $x^2 + y^2 + 2nx - \frac{1}{2}y + n^2 = 0$. Denote by C_n the set of all (x, y) with $x^2 + y^2 + 2nx + \frac{1}{2}y + n^2 = 0$. Moreover, denote by A_0^* the set of all (x, y) with $x = 0$, $y \geqq 0$ and by D the set of all (x, y) with $y = 0$ and x arbitrary. Put

$$P_{10} = \bigcup_{n=0}^{\infty} A_n \cup \bigcup_{n=1}^{\infty} (B_n \cup C_n) \cup D,$$

$$P_{11} = A_0^* \cup C_0 \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup C_n) \cup D.$$

Let D be the Cantor discontinuum. For $a \in \mathbf{E}_1$, $r > 0$ denote by $K_1(a, r)$ the set of all (x, y) with $(x - a)^2 + y^2 = r^2$, $y \geqq 0$; by $K_2(a, r)$ the set of all (x, y) with $(x - a)^2 + y^2 = r^2, y \leqq 0$. Denote by $\mathfrak{H}$ the system of all $K_1(a, r)$ with $a = \frac{1}{2}$, $a + r \in D$. For $n = 1, 2, 3, \ldots$ denote by $\mathfrak{H}_n$ the system of all $K_2(a, r)$ with $a = \frac{5}{2} \,.\, 3^{-n}$, $r \leqq \frac{1}{2} \,.\, 3^{-n}$, $a + r \in D$. Put $\mathfrak{K} = \mathfrak{H} \cup \bigcup_{n=1}^{\infty} \mathfrak{H}_n$. Put

$$P_{12} = \bigcup_{X \in \mathfrak{K}} X = (\bigcup_{X \in \mathfrak{H}} X) \cup (\bigcup_{n=1}^{\infty} \bigcup_{X \in \mathfrak{H}_n} X).$$

19.1. $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_{12}$ are continua.

19.2. P_8, P_9, P_{10}, P_{11} are connected spaces.

19.3. Put $a = (0, 0)$, $b = (0, 1)$. Let M be the set of all (x, y) with $x = 0$, $0 < y < 1$; let N be the set of all (x, y) such that either $0 < x \leqq 1$, $y = 0$, or $0 < x \leqq 1$, $y = 1$. Then $(a) \cup (b)$ is a quasicomponent of $P_1 - (M \cup N)$.

19.4. The point $(1, 0)$ separates P_9.

19.5. Let a, b_n ($n = 1, 2, 3, \ldots$) have the same meaning as in the description of P_7. The set $Q = [P_7 - \mathbf{S}(a, b_1)] \cup (a) \cup \bigcup_{n=1}^{\infty} (b_n)$ is connected. Every point b_n ($n = 1, 2, 3, \ldots$) separates Q.

The symbol σ in exercises 19.6—19.8 is used in the same sense as it was in exercises 18.20—18.23.

19.6. Let a, b_1, Q have the same meaning as in ex. 19.5. Let $M \subset Q$. Then M is not $\sigma(a, b_1)$.

19.7. Put $a_1 = (0, 0)$, $a_2 = (0, 1)$, $b_1 = (1, 0)$. Let $\emptyset \neq M \subset \mathbf{S}(a_1, a_2)$. Let $Q = [P_5 - \mathbf{S}(a_1, a_2)] \cup \cup M$. Then Q is $\sigma(b_1, x)$ for every $x \in M$.

19.8. Let A_1, A_2 have the same meaning as in the description of the space P_8. Let $a_1 = (\frac{1}{2}, 0)$, $z_1 \in A_1$, $z_2 \in A_2$. Then P_8 is $\sigma(a_1, z_1)$ and P_8 is $\sigma(a_1, z_2)$. If $M \subset P_8$, then M is not $\sigma(z_1, z_2)$.

19.9. There exists a one-to-one continuous mapping of P_{10} onto P_{11}. There exists a one-to-one continuous mapping of P_{11} onto P_{10}.

19.10. The point $a = (0, 0)$ of P_{11} has the following property. If U is a neighborhood of a and if $d(U) < \frac{1}{2}$, then the set $B(U)$ contains at least five points; for every $\varepsilon > 0$ there exists a neighborhood U of a with $d(U) < \varepsilon$ such that $B(U)$ contains exactly five points.

19.11. Considering ex. 19.10 we may prove that P_{10} and P_{11} are not homeomorphic.

19.12. Let D, $\mathfrak{K}$ have the same meaning as in the description of P_{12}. For every $X \in \mathfrak{K}$ there are exactly two points of the form $(x, 0)$ in X; in both cases $x \in D$. There is exactly one set $X \in \mathfrak{K}$ with $(0, 0) \in X$. If $x \in D$, $x \neq 0$, there are exactly two sets $X \in \mathfrak{K}$ with $(x, 0) \in X$.

There may be determined in exactly one way*) numbers $a_n \in D$ ($n = 0, 1, 2, \ldots$) and sets $X_n \in \mathfrak{K}$ ($n = 0, 1, 2, \ldots$) such that $a_0 = 0$, $a_n \neq a_{n+1}$, $(a_n, 0) \in X_n$, $(a_{n+1}, 0) \in X_n$ ($n = 0, 1, 2, \ldots$) and that $X_m \neq X_n$ for $m, n = 0, 1, 2, \ldots, m \neq n$. We have $X_n \cap X_{n+1} = [(a_{n+1}, 0)]$ for $n = 0, 1, 2, \ldots$, while $X_m \cap X_n = \emptyset$ for $m, n = 0, 1, 2, \ldots$, $|m - n| \geqq 2$.

Put $S(0) = \bigcup_{n=0}^{\infty} X_n$.

If $x \in D$ and if $(x, 0) \in S(0)$, put $S(x) = S(0)$.

Let $x \in D$ and $(x, 0)$ let not belong to $S(0)$. There may be determined in exactly one way numbers $a_n \in D$ ($n = 0, \pm 1, \pm 2, \ldots$) and sets $X_n \in \mathfrak{K}$ ($n = 0, \pm 1, \pm 2, \ldots$) such that $a_0 = x$, $a_n \neq a_{n+1}$, $(a_n, 0) \in X_n$, $(a_{n+1}, 0) \in X_n$ ($n = 0, \pm 1, \pm 2, \ldots$) and that $X_m \neq X_n$ for $m, n = 0, \pm 1, \pm 2, \ldots$, $m \neq n$. We have $X_n \cap X_{n+1} = [(a_{n+1}, 0)]$ for $n = 0, \pm 1, \pm 2, \ldots$, while $X_m \cap X_n = \emptyset$ for $m, n = 0, \pm 1, \pm 2, \ldots$, $|m - n| \geqq 2$. Put

$$S(x) = \bigcup_{n=0}^{\infty} X_n \cup \bigcup_{n=1}^{\infty} X_{-n}.$$

Thus, the set $S(x)$ is defined for every $x \in D$. If $x \in D$, $y \in D$, we have either $S(x) = S(y)$, or $S(x) \cap S(y) = \emptyset$. The system of all the sets $S(x)$ is uncountable and the space P_{12} is the union of all the $S(x)$ ($x \in D$).

Every $S(x)$ ($x \in D$) is connected and dense in P_{12}.

19.13. Let f be a continuous function on a connected domain P. Let $a \in P$ not separate P. Let there exist points $b_1 \in P$, $b_2 \in P$ with $f(b_1) < f(a) < f(b_2)$. Then there exists a point $c \in P$ such that $c \neq a$ and $f(c) = f(a)$.

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*) With the trivial exception of the interchange of a_n, a_{-n} ($n = 1, 2, 3, \ldots$).

19.14. Let P, Q be infinite connected spaces. Let f be a continuous function on $P \times Q$. Then there are at most two points $a \in P \times Q$ such that

$$x \in P \times Q, \quad x \neq a \quad \Rightarrow \quad f(x) \neq f(a).$$

19.15. Let $P \neq \emptyset$ not be connected. Then there exists a finite continuous function f on P such that f is not a constant and $f(P)$ is not an interval.

19.16. A space P has a finite number of components if and only if there is no finite continuous function f on P such that $f(P)$ is an infinite countable set.

19.17. Let P be a compact space. Let $\emptyset \neq M \subset P$. An ε-chain in M between any $a \in M$ and any $b \in M$ with any $\varepsilon > 0$ exists if and only if $\overline{M}$ is connected. Thus, M itself need not be connected.

19.18. Which properties must have spaces P, Q that the space $P \times Q$ may be [1] a continuum, [2] a semicontinuum?

19.19. A continuous image of a continuum is a continuum or a one-point set. A continuous image of a semicontinuum is a semicontinuum.

19.20. Let f be a finite function on $\mathbf{E}_1$. The set $\underset{(x,y)}{\mathrm{E}} [y = f(x)] \subset \mathbf{E}_2$ contains a continuum if and only if there is an interval J such that the partial function f_J is continuous.

19.21. Theorem 19.1.4 remains valid, if the assumption that P is compact is replaced by the assumption that $P \subset \mathbf{E}_1$.

19.22. One cannot omit in theorem 19.1.4 the assumption that P is compact. This may be shown using as an example $P = P_1 - (M \cup N)$ where M, N have the same meaning as in ex. 19.3.

19.23. One cannot replace in theorem 19.1.4 the assumption of compact P by the assumption that P is locally compact and $P \subset \mathbf{E}_2$.

19.24. One cannot replace in theorem 19.3.1 the assumption that P is a continuum by the assumption that P is connected. This may be shown by means of an example with $P \subset P_1$.

19.25. One cannot replace in theorem 19.3.2 the assumption that P is a continuum by the assumption that P is a connected space.

19.26. Let P be a continuum. Let $a \in P$. Let M be the set of all the $x \in P$ such that there is a continuum $K \neq P$ containing both the points a, x. Then the set M *is dense*.

19.27. Let P be a connected space. Let $H \subset P$, $K \subset P$ be continua. For any $x \in H$, $y \in K$, let there be a point z separating x from y. Then there is a point a such that $P - (a) = A \cup B$ with separated summands such that $H \subset A$, $K \subset B$.

19.28. Let P be a continuum. Let $A \subset P$, $B \subset P$ be non-void disjoint closed sets. Then there exists a continuum $K \subset P$ such that: [1] $K \cap A \neq \emptyset \neq K \cup B$, [2] if $H \subset K$ is a continuum and if $H \cap A \neq \emptyset \neq H \cap B$, then $H = K$.

19.29. Let P be a continuum. Let a set $M \subset P$ contain at least two points. Then there exists a continuum $K \subset P$ such that: [1] $M \subset K$, [2] if $H \subset K$ is a continuum and if $M \subset H$ then $H = K$.

19.30. Let P be a continuum. Let $a \in P$, $b \in P$, $a \neq b$. Let a set $M \subset P$ be connected between points a and b. Then $\overline{M}$ contains an irreducible continuum between the points a, b.

19.31. If $a = (1, 0)$, $b = (0, x)$, $0 \leqq x \leqq 1$, then P_5 is an irreducible continuum between the points a, b.

19.32. Let A_n, B_n have the same meaning as in the description of P_9. Then $A_n \cup B_n$ $(n = 1, 2, 3, \ldots)$ are constituants of P_9.

19.33. One cannot replace in theorem 19.5.12 cutting of the space between points a, b by separating the point a from the point b. This may be shown by means of an example with $P = P_2$, choosing $a = (0, 0)$, $b = (0, 1)$, $Q = (c) \cup \bigcup_{n=1}^{\infty} (c_n)$, where $c = (0, \frac{1}{2})$, $c_n = (1/2n, \frac{1}{2})$ $(n = 1, 2, 3, \ldots)$.

19.34. One cannot omit in theorem 19.1.8 the assumption that P is compact. E.g., one cannot put $P = \mathbf{E}_1$, not even under the assumption that f is one-to-one.

§ 20. Simple arc

20.1. Let $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$. A metric space P is said to be a *simple arc*, if it is homeomorphic with J. The notion of a simple arc is a topological notion (see 9.3).

20.1.1. *Any simple arc is a continuum.*

Proof: J is a continuum by 19.2.1. Thus, by 17.4.2 and 18.1.10, P is a continuum.

20.1.2. *A simple arc contains exactly two points which do not separate it.* These two points are called the *end points* of a simple arc. The notion of an end point is a topological notion.

Proof: Let f be a homeomorphic mapping of J onto P. The sets $J - (0)$, $J - (1)$ are connected by 19.2.2. Thus, the sets $P - f(0)$, $P - f(1)$ are connected by 18.1.10. Let $t \in J$ and let $P - f(t)$ be connected. By 18.1.10 the set $J - (t)$ is connected so that either $t = 0$ or $t = 1$ by 19.2.2.

20.1.3. *Let P be a simple arc with end points a, b. Then there is a homeomorphic mapping f of $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto P such that $f(0) = a$, $f(1) = b$.*

Proof: There exists a homeomorphic mapping φ of J onto P. By the previous proof the points $\varphi(0)$, $\varphi(1)$ do not separate P so that either $\varphi(0) = a$, $\varphi(1) = b$, or $\varphi(0) = b$, $\varphi(1) = a$. In the first case put $f = \varphi$. In the second one we may define f by $f(t) = \varphi(1 - t)$.

20.1.4. *A simple arc P is an irreducible continuum between its end points.*

Proof: If $K \subset P$ is a continuum, then $f_{-1}(K) \subset J$ is and interval by 18.1.10 and 19.2.2. If, moreover, $f(0) \in K$, $f(1) \in K$, then $0 \in f_{-1}(K)$, $1 \in f_{-1}(K)$, hence $f_{-1}(K) = J$ and hence $K = P$.

20.1.5. *Let P be a simple arc with end points a, b. Then $P - [(a) \cup (b)]$ is connected.*

Proof: $J - [(0) \cup (1)]$ is connected by 19.2.2; thus, $P - [(a) \cup (b)]$ is connected by 18.1.10.

Another proof follows by 19.4.2 and 20.1.4.

20.1.6. *Let P be a simple arc with end points a, b. Let $c \in P$, $a \neq c \neq b$. Then the set $P - (c)$ has exactly two components, one of them containing a and the other containing b.*

Proof: Let $a = f(0)$, $b = f(1)$, $c = f(\tau)$, hence $0 < \tau < 1$. The sets $A = \underset{t}{\mathrm{E}}[0 \leqq t < \tau]$, $B = \underset{t}{\mathrm{E}}[\tau < t \leqq 1]$ are connected by 19.2.2 and we have $0 \in A$,

$1 \in B$, $A \cup B = J - (\tau)$. Hence, the sets $f(A)$, $f(B)$ are connected by 18.1.10 and we have $a \in f(A)$, $b \in f(B)$, $f(A) \cup f(B) = P - (c)$. Let K_1 be the component of $P - (c)$ containing a (see 18.2.1); let K_2 be the component of $P - (c)$ containing b. By 18.2.5 $f(A) \subset K_1$, $f(B) \subset K_2$ and hence $K_1 \cup K_2 = P - (c)$ so that P has no components except K_1 and K_2. If there were $K_1 = K_2$, $K_1 = K_2 = P - (c)$ would be connected. This is a contradiction (see 20.1.2).

20.1.7. *Let P be a simple arc with end points a, b. Let $c \in P$, $a \neq c \neq b$. Then c separates a from b in P.*

Proof: Let us preserve the notation of the previous proof. The sets A, B are open in J. Since f_{-1} is continuous, $f(A)$, $f(B)$ are open in P by 9.2. Moreover, $A \cap B = 0$, so that $f(A) \cap f(B) = 0$. Hence, the sets $f(A)$, $f(B)$ are separated. Thus, $P - (c) = f(A) \cup f(B)$ with separated summands and $a \in f(A)$, $b \in f(B)$.

20.1.8. *Let P be a simple arc, $\alpha \in P$, $\beta \in P$, $\alpha \neq \beta$. Then there is exactly one simple arc $Q \subset P$ with end points α, β.* This simple arc Q will be denoted by either $P(\alpha, \beta)$ or $P(\beta, \alpha)$.

Proof: I. Let f be a homeomorphic mapping of J onto P. Let $u \in J$, $v \in J$, $\alpha = f(u)$, $\beta = f(v)$ and let, e.g., $u < v$. We see easily that $f(\underset{t}{\mathrm{E}}[u \leqq t \leqq v]) \subset P$ is a simple arc with end points α, β.

II. On the other hand, let $Q \subset P$ be a simple arc with end points α, β, By 18.1.10, $f_{-1}(Q) \subset J$ is a connected set. As $u \in f_{-1}(Q)$, $v \in f_{-1}(Q)$, we have, by 19.2.2, $\underset{t}{\mathrm{E}}[u \leqq t \leqq v] \subset f_{-1}(Q)$, hence, $f(\underset{t}{\mathrm{E}}[u \leqq t \leqq v]) \subset Q$. The set $f(\underset{t}{\mathrm{E}}[u \leqq t \leqq v])$ is a simple arc with end points α, β, hence, it is a continuum containing α and β so that $f(\underset{t}{\mathrm{E}}[u \leqq t \leqq v]) = Q$ by 20.1.4.

20.1.9. *Let P be a simple arc with end points a, b and let $c \in P$, $a \neq c \neq b$. Then*

$$P = P(a, c) \cup P(c, b), \qquad P(a, c) \cap P(c, b) = (c).$$

Proof: Let $a = f(0)$, $b = f(1)$, $c = f(\tau)$, hence $0 < \tau < 1$. Then $P(a, c) = f(\underset{t}{\mathrm{E}}[0 \leqq t \leqq \tau])$, $P(c, b) = f(\underset{t}{\mathrm{E}}[\tau \leqq t \leqq 1])$ and the statement is obvious.

20.1.10. *Let P be a metric space. Let $A \subset P$ be a simple arc with end points a, b. Let $B \subset P$ be a simple arc with end points b, c. Let $A \cap B = (b)$. Then $A \cup B$ is a simple arc with end points a, c.*

Proof: Let f_1 be a homeomorphic mapping of J onto A. Let $a = f_1(0)$, $b = f_1(1)$. Let f_2 be a homeomorphic mapping of J onto B. Let $b = f_2(0)$, $c = f_2(1)$. Define a mapping f of J into P as follows: First, $f(\frac{1}{2}) = b$, secondly, $f(t) = f_1(2t)$ for $0 \leqq t < \frac{1}{2}$, thirdly, $f(t) = f_2(2t - 1))$ for $\frac{1}{2} < t \leqq 1$. We verify easily that f

is a one-to-one continuous (and hence, by 17.4.6, homeomorphic) mapping of J onto $A \cup B$ and that $f(0) = a$, $f(1) = c$.

20.1.11. *Let P be a metric space. For $n = 1, 2, 3, \ldots$ let $C_n \subset P$ be a simple arc with end points a_n, a_{n+1}. Let [1] $C_n \cap C_{n+1} = (a_{n+1})$ $(n = 1, 2, 3, \ldots)$, [2] $d(C_n) \to 0$, [3] $a_n \to b \in P - \bigcup_{n=1}^{\infty} C_n$, [4] $C_n \cap C_m = \emptyset$ for $|n - m| \geqq 2$. Denote by Q the set $(b) \cup \bigcup_{n=1}^{\infty} C_n \subset P$. Then Q is a simple arc with end points a_1, b.*

Proof: For $n = 1, 2, 3, \ldots$ let f_n be a homeomorphic mapping of J onto C_n such that $f_n(0) = a_{n+1}$, $f_n(1) = a_n$. Define a mapping φ of J onto P as follows: First, put $\varphi(0) = b$. Further, put $\varphi(2^{-(n-1)}) = a_n$ $(n = 1, 2, 3, \ldots)$. If t is another number in J, there is exactly one n $(=1, 2, 3, \ldots)$ with $2^{-n} < t < 2^{-(n-1)}$; then put $\varphi(t) = f_n(2^n t - 1)$. It is easy to prove (in exercise 20.12) that φ is a homeomorphic mapping of J onto Q and that $\varphi(0) = b$, $\varphi(1) = a_1$.

20.1.12. *Let P be a simple arc with end points a, b. Let $\varepsilon > 0$. Then there is a one-to-one finite point sequence $\{c_i\}_{i=0}^m$ and a finite sequence of point sets $\{C_i\}_{i=1}^m$ such that [1] $c_0 = a$, $c_m = b$, [2] C_i is a simple arc with end points c_{i-1}, c_i $(1 \leqq i \leqq m)$, [3] $\bigcup_{n=1}^{m} C_i = P$, [4] $C_i \cap C_{i+1} = (c_i)$ $(1 \leqq i \leqq m - 1)$, [5] $C_i \cap C_j = \emptyset$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq m,\ |i - j| \geqq 2)$, [6] $d(C_i) < \varepsilon$ $(1 \leqq i \leqq m)$.*

Proof: Let f be a homeomorphic mapping of the interval J onto P such that $f(0) = a$, $f(1) = b$. By 9.6.1 and 17.4.4 there is a $\delta > 0$ such that

$$t_1 \in J, \quad t_2 \in J, \quad |t_1 - t_2| < \delta \ \Rightarrow\ \varrho[f(t_1), f(t_2)] < \varepsilon.$$

Evidently, it suffices to choose an m such that $m^{-1} < \delta$ and to put $c_i = f(i \cdot m^{-1})$ $(0 \leqq i \leqq m)$, $C_i = f(\mathrm{E}_t[(i - 1)\, m^{-1} \leqq t \leqq im^{-1}])$.

20.1.13. *Let P be a simple arc. Let $Q \subset P$ be a continuum. Then Q is a simple arc.*

This may be easily proved by theorem 19.2.2.

20.2. If P is an ordered set (see § 4) and if $\alpha \in P$, $\beta \in P$, $\alpha \neq \beta$, denote by $M(\alpha, \beta)$ the set of all $x \in P$ which are between α and β (see 4.1). Thus, $M(\beta, \alpha) = M(\alpha, \beta)$.

An *orientation of a simple arc P* is an ordering of the set P such that $(\alpha) \cup (\beta) \cup M(\alpha, \beta)$ is a simple arc whenever $\alpha \in P$, $\beta \in P$, $\alpha \neq \beta$.

The following theorem is evident.

20.2.1. *Let P be an oriented (i.e. endowed by an orientation) simple arc. Let $Q \subset P$ be a simple arc. The given orientation of P determines an ordering of Q (see 4.1). This ordering of the simple arc Q is an orientation.*

If simple arcs P and $Q \subset P$ are oriented by 20.2.1, we say that they are *coherently* oriented.

20.2.2. *Let P be an oriented simple arc. Then the oriented set P has both first and last elements. These two points are the end points of the simple arc P.*

The end point which is the first element in the given orientation is called the *initial point*, the other end point is called the *terminal point* of the oriented simple arc P.

Proof: Let a, b be the end points of the simple arc P, and let, e.g. a precede b. By the definition of orientation, the set $Q = (a) \cup (b) \cup M(a, b)$ is a simple arc, hence (see 20.1.1), $Q \subset P$ is a continuum containing both a and b, so that (see 20.1.4) $Q = P$. Thus, every $x \in P - [(a) \cup (b)]$ is between a and b, hence, it follows a and precedes b, i.e. a is the first element and b is the last one in P.

20.2.3. *Let P be an oriented simple arc and let $\alpha \in P$, $\beta \in P$, $\alpha \neq \beta$. Then $(\alpha) \cup (\beta) \cup$ $\cup\, M(\alpha, \beta) = P(\alpha, \beta)$ (see 20.1.8).*

Proof: The set $Q = (\alpha) \cup (\beta) \cup M(\alpha, \beta) \subset P$ is a simple arc. In the given ordering α is the first and β the last element in Q so that, by 20. 2.1 and 20.2.2, α and β are the end points of the simple arc Q; thus, $Q = P(\alpha, \beta)$.

20.2.4. *Let a be an end point of a simple arc P. Then there is exactly one orientation of P such that a is the initial point.*

Proof: I. Let (see 20.1.3) f be a homeomorphic mapping of the interval $J =$ $= \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto P such that $f(0) = a$. Define an ordering of P as follows: "x precedes y" if and only if $f_{-1}(x) < f_{-1}(y)$. Thus, we obtain an orientation of P with a as the initial point.

II. On the other hand, let P be oriented in such a way that $a = f(0)$ is the initial point and let $x \in P$, $y \in P$, x precede y. Hence, $x = f(u)$, $y = f(v)$, $0 \leqq u \leqq 1$, $0 < v \leqq 1$. We have to prove that $u < v$. Obviously $f(\underset{t}{\mathrm{E}}[0 \leqq t \leqq v]) = P(a, y)$ (see 20.18) so that, by 20.2.3, $f(\underset{t}{\mathrm{E}}[0 \leqq t < v]) = (a) \cup M(a, y)$. On the other hand, $x = f(u) \in (a) \cup M(a, y)$, so that $0 \leqq u < v$.

20.2.5. *Let P be a simple arc with end points a, b. Then P has exactly two orientations. In one of them a is the initial and b the terminal point, in the other one, a is the terminal and b the initial point. These two orientations are mutually inverse.*

Proof: In any orientation either a or b is the initial point by 20.2.2 and for any of these cases there is exactly one orientation by 20.2.4. Moreover, the inverse ordering to an orientation is evidently also an orientation.

20.2.6. *Let P be an oriented simple arc and let $\alpha \in P$ precede $\beta \in P$. Let the simple arc $P(\alpha, \beta) \subset P$* (see 20.1.8) *be oriented in such a way that α is the initial point. Then P and $P(\alpha, \beta)$ are coherently oriented.*

Proof: If $P(\alpha, \beta)$ is coherently oriented with P, the condition that α is the initial point is evidently satisfied. On the other hand, by 20.2.4, this condition determines uniquely an orientation of the simple arc $P(\alpha, \beta)$.

20.2.7. *Let P be a metric space. Let $C \subset P$ be an oriented simple arc. Let $F \subset P$ be a closed set. Let $C \cap F \neq \emptyset$. Then there are both first and last elements in the ordered set $C \cap F \subset C$.*

Proof: Let f be a homeomorphic mapping of $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto C such that $f(0)$ is the initial point. The set $C \cap F$ is closed in C (see 8.7.2) and hence (see 9.2) the set $f_{-1}(C \cap F)$ is closed in J. Of course, it is non-void and bounded, so that (see 17.4.1) there exist numbers $u = \min f_{-1}(C \cap F)$, $v = \max f_{-1}(C \cap F)$. Evidently, $f(u)$ is the first and $f(v)$ the last point of $C \cap F$.

20.3. (Converse of theorem 20.1.7.) *Let P be a continuum. Let $a \in P$, $b \in P$. Let every $x \in P - [(a) \cup (b)]$ separate the point a from the point b in P (so that $a \neq b$). Then P is a simple arc and a, b are its end points.*

Proof: I. Put $Q = P - [(a) \cup (b)]$. By 18.1.9 the set Q is uncountable.

II. For every $x \in Q$ there are sets $A(x)$, $B(x)$ such that [1] $A(x) \cup B(x) = P - (x)$, [2] $A(x)$, $B(x)$ are separated, [3] $a \in A(x)$, $b \in B(x)$. The sets $A(x)$, $B(x)$ are open in $P - (x)$ and hence (see 8 7.7) they are open in P.

III. Let $S \subset P$ be a connected set and let $a \in S$, $b \in S$. Then $S = P$. On the other hand, let $x \in P - S$ and hence $x \in Q$. We have $S \subset A(x) \cup B(x)$, $a \in S \cap A(x) \neq \emptyset$, $b \in S \cap B(x) \neq \emptyset$. This is a contradiction (see 18.1.2).

IV. If $x \in Q$ then $(x) \cup A(x)$, $(x) \cup B(x)$ are connected sets. This follows from 18.1.11.

V. If $x \in Q$, then $A(x)$, $B(x)$ are connected sets. Suppose, on the contrary, e.g. $A(x)$ not to be connected. We have $a \in A(x) \neq \emptyset$, hence $A(x) = A_1 \cup A_2$ with separated summands, $a \in A_1$, $A_2 \neq \emptyset$. Then $P - (x) = A_1 \cup [A_2 \cup B(x)]$ with separated summands. The sets $(x) \cup B(x)$, $(x) \cup A_1$ are connected (see 18.1.11), hence $S = (x) \cup A_1 \cup B(x)$ is connected by 18.1.4. On the other hand, $a \in A_1 \subset S$, $b \in B(x) \subset S$ and hence $S = P$ by III. Thus, $A_2 \subset S$ which is not possible.

VI. For every $x \in Q$, $P - (x)$ has exactly two components, namely $A(x)$ and $B(x)$. (Thus, the sets $A(x)$, $B(x)$ are uniquely determined by the point x.) Actually by 18.1.2, each component of $P - (x)$ is a part of either $A(x)$ or $B(x)$; by V and 18.2.5 each one of $A(x)$, $B(x)$ is a part of one component of $P - (x)$.

VII. Let $x \in Q$, $y \in Q$, $x \neq y$. Then exactly one of the following relations holds

$$A(x) \subset A(y), \qquad A(y) \subset A(x).$$

The sets $(y) \cup A(y)$, $(y) \cup B(y)$ are connected (see IV) and one of them is contained in $P - (x)$; hence (see VI and 18.2.5), one of them is a part of one of the sets $A(x)$, $B(x)$. Since $a \in A(y) - B(x)$, $b \in B(y) - A(x)$, we have either $(y) \cup A(y) \subset A(x)$, and hence $A(y) \subset A(x) \neq A(y)$ [as y does not belong to $A(y)$], or $(y) \cup B(y) \subset B(x)$, and hence $P - B(x) \subset P - [(y) \cup B(y)]$, i.e. $(x) \cup A(x) \subset A(y)$ and hence $A(x) \subset \subset A(y) \neq A(x)$.

VIII. Let us define an ordering **U** of P as follows: If $x \in P$, $y \in P$ then "x precedes y" means that $x \neq y$ and moreover either $x = a$ or $y = b$, or, finally, $x \in Q$, $y \in Q$, $A(x) \subset A(y)$. It is easy to verify that **U** is an ordering, i.e. that the properties [1], [2], [3] pronounced in 4.1 are satisfied. Moreover, a is the first and b the last element. Finally, for $x \in Q$, $A(x)$ is the set of all points preceding x and $B(x)$ is the set of all points following x.

Remark: If we interchange points a and b, we evidently have to replace the ordering **U** by its inverse ordering.

IX. Q is an infinite separable space (see 16.1.2 and 17.2.6), so that (see 16.1.3) there is an infinite countable set $M \subset Q$ dense in Q. Let $x \in P$, $y \in P$ and let x precede y, so that $x \neq b$, $y \neq a$. We shall prove that there is a point $z \in M$ between x and y. We have to distinguish four cases: [1] $x = a$, $y = b$, [2] $x = a$, $y \in Q$, [3] $x \in Q$, $y = b$, [4] $x \in Q$, $y \in Q$. In case [1] we may choose $z \in M$ arbitrarily. Secondly, let $x = a$, $y \in Q$. We have $a \in A(y)$ and $(a) \neq A(y)$, as (see IV) $(y) \cup A(y)$ is connected and hence $A(y) - (a)$ is a non-void open subset of Q, (see II) so that (see 12.1.2) there exists a $z \in M \cap A(y)$. As $z \in A(y)$, z precedes y and hence z is between a and y. Thirdly, let $x \in Q$, $y = b$. Now, $B(x) - (b)$ is a non-void open subset of Q so that there is a point $z \in M \cap B(x)$, which is between x and b. Finally, if we have $x \in Q$, $y \in Q$ and if x precedes y, we obtain $(x) \cup A(x) \subset A(y)$, hence $P - B(x) \subset \subset A(y)$, i.e. $[P - B(x)] \cap [P - A(y)] = \emptyset$. If $B(x) \cap A(y) = \emptyset$ then $P = [P - B(x)] \cup \cup [P - A(y)]$ with non-void separated summands, which is a contradiction. Thus, $B(x) \cap A(y)$ is a non-void open subset of Q, so that (see 12.1.2) there exists a $z \in M \cap \cap B(x) \cap A(y)$. Evidently, z is between x and y.

X. The ordering **U** of P determines an ordering of M. If $x \in M$, then, by IX, there are points $z_1 \in M$, $z_2 \in M$ such that z_1 is between a and x, and z_2 is between x and b. Thus, x is neither first nor last in M. If $x \in M$, $y \in M$ and if x precedes y, then, by IX, there is a $z \in M$ between x and y. Thus, M is densely ordered. Let R be the set of all rational numbers t such that $0 < t < 1$. By 4.7 there is a mapping φ of M onto R such that

$$x \in M, \quad y \in M, \quad x \text{ precedes } y \;\Rightarrow\; \varphi(x) < \varphi(y).$$

XI. Let us define a mapping f of P into the interval $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ as follows: First, $f(a) = 0, f(b) = 1$. Secondly, let $x \in Q$. Denote by $M_1(x)$ the set of all $z \in M$ which precede x and by $M_2(x)$ the set of all $z \in M$ which follow x. By IX, $M_1(x) \neq \emptyset \neq M_2(x)$. If $z_1 \in M_1(x)$, $z_2 \in M_2(x)$, then z_1 precedes z_2, hence, $0 < \varphi(z_1) < \varphi(z_2) < 1$. Hence

$$0 < \sup_{z \in M_1(x)} \varphi(z) \leqq \inf_{z \in M_2(x)} \varphi(z) < 1 \tag{1}$$

We shall prove that

$$\sup_{z \in M_1(x)} \varphi(z) = \inf_{z \in M_2(x)} \varphi(z). \tag{2}$$

If this were not true, there would exist rational numbers t_1, t_2 such that

$$z_1 \in M_1(x), \quad z_2 \in M_2(x) \quad \Rightarrow \quad \varphi(z_1) < t_1 < t_2 < \varphi(z_2). \tag{3}$$

There would exist points $y_1 \in M$, $y_2 \in M$ with $\varphi(y_1) = t_1$, $\varphi(y_2) = t_2$. By (3), y_1 would belong neither to $M_1(x)$ nor to $M_2(x)$ so that we would have $y_1 = x$ and similarly $y_2 = x$. Hence $y_1 = y_2$, $t_1 = t_2$ which would be a contradiction. Thus, (2) holds and we denote the common value of both sides by $f(x)$. By (1), $0 < f(x) < 1$.

XII. If $x \in P$ precedes $y \in P$, then $f(x) < f(y)$. This is obvious whenever at least one of x, y does not belong to Q. If $x \in Q$, $y \in Q$, then, by IX, there is a $z_1 \in M$ between x and y, and a $z_2 \in M$ between z_1 and y. By X, $\varphi(z_1) < \varphi(z_2)$. Moreover, $z_1 \in M_2(x)$, $z_2 \in M_1(y)$, hence $f(x) \leqq \varphi(z_1) < \varphi(z_2) \leqq f(y)$, so that $f(x) < f(y)$.

XIII. The mapping f is continuous. Let $x_n \in P$, $x \in P$, $x_n \to x$. We have to prove that $f(x_n) \to f(x)$. Let us assume the contrary. Then there is an $\varepsilon > 0$ such that $|f(x_n) - f(x)| > \varepsilon$ for infinitely many indices n. As $f(x_n) \in J$ and as J is compact, there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\lim f(y_n) = \tau$ exists, $0 \leqq \tau \leqq 1$, and such that $|f(y_n) - f(x)| > \varepsilon$ for every n, so that $\tau \neq f(x)$. E.g. let $\tau > f(x)$. There exist numbers $t_1 \in R$, $t_2 \in R$ with $f(x) < t_1 < t_2 < \tau$. There exist points $z_1 \in M$, $z_2 \in M$ with $t_1 = \varphi(z_1)$, $t_2 = \varphi(z_2)$. As $t_1 < t_2$, z_1 precedes z_2 by X. Since $f(x) < t_1 = \varphi(z_1)$, z_1 does not precede x, so that x precedes z_2, i.e. $x \in A(z_2)$. Since $A(z_2)$ is an open set, thre is a $\delta > 0$ such that $\Omega(x, \delta) \subset A(z)$. On the other hand, $y_n \to x$ by 7.1.2 so that there exists an index p such that $n > p$ implies $y_n \in \Omega(x, \delta) \subset A(z_2)$. Thus, if $n > p$ z_2 follows y_n, so that $f(y_n) \leqq \varphi(z_2) = t_2$. Hence also $\tau = \lim f(y_n) \leqq t_2 < \tau$, which is a contradiction.

XIV. $f(P)$ is connected by XIII and 18.1.10. It is contained in $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ and contains both $0 = f(a)$ and $1 = f(b)$. Thus, $f(P) = J$ by 19.2.2. Hence, f is a mapping of P *onto* the interval J. The mapping f is one-to-one by XII and continuous by XIII; hence, it is homeomorphic by 17.4.6. Thus, P is a simple arc and $a = f_{-1}(0)$, $b = f_{-1}(1)$ are its end points.

20.4. (Converse of theorem 20.1.2.) *Every continuum P contains at least two points which do not seaparate P. If P contains only two such points, then P is a simple arc.*

Proof: I. Let P be a continuum, let $a \in P$, $b \in P$, $a \neq b$. Let every $x \in P - [(a) \cup \cup (b)]$ separate P. We shall deduce that then P is a simple arc with end points a, b. It is easy to see that this yields a proof of the stated theorem.

II. By 20.3 it suffices to prove that every $x \in P$, $a \neq x \neq b$, separates a from b. Assume the contrary, i.e. the existence of a $c_0 \in P$, $a \neq c_0 \neq b$ such that it does not separate a from b in P. As $a \neq c_0 \neq b$, c_0 separates P, so that $P - (c_0) = A_0 \cup B_0$ with non-void separated summands. As c_0 does not separate a from b we may assume $a \in A_0$, $b \in A_0$.

III. Since P is a continuum, it is, by 18.1.9 and 17.2.6, an infinite separable space. Hence (see 16.1.3) there is an infinite countable set M dense in P. Let $\{x_k\}_1^\infty$ be a sequence containing exactly the points of M.

IV. For some n $(= 0, 1, 2, \ldots)$, let a point $c_n \in P$ and non-void separated sets A_n, B_n with $P - (c_n) = A_n \cup B_n$, $a \in A_n$, $b \in B_n$ be given. B_n is open in $P - (c_n)$ and hence in P, so that, by 12.1.2, there is an index k with $x_k \in B_n$. Let k_{n+1} be the *least* index k with $x_k \in B_n$. Put $c_{n+1} = x_{k_{n+1}}$. Thus, $c_{n+1} \in B_n$. Moreover, $a \neq \neq c_{n+1} \neq b$, so that (see I) c_{n+1} separates P and hence $P - (c_{n+1}) = A_{n+1} \cup B_{n+1}$ with non-void separated summands.

We may assume that $a \in A_{n+1}$. The set $(c_n) \cup A_n$ is connected by 18.1.11. Since $c_{n+1} \in B_n$, we have $(c_n) \cup A_n \subset P - (c_{n+1})$, so that, by 18.1.2, $(c_n) \cup A_n$ is contained in one of A_{n+1}, B_{n+1}. As $a \in A_n - B_{n+1} \neq \emptyset$, we have $(c_n) \cup A_n \subset A_{n+1}$. As $b \in A_n$, we obtain $b \in A_{n+1}$.

V. It follows, by II and IV, that we may construct a point sequence $\{c_n\}_0^\infty$ and set sequences $\{A_n\}_0^\infty$, $\{B_n\}_0^\infty$ such that: [1] A_n, B_n are separated $(n = 0, 1, 2, \ldots)$, [2] $a \in A_n$, $b \in A_n$, $B_n \neq \emptyset$ $(n = 0, 1, 2, \ldots)$, [3] $A_n \cup B_n = P - (c_n)$ $(n = 0, 1, 2, \ldots)$, [4] $c_n = x_{k_n}$ $(n = 1, 2, 3, \ldots)$ where k_n is the least index k with $x_k \in B_{n-1}$, so that $c_n \in B_{n-1}$ $(n = 1, 2, 3, \ldots)$, [5] $(c_n) \cup A_n \subset A_{n+1}$ $(n = 0, 1, 2, \ldots)$.

VI. The sequences $\{c_n\}$ and $\{k_n\}$ are one-to-one. Let $m < n$. By V [5], $c_m \in A_{m+1} \subset \subset A_n$, by V [3], $c_n \in P - A_n$, hence $c_m \neq c_n$, so that $k_m \neq k_n$ by V [4].

VII. We have $\bigcap\limits_{n=1}^{\infty} B_n \neq \emptyset$. By V [1] and 10.2.3, $A_n \cap \overline{B}_n = \emptyset$, and hence, by V [3], $\overline{B}_n \subset (c_n) \cup B_n$. By V [3] and V [5], $B_n = P - [(c_n) \cup A_n] \supset P - A_{n+1} = (c_{n+1}) \cup \cup B_{n+1}$ and hence $B_n \supset \overline{B}_{n+1}$. Thus, by V [2] and 17.5.1, $\bigcap\limits_{n=1}^{\infty} B_n \neq \emptyset$.

VIII. By VII there is a point $z \in \bigcap\limits_{n=1}^{\infty} B_n$. By V [1] and V [2], $a \in A_n \subset P - B_n$, hence $z \neq a$ and similarly $z \neq b$. Thus (see I), z separates P, i.e. $P - (z) = H \cup K$ with separated summands, $a \in H$, $K \neq \emptyset$. The sets $(c_n) \cup A_n$ are connected by V [1], V [3] and 18.1.11, and they contain the point a by V [2], so that $\bigcup_{n=1}^{\infty} [(c_n) \cup A_n]$ is connected by 18.1.5. Since $z \in B_n = P - [(c) \cup A_n]$, we have $\bigcup_{n=1}^{\infty} [(c_n) \cup A_n] \subset$ $\subset P - (z)$ and hence $\bigcup_{n=1}^{\infty} [(c_n) \cup A_n]$ is contained (see 18.1.2) in one of H, K. Consequently $\bigcup_{n=1}^{\infty} [(c_n) \cup A_n] \subset H$, as $a \in H$. K is open in $P - (z)$ and hence in P (see 8.7.7). Since $K \neq \emptyset$, there is, by 12.1.2, an index i with $x_i \in K$. By VI there is an index m with $k_{m+1} > i$. As $(c_m) \cup A_m \subset H$, we have $B_m = P - [(c_m) \cup A_m] \supset P - H \supset K$. Thus, $x_i \in K \subset B_m$, so that, by V [3], $i \leqq k_{m+1}$ which is a contradiction.

Exercises

$P_1, P_2, \ldots, P_{12}$ are the spaces from exercises to § 19.

20.1. If $P \times Q$ is a simple arc, then one of P, Q is a simple arc and the other is a one-point space.

20.2. If $C \subset \mathbf{E}_2$ is a simple arc, then $\mathbf{E}_2 - C$ is dense in $\mathbf{E}_2$.

20.3. Let $M \subset \mathbf{E}_2$ be the set of all (x, y) with $x \geqq 0$, $y \geqq 0$, $x + y \leqq 1$. Then there exists a disjoint system $\mathfrak{S}$ of simple arcs such that $\bigcup_{X \in \mathfrak{S}} X = M$. That system $\mathfrak{S}$ cannot be countable.

20.4. If a simple arc P is a union of a disjoint system $\mathfrak{S}$ of simple arcs, then $\mathfrak{S}$ contains only one element.

20.5. Let P be one of P_2, P_3, P_4, P_7. Let $a \in P$, $b \in P$, $a \neq b$. There is exactly one continuum $K \subset P$ irreducible between the points a, b. K is a simple arc.

20.6. Let $a \in P_1$, $b \in P_1$, $a \neq b$. There exist infinitely many simple arcs $C \subset P_1$ with the end points a, b.

20.7. Let $P = P_4$ or $P = P_6$. Let $\mathfrak{S}$ be a disjoint system of simple arcs in P. Let $\varepsilon > 0$. The system of all $C \in \mathfrak{S}$ with diameter greater than ε is finite.

20.8. Let P be one of P_9, P_{10}, P_{11}. Let $C \subset P$ be a simple arc. The set $P - C$ is not dense in P.

20.9. The following theorem may be deduced from 20.4 (see also 18.1.11). Let P be a continuum. If $S \subset P$ is a connected set, $S \neq P$, then there is a point $a \in P - S$ which does not separate P.

20.10. A space P is the image under a one-to-one continuous mapping of some of the three spaces

$$\mathbf{E}_1, \qquad \underset{t}{\mathrm{E}}[t \geqq 0], \qquad \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1],$$

if and only if $P = \bigcup_{n=1}^{\infty} C_n$ where C_n are simple arcs such that $C_n \subset C_{n+1}$ $(n = 1, 2, 3, \ldots)$.

20.11. Let the symbol $S(x)$ $(x \in D)$ have the same meaning as it had in ex. 19.12. There exists a one-to-one continuous mapping of $\underset{t}{\mathrm{E}}[t \geqq 0]$ onto $S(0)$; the inverse mapping is nowhere continuous.

If $S(x) \neq S(0)$ there exists a one-to-one continuous mapping of $\mathbf{E}_1$ onto $S(x)$; the inverse mapping is nowhere continuous.

20.12.* Complete the proof of theorem 20.1.11!

20.13. In theorem 20.4 we cannot replace the words "two points which do not separate P" by "two points x such that $P - (x)$ is a semicontinuum". In fact, P_5 contains only *one* such point x and it is easy to construct a similar space which contains *no* such point x.

§ 21. Simple loop

21.1. A metric space P is said to be a *simple loop*, if there exists a continuous mapping f of $\mathbf{E}_1$ onto P such that

$$f(u) = f(v) \Leftrightarrow u - v \text{ is an integer.} \tag{*}$$

21.1.1. *Simple loops are continua.*

Proof: Evidently $P = f(\underset{t}{\mathrm{E}}[0 \leqq t \leqq 1])$. The set $\underset{t}{\mathrm{E}}[0 \leqq t \leqq 1])$ is a continuum by 19.2.1 and hence P is a continuum by 17.4.2 and 18.1.10.

21.1.2. *Let P be a simple loop. Let $a \in P$, $b \in P$, $a \neq b$. Then P contains exactly two simple arcs with end points a, b. If A, B are these simple arcs, we have*

$$A \cup B = P, \qquad A \cap B = (a) \cup (b).$$

Proof: I. Let $a = f(u)$, $b = f(v)$. By (*) we may assume that $u < v < u + 1$. Put $J_1 = \underset{t}{\mathrm{E}}[u \leqq t \leqq v]$, $J_2 = \underset{t}{\mathrm{E}}[v \leqq t \leqq u + 1]$, $A = f(J_1)$, $B = f(J_2)$. The partial mappings f_{J_1} and f_{J_2} are continuous; moreover, by (*) they are one-to-one, so that they are, by 17.4.6, homeomorphic. Thus, A and B are simple arcs with end points a, b [since $f(u + 1) = a$ by (*)]. Moreover, we have, by (*),

$$A \cup B = P, \qquad A \cap B = (a) \cup (b).$$

II. Let $C \subset P$ be a simple arc with end points a, b and let $C \neq A$. We have to prove that $C = B$. As A and C are two different simple arcs with end points a, b, we obtain, by 20.1.1 and 20.1.4, $A - C \neq \emptyset$. Choose an $\alpha \in A - C$ and a number z with $\alpha = f(z)$. Obviously $\alpha \in P - B$ and $P = f(\underset{t}{\mathrm{E}}[z \leqq t \leqq z + 1])$. By 17.3.4, $\varrho(\alpha, B) > 0$, $\varrho(\alpha, C) > 0$, so that, by 9.1.1, there is an $\varepsilon > 0$ such that

$$|t - z| < \varepsilon \Rightarrow \varrho[f(t), \alpha] < \min[\varrho(\alpha, B), \varrho(\alpha, C)].^{*)}$$

Therefore $B \cup C \subset f(M)$, where $M = \underset{t}{\mathrm{E}}[z + \varepsilon \leqq t \leqq z + 1 - \varepsilon]$. Evidently $f(M)$ is a simple arc. As B and C are simple arcs with common end points and as $B \cup C \subset f(M)$, we have $B = C$ by 20.1.8.

21.1.3. *Let P be a metric space. Let $A \subset P$ and $B \subset P$ be two simple arcs with common end points a, b. Let*

$$A \cup B = P, \; A \cap B = (a) \cup (b).$$

Then P is a simple loop.

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*) This ε has to be less than $\frac{1}{2}$.

Proof: Let f_1 be a homeomorphic mapping of the interval $J = \underset{t}{\mathrm{E}}[0 \leqq t \leqq 1]$ onto A; let $a = f_1(0)$, $b = f_1(1)$. Let f_2 be a homeomorphic mapping of J onto B; let $b = f_2(0)$, $a = f_2(1)$. Define a mapping φ of J into P as follows: First, $\varphi(1/2) = b$; secondly, $\varphi(t) = f_1(2t)$ for $0 \leqq t < 1/2$, hence $\varphi(0) = a$; thirdly, $\varphi(t) = f_2(2t - 1)$ for $1/2 < t \leqq 1$ and hence $\varphi(1) = a = \varphi(0)$. Evidently, φ is a continuous mapping of J onto $A \cup B$ and, for $0 \leqq t_1 < t_2 \leqq 1$, $\varphi(t_1) = \varphi(t_2)$ only if $t_1 = 0$, $t_2 = 1$. If $t \in \mathbf{E}_1$, then there exists exactly one integer n with $0 \leqq t - n < 1$; put $f(t) = \varphi(t)$. It is easy to verify that f is a continuous mapping of P onto $A \cup B$ such that (*) holds.

21.1.4. *Let P be a simple loop. Then no point $a \in P$ separates P.*

Proof: Let f be a continuous mapping of $\mathbf{E}_1$ onto P such that (*) holds and let $a = f(z)$. Then $P - (a) = f(\underset{t}{\mathrm{E}}[z < t < z + 1])$ so that $P - (a)$ is connected by 18.1.10 and 19.2.2.

21.1.5. *Let P be a simple loop. Let $a \in P$, $b \in P$, $a \neq b$. Then $(a) \cup (b)$ separates P.*

Proof: By 21.1.2 there are simple arcs $A \subset P$, $B \subset P$ such that $A \cup B = P$, $A \cap B = (a) \cup (b)$. The sets A, B are closed by 17.2.2. Thus, $P - [(a) \cup (b)] = (P - A) \cup (P - B)$ with non-void separated (see 10.2.2) summands.

21.1.6. *Let P be a simple loop. Let $G \subset P$ be an open set. Let $P - G$ contain at least two points. Let T be a component of G. Then $\overline{T}$ is a simple arc and its end points form the set $\overline{T} - T = \overline{T} - G$.*

The reader can prove this without difficulties (see ex. 21.6).

21.1.7. *Let P be a simple loop. Let $\mathfrak{C}$ be a system of simple arcs $C \subset P$. If $C_1 \in \mathfrak{C}$, $C_2 \in \mathfrak{C}$, $C_1 \neq C_2$, $x \in C_1 \cap C_2$, let x be an end point of both simple arcs C_1, C_2. Let $\delta > 0$. Then $\mathfrak{C}$ contains only a finite number of simple arcs with diameter greater than δ.*

Proof: Let there be, on the contrary, a one-to-one sequence $\{C_n\}_1^\infty$ with $d(C_n) > \delta$ for every n. Denote by a_n, b_n the end points of the simple arc C_n. As C_n is connected and as $a_n \in C_n$, $b_n \in C_n$, $d(C_n) > \delta$, it is easy to prove that there is a point $c_n \in C_n$ with $\varrho(a_n, c_n) > \frac{1}{4}\delta$, $\varrho(b_n, c_n) > \frac{1}{4}\delta$. As P is compact, there are indices $i_1 < i_2 < i_3 < \ldots$ and a point $c \in P$ such that $c_{i_n} \to c$. It is easy to prove that there exist simple arcs A, B such that $P = A \cup B$, $c \in A - B$, $d(A) < \frac{1}{4}\delta$. Evidently there is an index p such that $n > p \Rightarrow c_{i_n} \in A \Rightarrow A \cap C_{i_n} \neq \emptyset$. Evidently neither a_{i_n} nor b_{i_n} belongs to A. On the other hand, it is easy to prove that $B(C_{i_n}) = (a_{i_n}) \cup (b_{i_n})$. Thus, $A \cap C_{i_n} \neq \emptyset = A \cap B(C_{i_n})$. Since A is connected, we have, by 18.1.8, $A \subset C_{i_n}$ for every $n > p$ and this is evidently impossible.

21.2. *Orientation* of a simple loop P is a cyclical ordering $\mathbf{C}$ (see § 5) of P such that $(a) \cup (b) \cup J(a, b)$ (see 5.3) is a simple arc whenever $a \in P$, $b \in P$, $a \neq b$. A simple loop endowed by an orientation is said to be *oriented.*

21.2.1. *Let P be an oriented simple loop. Let $a \in P$, $b \in P$, $a \neq b$ so that $Q = (a) \cup \cup (b) \cup J(a, b)$ is a simple arc. Then a, b are the end points of the simple arc Q. If Q is oriented* (see 20.2.4) *in such a way that a is the initial point, then the ordering of $J(a, b)$ determined,* (see 4.1,) *by the orientation of Q coincides with the ordering of $J(a, b)$ determined* (see 5.3) *by the given orientation of P. We say then that P and Q are coherently oriented.*

Proof: I. $(a) \cup (b) \cup J(a, b)$ is a simple arc, so that $J(b, a) \neq \emptyset$. Choose a $c \in J(b, a)$.

II. The cyclical ordering $\mathbf{C}$ of P determines by 5.2 an ordering $\mathbf{U}(c)$ of $P - (c)$. If $x \in P - (c)$, $y \in P - (c)$, $z \in P - (c)$, then (see Chapter I, 5.2.1) $(x, y, z) \in \mathbf{C}$ if and only if either

$$x \text{ precedes } y, \quad \text{and} \quad y \text{ precedes } z \quad \text{in } \mathbf{U}(c)$$

or

$$y \text{ precedes } z, \quad \text{and} \quad z \text{ precedes } x \quad \text{in } \mathbf{U}(c)$$

or

$$z \text{ precedes } x, \quad \text{and} \quad x \text{ precedes } y \quad \text{in } \mathbf{U}(c).$$

III. By 5.3, $Q \subset P - (c)$ so that the ordering $\mathbf{U}(c)$ of $P - (c)$ determines an ordering $\mathbf{V}$ of $Q = (a) \cup (b) \cup J(a, b)$. As $c \in J(b, a)$, we have (see 5.3) $(b, c, a) \in \mathbf{C}$, so that, by 5.1 [1], $(c, a, b) \in \mathbf{C}$; further, by 5.2 a precedes b in $\mathbf{U}(c)$ and hence in $\mathbf{V}$. If $x \in J(a, b)$, then $(a, x, b) \in \mathbf{C}$ so that, by II, x is between a and b in $\mathbf{U}(c)$ and hence in $\mathbf{V}$. Thus, a is the initial point and b the terminal one of Q (in the ordering $\mathbf{V}$).

IV. Since $(c, a, b) \in \mathbf{C}$ (see III), we have (see 5.1 [1] and 5.3) $(a, b, c) \in \mathbf{C}$, i.e. $b \in J(a, c)$, so that, by 5.5.1, $J(a, b) \subset J(a, c)$. By 5.3, $\mathbf{U}(c)$ and $\mathbf{U}(a)$ determine identical orderings of $J(a, c)$ and hence also of $J(a, b) \subset J(a, c)$. Consequently, the ordering of $J(a, b)$ determined by the ordering $\mathbf{V}$ of Q coincides with the ordering $\mathbf{U}(a, b)$, i.e. with the ordering of $J(a, b)$ determined (see 5.3) by the cyclical ordering $\mathbf{C}$ of the simple loop P.

V. Now, we are going to prove that $\mathbf{V}$ is an orientation of the simple arc Q. Let $x \in Q$, $y \in Q$, $x \neq y$, hence, e.g., let x precede y in $\mathbf{V}$ [and hence in $\mathbf{U}(c)$]. We have to prove that $(x) \cup (y) \cup M(x, y)$ is a simple arc, if $M(x, y)$ is the set of all the $z \in Q$ which are between x and y in $\mathbf{V}$. Since $\mathbf{C}$ is an orientation of the simple loop P, $(x) \cup (y) \cup J(x, y)$ is a simple arc, so that it suffices to prove that $M(x, y) = J(x, y)$. First, let $z \in M(x, y)$. Then x precedes z and z precedes y in $\mathbf{V}$ and hence in $\mathbf{U}(c)$, so that, by II, $(x, z, y) \in \mathbf{C}$ and hence $z \in J(x, y)$. Secondly, let $z \in J(x, y)$. Then $(x, z, y) \in \mathbf{C}$, so that, by 5.1 [2], (z, x, y) does not belong to $\mathbf{C}$. Since, on the other hand, x precedes y in $\mathbf{U}(c)$, we have $(c, x, y) \in \mathbf{C}$. Thus, $z \neq c$ and, of course, $x \neq$

$\neq c \neq y$. Since $(x, z, y) \in \mathbf{C}$ and since x precedes y in $\mathbf{U}(c)$, by II x precedes z and z precedes y in $\mathbf{U}(c)$. On the other hand, by III, either $x = a$, or a precedes x in $\mathbf{V}$ and hence in $\mathbf{U}(c)$; moreover, either $y = b$ or y precedes b in $\mathbf{V}$ and hence in $\mathbf{U}(c)$. Thus, a precedes z and z precedes b in $\mathbf{U}(c)$ so that, by II, $(a, z, b) \in \mathbf{C}$ and hence $z \in J(a, b) \subset Q$. Moreover, $x \in Q$, $y \in Q$ and x precedes z and z precedes y in $\mathbf{U}(c)$ and hence in $\mathbf{V}$. Thus, $\tau \in M(x, y)$.

VI. Since $\mathbf{V}$ is an orientation of the simple arc Q and since, by III, a is the initial point and b is the terminal point in $\mathbf{V}$, a and b are, by 20.2.2, the end points of the simple arc Q.

21.2.2. *Let P be an oriented simple loop. Let $a \in P$, $b \in P$, $a \neq b$, so that, by* 21.1.2, *P contains exactly two simple arcs with end points a, b. Let the two simple arcs be oriented coherently with the given orientation of the simple loop P* (see 21.2.1). *Then one of these simple arcs has initial point a and the other one has initial point b.*

We shall denote by $P(a, b)$ that one of the two simple arcs which has initial point a, so that the other will be denoted by $P(b, a)$.

Proof: Both simple arcs are evidently $(a) \cup (b) \cup J(a, b)$, $(b) \cup (a) \cup J(b, a)$. The validity of the statement follows by 21.2.1.

21.2.3. *Every simple loop P has exactly two orientations which are mutually inverse.*

Proof: I. For every couple (a, b) of distinct points there are in P, by 21.1.2, exactly two simple arcs A, B with end points a, b. Associate the sets $A^* = A - [(a) \cup (b)]$, $B^* = B - [(a) \cup (b)]$ with the couple (a, b). By 21.1.2, $A^* \cup B^* = P - [(a) \cup \cup (b)]$, $A^* \cap B^* = \emptyset$. An orientation of the simple loop P is a cyclical ordering such that, for every couple (a, b) the sets A^*, B^* associated with this pair are identical with the sets $J(a, b)$, $J(b, a)$.

II. Thus, by 5.5.2, it suffices to prove the following: If A^* and B^* are associated with a couple (a, b) and if $c \in A^*$ (thus, $c \in A$, $a \neq c \neq b$), then one of the two sets associated with (a, c) – denote it by C_1 – and one of the two sets associated with (c, b) – denote it by C_2 – are such that $A^* = (c) \cup C_1 \cup C_2$ with disjoint summands.

III. The statement, which has to be proved, may also be stated as follows: Let $a \in P$, $b \in P$, $a \neq b$. Let $A \subset P$ be a simple arc with end points a, b. Let $c \in A$, $a \neq c \neq b$. Then there exists a simple arc $C' \subset P$ with end points a, c and a simple arc $C'' \subset P$ with end points c, b such that $A - [(a) \cup (b)] = (c) \cup \{C' - [(a) \cup \cup (c)]\} \cup \{C'' - [(c) \cup (b)]\}$ with disjoint summands. This follows by 20.1.9.

Remark: If P is a simple loop and if $a \in P$, $b \in P$, $a \neq b$, then the notation $P(a, b)$, $P(b, a)$ for the pair of simple arcs with end points a, b contained in P is meaningful only if an orientation of P is chosen. Under a change of orientation the simple arcs $P(a, b)$, $P(b, a)$ interchange.

21.3. *Let P be a continuum. Let $C \subset P$ be a simple loop. Let K be the set of all $x \in C$ separating P. Then K is a countable set.*

Proof: I. By 18.1.9 and 17.2.6, P is an infinite separable space, so that, by 16.1.3, there is an infinite countable subset M dense in P. Let $\{z_n\}$ be a one-to-one sequence containing exactly the points of M.

II. Let $x \in K$. Then $x \in C$ and $P - (x) = A(x) \cup B(x)$ with non-void separated summands. The set $C - (x)$ is connected by 21.2.7, so that, by 18.1.2, we may assume $C - (x) \subset A(x)$, hence $C \cap B(x) = \emptyset$. Since $P - (x)$ is open, $B(x)$ is also open by 8.7.7. As $B(x) \neq \emptyset$, by 12.1.2 there is an index $n(x)$ with $z_{n(x)} \in B(x)$.

III. Obviously if suffices to prove that $n(x) \neq n(y)$ for $x \in K$, $y \in K$, $x \neq y$. Let, on the contrary, $n(x) = n(y)$. Then $B(x) \cap B(y) \neq \emptyset$. We have $y \in C - (x) \subset A(x)$ and similarly $x \in A(y)$. The set $(y) \cup B(y)$ is a connected (see 18.1.11) subset of $P - (x) = A(x) \cup B(x)$ and hence (see 18.1.2) it is a subset of one of the two sets $A(x)$, $B(x)$. As $y \in A(x)$, we have $(y) \cup B(y) \subset A(x)$. This is a contradiction, since $B(y) \cap B(x) \neq \emptyset = A(x) \cap B(x)$.

21.4. (Converse of theorem 21.1.5.) *Let P be a continuum. If every two-point set $M \subset P$ separates P, then P is a simple loop.*

Proof: I. Choose $a \in P$, $b \in P$, $a \neq b$. Then $P - [(a) \cup (b)] = A \cup B$ with non-void separated summands. The sets A, B are open (see 8.7.7), and hence $Q_1 = A \cup \cup (a) \cup (b) = P - B$ and $Q_2 = B \cup (a) \cup (b) = P - A$ are closed and hence (see 17.2.2) compact. Moreover, $Q_1 \cup Q_2 = P$, $Q_1 \cap Q_2 = (a) \cup (b)$.

II. Q_1 and Q_2 are continua. Let, on the contrary, e.g. Q_1 not be a continuum. Q_1 contains more than one point and it is compact. Thus, $Q_1 = H \cup K$ with non-void separated summands. We may assume that $a \in H$. If $b \in H$, we have $K \subset A$, so that (see 10.2.4) K and B are separated. Consequently (see 10.2.5), K and $H \cup B$ are separated, so that we have $P = K \cup (H \cup B)$ with non-void separated summands, which is a contradiction. Thus, $b \in K$. If H were not connected, we would have $H = H_1 \cup H_2$ with separated summands, $a \in H_1$, $H_2 \neq \emptyset$. We would have $H_2 \subset A$, so that (see 10.2.4) H_2 and B would be separated. H_2 and K would be also separated (again by 10.2.4), so that (see 10.2.5) H_2 and $H_1 \cup K \cup B$ would be separated. Thus, we would have $P = H_2 \cup (H_1 \cup K \cup B)$ with non-void separated summands, which would be a contradiction. Thus, H is connected. Similarly we can prove that K is connected. As $Q_1 = H \cup K$ with separated summands and as Q_1 is compact (and hence closed), the sets H, K are closed and hence compact. Thus, each of H, K is either a one-point set or a continuum. More precisely, either $H = (a)$ or H is a continuum and similarly either $K = (b)$ or K is a continuum. If $Q_2 = B \cup (a) \cup (b)$

is not connected, we have $Q_2 = H' \cup K'$ with separated summands, $a \in H'$, $K' \neq \emptyset$. H' and K' are closed. The point b cannot belong to H', since this yields $P = K' \cup \cup (H' \cup A)$ with non-void separated summands, so that $b \in K'$. This is also a contradiction, since then $P = (H \cup H') \cup (K \cup K')$ with non-void separated summands. Thus, Q_2 is connected.

We know that either $H = (a)$, or H is a continuum. Analogously either $K = (b)$ or K is a continuum. We cannot have simultaneously $H = (a)$ and $K = (b)$ since this yields $A = Q_1 - [(a) \cup (b)] = [H - (a)] \cup [K - (b)] = \emptyset$ and $A \neq \emptyset$ by I. Hence, we have either

[1] both H and K are continua, or:
[2] $H = (a)$, K is a continuum, or:
[3] H is a continuum, $K = (b)$.

In the first case there is an $x \in H$, $x \neq a$ and a $y \in K$, $y \neq b$ such that $H - (x)$ and $K - (y)$ are connected sets (this follows by 20.4). Then $P - [(x) \cup (y)] = = Q_2 \cup [H - (x)] \cup [K - (y)]$. Q_2 is connected and contains both points a and b. $H - (x)$ is connected and contains point a, $K - (y)$ is connected and contains point b. Thus, by 18.1.4, $P - [(x) \cup (y)]$ is connected, i.e. the two-point set $(x) \cup (y)$ does not separate P, which is a contradiction.

In the second case choose an $x \in Q_2$, $x \neq b$ such that $Q_2 - (x)$ is connected (this is possible by 20.4, as Q_2 is a continuum); then choose a $y \in K$, $y \neq b$ such that $K - (y)$ is connected (this is possible by 20.4). Then $P - [(x) \cup (y)] = = [Q_2 - (x)] \cup [K - (y)]$. We have $b \in Q_2 - (x)$, $b \in K - (y)$, $Q_2 - (x)$ and $K - (y)$ are connected. Thus, by 18.1.4, $P - [(x) \cup (y)]$ is connected, which is a contradiction.

The third case may be obtained from the second one by interchanging simultaneously a with b and H with K; this also yields a contradiction. Thus, Q_1 is connected.

III. Thus, there are continua $Q_1 \subset P$, $Q_2 \subset P$ such that $Q_1 \cup Q_2 = P$, $Q_1 \cap \cap Q_2 = (a) \cup (b)$. By 21.1.3 it remains to prove that Q_1 and Q_2 are simple arcs with end points a, b. Assume that this is not true e.g. for Q_1. Then, by 20.1.2 and 20.4 there is a point $c \in Q_1$, $a \neq c \neq b$, such that $Q_1 - (c)$ is connected. Choose an $x \in Q_2$, $a \neq x \neq b$, hence, $x \neq c$. If $Q_2 - (x)$ is connected, then, by 18.1.4, also $P - [(c) \cup (x)] = [Q_1 - (c)] \cup [Q_2 - (x)]$ is connected, and hence the two-point set $(c) \cup (x)$ does not separate P, which is a contradiction. Thus, $Q_2 - (x)$ is not connected. Hence, each $x \in Q_2 - [(a) \cup (b)]$ separates the continuum Q_2, so that Q_2 is, by 20.4, a simple arc and a, b are, certainly, its end points. Choose a $d \in Q_2$ (hence, $d \neq c$), $a \neq d \neq b$. By 20.1.9 we have $Q_2 = S \cup T$, where S is a simple arc with end points a, d, T is a simple arc with end points d, b and $S \cap T = (d)$. The sets $S - (d)$, $T - (d)$ are connected by 20.1.2. $Q_1 - (c)$ is also connected. Moreover, $P - [(c) \cup (d)] = [Q_1 - (c)] \cup [S - (d)] \cup [T - (d)]$, $a \in [Q_1 - (c)] \cap \cap [S - (d)]$, $b \in [Q_1 - (c)] \cap [T - (d)]$, so that, by 18.1.4, $P - [(c) \cup (d)]$ is connected, i.e. the two-point set $(c) \cup (d)$ does not separate P, which is a contradiction.

Exercises

21.1. A cartesian product $P \times Q$ where both P and Q contain more than one point, cannot be a simple loop.

21.2. Let P have the property that every two of its points belong to some simple arc $C \subset P$. Let P contain no simple loop. If $a \in P$, $b \in P$, $a \neq b$, there exists exactly one simple arc $C \subset P$ with end points a, b. If $C_1 \subset P$, $C_2 \subset P$ are simple arcs, then $C_1 \cap C_2$ is either void or connected.

21.3. Let $P = P_1$ or $P = P_6$ (see exercises to § 19). Let $a \in P$, $b \in P$, $a \neq b$. Then there is a simple loop $C \subset P$ with $a \in C$, $b \in C$.

21.4. The following more general theorem may be proved in a manner similar to 21.3: Let P be a separable connected space. Let $C \subset P$ be a connected set. Let K be the set of all $x \in C$ which separate P and do not separate C. Then K is countable.

21.5. Let P be a separable connected space. Let $C \subset P$ be a connected set. For $n = 1, 2, 3, \ldots$ let $A_n \subset P - C$ be connected sets. Let (see 8.8)

$$\underline{\operatorname{Lim}}\, A_n \supset C.$$

Let K be the set of all $x \in C$ which separate P. Then K is countable.

21.6.* Prove theorem 21.1.6!