Chapter III

SPECIAL METRIC SPACES

§ 15. Complete spaces

15.1. Let P be a metric space. Let $\{x_n\}$ be a sequence of points of P. We say that $\{x_n\}$ is a *Cauchy sequence* if, for every $\varepsilon > 0$, there is an index $p(\varepsilon)$ such that

$$m > p(\varepsilon)\,,\ n > p(\varepsilon) \Rightarrow \varrho(x_m, x_n) < \varepsilon\,.$$

The notion of a Cauchy sequence is metric, not topological.

15.1.1. *Any convergent sequence is a Cauchy sequence.*

Proof: Let $x_n \to x$. Let $\varepsilon > 0$. There is an index $p(\varepsilon)$ such that $n > p(\varepsilon)$ implies $\varrho(x_n, x) < \varepsilon/2$. Then

$$m > p(\varepsilon),\ \ n > p(\varepsilon) \Rightarrow \varrho(x_m, x_n) \leqq \varrho(x_m, x) + \varrho(x, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

P is said to be a *complete space* if P is a metric space such that every Cauchy sequence of points of P is convergent in P. Evidently, every finite metric space (e.g. $\emptyset$) is complete. Completeness is also a metric notion and not a topological one.

15.1.2. *If P and Q are complete spaces, then $P \times Q$ is a complete space.*

Proof: Let $\{(x_n, y_n)\}$ be a Cauchy sequence of points of $P \times Q$. As

$$\varrho(x_m, x_n) \leqq \varrho[(x_m, y_m), (x_n, y_n)]\,,$$

$\{x_n\}$ is evidently also a Cauchy sequence. Since P is complete, there exists $\lim x_n = x \in P$; and analogously $\lim y_n = y \in Q$. Obviously $(x_n, y_n) \to (x, y)$.

15.1.3. *The euclidean space $\mathbf{E}_m$ is complete.*

Proof: I. $\mathbf{E}_1$ is complete by the well-known Bolzano-Cauchy theorem from the calculus.

II. If $\mathbf{E}_m$ is complete, $\mathbf{E}_{m+1} = \mathbf{E}_m \times \mathbf{E}_1$ is complete by 15.1.2.

15.1.4. *The Hilbert space $\mathbf{H}$ is complete.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points $x_n = \{x_{ni}\}_{i=1}^{\infty} \in \mathbf{H}$. As $\varrho(x_{mi}, x_{ni}) \leqq \varrho(x_m, x_n)$, for every i $\{x_{ni}\}_{n=1}^{\infty}$ is a Cauchy sequence. Since the space $\mathbf{E}_1$

is complete, $y_i = \lim_{n\to\infty} x_{ni}$ exists for every i. Put $y = \{y_i\}_{i=1}^{\infty}$. Let us choose an $\varepsilon > 0$. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there is an index $p(\varepsilon)$ such that

$$m > p(\varepsilon), \quad n > p(\varepsilon) \Rightarrow \varrho(x_m, x_n) < \varepsilon .$$

We have, for $k = 1, 2, 3, \ldots,$

$$\sum_{i=1}^{k} (x_{mi} - x_{ni})^2 \leqq \sum_{i=1}^{\infty} (x_{mi} - x_{ni})^2 = [\varrho(x_m, x_n)]^2,$$

and hence

$$m > p(\varepsilon),\ n > p(\varepsilon) \Rightarrow \sum_{i=1}^{\infty} (x_{mi} - x_{ni})^2 < \varepsilon^2 .$$

On the other hand

$$\sum_{i=1}^{k} (y_i - x_{ni})^2 = \lim_{m\to\infty} \sum_{i=1}^{k} (x_{mi} - x_{ni})^2 ,$$

and therefore

$$n > p(\varepsilon) \Rightarrow \sum_{i=1}^{k} (y_i - x_{ni})^2 \leqq \varepsilon^2 .$$

Hence, the series $\sum_{i=1}^{\infty} (y_i - x_{ni})^2$ converges, and, by formula (2) in 6.1, the series $\sum_{i=1}^{\infty} y_i^2$ also converges, i.e. $y \in \mathbf{H}$. Moreover

$$[\varrho(y, x_n)]^2 = \lim_{k\to\infty} \sum_{i=1}^{k} (y_i - x_{ni})^2 ,$$

and hence $n > p(\varepsilon)$ implies $\varrho(y, x_n) \leqq \varepsilon$. Hence $y = \lim x_n$.

15.2. Let Q be a point set embedded into a metric space P; hence Q is also a metric space. If $\{x_n\}$ is a sequence of points of Q, then $\{x_n\}$ is a Cauchy sequence in the space Q if and only if it is a Cauchy sequence in the space P. On the other hand, a sequence may be convergent in the space P without being convergent in the space Q.

15.2.1. *Let $Q \subset P$. Let Q be a complete space. Then Q is closed in P.*

Proof: Let $\{x_n\}$ be a sequence of points of Q. Let there exist $x = \lim x_n \in P$. By 8.3.3, it suffices to show that $x \in Q$. But $\{x_n\}$ is a Cauchy sequence by 15.1.1. Since the space Q is complete, there exists $\lim x_n \in Q$. Hence, $x \in Q$.

15.2.2. *Let P be a complete space, let $Q \subset P$ be closed. Then Q is a complete space.*

Proof: Let $\{x_n\}$ be a Cauchy sequence of points of Q. Since P is a complete space, there exists $x = \lim x_n \in P$. As Q is closed, $x \in Q$ by 8.3.3. Hence, the sequence $\{x_n\}$ is convergent in Q.

15.3. Let P be an *arbitrary* space. Let $\mathbf{C}$ be the set of all *Cauchy sequences* of points of P, which are *not convergent* in P. If $\{x_n\}$ and $\{y_n\}$ are elements of $\mathbf{C}$, we shall call them (in this section only) *equivalent* if $\varrho(x_n, y_n) \to 0$ (ϱ denotes the distance function in P, of course). It is easy to prove that the set $\mathbf{C}$ may be divided into classes such that: [1] every sequence $\{x_n\} \in \mathbf{C}$ belongs to exactly one class, [2] two sequences $\{x_n\} \in \mathbf{C}$ and $\{y_n\} \in \mathbf{C}$ are equivalent if and only if they belong to the same class. Let us choose a subset Q of $\mathbf{C}$ containing exactly one element from any class. (Of course, if the space P is complete, $\mathbf{C} = Q = \emptyset$.)

In what follows, for convenience, lower case Latin letters denote the elements of P, lower case Greek letters denote the elements of Q.

We define a function ϱ_0 in the range $(P \cup Q) \times (P \cup Q)$ as follows:

[1] if $a \in P$, $b \in P$, then $\varrho_0(a, b) = \varrho(a, b)$;

[2] if $\alpha = \{a_n\} \in Q$, $b \in P$, then $\varrho_0(\alpha, b) = \varrho_0(b, \alpha) = \lim \varrho(a_n, b)$. Certainly we must prove that $\{\varrho(a_n, b)\}$ converges in $\mathbf{E}_1$. Since $\mathbf{E}_1$ is complete, it suffices to prove that $\{\varrho(a_n, b)\}$ is a Cauchy sequence. Let $\varepsilon > 0$. As $\{a_n\}$ is a Cauchy sequence, there is an index p such that $m > p$, $n > p$ imply $\varrho(a_m, a_n) < \varepsilon$. Let $m > p$, $n > p$; then $\varrho(a_m, b) \leqq \varrho(a_m, a_n) + \varrho(a_n, b) < \varrho(a_n, b) + \varepsilon$ and similarly $\varrho(a_n, b) < \varrho(a_m, b) + \varepsilon$. Consequently $m > p$, $n > p$ imply $|\varrho(a_m, b) - \varrho(a_n, b)| < \varepsilon$ and hence $\{\varrho(a_n, b)\}$ is indeed a Cauchy sequence.

[3] if $\alpha = \{a_n\} \in Q$, $\beta = \{b_n\} \in Q$, then $\varrho_0(\alpha, \beta) = \lim \varrho(a_n, b_n)$. Again, we have to prove that $\{\varrho(a_n, b_n)\}$ is convergent in $\mathbf{E}_1$ and again it suffices to prove that it is a Cauchy sequence. Let $\varepsilon > 0$. As $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, there is an index p such that $m > p$, $n > p$ imply $\varrho(a_m, a_n) < \varepsilon/2$ and $\varrho(b_m, b_n) < \varepsilon/2$. Let $m > p$, $n > p$; then $\varrho(a_m, b_m) \leqq \varrho(a_m, a_n) + \varrho(a_n, b_n) + \varrho(b_n, b_m) < \varrho(a_n, b_n) + \varepsilon$, and similarly $\varrho(a_n, b_n) < \varrho(a_m, b_m) + \varepsilon$. Consequently $m > p$, $n > p$ implies $|\varrho(a_m, b_m) - \varrho(a_n, b_n)| < \varepsilon$, and hence $\{\varrho(a_n, b_n)\}$ is indeed a Cauchy sequence.

We shall prove that ϱ_0 is a distance function in $P \cup Q$, i.e. that it possesses the properties [1], [2], [3] exhibited in section 6.1.

I. Evidently $\varrho_0(a, a) = 0$, $\varrho_0(\alpha, \alpha) = 0$ and also $\varrho_0(a, b) > 0$ for $a \neq b$. If $\alpha \neq \beta$, then $\varrho_0(\alpha, \beta) = \lim \varrho(a_n, b_n) \neq 0$ (and hence > 0), since, by the definition of the set Q, the sequences $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are not equivalent. Also $\varrho_0(\alpha, b) = \varrho_0(b, \alpha) = \lim \varrho(a_n, b) \neq 0$ (and hence > 0); the equality $\lim a_n = b$ cannot hold, as $\alpha = \{a_n\} \in Q \subset \mathbf{C}$ is not convergent in P.

II. $\varrho_0(a, b) = \varrho_0(b, a)$, $\varrho_0(\alpha, b) = \varrho_0(b, \alpha)$, $\varrho_0(\alpha, \beta) = \varrho_0(\beta, \alpha)$ is evident.

III. Let α, β, γ be three elements of $P \cup Q$. If $\alpha \in Q$, then $\alpha = \{a_n\}$, where $\{a_n\}$ is a sequence of points of P; if $\alpha \in P$, put $a_n = \alpha$ for every n. The sequences $\{b_n\}$, $\{c_n\}$ are defined analogously. Then $\varrho_0(\alpha, \beta) = \lim \varrho(a_n, b_n)$ and similarly for $\varrho_0(\alpha, \gamma)$, $\varrho_0(\beta, \gamma)$. However, $\varrho(a_n, c_n) \leqq \varrho(a_n, b_n) + \varrho(b_n, c_n)$ and hence $\lim \varrho(a_n, c_n) \leqq \lim \varrho(a_n, b_n) + \lim \varrho(b_n, c_n)$, i.e. $\varrho_0(\alpha, \gamma) \leqq \varrho_0(\alpha, \beta) + \varrho_0(\beta, \gamma)$.

Consequently, the set $P \cup Q$ endowed with the distance function ϱ_0 is a metric space. Since the partial distance function $(\varrho_0)_{P \times P}$ coincides with ϱ, the space $P = (P, \varrho)$ is a point set, embedded into $P \cup Q = (P \cup Q, \varrho_0)$.

Let $\alpha = \{a_n\} \in Q$, then $\varrho_0(\alpha, a_n) = \lim_{k \to \infty} \varrho(a_k, a_n)$. Let $\varepsilon > 0$. Then there is an index p such that $k > p$, $n > p$ imply $\varrho(a_k, a_n) < \varepsilon$. Hence $n > p$ implies $\lim_{k \to \infty} \varrho(a_k, a_n) \leqq \varepsilon$, i.e. $n > p$ implies $\varrho_0(\alpha, a_n) \leqq \varepsilon$. Hence $\varrho_0(\alpha, a_n) \to 0$, i.e. $a_n \to \alpha$. Consequently, by exercise 12.2, the set P is dense in the space $P \cup Q$.

The space $P \cup Q$ is complete. Let $\{\alpha_n\}$ be a Cauchy sequence in $P \cup Q$. Since the set P is dense in $P \cup Q$, there are points $a_n \in P$ such that $\varrho_0(\alpha_n, a_n) < 1/n$. For every $\varepsilon > 0$ there is an index $p(\varepsilon)$ such that $m > p(\varepsilon)$, $n > p(\varepsilon)$ imply $\varrho(\alpha_m, \alpha_n) < \varepsilon/3$; obviously, we may assume $p(\varepsilon) > 3/\varepsilon$. For $m > p(\varepsilon)$, $n > p(\varepsilon)$ we have $\varrho(a_m, a_n) = \varrho_0(a_m, a_n) \leqq \varrho_0(a_m, \alpha_m) + \varrho_0(\alpha_m, \alpha_n) + \varrho_0(\alpha_n, a_n) < 1/m + \varepsilon/3 + 1/n < \varepsilon$. Thus, $\{a_n\}$ is a Cauchy sequence of points of P. Now, it suffices to prove that $\{a_n\}$ converges in $P \cup Q$, since if $a_n \to \beta$ then $\varrho_0(\alpha_n, \beta) \leqq \varrho_0(\alpha_n, a_n) + \varrho_0(a_n, \beta) < \varrho_0(a_n, \beta) + 1/n$ and thus also $\alpha_n \to \beta$. In the case that $\{a_n\}$ is convergent in P there remains nothing to prove. In the other case $\{a_n\} \in \mathbf{C}$, and hence there is a $\beta = \{b_n\} \in Q$ equivalent with $\{a_n\}$. We know (see above) that $\varrho_0(b_n, \beta) \to 0$; as $\{a_n\}$ and $\{b_n\}$ are equivalent, we have $\varrho_0(a_n, b_n) = \varrho(a_n, b_n) \to 0$; as $\varrho_0(a_n, \beta) \leqq \varrho_0(a_n, b_n) + \varrho_0(b_n, \beta)$, there is $\varrho_0(a_n, \beta) \to 0$, i.e. $a_n \to \beta$, and hence $\{a_n\}$ is convergent in $P \cup Q$.

15.4. A metric space P_0 is called a *completion* of a metric space P if: [1] P is a point set embedded into P_0, [2] P_0 is complete, [3] P is dense in P_0. If $P = P_0$, then the space P is complete. On the other hand, if P is complete, then $P = \bar{P}$ by 15.2.1 ($\bar{P}$ denotes the closure of the set P in P_0). But $\bar{P} = P_0$ by [3] and hence $P = P_0$.

15.4.1. *Every metric space has a completion. If P_1 and P_2 are two completions of a space P, then there exists an isometric mapping f of P_1 onto P_2 such that $f(x) = x$ for every $x \in P$.*

Proof: In section 15.3 we constructed the metric space $P \cup Q$, which is obviously a completion of the metric space P. Let P_1 and P_2 be two completions of a metric space P, let ϱ, ϱ_1 and ϱ_2 be distance functions in P, P_1 and P_2 respectively; hence, $\varrho = (\varrho_1)_{P \times P}$. If $x \in P_1$ then by exercise 12.2 there is a sequence $\{a_n\}$ such that $a_n \in P$, $\varrho_1(a_n, x) \to 0$. The sequence $\{a_n\}$ is a Cauchy sequence by 15.1.1. Since the space P_2 is complete, there is a point $y \in P_2$ such that $\varrho_2(a_n, y) \to 0$. Preserving the original point $x \in P_1$, let us replace the sequence $\{a_n\}$ by a sequence $\{b_n\}$ having the same properties, i.e. $b_n \in P$, $\varrho_1(b_n, x) \to 0$. Instead of y we obtain some point $z \in P_2$ such that $\varrho_2(b_n, z) \to 0$. Then $\varrho_2(y, z) \leqq \varrho_2(y, a_n) + \varrho_2(a_n, b_n) + \varrho_2(b_n, z)$. But $\varrho_2(a_n, b_n) = \varrho(a_n, b_n) = \varrho_1(a_n, b_n) \leqq \varrho_1(a_n, x) + \varrho_1(b_n, x)$. Since $\varrho_2(y, a_n) \to 0$, $\varrho_2(b_n, z) \to 0$, $\varrho_1(a_n, x) \to 0$, $\varrho_1(b_n, x) \to 0$, we have $\varrho_2(y, z) = 0$ and hence $z = y$; i.e., the point $y \in P_2$ is uniquely determined by the point $x \in P_1$. On putting $y = f(x)$ we obtain a mapping f of the space P_1 into the space P_2. If $x \in P$, we may choose

$a_n = x$ for every n, and hence $f(x) = x$ for $x \in P$. If $x \in P_1$, $x' \in P_1$, we choose sequences $\{a_n\}$, $\{a'_n\}$ such that $a_n \in P$, $a'_n \in P$, $\varrho_1(a_n, x) \to 0$, $\varrho_1(a'_n, x') \to 0$. By definition of f, we have $\varrho_2[a_n, f(x)] \to 0$, $\varrho_2[a'_n, f(x')] \to 0$. Consequently $a_n \to x$, $a'_n \to x'$ in P_1 and $a_n \to f(x)$, $a'_n \to f(x')$ in P_2 and hence, by exercise 9.12, $\varrho_1(a_n, a'_n) \to \to \varrho_1(x, x')$, $\varrho_2(a_n, a'_n) \to \varrho_2[f(x), f(x')]$. Since $\varrho_1(a_n, a'_n) = \varrho(a_n, a'_n) = \varrho_2(a_n, a'_n)$, we have $\varrho_1(x, x') = \varrho_2[f(x), f(x')]$. Hence, the mapping f is isometric. It remains to show that f is a mapping of P_1 onto P_2, i.e. that for every $y \in P_2$ there is an $x \in P_1$ such that $f(x) = y$. Let $y \in P_2$. By exercise 12.2 there is a sequence $\{a_n\}$ such that $a_n \in P$, $\varrho_2(a_n, y) \to 0$. The sequence $\{a_n\}$ is a Cauchy sequence by 15.1.1. Since the space P_1 is complete, there is a point $x \in P_1$ such that $\varrho_1(a_n, x) \to 0$. Evidently $f(x) = y$.

15.5. A metric space P is said to be *absolutely closed*, if it has the following property: If P is embedded into any space P_0, then P is a closed subset of P_0.

15.5.1. *A metric space is absolutely closed if and only if it is complete.*

Proof: I. Let P be absolutely closed. Let P_0 be its completion. (cf. 15.1.1). By 15.2.2 P is complete.

II. Let P be complete. Then it is absolutely closed by 15.2.1.

A metric space P is said to be an *absolute* $\mathbf{G}_\delta$*-space*, if it possesses the following property: if P is embedded into any metric space P_0, then P is always a $\mathbf{G}_\delta$-set in P_0. For reasons which will be evident immediately (cf. 15.6.3), we shall use the term *topologically complete space* instead of absolute $\mathbf{G}_\delta$-space.

15.5.2. *A metric space P is topologically complete if and only if there is a complete space Q such that P is a $\mathbf{G}_\delta$-set in Q.*

Proof: I. Let P be an absolute $\mathbf{G}_\delta$-space. Let P_0 be its completion (cf. 15.4.1). Evidently P is *a* $\mathbf{G}_\delta$-set in P_0 and P_0 is complete.

II. Let there exist a complete space Q such that P is a $\mathbf{G}_\delta$-set in Q. Let R be a metric space into which P is embedded. We have to prove that P is a $\mathbf{G}_\delta$-set in R. Let R_0 be a completion of the space R (cf. 15.4.1). Hence Q and R_0 are complete spaces and P is embedded into both of them. Let $\bar{P}(Q)$ and $\bar{P}(R_0)$ be closures of the set P in Q and R_0 respectively. By 15.2.2 $\bar{P}(Q)$ and $\bar{P}(R_0)$ are complete spaces and P is embedded into both. Obviously, P is dense in both $\bar{P}(Q)$ and $\bar{P}(R_0)$, and hence $\bar{P}(Q)$ and $\bar{P}(R_0)$ are two completions of the space P. Hence, by 15.4.1, there exists an isometric mapping f of the space $\bar{P}(Q)$ onto the space $\bar{P}(R_0)$ such that $f(x) = x$ for $x \in P$. As P is a G_δ-set in Q and $P \subset \bar{P}(Q) \subset Q$, P is a $\mathbf{G}_\delta$-set in $\bar{P}(Q)$ by 13.6.1. Since the concept of a $\mathbf{G}_\delta$-set is metric (even topological), we conclude from the existence of the mapping f that P is a $\mathbf{G}_\delta$-set in $\bar{P}(R_0)$ too. By 13.2, $\bar{P}(R_0)$ is a $\mathbf{G}_\delta$-set in R_0 and hence, by exercise 13.10, P is a $\mathbf{G}_\delta$-set in R_0. As $P \subset R \subset R_0$, P is a $\mathbf{G}_\delta$-set in R by 13.6.1.

15.5.3. *Let P be a topologically complete space. Let A be a $\mathbf{G}_\delta$-set in P. Then A is a topologically complete space.*

Proof: By 15.5.2 there is a complete space Q such that P is a $\mathbf{G}_\delta$-set in Q. By exercise 13.10, A is a $\mathbf{G}_\delta$-set in Q and hence A is topologically complete by 15.5.2.

15.6. 15.6.1. *Let f be a homeomorphism of a metric space P onto a metric space Q. Then there exist topologically complete spaces P_0 and Q_0 such that:* [1] *P is embedded into P_0,* [2] *Q is embedded into Q_0,* [3] *there exists a homeomorphism φ of the space P_0 onto the space Q_0 such that $\varphi(x) = f(x)$ for $x \in P$.*

Proof: Let P_1 and Q_1 be completions of P and Q respectively (cf. 15.4.1). Denote by P_2 the set of all $x \in P_1$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$a \in P, \quad a' \in P, \quad \varrho(a, x) < \delta, \quad \varrho(a', x) < \delta \quad \Rightarrow \quad \varrho[f(a), f(a')] \leqq \varepsilon. \tag{1}$$

Let Q_2 denote the set of all elements $y \in Q_1$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ with

$$b \in Q, \quad b' \in Q, \quad \varrho(b, y) < \delta, \quad \varrho(b', y) < \delta \quad \Rightarrow \quad \varrho[f_{-1}(b), f_{-1}(b')] \leqq \varepsilon.$$

For positive integers m, n let A_{mn} be the set of all $x \in P_1$ satisfying

$$a \in P, \quad a' \in P, \quad \varrho(a, x) < \frac{1}{n}, \quad \varrho(a', x) < \frac{1}{n} \quad \Rightarrow \quad \varrho[f(a), f(a')] \leqq \frac{1}{m}$$

It is easy to see that

$$P_2 = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{mn}. \tag{2}$$

If $x \in \bigcup_{n=1}^{\infty} A_{mn}$, there is an index n such that $x \in A_{mn}$; if $x' \in P_1$ and $\varrho(x, x') < 1/2n$, then obviously $x' \in A_{m,2n}$ and hence $x' \in \bigcup_{n=1}^{\infty} A_{mn}$. Consequently, for every $x \in \bigcup_{n=1}^{\infty} A_{mn}$ there is a $\delta > 0$ such that $\Omega_{P_1}(x, \delta) \subset \bigcup_{m=1}^{\infty} A_{mn}$ and hence, by (2), P_2 is a $\mathbf{G}_\delta$-set in P_1. Similarly, Q_2 is a $\mathbf{G}_\delta$-set in Q_1.

The mapping f is continuous, since it is a homeomorphism. Hence, given a point $x \in P$, to every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$a \in P, \quad \varrho(a, x) < \delta \Rightarrow \varrho[f(a), f(x)] \leqq \frac{\varepsilon}{2}. \tag{3}$$

However, from (3) there follows (1), i.e. $x \in P_2$. Hence $P \subset P_2$. As the inverse mapping f_{-1} is also continuous, we obtain similarly that $Q \subset Q_2$.

Choose a point $x \in P_2$. As P_1 is the completion of P, the set P is dense in P_1 and P_1 contains P_2; hence, by exercise 12.2, there is a sequence $\{a_n\}$such that $a_n \in P$ and $a_n \to x$. Let $\varepsilon > 0$. Since $x \in P_2$, there is a $\delta > 0$ such that (1) holds. Since $a_n \to x$, there is an index p such that $n > p$ implies $\varrho(a_n, x) < \delta$. By (1) $m > p$ and $n > p$ imply $\varrho[f(a_m), f(a_n)] \leqq \varepsilon$. Hence, $\{f(a_n)\}$ is a Cauchy sequence. Since $f(a_n) \in Q \subset Q_1$ and Q_1 is a complete space, there is a point $y \in Q_1$ such that $f(a_n) \to y$. Preserving the point x, we replace the sequence $\{a_n\}$ by another sequence $\{a'_n\}$ with the same properties, i.e. $a'_n \in P$, $a'_n \to x$. Put $a''_{2n-1} = a_n$, $a''_{2n} = a'_n$; since $a''_n \in P$, $a''_n \to x \in P_2$, there is $\lim f(a''_n) \in Q_1$. By 7.1.2 $\lim f(a_n) = \lim f(a''_n)$, $\lim f(a'_n) = \lim f(a''_n)$ and consequently $\lim f(a_n) = \lim f(a'_n)$. Hence, the point $y = \lim f(a_n)$ depends on the point $x \in P_2$ only, and not on the choice of the sequence $\{a_n\}$. Hence, we may put $y = \varphi_1(x)$. If $x \in P$, we may choose $a_n = x$ for all n; consequently $\varphi_1(x) = f(x)$. Hence, φ_1 is a mapping of P_2 into Q_1 such that $\varphi_1(x) = f(x)$ for $x \in P$. Similarly we construct a mapping φ_2 of the set Q_2 into the set P_1 such that $y \in Q$ implies $\varphi_2(y) = f_{-1}(y)$.

Let $x_n \in P_2$, $x \in P_2$, $x_n \to x$. For every n there is a sequence $\{a_{ni}\}_{i=1}^{\infty}$ such that $a_{ni} \in P$, $\lim_{i \to \infty} a_{ni} = x_n$; hence, $\lim_{i \to \infty} f(a_{ni}) = \varphi_1(x_n)$. With every n we may associate an index $i(n)$ such that for $b_n = a_{n, i(n)}$ we have $\varrho(b_n, x_n) < 1/n$, $\varrho[f(b_n), \varphi_1(x_n)] < 1/n$. Now $b_n \in P$, $b_n \to x$ and hence $f(b_n) \to \varphi_1(x)$. As $\varrho[f(b_n), \varphi_1(x_n)] < 1/n$, we also have $\varphi_1(x_n) \to \varphi_1(x)$. This proves that the mapping φ_1 of P_2 into Q_1 is continuous. The continuity of the mapping φ_2 of Q_2 into P_1 may be proved in a similar manner.

Put $P_0 = \underset{x}{\mathrm{E}}[x \in P_2, \varphi_1(x) \in Q_2]$, $Q_0 = \underset{y}{\mathrm{E}}[y \in Q_2, \varphi_2(y) \in P_2]$. Evidently $P \subset P_0$, $Q \subset Q_0$. As Q_2 is a $\mathbf{G}_\delta$-set in Q_1 and as φ_1 is a continuous mapping of P_2 into Q_1, by exercise 13.7 the set P_0 is a $\mathbf{G}_\delta$-set in P_2. Since P_2 is a $\mathbf{G}_\delta$-set in P_1, the set P_0 is a $\mathbf{G}_\delta$-set in P_1 by exercise 13.10. As P_1 is a complete space, P_0 is topologically complete by 15.5.2. Similarly Q_0 is topologically complete.

Put $\varphi = (\varphi_1)_{P_0}$, $\psi = (\varphi_2)_{Q_0}$ (cf. 2.4). If $x \in P_0$, there is a sequence $\{a_n\}$ such that $a_n \in P$, $a_n \to x$, $f(a_n) \to \varphi_1(x) = \varphi(x)$. Since $x \in P_0$, we have $\varphi(x) \in Q_2$. Since $f(a_n) \in Q, f(a_n) \to \varphi(x)$, we have $a_n = f_{-1}[f(a_n)] \to \varphi_2[\varphi(x)]$ and hence $\varphi_2[\varphi(x)] = x$. Since $\varphi(x) \in Q_2$, $x \in P_2$, we have $\varphi(x) \in Q_0$. Hence $\varphi(P_0) \subset Q_0$. Since $\varphi(x) \in Q_0$, we have $\varphi_2[\varphi(x)] = \psi[\varphi(x)]$, hence $\psi[\varphi(x)] = x$ and consequently $\psi(Q_0) \supset P_0$. Similarly we prove that $\psi(Q_0) \subset P_0$, $\varphi(P_0) \supset Q_0$. Hence, $\varphi(P_0) = Q_0$, $\psi(Q_0) = P_0$, i.e. φ is a continuous mapping of P_0 onto Q_0 and ψ is a continuous mapping of Q_0 onto P_0. We have also seen that $\psi[\varphi(x)] = x$ for $x \in P_0$, and hence $\psi = \varphi_{-1}$ and φ is a homeomorphic mapping of P_0 onto Q_0. Of course $x \in P$ implies $\varphi(x) = f(x)$.

15.6.2. *Let P and Q be homeomorphic spaces. If P is topologically complete, then Q is also topologically complete.*

Thus, topological completeness is not only a metric property (which was obvious from the definition), but, moreover, a topological property.

Proof: Let f be a homeomorphism of a topologically complete space P onto a metric space Q. By 15.6.1 there are topologically complete spaces P_0, Q_0 containing P, Q respectively, and a homeomorphic mapping φ of Q_0 onto P_0 such that $\varphi(Q) = P$. As P is topologically complete and $P \subset P_0$, P is a $\mathbf{G}_\delta$-set in P_0. Since φ is a continuous mapping of the space Q_0 onto the space P_0 and $Q = \varphi_{-1}(P)$, Q is a $\mathbf{G}_\delta$-set in Q_0 by exercise 13.7. Let Q_1 be a completion of Q_0 (cf. 15.4.1). Since Q_0 is topologically complete, Q_0 is a $\mathbf{G}_\delta$-set in Q_1. Hence, by exercise 13.10, Q is a $\mathbf{G}_\delta$-set in Q_1 and consequently, by 15.5.2, Q is topologically complete.

15.6.3. *A metric space P is topologically complete if and only if there is a complete space homeomorphic to P.*

Proof: I. By 13.2, 15.5.1 and by the definition of topologically complete spaces, any complete space is topologically complete. Hence by 15.6.2, a space homeomorphic with a complete space is topologically complete.

II. Let $P = (P, \varrho)$ be a topologically complete space with the distance function ϱ. It suffices to prove (cf. 9.3) that there is a distance function ϱ_0 in P equivalent with the distance function ϱ and such that (P, ϱ_0) is complete. By 15.4.1, the space $P = (P, \varrho)$ may be embedded into a complete space Q. Without danger of misunderstanding, we may denote the distance function in Q by ϱ just as the previously given distance function in P. Since P is topologically complete, there exist open sets G_n in Q such that $P = \bigcap_{n=1}^{\infty} G_n$. If $P = Q$, the space (P, ϱ) is complete and there is nothing to prove. Hence we may suppose $P \neq Q$ and then, of course, we may suppose $G_n \neq Q$ for every n.*) For $x \in P$, $y \in P$, $n = 1, 2, \ldots$ put

$$f_n(x, y) = \varrho(x, y) + \varrho(x, Q - G_n) + \varrho(y, Q - G_n), \tag{1}$$

$$g_n(x, y) = \frac{\varrho(x, y)}{f_n(x, y)}, \tag{2}$$

$$\varrho_0(x, y) = \varrho(x, y) + \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(x, y). \tag{3}$$

As $x \in P \subset G_n$, $Q - G_n = \overline{Q - G_n}$ (where the right hand side denotes, of course, the closure in Q), we have $\varrho(x, Q - G_n) > 0$, and similarly $\varrho(y, Q - G_n) > 0$. Hence $0 \leqq \varrho(x, y) < f_n(x, y)$, and consequently

$$0 \leqq g_n(x, y) < 1; \tag{4}$$

thus the series on the right hand side of (3) is convergent. Moreover,

$$0 \leqq \varrho(x, y) \leqq \varrho_0(x, y). \tag{5}$$

*) There is an $a \in Q - P$ and hence $P = \bigcap_{n=1}^{\infty} [G_n - (a)]$ where $G_n - (a) \neq Q$ are open; consequently, we could take $G_n - (a)$ instead of G_n.

Evidently $\varrho_0(x, x) = 0$ and $\varrho_0(x, y) > 0$ for $x \neq y$. Obviously $\varrho_0(x, y) = \varrho_0(y, x)$. Since for any numbers $c > 0$, $t_1 \geqq 0$, $t_2 \geqq t_1$ one has the evident relation

$$\frac{t_1}{c + t_1} \leqq \frac{t_2}{c + t_2}$$

and since for $x \in P$, $y \in P$, $z \in P$ we have $\varrho(x, z) \leqq \varrho(x, y) + \varrho(y, z)$, we obtain

$$g_n(x, z) \leqq \frac{\varrho(x, y) + \varrho(y, z)}{\varrho(x, y) + \varrho(x, Q - G_n) + \varrho(y, z) + \varrho(z, Q - G_n)}. \tag{6}$$

By exercise 6.6,

$$\varrho(y, Q - G_n) \leqq \varrho(x, y) + \varrho(x, Q - G_n),$$
$$\varrho(y, Q - G_n) \leqq \varrho(y, z) + \varrho(z, Q - G_n),$$

and hence the denominator on the right-hand side in (6) is not less then either of the two following numbers

$$\varrho(x, y) + \varrho(x, Q - G_n) + \varrho(y, Q - G_n),$$
$$\varrho(y, z) + \varrho(y, Q - G_n) + \varrho(z, Q - G_n).$$

Hence, by (6), it follows that

$$g_n(x, z) \leqq g_n(x, y) + g_n(y, z)$$

and consequently, by (3), we obtain $\varrho_0(x, z) \leqq \varrho_0(x, y) + \varrho_0(y, z)$.

We have proved that ϱ_0 is a distance function in P. We shall show that the distance functions ϱ_0 and ϱ in P are equivalent, i.e. that for $x_n \in P$, $x \in P$ there is

$$\varrho(x_n, x) \to 0 \Leftrightarrow \varrho_0(x_n, x) \to 0.$$

If $\varrho_0(x_n, x) \to 0$ then $\varrho(x_n, x) \to 0$ by (5). Now let $\varrho(x_n, x) \to 0$. Choose an $\varepsilon > 0$. Find an index k such that $1/2^k < \varepsilon/2$. By (4), for every n there is

$$\sum_{i=k+1}^{\infty} \frac{1}{2^i} g_i(x_n, x) \leqq \sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k} < \frac{\varepsilon}{2},$$

hence

$$\varrho_0(x_n, x) < \varrho(x_n, x) + \sum_{i=1}^{k} \frac{1}{2^i} g_i(x_n, x) + \frac{\varepsilon}{2},$$

and thus, by (1) and (2),

$$\varrho_0(x_n, x) < \varrho(x_n, x) + \sum_{i=1}^{k} \frac{1}{2^i} \frac{\varrho(x_n, x)}{\varrho(x_n, x) + \varrho(x, Q - G_i)} + \frac{\varepsilon}{2}.$$

Putting

$$f(t) = t + \sum_{i=1}^{k} \frac{1}{2^i} \frac{t}{t + \varrho(x, Q - G_i)},$$

we obtain a continuous function f with domain $\underset{x}{\mathrm{E}}[t \mid t \geqq 0]$ such that $f(0) = 0$. Hence there is a $\delta > 0$ such that $0 \leqq t < \delta$ implies $f(t) < \varepsilon/2$. Since $\varrho(x_n, x) \to 0$, there is an index p such that the following sequence of implications holds:

$$n > p \Rightarrow 0 \leqq \varrho(x_n, x) < \delta \Rightarrow f[\varrho(x_n, x)] < \frac{\varepsilon}{2} \Rightarrow \varrho_0(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence indeed $\varrho_0(x_n, x) \to 0$.

It remains to prove that the space (P, ϱ_0) is complete. Let $\{x_n\}$ be a Cauchy sequence in this space. We have to show that there is a point $x \in P$ such that $\varrho_0(x_n, x) \to 0$; of course, it suffices to prove $\varrho(x_n, x) \to 0$, since the distance functions ϱ_0 and ϱ are equivalent. Since $\{x_n\}$ is a Cauchy sequence with respect to the distance function ϱ_0, it is, by (5), a Cauchy sequence with respect to the distance function ϱ. As (Q, ϱ) is complete, there is a point $x \in Q$ such that $\varrho(x_n, x) \to 0$. We are to prove that $x \in P$. Assume the contrary, that $x \in Q - P$. Since $P = \bigcap_{n=1}^{\infty} G_n$, there is an index k such that $x \in Q - G_k$. Hence $\varrho(x_n, Q - G_k) \leqq \varrho(x_n, x)$. Since $\varrho(x_n, x) \to 0$, we have

$$\varrho(x_n, Q - G_k) \to 0. \tag{7}$$

By (3),

$$\varrho_0(x_m, x_n) \geqq \frac{1}{2^k} g_k(x_m, x_n).$$

Hence, by (1) and (2),

$$\varrho_0(x_m, x_n) \geqq \frac{1}{2^k} \frac{\varrho(x_m, x_n)}{\varrho(x_m, x_n) + \varrho(x_m, Q - G_k) + \varrho(x_n, Q - G_k)}.$$

By exercise 6.6 we have

$$\varrho(x_m, Q - G_k) \leqq \varrho(x_m, x_n) + \varrho(x_n, Q - G_k);$$

hence,

$$\varrho_0(x_m, x_n) \geqq \frac{1}{2^{k+1}} \frac{\varrho(x_m, x_n)}{\varrho(x_m, x_n) + \varrho(x_n, Q - G_k)}.$$

Since $\{x_n\}$ is a Cauchy sequence with respect to ϱ_0, there is an index p such that $m > p$, $n > p$ imply $\varrho_0(x_m, x_n) < 1/2^{k+2}$. Hence the following implications hold

$$m > p,\quad n > p \Rightarrow \frac{\varrho(x_m, x_n)}{\varrho(x_m, x_n) + \varrho(x_n, Q - G_k)} < \frac{1}{2} \Rightarrow \varrho(x_m, x_n) < \varrho(x_m, Q - G_k).$$

Consequently

$$m > p \Rightarrow \lim_{n \to \infty} \varrho(x_m, x_n) \leqq \lim_{n \to \infty} \varrho(x_n, Q - G_k)$$

and hence by (7) and by exercise 9.10 it follows that

$$m > p \Rightarrow g(x_m, x) \leqq 0 \Rightarrow \varrho(x_m, x) = 0 \Rightarrow x_m = x\,,$$

which is a contradiction, since $x_m \in P$ and $x \in Q - P$.

15.7. 15.7.1. *Let P be a complete space. Let A_n $(n = 1, 2, 3, \ldots)$ be point sets embedded into P, such that $A_n \neq \emptyset$, $d(A_n) \to 0$, $A_n \supset \bar{A}_{n+1}$. Then the set $\bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.*

Proof: Let us choose $a_n \in A_n$. If $\varepsilon > 0$, there is an index p such that $d(A_p) < \varepsilon$. For $n > p$ we have $a_n \in A_n \subset A_p$. Consequently, $m > p$ and $n > p$ imply $\varrho(a_m, a_n) \leqq \leqq d(A_p) < \varepsilon$. Hence, $\{a_n\}$ is a Cauchy sequence. As the space P is complete, there is a point x_0 such that $a_n \to x_0$. Given a positive integer n, $a_i \in A_{n+1}$ for $i \geqq n + 1$; hence, by 8.2.1, $x_0 \in \bar{A}_{n+1} \subset A_n$. Consequently $x_0 \in \bigcap_{n=1}^{\infty} A_n$. Let $x \in \bigcap_{n=1}^{\infty} A_n$. For every n there is $\varrho(x, x_0) \leqq d(A_n)$. Since $d(A_n) \to 0$, we obtain $\varrho(x, x_0) = 0$, and hence $x = x_0$.

The theorem just proved is a (particularly important) special case of the following more general theorem:

15.7.2. *Let P be a complete space. For $n = 1, 2, 3, \ldots$, let $\delta_n > 0$ with $\delta_n \to 0$, $A_n \subset P$, $A_n \neq \emptyset$, $A_n \supset \bar{A}_{n+1}$. Let there exist finite sets $K_n \subset P$, $K_n \neq \emptyset$ such that $x \in A_n$ implies $\varrho(x, K_n) < \delta_n$. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.*

Proof: I. Let us choose $a_n \in A_n$. We shall prove that the sequence $\{a_n\}$ contains a convergent subsequence. For every n, $a_n \in A_n \subset A_1$, and hence there is a point $x \in K_1$ such that $\varrho(a_n, x) < \delta_1$. Since the set K_1 is finite, there is a point $x_1 \in K_1$ such that there is a subsequence $\{a_{1n}\}_{n=1}^{\infty}$ of $\{a_n\}$ with $\varrho(a_{1n}, x) < \delta_1$ for every n; as $A_n \supset A_{n+1}$, evidently $a_{1n} \in A_n$ for every n.

Now, suppose that for a given $i\,(= 1, 2, 3, \ldots)$ the same construction has been carried out as for $i = 1$; namely, that we have determined a point $x_i \in K_i$ and a sequence $\{a_{in}\}_{n=1}^{\infty}$ such that, for every n, $a_{in} \in A_n$, $\varrho(a_{in}, x_i) < \delta_i$. For $n > i$ we have $a_{in} \in A_n \subset \subset A_{i+1}$ and hence, for every $n > i$, there is a point $x \in K_{i+1}$ such that $\varrho(a_{in}, x) < \delta_{i+1}$. As the set K_{i+1} is finite, there is a point $x_{i+1} \in K_{i+1}$ and a subsequence $\{a_{i+1,n}\}_{n=1}^{\infty}$ of $\{a_{in}\}_{n=1}^{\infty}$ such that $\varrho(a_{i+1,n}, x_i) < \delta_{i+1}$ for every n. Evidently, $a_{i+1,n} \in A_n$. Hence, we may construct recursively the sequences $\{a_{in}\}_{n=1}^{\infty}$ for $i = 1, 2, 3, \ldots$

Put $b_n = a_{nn}$; hence $\{b_n\}$ is a subsequence of the sequence $\{a_n\}$. We have to prove that $\{b_n\}$ is convergent; since the space P is complete, it suffices to prove that $\{b_n\}$ is a Cauchy sequence. The sequence $\{b_n\}_{n=i}^{\infty}$ is a subsequence of $\{a_{in}\}_{n=1}^{\infty}$. Hence $n \geqq i$ implies $\varrho(b_n, x_i) < \delta_i$ and consequently $m \geqq i$, $n \geqq i$ imply $\varrho(b_m, b_n) < 2\delta_i$.

Since $\delta_i \to 0$, $\{b_n\}$ is a Cauchy sequence. Since P is complete, $\{b_n\}$ is convergent.

II. By I., there is a convergent sequence $\{a_n\}$ such that $a_n \in A_n$. Let $a_n \to x_0$. If n is given and $i \geqq n + 1$ then $a_i \in A_{n+1}$, and hence, by 8.2.1, $x_0 \in \bar{A}_{n+1} \subset A_n$. Consequently $x_0 \in \bigcap_{n=1}^{\infty} A_n$, and hence $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

15.8. 15.8.1.*) *Let $P \neq \emptyset$ be a topologically complete space. Let $G_n \subset P$ be open sets, G_n dense in P. Then $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$; moreover, the set $\bigcap_{n=1}^{\infty} G_n$ is dense in P.*

Proof: I. First show that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. From 15.6.3 it follows easily that it suffices to prove this under the assumption that P is complete. Choose a point $a_1 \in G_1$. Since the set G_1 is open, there is a real number δ_1 such that $0 < \delta_1 < 1/2$ and $\mathrm{E}_x[\varrho(a_1, x) \leqq \delta_1] \subset G_1$. More generally, for any given n $(= 1, 2, 3, \ldots)$ assume there have been found a point a_n and a number δ_n such that $a_n \in G_n$, $0 < \delta_n < 1/2^n$, $\mathrm{E}_x[\varrho(a_n, x) \leqq \delta_n] \subset G_n$. As the set G_{n+1} is dense, there is a point $a_{n+1} \in G_{n+1}$ such that $\varrho(a_n, a_{n+1}) < \delta_n$. As G_{n+1} and $\mathrm{E}_x[\varrho(a_n, x) < \delta_n]$ are open sets, there is a number δ_{n+1} such that $0 < \delta_{n+1} < 1/2^{n+1}$, $\mathrm{E}_x[\varrho(a_{n+1}, x) \leqq \delta_{n+1}] \subset G_{n+1} \cap \mathrm{E}_x[\varrho(a_n, x) \leqq \leqq \delta_n]$. Hence, the points a_n and the numbers δ_n may be constructed recursively. Put $S_n = \mathrm{E}_x[\varrho(a_n, x) \leqq \delta_n]$. Then $S_n \subset G_n$, $S_n \supset S_{n+1}$, $d(S_n) \leqq 2\delta_n \to 0$, $S_n \neq \emptyset$. Moreover, $S_n = \bar{S}_n$ (e.g. by 9.5 and exercise 9.10). Hence, by 15.7.1, $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$. As $S_n \subset G_n$, $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$.

II. Let $\Gamma \neq \emptyset$ be an open set. By 12.1.2, it suffices to show that $\Gamma \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$. By 13.1.1 and 15.5.3, Γ is a topologically complete space. The sets $\Gamma \cap G_n$ are open by 8.7.5, and dense in Γ by exercise 12.3. Hence, by I, $\bigcap_{n=1}^{\infty} (\Gamma \cap G_n) \neq \emptyset$, i.e. $\Gamma \cap \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$.

15.8.2. *Let $P \neq \emptyset$ be a topologically complete space. Let A be a set of the first category in P. Then $P - A \neq \emptyset$; moreover, the set $P - A$ is dense in P.*

Proof: We have $A = \bigcup_{n=1}^{\infty} A_n$, where A_n are nowhere dense sets, i.e. the (open) sets $G_n = P - \bar{A}_n$ are dense. Hence, by 15.8.1, the set $\bigcap_{n=1}^{\infty} G_n = P - \bigcup_{n=1}^{\infty} \bar{A}_n$ is dense. By 12.1.1 also the set $P - A \supset P - \bigcup_{n=1}^{\infty} \bar{A}_n$ is dense.

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*) **15.8.1** (**15.8.2**, resp.) is sometimes called Baire's theorem.

15.8.3. *Let $P \neq \emptyset$ be a topologically complete space. Let f be a function of the first class with domain P. Let C be the set of all $x \in P$ at which f is continuous. The set C is dense in P.*

Proof: By 14.5.2, $P - C$ is a set of the first category. Hence, by 15.8.2, the set C is dense.

15.8.4. *The set R of all rational numbers is not a $\mathbf{G}_\delta$-set in $\mathbf{E}_1$.*

Proof: If $a \in R$, then the set (a) is nowhere dense in R by exercise 12.5. Hence, by exercise 3.1, the set R is a set of the first category in R. Consequently, the space R is not topologically complete by 15.8.2. Hence, R is not a $\mathbf{G}_\delta$-set in $\mathbf{E}_1$ by 15.1.3 and 15.5.2.

Exercises

15.1. Let $P = A \cup B$. Let A and B be complete spaces. Then P is a complete space.

15.2. Let $P = A \cup B$. Let A and B be topologically complete spaces. Then P is a topologically complete space.

15.3. Let $\mathbf{C}$ be a non-void set. Let P be a metric space. For each $z \in \mathbf{C}$ let $A(z)$ be a complete space embedded into P. Then $\bigcap_{z \in \mathbf{C}} A(z)$ is a complete space.

15.4. Let P be a metric space. For $n = 1, 2, 3, \ldots$ let A_n be topologically complete spaces embedded into P. Then $\bigcap_{n=1}^{\infty} A_n$ is a topologically complete space.

15.5. Let P and Q be topologically complete spaces. Then $P \times Q$ is a topologically complete space.

15.6. Every absolutely open space (cf. analogous definitions in section 15.5) is void.

15.7. Let P be a complete space. Let $Q \subset P$. Then $\overline{Q}$ is a completion of the space Q.

15.8. Let P be a topologically complete space. Let A be a closed set of the first category in P. Then A is nowhere dense in P.

15.9. Let P be a topologically complete space. Let A_n $(n = 1, 2, 3, \ldots)$ be dense $\mathbf{G}_\delta$-sets in P. Then the set $\bigcap_{n=1}^{\infty} A_n$ is dense in P.

15.10. The spaces in exercises 6.5, 7.2, and 7.4 are complete.

15.11. In the proof of theorem 15.6.1 the following equalities hold: $P_0 = P_2 \cap \varphi_2(Q_2)$, $Q_0 = Q_2 \cap \varphi_1(P_2)$.

15.12. In theorem 15.6.1 we may put $P = \mathop{\mathrm{E}}_t[0 < t < 1] \cup \mathop{\mathrm{E}}_t[1 < t < 2] \cup \mathop{\mathrm{E}}_t[2 < t < 3] = Q$, $f(t) = 1 - t$ for $0 < t < 1$, $f(t) = 3 - t$ for $1 < t < 2$, $f(t) = t$ for $2 < t < 3$, $P_1 = Q_1 = P \cup (0) \cup (1) \cup (2) \cup (3)$. In the proof of the quoted theorem we have $P_2 = P \cup (0) \cup (3) = Q_2$, $P_0 = P \cup (3) = Q_0$.

15.13. Let P be a topologically complete space. Let f be a function of the first class with domain P. Let Q be a non-void $\mathbf{G}_\delta$-set in P. Then there is a point $x \in Q$ such that the partial function f_Q is continuous at x.

15.14. Let f be a function with domain $\mathbf{E}_1$. Let C be the set of all points $x \in \mathbf{E}_1$ such that f is continuous at x. Then C is not the set of all rational numbers. (This may be proved using 13.4.) Compare with the result of exercise 9.2.

15.15. For $x \in \mathbf{E}_1$ put $f(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n} m!\pi x$. The function f is nowhere continuous. Hence it is not a function of the first class.

15.16.* The set of all members of a Cauchy sequence is bounded.

§ 16. Separable spaces

16.1. An *open basis* of a metric space P is a system $\mathfrak{B}$ of open subsets of P such that for every neighborhood U of any point $x \in P$ there is a neighborhood V of the point x with $V \in \mathfrak{B}$ and $V \subset U$.

16.1.1. *Let $\mathfrak{B}$ be a system of subsets of a metric space P. $\mathfrak{B}$ is an open basis of P if and only if:* [1] *every set from $\mathfrak{B}$ is open*; [2] *for every open $G \subset P$, $G \neq \emptyset$, there is a system $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{A} \neq \emptyset$, such that $G = \bigcup_{X \in \mathfrak{A}} X$.*

Proof: I. Let the system $\mathfrak{B}$ have the properties [1] and [2]. Let U be a neighborhood of a point $x \in P$. Then there is a system $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{A} \neq \emptyset$, such that $U = \bigcup_{X \in \mathfrak{A}}^{\infty} X$. Since $x \in U$, there is a $V \in \mathfrak{A}$ with $x \in V$. The set V is a neighborhood of x and we have $V \subset U$.

II. Let $\mathfrak{B}$ be an open basis of the space P. Let $G \subset P$ be a non-void open set. As $G \neq \emptyset$, there is a point $a \in G$. G is a neighborhood of the point a and hence there is a set $H \in \mathfrak{B}$ with $a \in H \subset G$. Thus, the system $\mathfrak{A}$ of all the $X \in \mathfrak{B}$ such that $X \subset G$ is non-void. Evidently $\bigcup_{X \in \mathfrak{A}} X \subset G$. If $x \in G$, G is a neighborhood of the point a, so that there is a set $U \in \mathfrak{B}$ with $x \in U \subset G$; thus, $U \in \mathfrak{A}$ and consequently $x \in \bigcup_{X \in \mathfrak{A}} X$. Thus, $G \subset \bigcup_{X \in \mathfrak{A}} X$, i.e. $G = \bigcup_{X \in \mathfrak{A}} X$.

A *separable* space is a metric space which has (at least one) countable open basis. This is obviously a topological property.

16.1.2. *Let P be a separable space. Let $Q \subset P$. Then Q is separable.*

Proof: If $\mathfrak{B}$ is an open basis of P and if we replace every set $X \in \mathfrak{B}$ by the set $Q \cap X$, we obtain a system $\mathfrak{B}_0$. 8.7.5 yields that $\mathfrak{B}_0$ is an open basis of the space Q. If $\mathfrak{B}$ is countable, the system $\mathfrak{B}_0$ is evidently also countable.

16.1.3. *A metric space P is separable if and only if there is a countable $A \subset P$ dense in P.*

Proof: I. Let $\mathfrak{B}$ be a countable open basis of the space P. Let us choose one point in each non-void $X \in \mathfrak{B}$. Let A be the set of all chosen points. Then A is a countable set. If G is non-void and open, choose an $x \in G$. As $\mathfrak{B}$ is a basis, there exists a $U \in \mathfrak{B}$

with $x \in U \subset G$. If $a \in A$ is the point chosen in U, we have $a \in A \cap U \subset A \cap G$. Thus, $A \cap G \neq \emptyset$ for every non-void open G, so that the set A is dense by 12.1.2.

II. Let A be a dense countable subset of P. Let $\mathfrak{B}$ be the system of all $\Omega(a, r)$, where a varies over all the points of A and r varies over all the positive rational numbers. By 3.5.2 and 3.6 we see easily that the system $\mathfrak{B}$ is countable. By 8.6.1 we see easily that $\mathfrak{B}$ is an open basis of P.

16.1.4. *The Hilbert space* $\mathbf{H}$ *is separable.*

Proof: Let A be the set of all $r = \{r_n\}_1^\infty$ such that: [1] every r_n is a rational number; [2] there exists an index p such that $r_n = 0$ for every $n > p$. Evidently $A \subset \mathbf{H}$. Let $x = \{x_n\}_1^\infty \in \mathbf{H}$. Choose an $\varepsilon > 0$. There exists an index p such that $\sum_{n=p+1}^{\infty} x_n^2 < \varepsilon^2/2$. For $1 \leqq n \leqq p$ there are rational numbers r_n such that $\sum_{n=1}^{p} (x_n - r_n)^2 < \varepsilon^2/2$. For $n > p$ put $r_n = 0$. If $r = \{r_n\}_1^\infty$, we have $r \in A$, $\varrho(x, r) < \varepsilon$. Thus, $\varrho(x, A) < \varepsilon$. As $\varepsilon > 0$ was arbitrary, we have $\varrho(x, A) = 0$, i.e. $x \in \bar{A}$. Thus, $\bar{A} = \mathbf{H}$, i.e. the set A is dense in $\mathbf{H}$. The set A is countable by ex. 3.1 and 3.14.

16.1.5. *The euclidean space* $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ *is separable.*

Proof: Let Q_m be the set of all points $x = \{x_n\}_1^\infty \in \mathbf{H}$ with $x_n = 0$ for $n > m$. Q_m is separable by 16.1.2 and 16.1.4. The spaces $\mathbf{E}_m$ and Q_m are evidently isometric, so that $\mathbf{E}_m$ is also separable.

16.1.6. *Let P be a metric space. For every $\delta > 0$ let there be a countable set $A(\delta) \subset P$ such that $\varrho[x, A(\delta)] < \delta$ for every $x \in P$. Then P is separable.*

Proof: Put $B = \bigcup_{n=1}^{\infty} A(1/n)$. The set B is countable by 3.6. For every point $x \in P$ we have $\varrho(x, B) \leqq \varrho[x, A(1/n)] < 1/n$, hence $\varrho(x, B) = 0$, i.e. $x \in \bar{B}$. Thus, $\bar{B} = P$, i.e. the set B is dense, so that P is separable by 16.1.3.

16.1.7. *Let P be a metric space. Let there exist a number $\delta > 0$ and an uncountable set $A \subset P$ such that*

$$x \in A, \quad y \in A, \quad x \neq y \text{ imply } \varrho(x, y) > \delta.$$

Then P is not separable.

Remark: This theorem is a useful criterion for proving that a given space is not separable. Its converse is valid; however, it cannot be prowed without a use of the theorem that the set P may be well ordered. (Which is not proved in this book; see 4.3.)

Proof: Let $\mathfrak{B}$ be an open basis of the space P. For every $x \in A$ there is a set $B(x) \in \mathfrak{B}$ with $x \in B(x) \subset \Omega(x, \delta)$. If $x \in A$, $y \in A$, $x \neq y$, we have $y \in B(y)$, while x is not in $B(y)$, hence, $B(x) \neq B(y)$. As the set A is uncountable, the system of all the $B(x)$ is uncountable. Thus, the system $\mathfrak{B}$ is uncountable.

16.2. 16.2.1. *Let P be a separable space. Let $\mathfrak{A}$ be a disjoint system of open subsets of P. Then the system $\mathfrak{A}$ is countable.*

Proof: By 16.1.3 there is a countable dense subset A. By 12.1.2, we may choose in every $G \in \mathfrak{A}$ – with the exception of $G = \emptyset$, which may be also an element of $\mathfrak{A}$ – a point $\varphi(G) \in A \cap G$. The set of all points $\varphi(G)$ is countable by 3.4.1. Since the system $\mathfrak{A}$ is disjoint, we have $\varphi(G_1) \neq \varphi(G_2)$ for $G_1 \neq G_2$. Thus, the system $\mathfrak{A}$ is also countable.

16.2.2. *A necessary and sufficient condition for P to be a separable space is the following: For every system $\mathfrak{A}$ of open sets with $\bigcup_{X \in \mathfrak{A}} X = P$ there is a countable system $\mathfrak{A}_0 \subset \mathfrak{A}$ such that $\bigcup_{X \in \mathfrak{A}_0} X = P$.*

Proof: I. Let the condition be satisfied. For $n = 1, 2, 3, \ldots$ denote by $\mathfrak{S}_n$ the system of all $\Omega(x, 1/n)$ with $x \in P$. There is a countable $\mathfrak{T}_n \subset \mathfrak{S}_n$ such that $\bigcup_{X \in \mathfrak{T}_n} X = P$. Put $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{T}_n$. Then $\mathfrak{B}$ is a countable (see 3.6) system of open sets. It suffices to prove that $\mathfrak{B}$ is an open basis of the space P, i.e. that for any given neighborhood U of a given point $a \in P$ there is a set $V \in \mathfrak{B}$ with $a \in V \subset U$. There is a number $r > 0$ with $\Omega(a, r) \subset U$. Choose an index $n > 2/r$. Since $\mathfrak{T}_n \subset \mathfrak{S}_n$, $\bigcup_{X \in \mathfrak{T}_n} X = P$, there is a point $b \in P$ such that $\Omega(b, 1/n) \in \mathfrak{T}_n \subset \mathfrak{B}$ and $a \in \Omega(b, 1/n)$. Then the following sequence of implications holds

$$x \in \Omega(b, 1/n) \Rightarrow \varrho(b, x) < 1/n \Rightarrow \varrho(a, x) \leqq \varrho(a, b) +$$
$$+ \varrho(b, x) \leqq 2/n < r \Rightarrow x \in \Omega(a, r) \subset U,$$

hence $\Omega(b, 1/n) \subset U$. This $\Omega(b, 1/n)$ is an element of $\mathfrak{B}$.

II. Let P be separable. Let $\mathfrak{A}$ be a system of open sets with $\bigcup_{X \in \mathfrak{A}} X = P$. Let $\mathfrak{B}$ be a countable basis of the space P. With every $x \in P$ we may associate a set $A_x \in \mathfrak{A}$ such that $x \in A_x$; then we choose a set $B_x \in \mathfrak{B}$ such that $x \in B_x \subset A_x$. Evidently $\bigcup_{x \in P} B_x = P$. Since the system $\mathfrak{B}$ is countable, there is a countable $C \subset P$ such that $\bigcup_{x \in C} B_x = \bigcup_{x \in P} B_x$ i.e. $\bigcup_{x \in C} B_x = P$. As $A_x \supset B_x$, we also have $\bigcup_{x \in C} A_x = P$. Thus, the system $\mathfrak{A}_0$ of all A_x with $x \in C$ is countable and such that $\mathfrak{A}_0 \subset \mathfrak{A}$, $\bigcup_{X \in \mathfrak{A}_0} X = P$.

16.2.3. *A necessary and sufficient condition for P to be separable is the following: Every open basis contains a countable open basis.*

Proof: I. Let the condition be satisfied. Since there is at least one open basis (namely the system of all the open sets), there is a countable open basis, i.e. P is separable.

II. Let P be separable. Let $\mathfrak{B}$ be a countable open basis. Let $\mathfrak{A}$ be an arbitrary open basis. If there is given a point $x \in P$ and an index $n = 1, 2, 3, \ldots$, there is a set $A_n(x) \in \mathfrak{A}$ such that $x \in A_n(x) \subset \Omega(x, 1/n)$; further, there is a set $B_n(x) \in \mathfrak{B}$ such that $x \in B_n(x) \subset A_n(x)$. Since the system $\mathfrak{B}$ is countable, there is, for every n, a countable set $C_n \subset P$ such that $\bigcup_{x \in C_n} B_n(x) = \bigcup_{x \in P} B_n(x)$, i.e. $\bigcup_{x \in C_n} B_n(x) = P$. Since $B_n(x) \subset \subset A_n(x)$, we have $\bigcup_{x \in C_n} A_n(x) = P$. Let $\mathfrak{A}_0$ be the system of all $A_n(x)$ ($n = 1, 2, 3, \ldots \ldots, x \in C_n$). Then $\mathfrak{A}_0 \subset \mathfrak{A}$ and the system $\mathfrak{A}_0$ is countable by 3.4.1 and 3.6. It suffices to prove that the system $\mathfrak{A}_0$ is an open basis, i.e. that for every neighborhood U of any point $a \in P$ there is a set $V \in \mathfrak{A}_0$ with $a \in V \subset U$. There is a number $r > 0$ such that $\Omega(a, r) \subset U$. Choose an index $n > 2/r$. Since $\bigcup_{x \in C_n} A_n(x) = P$, there is a point $b \in C_n$ with $a \in A_n(b)$. We have $A_n(b) \subset \Omega(b, 1/n)$, hence $\varrho(a, b) < 1/n$; thus $x \in \Omega(b, 1/n)$ implies $\varrho(b, x) < 1/n$ which implies $\varrho(a, x) \leqq \varrho(a, b) + \varrho(b, x) < < 2/n < r$, hence $\Omega(b, 1/n) \subset \Omega(a, r) \subset U$. Hence $a \in A_n(b) \subset U$. Since $b \in C_n$, we have $A_n(b) \in \mathfrak{A}_0$.

16.3. 16.3.1. *Let P be an uncountable separable space. Let Q be the set of all the $x \in P$ such that every neighborhood of x is uncountable. Then:* [1] *$P - Q$ is countable, hence, the set Q is uncountable,* [2] *the set Q is dense-in-itself.*

Proof: I. With every $x \in P - Q$ we may associate a countable neighborhood $U(x)$. The sets $U(x) - Q$ are (see 8.7.5) open in $P - Q$ and we have $\bigcup_{x \in P - Q} (U(x) - - Q) = P - Q$. $P - Q$ is a separable space by 16.1.2. Thus, by 16.2.2, there is a countable $A \subset P - Q$ such that $\bigcup_{x \in P - Q} [U(x) - Q] = \bigcup_{x \in A} [U(x) - Q]$, i.e. $\bigcup_{x \in A} [U(x) - - Q] = P - Q$. Hence, the set $P - Q$ is countable by 3.6. Q is uncountable, since otherwise the set $P = (P - Q) \cup Q$ would be also countable.

II. If $x \in Q$ and $\varepsilon > 0$, the set $\Omega(x, \varepsilon)$ is a neighborhood of the point x and it is, consequently, uncountable. Since $P - Q$ is countable, the set $Q \cap \Omega(x, \varepsilon) = = \Omega(x, \varepsilon) - (P - Q)$ is uncountable. Hence, there is a $y \in Q$, $y \neq x$ with $\varrho(x, y) < \varepsilon$. Thus, x is not an isolated point of the set Q. Hence, Q is dense-in-itself.

16.3.2. *Every dispersed separable space P is countable.*

Proof: If P were uncountable, it would contain, by 16.3.1, a dense-in-itself set Q.

16.4. *Let P be a separable space. Let a non-void system $\mathfrak{A}$ of closed subsets of P have the following property: If, for $p = 1, 2, 3, \ldots$, $A_n \in \mathfrak{A}$, $A_n \supset A_{n+1}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$.*

Then there is at least one minimal set $M \in \mathfrak{A}$, i.e. a set M such that $A \in \mathfrak{A}$, $A \subset M$ imply $A = M$.

Proof: Let $\{B_n\}_1^\infty$ be a sequence the members of which are exactly all the elements of a (countable) open basis $\mathfrak{B}$ of the space P. Choose arbitrarily a set $A_1 \in \mathfrak{A}$. If (for $n = 1, 2, 3, \ldots$) the set $A_n \in \mathfrak{A}$ is chosen, choose, *if it is possible*, an $A_{n+1} \in \mathfrak{A}$ with $A_{n+1} \subset A_n - B_n$; if it is not possible, put $A_{n+1} = A_n$. Then we always have $A_n \in \mathfrak{A}$, $A_{n+1} \subset A_n$ and hence $M = \bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$. Let us prove that M is minimal in $\mathfrak{A}$. Let there be, on the contrary, a set $C \in \mathfrak{A}$ with $C \subset M \neq C$. Choose a point $a \in M - C$. The set $P - C$ is a neighborhood of the point a. Since $\mathfrak{B}$ is an open basis, there is an index n such that $a \in B_n \subset P - C$. We have $C \in \mathfrak{A}$, $C \subset M - B_n \subset A_n - B_n$. Hence, $A_{n+1} \subset A_n - B_n$, so that $a \in P - A_{n+1}$ for $a \in B_n$. This is a contradiction, since $a \in M \subset A_{n+1}$. (The theorem just proved is called the *Brouwer reduction theorem.*)

16.5. *A metric space P is separable if and only if there is a point set Q embedded into the Urysohn space $\mathbf{U}$ which is homeomorphic with P.*

Proof: I. Since $\mathbf{U} \subset \mathbf{H}$, the space Q embedded into $\mathbf{U}$ is, by 16.1.2, and 16.1.4, separable; thus, a space P homeomorphic with Q is also separable.

II. Let P be separable. We may assume that $P \neq \emptyset$. By 16.1.3 there is a countable set A dense in P. Let T be the set of all the triples (a, r, s) where $a \in A$ and r, s are rational numbers such that $0 < r < s$. Evidently (see 3.5.2 and 3.6) T is a non-void countable set, so that we may form a one-to-one sequence $\{(a_n, r_n, s_n)\}_1^\infty$ consisting exactly of all the elements of T. For $x \in P$, $n = 1, 2, 3, \ldots$ put

$$f_n(x) = \frac{\varrho[x, \Omega(a_n, r_n)]}{\varrho[x, \Omega(a_n, r_n)] + \varrho[x, P - \Omega(a_n, s_n)]} \tag{1}$$

if $\Omega(a_n, s_n) \neq P$,

$$f_n(x) = 0 \quad \text{if} \quad \Omega(a_n, s_n) = P\,.$$

The denominator on the right-hand side in (1) could be zero only in the case of $x \in \bar{\Omega}(a_n, r_n) - \Omega(a_n, s_n)$; in such a case we would have simultaneously $\varrho(a_n, x) \leqq r_n$ and $\varrho(a_n, x) \geqq s_n$, which is impossible, as $r_n < s_n$. Thus, f_n is a finite continuous function (see ex. 9.3) on P. Evidently: [1] for $x \in P$ we have $0 \leqq f_n(x) \leqq 1$, so that $\{(1/n) f_n(x)\}_1^\infty \in \mathbf{U}$; [2] $\varrho(a_n, x) < r_n$ implies $f_n(x) = 0$, [3] $\varrho(a_n, x) > s_n$ implies $f_n(x) = 1$. Put

$$F(x) = \left\{\frac{1}{n} f_n(x)\right\}_1^\infty \quad \text{and} \quad Q = F(P)\,.$$

Then F is a mapping of the space P onto the space Q. We shall prove that F is a homeomorphic mapping, i.e. that: [1] F is one-to-one, [2] F is continuous, [3] F_{-1} is continuous.

Let $x \in P$, $y \in P$, $x \neq y$. Since A is dense in P, there is an $a \in A$ such that $\varrho(a, x) < \frac{1}{2}\varrho(x, y)$. Since $\varrho(x, y) \leqq \varrho(a, x) + \varrho(a, y)$, we have $\varrho(a, y) > \varrho(a, x)$. Hence there exist rational numbers r, s with $0 \leqq \varrho(a, x) < r < s < \varrho(a, y)$. There is an index n such that $a = a_n$, $r = r_n$, $s = s_n$. We have $f_n(x) = 0$, $f_n(y) = 1$, hence $f_n(x) \neq f_n(y)$, hence $F(x) \neq F(y)$. Thus, the mapping F is one-to-one.

Let $x_i \in P$, $x \in P$, $x_i \to x$. Since the functions f_n are continuous, we have, for every n, $\lim_{i \to \infty} f_n(x_i) = f_n(x)$ so that, by 7.3.1, $\lim_{i \to \infty} F(x_i) = F(x)$. Thus, the mapping F is continuous.

Let $x_i \in P$, $x \in P$, $\lim_{i \to \infty} F(x_i) = F(x)$. By 7.3.1, $\lim_{i \to \infty} f_n(x_i) = f_n(x)$ for every index n. Let us assume that $\lim_{i \to \infty} x_i \neq x$. Then there is a number $\varepsilon > 0$ and an infinite set M of indices i such that $i \in M$ implies $\varrho(x_i, x) > \varepsilon$. As A is dense in P, there is an $a \in A$ with $\varrho(a, x) < \varepsilon/2$. There are rational numbers r, s with $0 \leqq \varrho(a, x) < r < s < \varepsilon/2$. There is an index n such that $a = a_n$, $r = r_n$, $s = s_n$. Then $\varrho(a_n, x) < r_n$ and, for $i \in M$, $\varrho(a_n, x_i) > s_n$, so that $f_n(x) = 0$ and, for $i \in M$, $f_n(x_i) = 1$. Since the set M is infinite, $f_n(x_i)$ does not converge to $f_n(x)$; this is a contradiction. Thus, $\lim_{i \to \infty} x_i = x$. Hence, the mapping F_{-1} is continuous.

16.6. 16.6.1. *Let P be a separable space. Let ε be a positive number. Let f be a finite function on P. For every $a \in P$ let there be a number $\delta^{(a)} > 0$ and a finite function $\varphi^{(a)}$ of the first class on $\Omega(a, \delta^{(a)})$ such that $|\varphi^{(a)}(x) - f(x)| < \varepsilon$ for every $x \in \Omega(a, \delta^{(a)})$. Then there is a finite function φ of the first class on P such that $|\varphi(x) - f(x)| < \varepsilon$ for every $x \in P$.*

Proof: The sets $\Omega(a, \delta^{(a)})$ are open and we have $\bigcup_{a \in P} \Omega(a, \delta^{(a)}) = P$. Hence, by 16.2.2, there are (with the exception of the trivial case of $P = \emptyset$) sequences $\{a_n\}_1^\infty$ and $\{\delta_n\}_1^\infty$ such that $a_n \in P$, $\delta_n = \delta^{(a_n)}$, $\bigcup_{n=1}^{\infty} \Omega(a_n, \delta_n) = P$. Put $\varphi_n = \varphi^{(a_n)}$, $A_1 = \Omega(a_1, \delta_1)$, $A_{n+1} = \Omega(a_{n+1}, \delta_{n+1}) - \bigcup_{i=1}^{n} \Omega(a_i, \delta_i)$ $(n = 1, 2, 3, \ldots)$. The sets A_n are $\mathbf{F}_\sigma$ (see 13.3.2, 13.3.4 and 13.3.5) and we have $P = \bigcup_{n=1}^{\infty} A_n$ with disjoint summands. Hence there is a finite function φ on P such that $x \in A_n$ implies $\varphi(x) = \varphi_n(x)$. Evidently $|\varphi(x) - f(x)| < \varepsilon$ for every $x \in P$. Thus it suffices to prove that φ is a function of the first class.

Let $c \in \mathbf{E}_1$. We have

$$\underset{x}{\mathrm{E}}[\varphi(x) > c] = \bigcup_{n=1}^{\infty} A_n \cap \underset{x}{\mathrm{E}}[x \in \Omega(a_n, \delta_n),\ \varphi_n(x) > c]\,.$$

Since φ_n is a function of the first class on $\Omega(a_n, \delta_n)$, the set $B_n = \underset{x}{\mathrm{E}}[x \in \Omega(a_n, \delta_n),$ $\varphi_n(x) > c]$ is, by 14.3.1, $\mathbf{F}_\sigma[\Omega(a_n, \delta_n)]$. The set $\Omega(a_n, \delta_n)$ is open in P and hence it is $\mathbf{F}_\sigma(P)$ by 13.3.5. Hence, B_n is $\mathbf{F}_\sigma(P)$ by ex. 13.10. Thus, $A_n \cap B_n$ is $\mathbf{F}_\sigma(P)$ by 13.3.4, so that $\underset{x}{\mathrm{E}}[\varphi(x) > c] = \bigcup_{n=1}^{\infty} (A_n \cap B_n)$ is $\mathbf{F}_\sigma(P)$ by 13.3.3. Similarly we may prove that $\underset{x}{\mathrm{E}}[\varphi(x) < c]$ is $\mathbf{F}_\sigma(P)$. Thus, φ is a function of the first class by 14.3.1.

16.6.2. *Let P be a separable space. Let f be a function on P with the following property: in every non-void closed set $A \subset P$ there is at least one point at which the partial function f_A is continuous. Then f is a function of the first class.*

Proof: I. First, let us assume that the function f is finite. It suffices to prove that for every $\varepsilon > 0$ there is a finite function F_ε of the first class such that $|f(x) - F_\varepsilon(x)| < \varepsilon$. Then f is the uniform limit of the sequence $\{F_{1/n}\}$, so that, by 14.2.1, f is a function of the first class. Let us assume that the function F_ε does not exist for some $\varepsilon > 0$. Let us denote by G the set of all the $a \in P$ for which there is a number $\delta^{(a)} > 0$ and a finite function $\varphi^{(a)}$ of the first class on $\Omega(a, \delta^{(a)})$ such that $|f(x) - \varphi^{(a)}(x)| < \varepsilon$ for every $x \in \Omega(a, \delta^{(a)})$. If $a \in G$ we see easily that $\Omega(a, \delta^{(a)}) \subset G$. Hence $G = \bigcup_{a \in G} \Omega(a, \delta^{(a)})$, so that the set G is open. Since we assume that F_ε does not exist, we have, by 16.6.1, $G \neq P$. Thus, $P - G$ is a non-void closed set, so that, by the assumed property of the function f there is a point $a \in P - G$ in which the partial function $(f)_{P-Q}$ is continuous. Since the function $(f)_{P-Q}$ is finite and continuous at the point a, there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for every $x \in (P - G) \cap \Omega(a, \delta)$.

By 16.1.2, G is a separable space. $(f)_G$ is a finite function on G. By the definition of the set G and by theorem 16.6.1, where we replace P by G, there is a finite function ψ of the first class on G such that $|f(x) - \psi(x)| < \varepsilon$ for every $x \in G$.

Let us define a finite function φ on $\Omega(a, \delta)$ as follows: For $x \in G \cap \Omega(a, \delta)$ put $\varphi(x) = \psi(x)$, for $x \in (P - G) \cap \Omega(a, \delta)$ put $\varphi(x) = f(a)$. Thus, $x \in \Omega(a, \delta)$ implies $|\varphi(x) - f(x)| < \varepsilon$. It suffices to prove that φ is a function of the first class on $\Omega(a, \delta)$, as then it follows from the definition of the set G that $a \in G$, which is a contradiction.

Let $c \in \mathbf{E}_1$. Since ψ is a function of the first class on G, the partial function $(\psi)_{G \cap \Omega(a,\delta)}$ is of the first class on $G \cap \Omega(a, \delta)$, so that, by 14.3.1, the set

$$\underset{x}{\mathrm{E}}[x \in G \cap \Omega(a, \delta), \psi(x) > c] \tag{1}$$

is $\mathbf{F}_\sigma[G \cap \Omega(a, \delta)]$. The set $G \cap \Omega(a, \delta)$ is open in $\Omega(a, \delta)$ and hence it is $\mathbf{F}_\sigma[\Omega(a, \delta)]$ by 13.3.5, so that the set (1) is also $\mathbf{F}_\sigma[\Omega(a, \delta)]$ by ex. 13.10.

If $c \geqq f(a)$, we have

$$\underset{x}{\mathrm{E}}[x \in \Omega(a, \delta), \varphi(x) > c] = \underset{x}{\mathrm{E}}[x \in G \cap \Omega(a, \delta), \psi(x) > c]$$

so that $\underset{x}{\mathrm{E}}[x \in \Omega(a, \delta), \varphi(x) > c]$ is $\mathbf{F}_\sigma[\Omega(a, \delta)]$. If $c < f(a)$, we have

$$\underset{x}{\mathrm{E}}[x \in \Omega(a, \delta), \varphi(x) > c] =$$
$$= \underset{x}{\mathrm{E}}[x \in G \cap \Omega(a, \delta), \psi(x) > c] \cup [(P - G) \cap \Omega(a, \delta)] \tag{2}$$

and the first summand is $\mathbf{F}_\sigma[\Omega(a, \delta)]$. The set $(P - G) \cap \Omega(a, \delta)$ is closed in $\Omega(a, \delta)$, hence, by 13.3.2, it is $\mathbf{F}_\sigma[\Omega(a, \delta)]$. Thus, by (2) and 13.3.3, the set $\underset{x}{\mathrm{E}}[x \in \Omega(a, \delta), \varphi(x) > c]$ is $\mathbf{F}_\sigma[\Omega(a, \delta)]$.

Similarly we can prove that, for every $c \in \mathbf{E}_1$, the set $\underset{x}{\mathrm{E}}[x \in \Omega(a, \delta), \varphi(x) < c]$ is $\mathbf{F}_\sigma[\Omega(a, \delta)]$. Hence, by 14.3.1, φ is a function of the first class on $\Omega(a, \delta)$.

II. There remains the case of f which is not finite. By ex. 9.18, there is a homeomorphic mapping φ of the set $\mathbf{R}$ onto the interval $\underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$. Put $F(x) = \varphi[f(x)]$. Then F is a finite function on P. If a set $A \subset P$, $A \neq \emptyset$ is closed, there is a point $a \in A$ such that the partial function f_A is continuous at a. Evidently the function F_A is also continuous at a. By I, F is a function of the first class. As $f(x) = \varphi_{-1}[F(x)]$, f is also a function of the first class.

16.6.3. *Let P be a topologically complete separable space. A nesessary and sufficient condition for a function f on P to be of the first class is the following: In every non-void closed set $A \subset P$ there is at least one point such that the partial function f_A is continuous at it.**)

Proof: I. The condition is sufficient by 16.6.2.

II. Let f be a function of the first class on P. Let $A \subset P$ by a non-void closed set. By 13.2 and 15.5.3 A is a topologically complete space. Let C be the set of all $x \in A$ at which the function f_A is continuous. f_A is a function of the first class on P, so that, by 15.8.3, the set C is dense in A. Since $A \neq \emptyset$, we have $C \neq \emptyset$.

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*) This necessary and sufficient condition may be replaced by several others. Let P be a topologically complete separable space. Let f be a function on P. For $A \subset P$ let S_A be the set of all $x \in A$ at which the partial function f_A is continuous; put $D_A = A - S_A$. Then every one of the following conditions [1], [2], [3], [4] is a necessary and sufficient condition for f to be of the first class:

[1] For every non-void closed $A \subset P$, $S_A \neq \emptyset$.
[2] For every non-void closed $A \subset P$, S_A is dense in A.
[3] For every non-void closed $A \subset P$, D_A is of the first category in A.
[4] For every $A \subset P$, D_A is of the first category in A.

Proof: By 16.6.3 it suffices to prove that conditions [1], [2], [3], [4] are equivalent. i.e. that [1] $\Rightarrow$ [4] $\Rightarrow$ [3] $\Rightarrow$ [2] $\Rightarrow$ [1]. If [1] holds, f is of the first class by 16.6.3 and hence, by 14.5.3 (see the footnote to theorem 14.5.2), [4] holds. Evidently [4] $\Rightarrow$ [3]. If [3] holds, [2] holds by 13.2, 15.5.3, 15.8.2. Finally, obviously [2] $\Rightarrow$ [1].

16.6.4. *Countable metric spaces are topologically complete if and only if they are dispersed.*

Proof: I. Let P be a countable topologically complete space. Let us assume that P is not dispersed. Let Q be its kernel (see 11.1). Then $Q \neq \emptyset$, $Q = \bar{Q}$ and Q is dense-in-itself, i.e. it has no isolated points. By 13.2 and 15.5.3, Q is a topologically complete space. Since Q is countable and has no isolated points, Q is, by ex. 12.13, of the first category in Q, in contradiction with 15.8.2.

II. Let P be a countable dispersed space. Let P_0 be its completion (see 15.4.1). As P is dense in P_0, P_0 is separable by 16.13. Let us define a function f on P_0 as follows: $f(x) = 1$ for $x \in P$, $f(x) = 0$ for $x \in P_0 - P$. Let A be a non-void closed subset of P_0. If $A \cap P = \emptyset$, the partial function f_A is continuous (since it is a constant). As P is dispersed, if $A \cap P \neq \emptyset$, there exists an isolated point a of the set $A \cap P$. There is a $\delta > 0$ such that $x \in A \cap P$, $\varrho(a, x) < 2\delta$ imply $x = a$. If a is an isolated point of the set A, then f_A is obviously continuous at the point a. In the converse case there is a point $b \in A$ such that $a \neq b$, $\varrho(a, b) = \delta_1 < \delta$. If $x \in A$ and $\varrho(b, x) < \varrho(a, b)$, we have $\varrho(a, x) \leqq \varrho(a, b) + \varrho(b, x) < 2\varrho(a, b) < 2\delta$ and, moreover, $x \neq a$, so that, by the choice of the number δ, x is not an element of $A \cap P$. Thus, $A \cap \Omega(b, \delta_1) \subset P_0 - P$, so that $x \in A \cap \Omega(b, \delta_1)$ implies $f(x) = 0$, and hence f_A is continuous at the point b. Thus, in all cases, there is a point $a \in A$ at which f_A is continuous. Since P_0 is separable, f is, by 16.6.2, of the first class, so that, by 14.3, the set $P = \underset{x}{\mathrm{E}}[f(x) \geqq 1]$ is $\mathbf{G}_\delta(P_0)$. As P_0 is complete, P is topologically complete by 15.5.2.

16.7. *Let P be a separable space. Let $A_n \subset P$ $(n = 1, 2, 3, \ldots)$. Then there is a subsequence $\{C_n\}$ of $\{A_n\}$ such that* $\operatorname{Lim} C_n$ *(see* 8.8*) exists.*

Proof: As P is separable, there is a sequence $\{B_n\}_{n=1}^{\infty}$ such that its terms form an open basis of the space P. Put $A_n^{(0)} = A_n$. If, for some i $(= 1, 2, 3, \ldots)$, the sequence $\{A_n^{(i-1)}\}_1^{\infty}$ is chosen, we choose, *if it is possible*, some subsequence $\{A_n^{(i)}\}_1^{\infty}$ for which $B_i \cap \overline{\operatorname{Lim}}_{n \to \infty} A_n^{(i)} = \emptyset$; if it is not possible, put $A_n^{(i)} = A_n^{(i-1)}$ for every n. Put $C_n = A_n^{(n)}$, so that the sequence $\{C_n\}$ is a subsequence of $\{A_n\}$. We have to prove that $\operatorname{Lim} C_n$ exists. Let us assume the contrary. Hence, $\underline{\operatorname{Lim}}\, C_n \neq \overline{\operatorname{Lim}}\, C_n$, so that there exists a point

$$x \in \overline{\operatorname{Lim}}\, C_n - \underline{\operatorname{Lim}}\, C_n .$$

By ex. 8.16, $\varrho(x, C_n)$ does not converge to zero. Thus, there is a number $\delta > 0$ and indices $j_1 < j_2 < j_3 < \ldots$ such that $\varrho(x, C_{j_n}) > \delta$ for every n. If $\varrho(x, y) < \delta$, by ex. 6.6 we have $\varrho(y, C_{j_n}) > \delta - \varrho(x, y) > 0$ for every n, so that, by ex. 8.16, y is not an element of $\overline{\operatorname{Lim}}\, C_{j_n}$. Thus, $\Omega(x, \delta) \cap \overline{\operatorname{Lim}}\, C_{j_n} = \emptyset$. Since $\Omega(x, \delta)$ is a neighbourhood of x, there is, by definition of the sequence $\{B_n\}$, an index i such that

$x \in B_i \subset \Omega(x, \delta)$, hence, $B_i \cap \overline{\text{Lim}}\, C_{j_n} = \emptyset$. For $n \geqq i - 1$ we have $j_n \geqq i - 1$, so that $C_{j_n} = A_{j_n}^{(j_n)}$ is a term of the sequence $\{A_n^{(i-1)}\}_1^\infty$. Hence, there is a subsequence $\{C_{j_n}\}_{n=1}^\infty$ of $\{A_n^{(i-1)}\}$ such that the set B_i contains no point of the upper limit of the subsequence. Thus, $B_i \cap \overline{\text{Lim}}\, A_n^{(i)} = \emptyset$. Since $\{C_n\}_{n=1}^\infty$ is a subsequence of $\{A_n^{(i)}\}$, we have, by ex. 8.20, $\overline{\text{Lim}}\, C_n \subset \overline{\text{Lim}}\, A_n^{(i)}$ and hence $B_i \cap \overline{\text{Lim}}\, C_n = \emptyset$. This is a contradiction, as $x \in B_i \cap \overline{\text{Lim}}\, C_n$.

Exercises

16.1. Let A be a dense subset of a metric space P. Let A be a separable space. Then P is separable.

16.2. Let A_n $(n = 1, 2, 3, \ldots)$ be separable spaces embedded into a metric space P. Let $\bigcup_{n=1}^{\infty} A_n = P$. Then P is separable.

16.3. Let A be a separable space embedded into a metric space P. Then the closure $\bar{A}$ and the derived set A' of A are separable spaces.

16.4. Let P and Q be separable spaces. Then $P \times Q$ is a separable space.

16.5. The spaces from exercises 7.2 and 7.4 are separable.

16.6. The space from exercise 6.5 is not separable.

16.7. A system $\mathfrak{B}$ of open subsets of a metric space P is an open basis of the space P if and only if for every $\varepsilon > 0$

$$\bigcup_{X \in \mathfrak{B}_\varepsilon} X = P \quad \text{where} \quad \mathfrak{B}_\varepsilon = \mathop{\mathrm{E}}_X [X \in \mathfrak{B}, d(X) < \varepsilon]\,.$$

16.8.* Let $\mathfrak{B}_1$ be an open basis of a metric space P. Let $\mathfrak{B}_2$ be an open basis of a metric space Q. Let $\mathfrak{B}_{12}$ be the system of all the sets of form $G_1 \times G_2$ where $G_1 \in \mathfrak{B}_1$, $G_2 \in \mathfrak{B}_2$. Then $\mathfrak{B}_{12}$ is an open basis of the space $P \times Q$.

§ 17. Compact spaces

17.1. A *totally bounded* space is a metric space P such that every sequence of points of P has a Cauchy subsequence. This is obviously a *metric property;* however, it is not a topological property (see 17.2.5). Since a point set embedded into a metric space is a metric space, we need not define the notion of totally bounded point set. Evidently:

17.1.1. *Point sets embedded into a totally bounded space are totally bounded.*

17.1.2. *Each totally bounded space P is bounded.*

Proof: If $d(P) = \infty$, there is a sequence $\{x_n\}$ such that $x_n \in P$, $\varrho(x_i, x_n) > n$ for $i < n$. $\{x_n\}$ has no bounded subsequence, while every Cauchy sequence is bounded (ex. 15.16).

17.1.3. *Let P be a metric space. Let there be an infinite set $A \subset P$ and a number $\delta > 0$ such that*

$$x \in A,\ y \in A,\ x \neq y \quad \text{imply} \quad \varrho(x, y) \geqq \delta\,.$$

Then P is not totally bounded.

Proof: There is a one-to-one sequence $\{x_n\}$, $x_n \in A$. $\{x_n\}$ has no Cauchy subsequence.

17.1.4. *A metric space P is totally bounded if and only if for every $\delta > 0$ there is a finite set $A(\delta) \subset P$ such that $\varrho[x, A(\delta)] < \delta$ for every $x \in P$.*

Proof: I. Let the sets $A(\delta)$ exist. Let $x_n \in P$ $(n = 1, 2, 3, \ldots)$. Put $x_n^{(0)} = x_n$ and construct recursively sequences $[x_n^{(i)}]_{n=1}^{\infty}$ $(i = 0, 1, 2, \ldots)$ as follows: Since $A(1/i)$ is finite and is less than $1/i$ in distance from every $x_n^{(i-1)}$, there is a point $y_i \in A(1/i)$ and a subsequence $\{x_n^{(i)}\}_{n=1}^{\infty}$ of $\{x_n^{(i-1)}\}_{n=1}^{\infty}$ such that $\varrho(y_i, x_n^{(i)}) < 1/i$ for every n. Put $z_n = x_n^{(n)}$. Then $\{z_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_1^{\infty}$. It suffices to prove that $\{z_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Choose an index i such that $1/i < \varepsilon/2$. Then $\{z_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n^{(i)}\}_{n=1}^{\infty}$. Hence

$$m > i, n > i \Rightarrow \varrho(y_i, z_m) < 1/i, \quad \varrho(y_i, z_n) < 1/i \Rightarrow \varrho(z_m, z_n) < \varepsilon\,.$$

II. Let P be totally bounded. Let $\delta > 0$. Choose an arbitrary $x_1 \in P$. If points x_i $(1 \leqq i \leqq n)$ are chosen, choose a point $x_{n+1} \in P$, *if it is possible*, such that $\varrho(x_i, x_{i+1}) \geqq \delta$. By 17.1.3 there is an index n such that $x_1, x_2, \ldots, x_n$ exist, while there is no x_{n+1}. The points $x_1, \ldots, x_n$ form a finite set $A(\delta) \subset P$ such that $\varrho[x, A(\delta)] < \delta$ for every $x \in P$.

17.1.5. *A point set Q embedded into the euclidean $\mathbf{E}_m$ is totally bounded if and only if it is bounded.*

Proof: I. Totally bounded Q is bounded by 17.1.2.

II. Let Q be bounded. There exists a c $(= 1, 2, 3, \ldots)$ such that $Q \subset R$ where

$$R = \mathop{\mathrm{E}}_{(x_1, \ldots, x_n)} [|x_1| \leqq c, \ldots, |x_m| \leqq c]\,.$$

If $\delta > 0$ is given, choose an index k such that $\sqrt{m/k} < \delta$ and denote by $A(\delta)$ the set of all $(x_1, \ldots, x_m)$ with $kx_i = \gamma_i$ $(1 \leqq i \leqq m)$, where γ_i are integers and $|\gamma_i| \leqq$ $\leqq ck$. Then $A(\delta)$ is a finite set, $A(\delta) \subset R$ and $\varrho[x, A(\delta)] < \delta$ for every $x \in R$. Thus, R is totally bounded by 17.1.4. Thus, Q is totally bounded by 17.1.1.

17.1.6. *Every totally bounded point set Q embedded into the Hilbert space $\mathbf{H}$ is nowhere dense in $\mathbf{H}$.*

Proof: Let Q not be nowhere dense. By 12.2.3 there is an open $G \neq \emptyset$ such that $Q \cap \Gamma \neq \emptyset$ for every open Γ such that $\emptyset \neq \Gamma \subset G$. Choose an $a = \{a_n\}_1^{\infty} \in G$.

There is a $\delta > 0$ such that $\Omega(a, \delta) \subset G$. Put $b_{in} = a_n$ for $i \neq n$, $b_{nn} = a_n + \delta/2$, $b_i = \{b_{in}\}_{n=1}^{\infty}$. Then $\varrho(a, b_i) = \delta/2$ so that $\Omega(b_i, \delta/4) \subset \Omega(a, \delta) \subset G$. Since the set $\Gamma_i = \Omega(b_i, \delta/4)$ is open and $\emptyset \neq \Gamma_i \subset G$, there is a point $c_i \in Q \cap \Gamma_i$. For $i \neq k$ we have

$$\delta \frac{\sqrt{2}}{2} = \varrho(b_i, b_k) \leqq \varrho(b_i, c_i) + \varrho(c_i, c_k) + \varrho(c_k, b_k) < \frac{\delta}{4} + \varrho(c_i, c_k) + \frac{\delta}{4},$$

hence,

$$\varrho(c_i, c_k) > \frac{\sqrt{2} - 1}{\sqrt{2}} \delta > 0 \quad \text{for} \quad i \neq k.$$

Thus, Q is not totally bounded by 17.1.3.

17.2. A *compact space* is a metric space such that every sequence of its points has a convergent subsequence. This is evidently a topological property. As a point set Q embedded into a metric space P is a metric space, we need not define the notion of compact point set.

Many authors use the term compact for every point set embedded into a compact (in our sense) space, and, for compact (in our sense) point sets, use the term *compact in itself.*

17.2.1. *A metric space P is compact if and only if it is complete and totally bounded.*

Proof: I. Every compact space is complete. Let P be a compact space embedded into a metric space Q. Let $x_n \in P$, $x \in Q$, $x_n \to x$. As P is compact, we may find a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\lim y_n \in P$ exists. By 7.1.2, we have $\lim y_n = x$. Thus, $x \in P$. Hence, by 8.3.1, P is a closed subset of Q. Thus, P is complete by 15.5.1.

II. Every compact space is totally bounded by 15.1.1.

III. Let P be a complete totally bounded space. If $x_n \in P$, $\{x_n\}$ has a Cauchy subsequence. Any Cauchy sequence in P is convergent. Thus, P is compact.

17.2.2. *A point set Q embedded into a compact space P is compact if and only if it is closed in P.*

Proof: I. Let Q be compact. Q is closed in P by 15.2.1 and 17.2.1.

II. Let Q be closed in P. By 17.1.1 and 17.2.1 Q is totally bounded. By 15.2.2 and 17.2.1, Q is a complete space. Thus, Q is compact by 17.2.1.

17.2.3. *A point set Q embedded into the euclidean $\mathbf{E}_m$ is compact if and only if it is bounded and closed in $\mathbf{E}_m$.*

Proof: I. Let Q be compact. Q is bounded by 17.1.5 and 17.2.1. Q is closed in $\mathbf{E}_m$ by 15.5.1 and 17.2.1.

II. Let Q be bounded and closed in $\mathbf{E}_m$. Q is totally bounded by 17.1.5. Q is a complete space by 15.1.3 and 15.2.2. Thus, Q is compact by 17.2.1.

17.2.4. *The Urysohn space* $\mathbf{U}$ *is compact.*

Proof: Let $n = 1, 2, 3, \ldots$. Denote by A_n the set of all the sequences $\{x_i\}_{i=1}^{\infty}$ such that: for $i > n$, $x_i = 0$; for $1 \leqq i \leqq n$, $x_i = \gamma_i/in$, where γ_i is an integer with $|\gamma_i| \leqq n$. We have $A_n \subset \mathbf{U}$ and A_n is a finite set.

If $\delta > 0$ is given, choose an n such that $\sum_{i=n+1}^{\infty} 1/i^2 < \delta^2/2$, $\sum_{i=1}^{\infty} 1/i^2 < n^2(\delta^2/2)$. Then we prove easily that $\varrho(x, A_n) < \delta$ for every $x \in \mathbf{U}$. Thus, $\mathbf{U}$ is totally bounded by 17.1.4. It follows easily by 7.3.1 and 8.3.3 that $\mathbf{U}$ is a closed subset of $\mathbf{H}$, so that $\mathbf{U}$ is a complete space by 15.1.4 and 15.2.2. Thus, $\mathbf{U}$ is compact by 17.2.1.

17.2.5. *A metric space P is separable if and only if there is a totally bounded space Q homeomorphic with P.*

Proof: I. Let Q be totally bounded. Since every finite set is countable, Q is separable by 16.1.6 and 17.1.4. Since separability is a topological property, the space P homeomorphic with Q is also separable.

II. Let P be separable. By 16.5 there is a point set $Q \subset \mathbf{U}$ homeomorphic with P. Q is totally bounded by 17.1.1, 17.2.1 and 17.2.4.

17.2.6. *Every compact space is separable.*

This is an important corollary of theorem 17.2.5.

17.3. 17.3.1. Let P be a metric space. Let $A \subset P$, $B \subset P$, $A \neq \emptyset \neq B$. Let A be compact. Then there are points $y \in A$, $z \in A$ such that

$$\varrho(y, B) = \min_{x \in A} \varrho(x, B) = \varrho(A, B),$$

$$\varrho(z, B) = \max_{x \in A} \varrho(x, B).$$

If $d(B) < \infty$, there are points $u \in A$, $v \in A$ such that

$$d(u, B) = \min_{x \in A} d(x, B),$$

$$d(v, B) = \max_{x \in A} d(x, B) = d(A, B).$$

Proof: There exist sequences $\{y_n\}$ and $\{z_n\}$ such that

$$y_n \in A,\ z_n \in A,\ \varrho(y_n, B) \to \inf_{x \in A} \varrho(x, B),\ \varrho(z_n, B) \to \sup_{x \in A} \varrho(x, B).$$

As A is compact, there are subsequences $\{y'_n\}$ of $\{y_n\}$, $\{z'_n\}$ of $\{z_n\}$ and points $y \in A$, $z \in A$ such that $y'_n \to y$, $z'_n \to z$, so that, by ex. 9.10, $\varrho(y'_n, B) \to \varrho(y, B)$, $\varrho(z'_n, B) \to \to \varrho(z, B)$. By 7.1.2 we have $\lim \varrho(y'_n, B) = \lim \varrho(y_n, B)$, $\lim \varrho(z'_n, B) = \lim \varrho(z_n, B)$. Hence, $\varrho(y, B) = \inf_{x \in A} \varrho(x, B) = \min_{x \in A} \varrho(x, B)$, $\varrho(z, B) = \sup_{x \in A} \varrho(x, B) = \max_{x \in A} \varrho(x, B)$. The existence of the points u and v can be proved similarly, using $d(x, B)$ instead of $\varrho(x, B)$ and ex. 9.11 instead of ex. 9.10.

17.3.2. *Let P be a metric space. Let $A \subset P$, $B \subset P$, $A \neq \emptyset \neq B$. Let A and B be compact. Then there are points $y_1 \in A$, $y_2 \in B$, $z_1 \in A$, $z_2 \in B$ such that*

$$\varrho(y_1, y_2) = \min_{\substack{x_1 \in A \\ x_2 \in B}} \varrho(x_1, x_2) = \varrho(A, B),$$

$$\varrho(z_1, z_2) = \max_{\substack{x_1 \in A \\ x_2 \in B}} \varrho(x_1, x_2) = d(A, B).$$

Proof: By 17.3.1 (see also 17.1.2) there are points $y_1 \in A$, $z_1 \in A$ such that

$$\varrho(y_1, B) = \varrho(A, B), \quad d(z_1, B) = d(A, B).$$

By 17.3.1 there are points $y_2 \in B$, $z_2 \in B$ such that $\varrho(y_1, y_2) = \varrho(y_1, B)$, $\varrho(z_1, z_2) = = d[z_1, (z_2)] = d(z_1, B)$.

17.3.3. *Let P be a compact space. There exist points $y \in P$, $z \in P$ such that*

$$\varrho(y, z) = \max_{\substack{x_1 \in P \\ x_2 \in P}} \varrho(x_1, x_2) = d(P).$$

This is a particular case of theorem 17.3.2, as $d(P) = d(P, P)$.

17.3.4. *Let P be a metric space. Let $A \subset P$, $B \subset P$, $A \neq \emptyset \neq B$, $A \cap B = \emptyset$. Let A be compact and let B be closed in P. Then $\varrho(A, B) > 0$.*

Proof: Let, on the contrary, $\varrho(A, B) = 0$. By 17.3.1 there is a point $y \in A$ such that $\varrho(y, B) = 0$ and hence $y \in \bar{B}$. This is a contradiction, since $y \in A$, $B = \bar{B}$, $A \cap B = \emptyset$.

17.4. 17.4.1. *Let $A \subset \mathbf{E}_1$ be a non-void bounded and closed set. Then there exist numbers* $\min A$ *and* $\max A$.

Proof: Choose a number $c \in \mathbf{E}_1$ such that $A \subset \underset{x}{\mathrm{E}}[x > c]$. By 17.2.3 and 17.3.1 there exists a number $y \in A$ such that $\varrho(c, y) = \min_{x \in A} \varrho(c, x)$. We have $\varrho(c, x) = = x - c$, $\varrho(c, y) = y - c$. Hence, $y - c = \min_{x \in A} (x - c)$, and hence $y = \min_{x \in A} x$. Similarly for the maximum.

17.4.2. *Let f be a continuous mapping of a compact space P onto a metric space Q. Then Q is compact.*

Proof: Let $y_n \in Q$ ($n = 1, 2, 3, \ldots$). There exist points $x_n \in P$ such that $f(x_n) = y_n$. Since P is compact, there are indices $i_1 < i_2 < i_3 < \ldots$ such that $\lim_{n \to \infty} x_{i_n} = x$ exists. Since f is continuous, we have $\lim_{n \to \infty} y_{i_n} = f(x) \in Q$. Hence, $\{y_n\}$ has a convergent subsequence $\{y_{i_n}\}$.

17.4.3. *Let P be a compact space. Let f be a finite continuous function on P. The set $f(P)$ is bounded and closed. There exist numbers* $\min f(A)$ *and* $\max f(A)$.

Proof: $f(P)$ is compact by 17.4.2. Hence, the statement follows from 17.2.3 and 17.4.1.

17.4.4. *Let f be a continuous mapping of a compact space P into a metric space Q. Then f is uniformly continuous.*

Proof: Let $x_n \in P$, $y_n \in P$, $\varrho(x_n, y_n) \to 0$. We have to prove that $\varrho[f(x_n), f(y_n)] \to 0$. Let us assume the contrary. Then there is a number $\delta > 0$ and indices $i_1 < i_2 < < i_3 < \ldots$ such that $\varrho[f(x_{i_n}), f(y_{i_n})] > \delta$ for every n. Since P is compact, there is a subsequence $\{j_n\}$ of the sequence $\{i_n\}$ such that $\lim x_{j_n} = z \in P$ exists. Since $\varrho(x_n, y_n) \to 0$, we also have $\lim y_{j_n} = z$. Since the mapping f is continuous, we have $\lim f(x_{j_n}) = f(z)$, $\lim f(y_{j_n}) = f(z)$, hence (see ex. 9.12) $\lim \varrho[f(x_{j_n}), f(y_{j_n})] = 0$, which is a contradiction.

17.4.5. *Let P be a compact space. Let f be a finite continuous function on P. Then f is uniformly continuous.*

This is a particular case of theorem 17.4.4.

17.4.6. *Let f be a one-to-one continuous mapping of a compact space P onto a metric space Q. Then the inverse mapping f_{-1} is continuous, i.e. f is a homeomorphic mapping.*

Proof: If A is a closed set in P, it is compact by 17.2.2. Hence, the set $f(A)$ is compact by 17.4.2. Thus, $f(A)$ is closed in Q by 15.5.1 and 17.2.1. Thus, for every A closed in P, $f(A)$ is closed in Q so that f_{-1} is continuous by 9.2.

17.5. 17.5.1. *Let P be a compact space. Let, for $n = 1, 2, 3, \ldots$, $A_n \subset P$, $A_n \neq \emptyset$, $A_n \supset \bar{A}_{n+1}$. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.*

Proof: For $n = 1, 2, 3, \ldots$ there is, by 17.1.4, a finite set $K_n \subset P$ such that $\varrho(x, K_n) < 1/n$ for every $x \in P$. Thus, $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ by 15.7.2 and 17.2.1.

17.5.2. *The statement of theorem* 15.7.2 *may be supplemented by the proposition that* $\bigcap_{n=1}^{\infty} A_n$ *is compact.**)

Proof: Let $a_n \in \bigcap_{i=1}^{\infty} A_i$ $(n = 1, 2, 3, \ldots)$. Then $a_n \in A_n$, so that, by the proof of theorem 15.7.2, we may find a convergent subsequence $\{b_n\}$ of $\{a_n\}$. If $n \geqq i + 1$ we have $b_n \in A_{i+1}$, hence $\lim b_n \in \bar{A}_{i+1} \subset A_i$, hence $\lim b_n \in \bigcap_{i=1}^{\infty} A_i$. Hence, for every sequence $\{a_n\}$ in the space $\bigcap_{i=1}^{\infty} A_i$ there is a subsequence $\{b_n\}$ which has a limit in $\bigcap_{i=1}^{\infty} A_i$.

17.5.3. *Let a metric space P not be compact. Then there exist closed sets* $A_n \subset P$ $(n = 1, 2, 3, \ldots)$ *such that* $A_n \neq \emptyset$, $A_n \supset A_{n+1}$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Proof: There is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of P which has no convergent subsequence. By 8.3.3 we conclude easily that the sets $A_n = \bigcup_{i=n}^{\infty} (x_i)$ are closed. Evidently $A_n \neq \emptyset$, $A_n \supset A_{n+1}$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

17.5.4. *A necessary and sufficient condition for a metric space P to be compact is the following: For every system* $\mathfrak{A}$ *of open sets such that* $\bigcup_{X \in \mathfrak{A}} X = P$ *there is a finite system* $\mathfrak{A}_0 \subset \mathfrak{A}$ *such that* $\bigcup_{X \in \mathfrak{A}_0} X = P$.

Proof: I. Let P be compact. By 16.2.2 and 17.2.6 there is a sequence $\{X_n\}$ such that $X_n \in \mathfrak{A}$, $\bigcup_{n=1}^{\infty} X_n = P$. Put $A_n = P - \bigcup_{i=1}^{n} X_i$. We have $A_n = \bar{A}_n$, $A_n \supset A_{n+1}$. We have $\bigcap_{n=1}^{\infty} A_n = P - \bigcup_{n=1}^{\infty} X_n = \emptyset$, so that by 17.5.1 there exists an index n such that $A_n = \emptyset$, hence, $\bigcup_{i=1}^{n} X_i = P$.

II. Let P not be compact. By 17.5.3 there are closed sets $A_n \subset P$ such that $A_n \neq \emptyset$, $A_n \supset A_{n+1}$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Put $G_n = P - A_n$. Then the sets G_n are open and we have $\bigcup_{n=1}^{\infty} G_n = P - \bigcap_{n=1}^{\infty} A_n = P$, while, for $m = 1, 2, 3, \ldots$, $\bigcup_{n=1}^{m} G_n = $ $= P - \bigcap_{n=1}^{m} A_n = P - A_m \neq P$.

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*) We do not assume that the space is compact. Similarly as in 15.7.2, we assume the completeness of P only.

17.6. Let P be an arbitrary metric space. Let us denote by P^* the system of all compact subsets of P with the exception of the set $\emptyset$. If $A \in P^*$, $B \in P^*$, there exist (see 17.3.1) real numbers

$$u(A, B) = \max_{x \in A} \varrho(x, B),$$

$$u(B, A) = \max_{y \in B} \varrho(y, A).$$

Put

$$\varrho^*(A, B) = \max [u(A, B), u(B, A)].$$

If $A = B$, evidently $\varrho^*(A, B) = 0$. If $A \neq B$, we have either $A - B \neq \emptyset$, so that $u(A, B) > 0$ (as $B = \bar{B}$ by 15.2.1 and 17.2.1) or $B - A = \emptyset$ so that $u(B, A) > 0$. Thus, for $A \neq B$ we always have $\varrho^*(A, B) > 0$. Obviously always $\varrho^*(A, B) = \varrho^*(B, A)$. If also $C \in P^*$, then by ex. 6.6 we have for $x \in A$ and $y \in B$:

$$\varrho(x, C) \leqq \varrho(x, y) + \varrho(y, C) \leqq \varrho(x, y) + u(B, C),$$

hence

$$\varrho(x, C) \leqq \min_{y \in B} \varrho(x, y) + u(B, C) = \varrho(x, B) + u(B, C) \leqq$$
$$\leqq u(A, B) + u(B, C) \leqq \varrho^*(A, B) + \varrho^*(B, C),$$

hence

$$u(A, C) \leqq \varrho^*(A, B) + \varrho^*(B, C)$$

and similarly

$$u(C, A) \leqq \varrho^*(A, B) + \varrho^*(B, C);$$

thus

$$\varrho^*(A, C) \leqq \varrho^*(A, B) + \varrho^*(B, C).$$

Thus, ϱ^* is a distance function in P^*. The metric space (P^*, ϱ^*) is called the *Hausdorff hyperspace* of the space P.

17.6.1. *If* $A_n \in P^*$ $(n = 1, 2, 3, \ldots)$, $A \in P^*$, *then*

$$u(A, A_n) \to 0 \quad \textit{if and only if} \quad A \subset \underline{\mathrm{Lim}}\, A_n.$$

Proof: I. Let $u(A, A_n) \to 0$. Let $a \in A$. We shall prove that $a \in \overline{\mathrm{Lim}}\, A_n$. We have $\varrho(a, A_n) \leqq \max_{x \in A} \varrho(x, A_n) = u(A, A_n)$, hence $\varrho(a, A_n) \to 0$. By 17.3.1 there is a point $a_n \in A_n$ such that $\varrho(a, a_n) = \varrho(a, A_n)$. We have $\varrho(a, a_n) \to 0$, hence $a_n \to a$. As $a_n \in A_n$, we have $a \in \underline{\mathrm{Lim}}\, A_n$.

II. Let $A \subset \underline{\mathrm{Lim}}\, A_n$. We shall prove that $u(A, A_n) \to 0$. Let us assume the contrary. Then there are a number $\delta > 0$ and indices $i_1 < i_2 < i_3 < \ldots$ such that $u(A, A_{i_n}) > \delta$ for every n. There are points $b_n \in A$ with $\varrho(b_n, A_n) = \max_{x \in A} \varrho(x, A_n) = u(A, A_n)$. Since A is compact, there is a subsequence $\{j_n\}$ of $\{i_n\}$ such that $\lim b_{j_n} = a \in A$ exists. Since $A \subset \underline{\mathrm{Lim}}\, A_n$, there are points $a_n \in A_n$ such that $a_n \to a$. As $b_{j_n} \to a$,

$a_n \to a$, we have $\varrho(b_{j_n}, a_{j_n}) \to 0$. Hence, there is an index m such that $\varrho(b_{j_m}, a_{j_m}) < \delta$. We have $u(A, A_{j_m}) = \varrho(b_{j_m}, A_{j_m}) = \min_{x \in A_{j_m}} \varrho(b_{j_m}, x) \leqq \varrho(b_{j_m}, a_{j_m}) < \delta$. This is a contradiction as $u(A, A_{j_m}) > \delta$, since j_m is a member of the sequence $\{i_n\}$.

17.6.2. *Let $A_n \in P^*$ $(n = 1, 2, 3, \ldots)$, $A \in P^*$. Then always*

$$u(A_n, A) \to 0 \quad \textit{implies} \quad \overline{\mathrm{Lim}}\, A_n \subset A$$

and if P is compact, also

$$\overline{\mathrm{Lim}}\, A_n \subset A \quad \textit{implies} \quad u(A_n, A) \to 0\,.$$

Proof: I. Let $u(A_n, A) \to 0$. Let $a \in \overline{\mathrm{Lim}}\, A_n$; we shall prove that $a \in A$. Since $a \in \overline{\mathrm{Lim}}\, A_n$, there exist indices $i_1 < i_2 < i_3 < \ldots$ and points $a_n \in A_{i_n}$ such that $a_n \to a$. We have $\varrho(a_n, A) \leqq \max_{x \in A_{i_n}} \varrho(x, A) = u(A_{i_n}, A)$, hence $\varrho(a_n, A) \to 0$, so that, by ex. 9.10, $\varrho(a, A) = 0$, i.e. $a \in \bar{A}$. By 15.2.1 and 17.2.1, $\bar{A} = A$.

II. Let P be compact and let $\overline{\mathrm{Lim}}\, A_n \subset A$. We shall prove that $u(A_n, A) \to 0$. Let us assume the contrary. Then there are a number $\delta > 0$ and indices $i_1 < i_2 < i_3 < \ldots$ such that $u(A_{i_n}, A) > \delta$ for every n. There exist points $a_n \in A_n$ such that $\varrho(a_n, A) = \max_{x \in A_n} \varrho(x, A) = u(A_n, A)$. Since P is compact, there is a subsequence $\{j_n\}$ of $\{i_n\}$ such that $\lim a_{j_n} = a$ exists. Since $a_{j_n} \in A_{j_n}$, $\overline{\mathrm{Lim}}\, A_n \subset A$, we have $a \in A$ and hence $\varrho(a, A) = 0$, so that, by ex. 9.10, $\varrho(a_{j_n}, A) \to 0$ i.e. $u(A_{j_n}, A) \to 0$. This is a contradiction, since $\{j_n\}$ is a subsequence of $\{i_n\}$ and $u(A_{i_n}, A) > \delta > 0$ for every n.

17.6.3. *Let $A_n \in P^*$ $(n = 1, 2, 3, \ldots)$ $A \in P^*$. If the space P is compact, then $A_n \to A$ (with respect to the distance function ϱ^*) if and only if* $\mathrm{Lim}\, A_n = A$ *(in the sense of section* 8.8*). If P is an arbitrary metric space, then $A_n \to A$ if and only if:* [1] $\mathrm{Lim}\, A_n = A$, [2] *the set $A \cup \bigcup_{n=1}^{\infty} A_n$ is compact.*

Proof: I. Let $A_n \to A$. Then $\varrho^*(A_n, A) \to 0$, hence on the one hand $u(A, A_n) \to 0$, so that, by 17.6.1, $A \subset \underline{\mathrm{Lim}}\, A_n$, on the other hand $u(A_n, A) \to 0$, so that, by 17.6.2, $\overline{\mathrm{Lim}}\, A_n \subset A$. Since always $\underline{\mathrm{Lim}}\, A_n \subset \overline{\mathrm{Lim}}\, A_n$, $\mathrm{Lim}\, A_n = A$.

II. Let $A_n \to A$. Let $x_n \in A \cup \bigcup_{i=1}^{\infty} A_i$. If $x_n \in A$ for infinitely many indices n, or if there exists an index i such that $x_n \in A_i$ for infinitely many indices n, then there is a subsequence of $\{x_n\}$, which has a limit in $A \cup \bigcup_{i=1}^{\infty} A_i$, as the sets A and A_i are compact. If none of the cases occur, there are indices $i_1 < i_2 < i_3 < \ldots$ such that there exists a subsequence $\{y_n\}$ of $\{x_n\}$ with $y_n \in A_{i_n}$ for every n. We have

$\varrho(y_n, A) \leqq \max_{x \in A_{i_n}} \varrho(x, A) = u(A_{i_n}, A) \leqq \varrho^*(A_{i_n}, A)$. Since $A_n \to A$, we have $\varrho(y_n, A) \to \to 0$. By 17.3.1 there exist points $z_n \in A$ such that $\varrho(y_n, z_n) = \varrho(y_n, A)$, hence $\varrho(y_n, z_n) \to 0$. Since A is compact, there are indices $n_1 < n_2 < n_3 < \ldots$ and a point $a \in A$ such that $\lim_{k\to\infty} z_{n_k} = a$. Since $\varrho(y_n, z_n) \to 0$, we have $\lim_{k\to\infty} y_{n_k} = a$. Hence there is a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$, which has a limit in $A \subset A \cup \bigcup_{i=1}^{\infty} A_i$. Thus, the set $A \cup \bigcup_{i=1}^{\infty} A_i$ is compact.

III. Let $\operatorname{Lim} A_n = A$ and let either P or $A \cup \bigcup_{i=1}^{\infty} A_i$ be compact. By 17.6.1, $u(A, A_n) \to 0$. In the proof of theorem 17.6.2 we used the assumption of compact P only in the assertion that a sequence $\{a_n\}$ with $a_n \in A_n$ has a convergent subsequence; this, however, follows from the assumption that $A \cup \bigcup_{n=1}^{\infty} A_n$ is compact. Hence, $u(A_n, A) \to 0$. Since $u(A, A_n) \to 0$, $u(A_n, A) \to 0$, we have $\varrho^*(A_n, A) \to 0$, i.e. $A_n \to A$.

17.6.4. *Let metric spaces P and Q be homeomorphic. Then their Hausdorff hyperspaces P^* and Q^* are homeomorphic. More precisely: Let f be a homeomorphic mapping of the space P onto the space Q. For $X \in P^*$ put $\varphi(X) = f(X)$; then φ is a homeomorphic mapping of P^* onto Q^*.*

This is a corollary of theorem 17.6.3 (see also ex. 9.21 and theorem 17.4.2).

17.6.5. *If P is a complete space, then P^* is also a complete space.*

Proof: Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence with respect to the distance function ϱ^*. Put $B_n = \overline{\bigcup_{i=n}^{\infty} A_i}$. Then $B_n \neq \emptyset$, $B_n \supset B_{n+1}$, $B_n = \bar{B}_n$. Choose an index m and a number $\delta > 0$. Since the sets A_i are compact, there is, by 17.1.4, for every i a finite set K_i such that $x \in A_i$ implies $\varrho(x, K_i) < \frac{1}{2}\delta$. Since $\{A_n\}$ is a Cauchy sequence, there is an index $p > m$ such that for $n > p$ we have $u(A_n, A_p) < \frac{1}{2}\delta$. If $x \in A_n$, $n > p$, we have $\varrho(x, A_p) \leqq u(A_n, A_p) < \frac{1}{2}\delta$, hence there is a point $y \in A_p$ with $\varrho(x, y) < \frac{1}{2}\delta$. We obtain easily that $\varrho(x, \bigcup_{i=m}^{p} K_i) \leqq \delta$ for every $x \in B_m$. Hence, by 15.7.2 the set $A = \bigcap_{n=1}^{\infty} B_n$ is non-void. By 17.5.2 A is compact. Hence, $A \in P^*$. Choose an $\varepsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence, there is an index q such that for $i > q$, $j > q$ we have $u(A_i, A_j) < \frac{1}{2}\varepsilon$.

Choose an $n > q$. If $x \in A$, we have $x \in B_n = \overline{\bigcup_{i=n}^{\infty} A_i}$ so that there is a point $x' \in \bigcup_{i=n}^{\infty} A_i$ such that $\varrho(x, x') < \frac{1}{2}\varepsilon$. There exists an index $i \geqq n > q$ with $x' \in A_i$. We have $\varrho(x', A_n) \leq u(A_i, A_n) < \frac{1}{2}\varepsilon$, so that, by ex. 6.6, $\varrho(x, A_n) < \varrho(x, x') +$

$+ \varrho(x', A_n) < \varepsilon$. Thus, for $n > q$ and $x \in A$ we have $\varrho(x, A_n) < \varepsilon$, so that, for $n > q$, $u(A, A_n) \leqq \varepsilon$, i.e. $u(A, A_n) \to 0$. Choose again an $n > q$. If $x \in A_n$ then, for every $i \geqq n$, $\varrho(x, A_i) \leqq u(A_n, A_i) < \frac{1}{2}\varepsilon$, so that for every $i \geqq n$ there is a point $y_i \in A_i \subset B_i$ with $\varrho(x, y_i) < \frac{1}{2}\varepsilon$. By the assertion expressed (and then proved) at the beginning of the proof of theorem 15.7.2 we may choose a subsequence of $\{y_i\}_n^\infty$ which has a limit $z \in A$. As $\varrho(x, y_i) < \frac{1}{2}\varepsilon$, we have, by ex. 9.12, $\varrho(x, z) < \varepsilon$, hence $\varrho(x, A) < \varepsilon$. Thus, $n > q$, $x \in A_n$ imply $\varrho(x, A) < \varepsilon$, so that for $n > q$ we have $u(A_n, A) \leqq \varepsilon$, i.e. $u(A_n, A) \to 0$. Since also $u(A, A_n) \to 0$, we have $\varrho^*(A_n, A) \to 0$, i.e. $A_n \to A$, so that the sequence $\{A_n\}$ is convergent (with respect to the distance function ϱ^*).

17.6.6. *If P is a totally bounded space, then P^* is also totally bounded.*

Proof: Choose a number $\delta > 0$. By 17.1.4 there is a finite set $K \subset P$ such that $\varrho(x, K) < \delta$ for every $x \in P$. Denote by $\mathfrak{K}$ the system of all the subsets of K, with the exception of the set $\emptyset$. Evidently $\mathfrak{K}$ is a finite subset of P^*. Choose an $A \in P^*$. Put $B = \underset{x}{\mathrm{E}}[x \in K, \varrho(x, A) < \delta]$. We may prove easily that $B \in \mathfrak{K}$ and that $\varrho^*(A, B) < \delta$. Hence, the space P^* is totally bounded by 17.1.4.

17.6.7. *If P is a separable space, then P^* is also separable.*

This is a corollary of theorems 17.2.5, 17.6.4 and 17.6.6.

17.6.8. *If P is a compact space, then P^* is also compact.*

This is a corollary of theorems 17.2.1, 17.6.5 and 17.6.6.

17.7. Let $K \neq \emptyset$ be a given compact space. Let P be a given metric space. Let us denote by P^K the set of all continuous mappings f of K into P.

If $f \in P^K$, $g \in P^K$, put $\varphi(x) = \varrho[f(x), g(x)]$ for $x \in K$. By ex. 9.12 we deduce easily that φ is a finite continuous function on K. By 17.4.3 there exists a number $\max \varrho[f(x), g(x)]$; denote this number by $\varrho^+(f, g)$. If $f = g$, evidently $\varrho^+(f, g) = 0$; if $f \neq g$, evidently $\varrho^+(f, g) > 0$. Obviously we always have $\varrho^+(f, g) = \varrho^+(g, f)$. If also $h \in P^K$, then, for every $x \in K$, $\varrho[f(x), h(x)] \leqq \varrho[f(x), g(x)] + \varrho[g(x), h(x)] \leqq$ $\leqq \varrho^+(f, g) + \varrho^+(g, h)$, hence $\varrho^+(f, h) \leqq \varrho^+(f, g) + \varrho^+(g, h)$. Hence, ϱ^+ is a distance function in P^K. Whenever we speak about P^K, we shall mean the metric space (P^K, ϱ^+). The following three theorems are evident:

17.7.1. *If K consists of a single point, then the spaces P and P^K are isometric.*

17.7.2. *If compact spaces $K \neq \emptyset$ and L are homeomorphic, then the spaces P^K and P^L are isometric.*

17.7.3. *If spaces P and Q are isometric, then the spaces P^K and Q^K are isometric.*

17.7.4. *If spaces P and Q are homeomorphic, then the spaces P^K and Q^K are homeomorphic.*

Proof: Let φ be a homeomorphic mapping of P onto Q. Let us associate, with every $f \in P^K$, a mapping $\Phi(f)$ of K into Q as follows: the image of a point $x \in K$ under the mapping $\Phi(f)$ is the point $\varphi[f(x)]$. We see easily that Φ is a one-to-one mapping of P^K onto Q^K. We have to prove that both the mappings Φ and Φ_{-1} are continuous. Thus, let $f_n \in P^K$, $f \in P^K$; we have to prove that

$$f_n \to f \text{ if and only if } \Phi(f_n) \to \Phi(f).$$

Denote by H_1 and H_2, respectively, the Hausdorff hyperspaces of $K \times P$ and $K \times Q$. For $x \in K$, $y \in P$ put $\psi(x, y) = [x, \varphi(y)]$. It is easy to see that ψ is a homeomorphic mapping of $K \times P$ onto $K \times Q$. For $Z \in H_1$ put $\Psi(Z) = \psi(Z)$. By 17.6.4, Ψ is a homeomorphic mapping of H_1 onto H_2.

Put

$$F_n = \mathop{\mathrm{E}}_{(x,y)} [x \in K,\ y = f_n(x)], \qquad F = \mathop{\mathrm{E}}_{(x,y)} [x \in K,\ y = f(x)],$$

$$G_n = \mathop{\mathrm{E}}_{(x,y)} \{x \in K,\ z = \varphi[f_n(x)]\}, \qquad G = \mathop{\mathrm{E}}_{(x,z)} \{x \in K,\ z = \varphi[f(x)]\}.$$

We can prove easily (see 17.4.2) that $F_n \in H_1$, $F \in H_1$, $G_n \in H_2$, $G \in H_2$, and that $\Psi(F_n) = G_n$, $\Psi(F) = G$. Since Ψ is a homeomorphic mapping, we have $F_n \to F$ if and only if $G_n \to G$. We shall prove that $f_n \to f$ if and only if $F_n \to F$. Similarly we may prove that $\Phi(f_n) \to \Phi(f)$ if and only if $G_n \to G$, hence, we prove in fact that $f_n \to f$ if and only if $\Phi(f_n) \to \Phi(f)$.

First, let $f_n \to f$ in P^K. Choose an $\varepsilon > 0$. There is an index p such that, for $n > p$, $\varrho^+(f_n, f) < \varepsilon$, hence $\varrho[f_n(x), f(x)] < \varepsilon$ for every $x \in K$. If $x \in K$, we have $[x, f_n(x)] \in F_n$ and $[x, f(x)] \in F$, and the distance of the points $[x, f_n(x)]$, $[x, f(x)]$ in the space $K \times P$ is equal to $\varrho[f_n(x), f(x)]$. Hence, for $n > p$: $z \in F_n$ implies $\varrho(z, F) < \varepsilon$, $z \in F$ implies $\varrho(z, F_n) < \varepsilon$, so that for $n > p$ the distance of F_n from F in H_1 is less than ε. Thus, $F_n \to F$ in H_1.

Secondly, let $F_n \to F$ in H_1. Choose an $\varepsilon > 0$. By 9.6.1 and 17.4.4 there is a $\delta > 0$ such that

$$x \in K, \quad y \in K, \quad \varrho(x, y) < \delta \quad \text{imply} \quad \varrho[f(x), f(y)] < \varepsilon/2.$$

We may suppose that $\delta < \varepsilon/4$. There exists an index p such that for $n > p$ the distance of F_n from F in P^K is less than δ. Let $n > p$, $x \in K$. Then $[x, f_n(x)] \in F_n$, so that there is a point $[y, f(y)] \in F$ (hence, $y \in K$) such that

$$\varrho\{[x, f_n(x)], [y, f(y)]\} = \sqrt{\{[\varrho(x, y)]^2 + \varrho[f_n(x), f(y)]^2\}} < \delta < \varepsilon/4, \tag{1}$$

so that $\varrho(x, y) < \delta$, hence $\varrho[f(x), f(y)] < \varepsilon/2$ and hence

$$\varrho\{[x, f(x)], [y, f(y)]\} \leqq \sqrt{\{[\varrho(x, y)]^2 + [\varrho(f(x), f(y)]^2\}} < \\ < \sqrt{[\delta^2 + (\varepsilon/2)^2]} < \sqrt{[(\varepsilon/4)^2 + (\varepsilon/2)^2]} < 3\varepsilon/4. \tag{2}$$

By (1) and (2) we obtain

$$\varrho[f_n(x), f(y)] < \varepsilon/4, \quad \varrho[f(x), f(y)] < 3\varepsilon/4$$

and hence $\varrho[f_n(x), f(x)] < \varepsilon$. Thus, for $n > p$ we have $\varrho^+(f_n, f) < \varepsilon$ so that $f_n \to f$ in P^K.

17.7.5. *If $K \neq \emptyset$ is a compact space and if P is a complete space, then P^K is a complete space.*

Proof: Let $\{f_n\}$ be a Cauchy sequence in P^K. For every $\varepsilon > 0$ there is an index $p(\varepsilon)$ such that for $m > p(\varepsilon)$, $n > p(\varepsilon)$ we have $\max \varrho[f_m(x), f_n(x)] < \varepsilon$. Consequently, for every $x \in K$, $\{f_n(x)\}$ is a Cauchy sequence in P. As the space P is complete, we obtain, for every $x \in K$, a point $f(x) \in P$ such that $f_n(x) \to f(x)$. Thus, f is a mapping of K into P. For every $\varepsilon > 0$

$$x \in K, \quad m > p(\varepsilon), \quad n > p(\varepsilon) \quad \text{imply} \quad \varrho[f_m(x), f_n(x)] < \varepsilon,$$

hence, by ex. 9.12

$$x \in K, \quad m > p(\varepsilon) \quad \text{imply} \quad \varrho[f_m(x), f(x)] \leqq \varepsilon.$$

Choose an index $m > p(\varepsilon/3)$. By 9.6.1 and 17.4.4, there is a $\delta > 0$ such that

$$x \in K, \quad y \in K, \quad \varrho(x, y) < \delta \quad \text{imply} \quad \varrho[f_m(x), f_m(y)] < \varepsilon/3.$$

Let $x \in K$, $y \in K$, $\varrho(x, y) < \delta$. Then

$$\varrho[f(x), f(y)] \leqq \varrho[f(x), f_m(x)] + \varrho[f_m(x), f_m(y)] + \\ + \varrho[f_m(y), f(y)] < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence, the mapping f is continuous, so that $f \in P^K$. Moreover, $n > p(\varepsilon)$ implies $\varrho^+(f_n, f) \leqq \varepsilon$, hence $f_n \to f$, i.e. the sequence $\{f_n\}$ is convergent in P^K.

17.7.6. *If $K \neq \emptyset$ is a compact space and if P is a separable space, then P^K is a separable space.*

Proof: By 16.1.3 there is a countable set A dense in P. Choose a $\delta > 0$. For $n = 1, 2, 3, \ldots$ denote by Φ_n the set of all $f \in P^K$ such that

$$x \in K, \quad y \in K, \quad \varrho(x, y) < 1/n \quad \text{imply} \quad \varrho[f(x), f(y)] < \tfrac{1}{4}\delta.$$

By 9.6.1 and 17.4.4, $\bigcup_{n=1}^{\infty} \Phi_n = P^K$. By 17.1.4, for every n there is a finite sequence $\{c_i\}_{i=1}^m$ (the points c_i and the number m depend on n) of points of K such that for every $x \in K$ there is an index i such that $\varrho(x, c_i) < 1/n$. Let us denote by $\mathfrak{A}_n$ the set of all the sequences $\{a_i\}_{i=1}^m$ with $a_i \in A$. By ex. 3.14, the set $\mathfrak{A}_n$ is countable. Let us associate with every $\{a_i\}_{i=1}^m \in \mathfrak{A}_n$ exactly one mapping $f \in \Phi_n$, where, *if it is possible*, this f is chosen in such a way that $\varrho[f(c_i), a_i] < \frac{1}{4}\delta$ for $1 \leqq i \leqq m$. Let Ψ_n be the set of all the mappings associated with the sequences $\{a_i\}_{i=1}^m \in \mathfrak{A}_n$. The set Ψ_n is countable by 3.4.1, hence, by 3.6, the set $\Psi = \bigcup_{n=1}^{\infty} \Psi_n$ is also countable.

Now, let $f \in P^K$ be arbitrary. There is an index n such that $f \in \Phi_n$. Since A is dense in P, there is a sequence $\{a_i\}_{i=1}^m$ such that $\varrho[f(c_i), a_i] < \frac{1}{4}\delta$ for $1 \leqq i \leqq m$. Let $g \in \Psi_n$ be the mapping which is associated with the sequence $\{a_i\}_{i=1}^m$. For $1 \leqq$ $\leqq i \leqq m$ we have $\varrho[g(c_i), a_i] < \frac{1}{4}\delta$, hence $\varrho[f(c_i), g(c_i)] < \frac{1}{2}\delta$. If x is an arbitrary point of the space K, there is an index i with $\varrho(x, c_i) < 1/n$. Since $f \in \Phi_n$, $g \in \Psi_n \subset \Phi_n$ we have $\varrho[f(x), f(c_i)] < \frac{1}{4}\delta$, $\varrho[g(x), g(c_i)] < \frac{1}{4}\delta$, hence $\varrho[f(x), g(x)] \leqq \varrho[f(x), f(c_i)] +$ $+ \varrho[f(c_i), g(c_i)] + \varrho[g(x), g(c_i)] < \delta$; thus, $\varrho^+(f, g) < \delta$. Hence, for every $f \in P^K$ there is a $g \in \Psi$ with $\varrho^+(f, g) < \delta$. Since Ψ is countable, P^K is separable by 16.1.6.

If P is compact, P^K need not be compact (see ex. 17.17).

17.8. Let K be a compact point set embedded into the space $\mathbf{E}_1$. Let us assume that K contains at least two distinct points. By 17.2.3 and 17.4.1 there exist points

$$a = \min K, \qquad b = \max K.$$

We have $a < b$. Put $J = \underset{t}{\mathrm{E}}[a \leqq t \leqq b]$. Evidently $K \subset J$; we may have $K = J$.

A *contiguous interval* of the set K is any interval $S = \underset{t}{\mathrm{E}}[u < t < v]$ ($u \in \mathbf{E}_1$, $v \in \mathbf{E}_1$, $u < v$) such that [1] $S \cap K = \emptyset$, [2] $u \in K$, $v \in K$.

17.8.1. *$J - K$ is a disjoint union of all the contiguous intervals of the set K.*

Proof: I. Let $S = \underset{t}{\mathrm{E}}[u < t < v]$ be a contiguous interval. Evidently $a \leqq u <$ $> v \leqq b$. hence, $S \subset J$. As $S \cap K = \emptyset$, we have $S \subset J - K$.

II. Let $S_1 = \underset{t}{\mathrm{E}}[u_1 < t < v_1]$, $S_2 = \underset{t}{\mathrm{E}}[u_2 < t < v_2]$ be two contiguous intervals. Let $c \in S_1 \cap S_2$. Since $S_1 \cap K = \emptyset$, $v_1 \in K$, we have $v_1 = \min K \cap \underset{t}{\mathrm{E}}[t > c]$. Since $S_2 \cap K = \emptyset$, $v_2 \in K$, we have $v_2 = \min K \cap \underset{t}{\mathrm{E}}[t > c]$. Hence, $v_1 = v_2$ and similarly we may prove that $u_1 = u_2$. Thus, $S_1 = S_2$. Hence, the system of contiguous intervals is disjoint.

III. Let $c \in J - K$. The sets $K' = K \cap \underset{t}{\mathrm{E}}[t \geqq c]$ and $K'' = K \cap \underset{t}{\mathrm{E}}[t \leqq c]$ are compact (see 17.2.3). We have $b \in K'$, $a \in K''$, hence $K' \neq \emptyset \neq K''$. By 17.4.1 there exist $v = \min K'$, $u = \max K''$. We have $u \leqq c \leqq v$. Since $c \in J - K$, $u \in K$, $v \in K$, we have $u < c < v$, i.e. $c \in S = \underset{t}{\mathrm{E}}[u < t < v]$. Obviously S is a contiguous interval.

17.8.2. *Systems of contiguous intervals are countable.*

This follows by 16.1.5, 16.2.1 and 17.8.1.

17.8.3. *Let* $a \in \mathbf{E}_1$, $b \in \mathbf{E}_1$, $a < b$. *Let* $\mathfrak{M}$ *be a disjoint* (possibly void) *system of intervals of the form* $\mathop{E}_t[u < t < v]$, *where* $a \leqq u < v \leqq b$. *Then there is exactly one compact set* $K \subset \mathbf{E}_1$ *with* $a = \min K$, $b = \max K$, *such that* $\mathfrak{M}$ *is the system of all its contiguous intervals.*

Proof: Put $J = \mathop{E}_t[a \leqq t \leqq b]$, $M = \bigcup_{X \in \mathfrak{M}} X$. Evidently $M \subset J$. If the required set exists, it must be identical with $J - M$ by 17.8.1. Thus, put $K = J - M$. The set M is (see 8.5.3) open in $\mathbf{E}_1$, so that K is closed in $\mathbf{E}_1$ and bounded, and hence compact. Obviously $a = \min K$, $b = \max K$. It remains to be shown that $\mathfrak{M}$ is the system of all contiguous intervals of K.

Let $S = \mathop{E}_t[u < t < v] \in \mathfrak{M}$. We have $S \subset M$, hence $S \cap K = \emptyset$. If there were a $v \in M$, there would be an interval $S_1 \in \mathfrak{M}$, $v \in S_1$. We see easily that $S_1 \cap S \neq \emptyset$, $S \neq S_1$, which is a contradiction. Thus, $v \in K$, and similarly $u \in K$. Hence, every $S \in \mathfrak{M}$ is a contiguous interval. Since, by 17.8.1, M is the disjoint union of all contiguous intervals, we deduce easily that every contiguous interval is in $\mathfrak{M}$.

For a moment, let us denote by M_3 the set of all the sequences $\{j_n\}_{n=1}^{\infty}$ such that their terms are 0, 1 or 2, and by M_2 the set of all $\{i_n\}_{n=1}^{\infty}$ such that their terms are either 0 or 2. It is well-known that: [1] if $\{j_n\} \in M_3$, then $\sum_{n=1}^{\infty} j_n/3^n \in J$, where $J = \mathop{E}_t[0 \leqq t \leqq 1]$; [2] if $t \in J$, $t \neq 0$, $t \neq 1$, and if there is an index m such that $t \,.\, 3^m$ is an integer, then there are exactly two sequences $\{j_m\} \in M_3$ with $\sum_{n=1}^{\infty} j_n/3^n = t$ (if we find the least possible m, then exactly one from the two numbers j_m is equal to 1, and for $n > m$ there is always in one sequence a $j_n = 0$ and in the other always a $j_n = 2$); [3] if $t \in J$ and if no number $t \,.\, 3^m$ is an integer, then there is exactly one sequence $\{j_n\} \in M_3$ with $\sum_{n=1}^{\infty} j_n/3^n = t$ (and there is infinitely many n such that $j_n \neq 0$ and infinitely many n such that $j_n \neq 2$). Denote by D the set of all the numbers $\sum_{n=1}^{\infty} i_n/3^n$ with $\{i_n\} \in M_2$. The set D is called the (*Cantor*) *discontinuum.* Put

$$S = \mathop{E}_t[1/3 < t < 2/3]; \tag{1}$$

if $n = 1, 2, 3, \ldots$ and if every one of the indices $i_1, i_2, \ldots, i_n$ has either the value 0 or the value 2, put

$$S_{i_1 i_2 \ldots i_n} = \mathop{E}_t\left[\sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{1}{3^{n+1}} < t < \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{2}{3^{n+1}}\right]. \tag{2}$$

Denote by $\mathfrak{M}$ the system consisting of the interval (1) and all the intervals (2). We see easily (see 17.8.3) that the set D is compact, $\min D = 0$, $\max D = 1$, and that $\mathfrak{M}$ is the system of all the contiguous intervals of the set D. Put

$$H_0 = \mathop{E}_t[0 \leqq t \leqq \tfrac{1}{3}], \qquad H_2 = \mathop{E}_t[\tfrac{2}{3} \leqq t \leqq 1];$$

if $n = 2, 3, 4, \dots$ and if every one of the indices $i_1, i_2, \dots, i_n$ has either the value 0 or the value 2, put

$$H_{i_1 i_2 \dots i_n} = \underset{t}{\mathrm{E}}\left[\sum_{k=1}^{n} \frac{i_k}{3^k} \leqq t \leqq \sum_{k=1}^{n-1} \frac{i_k}{3^k} + \frac{i_n + 1}{3^n}\right].$$

Then we have

$$J - S = H_0 \cup H_2$$

with disjoint summands on the right-hand side, and, for $n = 1, 2, 3, \dots$

$$J - (S \cup \bigcup S_{i_1} \cup - S_{i_1 i_2} \cup \dots \cup \bigcup S_{i_1 i_2 \dots i_n}) = \bigcup H_{i_1 i_2 \dots i_n i_{n+1}},$$

hence

$$D = \bigcap_{n=1}^{\infty} \bigcup H_{i_1 i_2 \dots i_n}.$$

For every $x \in D$ there is exactly one sequence $\{i_n\} \in M_2$ such that $x = \sum_{n=1}^{\infty} i_n/3^n$. We see easily that then

$$x = \sum_{n=1}^{\infty} \frac{i_n}{3^n} = \bigcap H_{i_1 i_2 \dots i_n}.$$

17.8.4. *Let $P \neq \emptyset$ be a compact space. Then there exists a continuous mapping f of the discontinuum D onto P.*

Proof: I. Choose a $\delta > 0$. By 17.1.4 there exists a finite number of points $a_k \in P$ $(1 \leqq k \leqq m)$ such that

$$P = \bigcup_{k=1}^{m} \bar{\Omega}(a_k, \delta).$$

Choose a $h = 1, 2, 3, \dots$ with $m \leqq 2^h$ (the number h may be chosen greater than a prescribed number) and put $a_k = a_m$ for $m + 1 \leqq k \leqq 2^h$. Then

$$P = \bigcup_{k=1}^{2^h} \bar{\Omega}(a_k, \delta).$$

The points a_k $(1 \leqq k \leqq 2^h)$ may be denoted by $b_{i_1 i_2 \dots i_h}$ where each one of the indices $i_1, i_2, \dots, i_h$ has either the value 0 or the value 2. Put

$$\bar{\Omega}(b_{i_1 i_2 \dots i_h}, \delta) = P_{i_1 i_2 \dots i_h}.$$

Then

$$P = \bigcup P_{i_1 i_2 \dots i_h}, \qquad d(P_{i_1 i_2 \dots i_h}) < 2\delta$$

and the sets $P_{i_1 i_2 \dots i_h}$ are non-void and compact (see 17.2.2).

II. Let us carry out the construction just described with the given space P and $\delta = 1/2^2$. Let us denote the number h by h_1. Now, let us carry out the construction again with any one from the 2^{h_1} spaces $P_{i_1 i_2 \dots i_n}$ and $\delta = 1/2^3$; we may assume that the number h has in all 2^{h_1} cases the same value, which we denote by $h_2 - h_1$. For every $P_{i_1 i_2 \dots i_{h_1}}$ we obtain $2^{h_2 - h_1}$ spaces $P_{i_1 i_2 \dots i_{h_2}}$. With any $P_{i_1 i_2 \dots i_{h_2}}$ and $\delta = 1/2^4$ we again carry out the construction, choosing always the same value $h_3 - h_2$ for h. Proceeding this way we obtain natural numbers $h_1 < h_2 < h_3 < \dots$ and compact sets $P_{i_1 i_2 \dots i_{h_n}}$ (every index has the value 0 or 2) such that

$$d(P_{i_1 i_2 \dots i_{h_n}}) < 1/2^n, \tag{1}$$

$$P = \bigcup P_{i_1 i_2 \dots i_{h_1}}, \tag{2}$$

$$P_{i_1 i_2 \dots i_{h_n}} = \bigcup P_{i_1 i_2 \dots i_{h_{n+1}}} \tag{3}$$

where, on the right-hand sides, the summation indices are $i_{h_n+1}, i_{h_n+2}, \dots, i_{h_{n+1}}$.

For every point $t \in D$ there is exactly one sequence $\{i_n\} \in M_2$ with $t = \sum_{n=1}^{\infty} i_n/3^n$. The set

$$\bigcap_{n=1}^{\infty} P_{i_1 i_2 \dots i_{h_n}} \tag{4}$$

consists, by (1) and (3), of exactly one point (see 15.7.1, 17.2.1 and 17.2.2), which we denote by $f(t)$. In this way we obtain a mapping f of D into P.

For every $x \in P$ there is, by (2) and (3), at least one sequence $\{i_n\} \in M_2$ such that x belongs to the set (4). Thus, f is a mapping of D onto P.

Choose a point $t_0 = \sum_{n=1}^{\infty} i_n^{(0)}/3^n \in D$; hence $\{i_n^{(0)}\} \in M_2$. Let $\varepsilon > 0$. Determine an m with $2^{-m} < \varepsilon$. We may prove easily that [1] $t_0 \in H_{i_1^{(0)} i_2^{(0)} \dots i_{h_m}^{(0)}}$, [2] if $(i_1, i_2, \dots \dots, i_{h_m}) \neq (i_1^{(0)}, i_2^{(0)}, \dots, i_{h_m}^{(0)})$ then $\varrho(t_0, H_{i_1 i_2 \dots i_{h_m}}) \geqq 1/3^{h_m}$. Hence, for $t \in D$, $|t - t_0| < < 1/3^{h_m}$, we have $t \in H_{i_1^{(0)} i_2^{(0)} \dots i_{h_m}^{(0)}}$; hence, for $t \in D$, $|t - t_0| < 1/3^{h_m}$, $t = \sum_{n=1}^{\infty} i_n/3^n$, $\{i_n\} \in M_2$ we have $f(t) \in P_{i_1^{(0)} i_2^{(0)} \dots i_{h_m}^{(0)}}$, hence $\varrho[f(t), f(t_0)] \leqq d(P_{i_1^{(0)} i_2^{(0)} \dots i_{h_m}^{(0)}}) < < 2^{-m} < \varepsilon$. Thus, the mapping f is continuous.

17.9. We say that P is a *locally compact* space, if P is a metric space and if for every $x \in P$ there is a neighborhood U such that its closure $\overline{U}$ is compact. Local compactness is obviously a topological property.

17.9.1. *A metric space is separable and locally compact if and only if it is homeomorphic with an open subset of a compact space.*

Proof: I. Let G be an open subset of a compact space Q. Let P be homeomorphic with G. We have to prove that P is separable and locally compact. Since both properties are topological ones, it suffices to deduce this for G (instead of P). G is separable by 16.1.2 and 17.2.6. Let $x \in G$. Then G is a neighborhood of the point x (in the space Q), so that, by 10.1.2, there is a neighborhood U of x such that $\overline{U} \subset G$. The set $\overline{U}$ is compact by 17.2.2. Since $\overline{U} \subset G$, we have $U = G \cap U$, $\overline{U} =$

$= G \cap \overline{U}$, i.e., U is a neighborhood of x in G and $\overline{U}$ is the closure of U in G. Thus, G is locally compact.

II. Let P be separable and locally compact. Since P is separable, there is, by 16.5, a subset G of the Urysohn space $\mathbf{U}$ homeomorphic with P and hence locally compact. The closure $\overline{G}$ of the set G in $\mathbf{U}$ is compact by 17.2.2 and 17.2.4. It remains to show that the set G is open in $\overline{G}$, hence, that $\overline{G} - G$ is closed in $\overline{G}$. Let us assume the contrary. Then (see 8.3.3) there is a sequence $\{x_n\}$ with $x_n \in \overline{G} - G$ such that $\lim x_n = x \in \overline{G}$ exists and does not belong to $\overline{G} - G$, so that $x \in G$. Since G is locally compact, there is a set V open in G, containing the point x and such that its closure in G, V_0, is compact. By 8.7.1, $V_0 = G \cap \overline{V}$, where (similarly as in the following) the bar denotes the closure in $\mathbf{U}$. By 8.7.5, $V = G \cap W$, where W is open in $\mathbf{U}$. We have $G = (G \cap V) \cup (G - V) = (G \cap V) \cup (G - W) \subset (G \cap \overline{V}) \cup (\mathbf{U} - W)$, i.e.,

$$G \subset V_0 \cup (\mathbf{U} - W). \tag{1}$$

The set V_0 is closed in $\mathbf{U}$ by 17.2.2. The set $\mathbf{U} - W$ is also closed in $\mathbf{U}$ as W is open in $\mathbf{U}$. Hence, the set on the right-hand side in (1) is closed in $\mathbf{U}$, so that (see 8.4) $\overline{G} \subset V_0 \cup (\mathbf{U} - W)$, hence $\overline{G} \cap W \subset V_0 \subset G$. Since $x_n \to x \in W$ and since W is open in $\mathbf{U}$, there is an index p such that for $n \geqq p$ we have $x_n \in W$, i.e. $x_n \in \overline{G} \cap W$, hence $x_n \in G$. This is a contradiction.

17.9.2. *A metric space P is separable and locally compact if and only if there is a compact space Q and a point $a \in Q$ such that the set $Q - (a)$ is homeomorphic with P.*

Proof: I. Let Q be a compact space. Let $a \in Q$. Let $Q - (a)$ be homeomorphic with P. The set $Q - (a)$ is open in Q. Thus, P is separable and locally compact by 17.9.1.

II. Let P be separable and locally compact. By 17.9.1 there exists a compact space $K = (K, \varrho)$ and an open $G \subset K$ homeomorphic with P. Denote by Q the set consisting of all points of the set G and one new element, which will be denoted by a. Let us distinguish two cases:

IIα. Let P be compact so that G is also compact. By 17.1.2, $d(G) < \infty$. Let us define a finite function ϱ_0 on $Q \times Q$ as follows: for $x \in G$, $y \in G$ put $\varrho_0(x, y) = \varrho(x, y)$, for $x \in G$ put $\varrho_0(a, x) = \varrho_0(x, a) = 1 + d(G)$, finally, put $\varrho_0(a, a) = 0$. We see easily that ϱ_0 is a distance function in Q and that the space (Q, ϱ_0) is compact. Since the partial distance functions in G determined on the one hand by the distance function ϱ on $K \supset G$, on the other hand by the distance function ϱ_0 in $Q \supset G$ coincide, P is homeomorphic (moreover, identical) with the set $Q - (a)$ embedded into Q.

IIβ. Let P not be compact, so that G is not compact either. By 17.2.2. $G \neq \overline{G}$ so that $K - G \neq 0$; since G is open in K, $K - G$ is closed in K and hence compact by 17.2.2.

Let us define a finite function ϱ_0 on $Q \times Q$ as follows: for $x \in G$, $y \in G$ put

$$\varrho_0(x, y) = \min [\varrho(x, y), \varrho(x, K - G) + \varrho(y, K - G)], \tag{1}$$

for $x \in G$ put $\varrho_0(x, a) = \varrho_0(a, x) = \varrho(x, K - G)$; finally, put $\varrho_0(a, a) = 0$. For $x \in Q$, $y \in Q$, evidently $\varrho_0(x, y) = \varrho_0(y, x)$. Further, $\varrho_0(x, y) = 0$ if $x = y$ and $\varrho_0(x, y) > 0$ if $x \neq y$, since $\varrho(x, K - G) > 0$ for $x \in G$ and since $K - G = \overline{K - G}$.

Let us define a finite function ϱ_1 on $K \times K$ as follows: for $x \in G$, $y \in G$ put $\varrho_1(x, y) = \varrho(x, y)$; for $x \in G$, $y \in K - G$ put $\varrho_1(x, y) = \varrho_1(y, x) = \varrho(x, K - G)$; for $x \in K - G$, $y \in K - G$ put $\varrho_1(x, y) = 0$. For $x \in Q$, $y \in Q$ we define, for the moment, a chain from x to y to be every finite sequence $\{u_i\}_{i=1}^m$ such that: [1] $u_i \in K$ for $1 \leqq i \leqq m$, [2] $u_1 = x$ if $x \in G$ and $u_1 \in K - G$ if $x = a$, [3] $u_m = y$ if $y \in G$ and $u_m \in K - G$ if $y = a$. The number

$$\sum_{i=1}^{m-1} \varrho_1(u_i, u_{i+1}) \qquad \text{(equal to 0 for } m = 1)$$

is called the length of the chain $\{u_i\}_{i=1}^m$. We may prove easily that for $x \in Q$, $y \in Q$ there are chains from x to y and that the number $\varrho_0(x, y)$ is the least length of such chains.

Let $x \in Q$, $y \in Q$, $z \in Q$. There is a chain $\{u_i\}_{i=1}^m$ from x to y with length $\varrho_0(x, y)$. There is a chain $\{u_i\}_{i=m+1}^{m+n}$ from y to z with length $\varrho_0(y, z)$. Then $\{u_i\}_{i=1}^{m+n}$ is a chain from x to z with length on the one hand greater than or equal to $\varrho_0(x, z)$, on the other hand equal to $\varrho_0(x, y) + \varrho_0(y, z)$. Thus, $\varrho_0(x, y) + \varrho_0(y, z) \geqq \varrho_0(x, z)$.

This proves that ϱ_0 is a distance function in Q. Let us prove that the space (Q, ϱ_0) is compact. Thus, let $\{x_n\}_1^\infty$ be a point sequence in Q. We have to prove that there is a subsequence of $\{x_n\}$ convergent with respect to the distance function ϱ_0. This is evident if $x_n = a$ for infinitely many indices n. In the contrary case we may find a subsequence $\{x'_n\}_1^\infty$ of $\{x_n\}$ such that $x'_n \in G$ for every n. It may occur that there is a number $\varepsilon > 0$ such that, for infinitely many indices $n_1 < n_2 < n_3 < \ldots$, $\varrho(x'_{n_i}, K - G) \geqq \varepsilon$. Then we have for every i

$$x'_{n_i} \in K - \Omega_K(K - G, \varepsilon) = L.$$

The set $\Omega_K(K - G, \varepsilon)$ is open in K. Hence, L is closed in K. Hence, by 17.2.2 L is a compact space (with respect to the partial distance function determined in L by the distance function ϱ of the space K). Thus, there is a subsequence $\{y_n\}_1^\infty$ of $\{x'_{n_i}\}_{i=1}^\infty$ such that there is a point $y \in L$ with $\varrho(y_n, y) \to 0$. As $\varrho_0(y_n, y) \leqq \varrho(y_n, y)$, we also have $\varrho_0(y_n, y) \to 0$, i.e. the sequence $\{y_n\}$ is convergent with respect to the distance function ϱ_0. There remains the case where for every $\varepsilon > 0$ there is an index p such that for $n \geqq p$ we always have $\varrho(x'_n, K - G) < \varepsilon$. Then $\varrho_0(x'_n, a) = \varrho(x'_n, K - G) \to 0$; hence, $x'_n \to a$ with respect to the distance function ϱ_0.

It remains to be shown that both the partial distance functions determined in G, on the one hand by the distance function ϱ in $K \supset G$ and on the other hand by the distance function ϱ_0 in $Q \supset G$, are equivalent, i.e. that, for $x_n \in G$, $x \in G$ we have

$$\varrho(x_n, x) \to 0 \quad \text{if and only if} \quad \varrho_0(x_n, x) \to 0.$$

First, if $\varrho(x_n, x) \to 0$, we have $\varrho_0(x_n, x) \to 0$, since $\varrho_0(x_n, x) \leqq \varrho(x_n, x)$. Let, secondly, $\varrho_0(x_n, x) \to 0$. As $x \in G$ and $K - G = \overline{K - G}$, we have $\varrho(x, K - G) > 0$, so that there is an index p such that, for $n \geqq p$, $\varrho_0(x_n, x) < \varrho(x, K - G) \leqq \varrho(x_n, K - G) + \varrho(x, K - G)$. By (1), for $n \geqq p$ we have $\varrho_0(x_n, x) = \varrho(x_n, x)$, so that $\varrho(x_n, x) \to 0$.

17.10. 16.1.5 and 17.2.3 yield:

17.10.1. *The euclidean space* $\mathbf{E}_m$ $(m = 1, 2, 3, \ldots)$ *is separable and locally compact: however, it is not compact.*

By 17.9.2 there is a compact space Q and a point $a \in Q$ such that $\mathbf{E}_m$ is homeomorphic with $Q - (a)$. We are going to construct such a space by means of the elementary calculus.

The set of all points $x = (x_0, x_1, \ldots, x_m)$ of the euclidean space $\mathbf{E}_{m+1}$ with $\sum_{i=0}^{\infty} x_i^2 = 1$ will be called the *m-dimensional spherical space* $(m = 0, 1, 2, 3, \ldots)$ and denoted by $\mathbf{S}_m$. The distance function in $\mathbf{S}_m$ is certainly the partial distance function of the usual one in $\mathbf{E}_{m+1}$. The space $\mathbf{S}_0$ consists of exactly two points, while the spaces $\mathbf{S}_m$ $(m = 1, 2, 3, \ldots)$ are infinite.

9.5 and 17.2.3 yield:

17.10.2. *The spherical space* $\mathbf{S}_m$ $(m = 0, 1, 2, \ldots)$ *is compact.*

17.10.3. *Let* $a \in \mathbf{S}_m$, $b \in \mathbf{S}_m$ $(m = 0, 1, 2, \ldots)$. *There exists an isometrical mapping* f *of* $\mathbf{S}_m$ *onto* $\mathbf{S}_m$ *such that* $f(a) = b$.

Proof: I. We shall prove that, for $-1 \leqq i \leqq m - 1$ there is an isometrical mapping f_i of $\mathbf{S}_m$ onto $\mathbf{S}_m$ such that if $f_i(a) = c_i = (c_{i0}, c_{i1}, \ldots, c_{im})$, then, for $0 \leqq j \leqq i$, $c_{ij} = 0$. This statement is trivial for $i = -1$. Let it hold for some i $(-1 \leqq i \leqq m - 2)$. It suffices to prove that then it also holds for $i + 1$. This is evident if $c_{i,i+1} = 0$. In the contrary case put, for $(x_0, x_1, \ldots, x_m) \in \mathbf{S}_m$: $\varphi(x_0, x_1, \ldots, x_m) = (x'_0, x'_1, \ldots, x'_m)$, where

$$x'_{i+1} = \frac{c_{i,i+2}x_{i+1} + c_{i,i+1}x_{i+2}}{\sqrt{(c_{i,i+1}^2 + c_{i,i+2}^2)}},$$

$$x'_{i+2} = \frac{-c_{i,i+1}x_{i+1} + c_{i,i+2}x_{i+2}}{\sqrt{(c_{i,i+1}^2 + c_{i,i+2}^2)}},$$

$$x'_j = x_j, \qquad 0 \leqq j \leqq m, \qquad i + 1 \neq j \neq i + 2.$$

If we put $c_{i+1} = (c_{i+1,0}, c_{i+1,1}, \ldots, c_{i+1,m})$, where $c_{i+1,i+1} = 0$, $c_{i+1,i+2} = \sqrt{(c_{i,i+1}^2 + c_{i,i+2}^2)}$, $c_{i+1,j} = c_{ij}$ for $0 \leqq j \leqq m$, $i+1 \neq j \neq i+2$, we see easily that $c_{i+1} \in \mathbf{S}_m$, that $c_{i+1,j} = 0$ for $0 \leqq j \leqq i+1$, that φ is an isometrical mapping of $\mathbf{S}_m$ onto $\mathbf{S}_m$ and that $\varphi(c_{i+1}) = c_i$. The required isometrical mapping f_{i+1} will be evidently obtained putting $f_{i+1}(x) = \varphi_{-1}[f_i(x)]$ for $x \in \mathbf{S}_m$.

II. By I (where we put $i = m - 1$) there is an isometrical mapping f' of $\mathbf{S}_m$ onto $\mathbf{S}_m$ with either $f'(a) = (0, \ldots, 0, 1)$ or $f'(a) = (0, \ldots, 0, -1)$. Since there is an isometrical mapping h of $\mathbf{S}_m$ onto $\mathbf{S}_m$ such that $h(0, \ldots, 0, -1) = (0, \ldots, 0, 1)$ [it suffices to put $h(x_0, x_1, \ldots, x_m) = (-x_0, -x_1, \ldots, -x_m)$], we may assume that $f'(a) = (0, \ldots, 0, 1)$. Similarly, there is an isometrical mapping f'' of $\mathbf{S}_m$ onto $\mathbf{S}_m$ such that $f''(b) = (0, \ldots, 0, 1)$. Putting $f(x) = f''_{-1}[f'(x)]$ we obtain an isometrical mapping f such that $f(a) = b$.

17.10.4. *Let $a \in \mathbf{S}_m$ $(m = 1, 2, 3, \ldots)$. The spaces $\mathbf{E}_m$ and $\mathbf{S}_m - (a)$ are homeomorphic.*

Proof: By 17.10.3 we may assume that $a = (1, 0, \ldots, 0)$. For $(x_0, x_1, \ldots, x_m) \in \mathbf{S}_m - (a)$ put $f(x_0, x_1, \ldots, x_m) = (y_1, y_2, \ldots, y_m)$ where

$$y_i = \frac{x_i}{1 - x_0}, \quad (1 \leqq i \leqq m). \tag{1}$$

We calculate easily that equations (1) are equivalent to the equations

$$x_0 = \frac{\sum_{i=1}^{m} y_i^2 - 1}{\sum_{i=1}^{m} y_i^2 + 1}, \quad x_j = \frac{2y_j}{\sum_{i=1}^{m} y_i^2 + 1}, \qquad (1 \leqq j \leqq m). \tag{2}$$

It follows easily that f is a one-to-one mapping of $\mathbf{S}_m - (a)$ onto $\mathbf{E}_m$ and that both the mappings f and f_{-1} are continuous.

Exercises

17.1. If P and Q are totally bounded spaces, then $P \times Q$ is a totally bounded space.
17.2.* If P and Q are compact spaces, then $P \times Q$ is compact.
17.3. If sets $A \subset P$ and $B \subset P$ are totally bounded, then $A \cup B$ is totally bounded.
17.4.* If $A \subset P$ and $B \subset P$ are compact sets, then $A \cup B$ is compact.
17.5. Let $A \subset P$. The closure $\bar{A}$ is compact if and only if every point sequence $\{x_n\}$ in A has a convergent subsequence (in P; the limit need not belong to A).
17.6. Let P be a compact space. Let $A_n \subset P$, $A_n \supset \bar{A}_{n+1}$. Let G be a neighborhood of the set $\bigcap_{n=1}^{\infty} A_n$. Then there is an index m such that $A_n \subset G$ for every $n > m$.

17.7. Let Q be the completion of a metric space P. P is totally bounded if and only if Q is compact.

A metric space is said to be *σ-compact*, if $P = \bigcup_{n=1}^{\infty} A_n$, where every summand is compact.

17.8. Let P be a σ-compact space. A point set $A \subset P$ is σ-compact if and only if it is $\mathbf{F}_\sigma(P)$.
17.9. An isolated metric space is compact if and only if it is finite.
17.10. Let $A \subset \mathbf{E}_m$, $B \subset \mathbf{E}_m$. Let $A \neq 0 \neq B$. Let A be closed and B bounded. Then there exists a point $y \in A$ such that

$$\varrho(y, B) = \min_{x \in A} \varrho(x, B) = \varrho(A, B).$$

17.11. Let $A \subset \mathbf{E}_m$, $B \subset \mathbf{E}_m$. Let $A \neq 0 \neq B$. Let A and B be closed; let A be bounded. Then there exist points $y_1 \in A$, $y_2 \in B$ such that

$$\varrho(y_1, y_2) = \min_{\substack{x_1 \in A \\ x_2 \in B}} \varrho(x_1, x_2) = \varrho(A, B).$$

17.12. Let f be a continuous mapping of a metric space P into a metric space Q. Let $A \subset P$ be compact. Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$x \in A, \quad y \in P, \quad \varrho(x, y) < \delta \quad \Rightarrow \quad \varrho[f(x), f(y)] < \varepsilon.$$

In the exercises 17.13—17.16, P^* is the Hausdorff hyperspace of P.

17.13. If P is not complete, then P^* is not complete.
17.14. If P is not totally bounded, then P^* is not totally bounded.
17.15. If P is not separable, then P^* is not separable.
17.16. If P is not compact, then P^* is not compact.
17.17. If $P = K = \mathop{\mathrm{E}}_t[0 \leqq t \leqq 1]$ and if $f_n(t) = t^n$, then there is no subsequence of $\{f_n\}$ convergent in P^K. Thus, P^K is not compact, while K and P are compact.
17.18. Deduce theorem 17.6.7 directly, without use of theorems 17.2.5, 17.6.4 and 17.6.6.
17.19. Deduce theorem 16.7 from theorems 16.5, 17.2.4 and 17.6.8.
17.20.* Every open subset of a locally compact space is a locally compact space.
17.21. A locally compact space is σ-compact if and only if it is separable.
17.22. Let the assumptions and notation of 17.9.2 be preserved. Let f be a homeomorphic mapping of P onto $Q - (a)$. Let $\{x_n\}$ be a point sequence in P. We have $f(x_n) \to a$ if and only if there is no convergent subsequence of $\{x_n\}$.
17.23.* $\mathbf{R}$ (see 9.4) is a compact space.
17.24. Let $a \in \mathbf{E}_1$, $b \in \mathbf{E}_1$, $a < b$, $P = \mathop{\mathrm{E}}_t[a \leqq t \leqq b]$. If $c > 0$, $\alpha > 0$, denote by $\Psi(\alpha, c)$ the system of all the finite functions f on P such that

$$x \in P, \quad y \in P \quad \text{imply} \quad | f(x) - f(y) | \leqq c \, | x - y |^\alpha.$$

If $\alpha > 0$, put $\Phi(\alpha) = \bigcup_{c>0} \Psi(\alpha, c)$. We say that a function f on P satisfies the Lipschitz condition of the order α if $f \in \Phi(\alpha)$. If $f \in \Phi(\alpha)$, $\alpha > 1$, then f is a constant. Let $0 < \alpha < \beta \leqq 1$, so that $\Phi(\alpha) \supset \Phi(\beta)$. Let $c > 0$. If $f_1 \in \Psi(\alpha, c)$, $f_2 \in \Psi(\alpha, c)$, put

$$\varrho(f_1, f_2) = \max_{x \in P} | f_1(x) - f_2(x) | .$$

Then $\Psi(\alpha, c) = [\Psi(\alpha, c), \varrho]$ is a complete space. The set

$$\Phi(\beta) \cap \Psi(\alpha, c)$$

is of the first category in $\Psi(\alpha, c)$. Thus, by 15.8.2, there is a function $f \in \Phi(\alpha)$ such that $f \in \Phi(\beta)$ for no $\beta > \alpha$. Moreover, it may be shown that there is a function which satisfies the Lipschitz condition of order α while for no $\beta > \alpha$ and no interval $Q = \underset{t}{\mathrm{E}}[a_1 \leqq t \leqq b_1] \subset P$ does the partial function f_Q satisfy the Lipschitz condition of order β.

17.25. State the so called Borel (Heine-Borel) theorem. This is obtained from theorem 17.2.3 interpreting the word "compact" in the sense of theorem 17.5.4.