Chapter I

INTRODUCTION

§ 1. Sets and set operations

1.1. The theory of *sets* has a central position in the whole of modern mathematics. A detailed logical analysis of the notion of a set would hardly be useful for a beginner, for whom the present book is primarily intended. For our purposes, it suffices to say simply that a set is a collection of some things, which are called the *elements* of the set. A set is fully determined by its elements; if two sets A and B have the same elements, they are *identical* or *equal* and we write $A = B$.

As a rule, sets will be denoted by upper case Roman letters and their elements will be denoted by lower case Roman letters. However, there will be frequent exceptions from this rule. I shall describe one of these at once. The object of our considerations will be, as a rule, some family S of sets, which will be denoted by upper case Roman letters; besides the sets of the family S, we frequently meet with sets, the elements of which are sets themselves, namely sets from the family S. These *sets of sets* will be, as a rule, denoted by upper case German letters. Instead of a *set of sets* we shall rather use the term *system of sets.*

Examples of sets: [1] the set of all the words printed in the present book; [2] the set of all the primes with five figures; [3] the set of all primes of the form $2^{2^n} + 1$ $(n = 5, 6, 7, \ldots)$; [4] the set of all *natural* numbers $1, 2, 3, 4, \ldots$; [5] the set of all tangents to a given circle; [6] the set of all points common to two given spheres.

It is useful to consider also the *void* set, which has no elements at all. Since any set is fully determined by its elements, there is only one void set; we shall denote it by the symbol $\emptyset$. Many authors denoted the void set by the cipher 0; some of them by the symbol Λ. If we did not introduce the void set, we would not know whether the set [3] exists, and the set [6] would exist for certain positions of the spheres only.

If a is an arbitrary thing, (a) designates the set consisting of the unique element, namely the element a. The set $\emptyset$ has no elements, while the set $(\emptyset)$ has one element, namely the set $\emptyset$.

If **A** and **B** are any propositions, there are four possibilities excluding each other: [1] **A** is valid, **B** is valid, [2] **A** is not valid, **B** is not valid, [3] **A** is not valid, **B** is valid; [4] **A** is valid, **B** is not valid. Whenever any of the first three possibilities occur, we say that the proposition **A** *implies* the proposition **B** and write

$$\mathbf{A} \Rightarrow \mathbf{B}, \tag{1}$$

e.g. $1 < 2 \Rightarrow 1 < 2$; $1 > 2 \Rightarrow 1 > 2$; $1 > 2 \Rightarrow 1 < 2$; but not $1 < 2 \Rightarrow 1 > 2$. Whenever one of the first two possibilities occurs, we write

$$\mathbf{A} \Leftrightarrow \mathbf{B}. \tag{2}$$

Thus, $\mathbf{A} \Leftrightarrow \mathbf{B}$ means that simultaneously both $\mathbf{A} \Rightarrow \mathbf{B}$ and $\mathbf{B} \Rightarrow \mathbf{A}$. If $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}$ are three propositions, then

$$\mathbf{A}_1, \mathbf{A}_2 \Rightarrow \mathbf{B}$$

means that the simultaneous validity of the propositions $\mathbf{A}_1$ and $\mathbf{A}_2$ implies the proposition $\mathbf{B}$; similarly for an arbitrary number of propositions.

If $\mathbf{A}$ and $\mathbf{B}$ are two propositions and (1) holds, we also say that *the validity of* $\mathbf{A}$ *is a sufficient condition for the validity of* $\mathbf{B}$, or that *the validity of* $\mathbf{B}$ *is a necessary condition for the validity of* $\mathbf{A}$. Since (2) means that simultaneously $\mathbf{A} \Rightarrow \mathbf{B}$ and $\mathbf{B} \Rightarrow \mathbf{A}$, we read (2) also as follows: *the validity of* $\mathbf{A}$ *is a necessary and sufficient condition for the validity of* $\mathbf{B}$; (2) is also read: $\mathbf{A}$ *is valid if and only if* $\mathbf{B}$ *is valid.*

1.2. To indicate briefly that a thing a is an element of a set A, we write

$$a \in A.$$

The Greek letter epsilon appears in the present book in two types, $\in$ and ε. The first type is reserved for the use just described.

If A and B are two sets, we say that the former set is a *subset* or *part* of the latter one, if

$$x \in A \Rightarrow x \in B,$$

i.e., if the set A has no element which is not also an element of the set B. To expresss this concisely, we write

$$A \subset B \quad \text{or} \quad B \supset A.$$

For arbitrary sets, the following simple rules hold

$$\emptyset \subset A; \quad A \subset A; \quad A \subset \emptyset \Rightarrow A = \emptyset;$$

$$A \subset B \subset C \Rightarrow A \subset C; \ ^{*)}$$

$$A \subset B \subset A \Rightarrow A = B.$$

The last rule is, in spite of its simplicity, very important. If we have to prove that two sets A and B are identical, we proceed usually in the following way: we prove first that

$$x \in A \Rightarrow x \in B$$

and then that

$$x \in B \Rightarrow x \in A.$$

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*) $A \subset B \subset C$ means, of course, that simultaneously $A \subset B$ and $B \subset C$.

Many authors write $A \subseteq B$ instead of $A \subset B$, using the symbol $A \subset B$ for the case os a subset A distinct from the set B.

1.3. The object of consideration is frequently a fixed set P and its subsets. If $\mathbf{V}(x)$ is a property which is meaningful for every $x \in P$ (i.e., for every $x \in P$, $\mathbf{V}(x)$ is either valid or not) we denote by

$$\underset{x}{\mathrm{E}}\,[\mathbf{V}(x)]$$

the subset of all $x \in P$ which have the property $\mathbf{V}(x)$. Similarly

$$\underset{x}{\mathrm{E}}\,[\mathbf{V}(x), \mathbf{W}(x)]$$

is the subset of all $x \in P$ which have simultaneously both properties $\mathbf{V}(x)$ and $\mathbf{W}(x)$. If there is danger of confusion regarding P, we write more precisely

$$\underset{x}{\mathrm{E}}[x \in P, \mathbf{V}(x)] \quad \text{instead of} \quad \underset{x}{\mathrm{E}}[\mathbf{V}(x)]\,.$$

E.g. if P is the set of all real numbers, then

$$\underset{x}{\mathrm{E}}[0 < x < 1]$$

is the open interval with the end points 0 and 1,

$$\underset{x}{\mathrm{E}}[0 \leqq x \leqq 1]$$

is the closed interval with the end points 0 and 1, while

$$\underset{x}{\mathrm{E}}[0 > x > 1] = \emptyset\,.$$

1.4. If A and B are given sets,*) then: [1] the set of all the elements x such that either $x \in A$ or $x \in B$ **) is called their *union* and denoted by $A \cup B$; [2] the set of all the elements x such that simultaneously both $x \in A$ and $x \in B$ is called their *intersection* and denoted by $A \cap B$.

More generally, with every element z of a given set $\mathbf{C} \neq \emptyset$ let there be associated a set $A(z)$. Then: [1] the set consisting of all the elements x such that there is a $z \in \mathbf{C}$ with $x \in A(z)$ is called the *union* of all the sets $A(z)$ and denoted by

$$\bigcup A(z) \quad \text{or} \quad \bigcup_{z} A(z) \quad \text{or} \quad \bigcup_{z \in \mathbf{C}} A(z)\,;$$

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*) We do not assume $A \neq B$; such an assumption would be stated expressly.

) $A \cup B$ contains also the elements for which simultaneously both $x \in A$ and $x \in B$. Generally, if **A and **B** are propositions, then "**A** or **B**" means that out of the four possibilities stated in 1.1, the first, third or fourth holds.

[2] the set consisting of all the elements x for which $x \in A(z)$ for every $z \in \mathbf{C}$ is called the *intersection* of all the sets $A(z)$ and denoted by

$$\bigcap A(z) \quad \text{or} \quad \bigcap_{z} A(z) \quad \text{or} \quad \bigcap_{z \in \mathbf{C}} A(z)\,.$$

The case of $\mathbf{C}$ being the set of all natural numbers is particularly frequent; if A_n is the set associated with the number n, then the union and intersection of all the sets A_n is denoted by

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n \quad \text{respectively.}$$

With this notation, it is not necessary to write out in detail what we mean by e.g.

$$\bigcup_{n=1}^{3} A_n = A_1 \cup A_2 \cup A_3$$

or

$$\bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4\,.$$

Sets A and B are said to be *disjoint* if $A \cap B = \emptyset$. A system of sets $\mathfrak{A}$ is said to be *disjoint*, if

$$A \in \mathfrak{A}\,,\; B \in \mathfrak{A}\,,\; A \neq B \;\Rightarrow\; A \cap B = \emptyset\,.$$

A union $\bigcup_{z \in \mathbf{C}} A(z)$ is said to be *disjoint*, or, to have *disjoint summands*, if

$$z \in \mathbf{C}\,,\; z' \in \mathbf{C}\,,\; z \neq z' \;\Rightarrow\; A(z) \cap A(z') = \emptyset\,.$$

1.5. If A and B are given sets, then the set consisting of all the elements belonging to A and not belonging to B is called the *difference* of A and B and denoted by $A - B$. Many authors use the symbol $A - B$ only in the case of $A \supset B$.

Exercises*)

1.1. $A \cup B = B \cup A$; $A \cap B = B \cap A$; $(A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C$; $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$.

1.2. $A \cup A = A \cap A$; $A \cup \emptyset = A - \emptyset = A$; $A \cap \emptyset = \emptyset - A = \emptyset$.

1.3. $A \subset B \Leftrightarrow A = A \cap B \Leftrightarrow A \cup B = B \Leftrightarrow A - B = \emptyset$.

1.4. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.

1.5. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

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*) The exercises referred to later in the text are denoted by an asterisk. However, a beginner reader should go carefully through *all* the exercises, in order to penetrate into the discussed material. The simple results of the exercises for § 1 are used in the text without explicit reference; therefore they are not marked by an asterisk.

1.6. $A - B = (A \cup B) - B = A - A \cap B$; $A - (A - B) = A \cap B$.
1.7. $C - (A \cup B) = (C - A) \cap (C - B)$; $C - A \cap B = (C - A) \cup (C - B)$.
1.8. $C - \bigcup A(z) = \bigcap [C - A(z)]$; $C - \bigcap A(z) = \bigcup [C - A(z)]$.
1.9. $A - (B \cup C) = (A - B) - C$; $(A - B) \cap C = A \cap C - B \cap C = A \cap C - B$;
$(A \cup B) - C = (A - C) \cup (B - C)$; $A \cap B - C = A \cap (B - C) =$
$= (A - C) \cap (B - C)$.
1.10. $A - (B - C) = (A - B) \cup A \cap C$.
1.11. $C - A \neq 0 \Rightarrow A - (B - C) \neq (A - B) \cup C$.
1.12. $A \supset C \Rightarrow A - (B - C) = (A - B) \cup C$.
1.13. $A = (A - B) \cup (A \cap B)$; $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$; the unions on the right-hand side are disjoint.
1.14. $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} C_n$ where $B_n = \bigcup_{n=1}^{n} A_i$, $C_n = \bigcap_{n=1}^{n} A_i$.
1.15. Sets A and 0 are disjoint.
1.16. Sets A and $B \supset A$ are disjoint if and only if $A = 0$.
1.17. $A(z) \subset B(z) \Rightarrow \bigcup A(z) \subset \bigcup B(z)$, $\bigcap A(z) \subset \bigcap B(z)$.
1.18. $A \subset B \Rightarrow C - A \supset C - B$.
1.19. If a set A has a finite number $n \geqq 0$ of elements, then A has 2^n subsets.

§ 2. Mappings

2.1. Let A and B be two given sets. The set of all pairs (x, y), where $x \in A$, $y \in B$ (if $x \in A \cap B$, $y \in A \cap B$, $x \neq y$, the pair (x, y) is taken as distinct from the pair (y, x)) is said to be their *cartesian product* and denoted by $A \times B$. The term cartesian product, introduced by Kuratowski,*) reminds us of the important particular case where $A = B$ is the set of all real numbers and $A \times B$ is the arithmetical equivalent of the plane: every point is represented by its cartesian coordinates. The term *combinatorial product* for $A \times B$ is also customary.

Similarly, if A, B, C are three given sets, we say that their cartesian product, denoted by $A \times B \times C$, is the set of all triples (x, y, z), where $x \in A$, $y \in B$, $z \in C$. Similarly for an arbitrary finite number of given sets.

Remark: The sets $(A \times B) \times C$, $A \times (B \times C)$ and $A \times B \times C$ are distinct. The first one consists of the pairs (ξ, z), where ξ is a pair (x, y), $x \in A$, $y \in B$, and $z \in C$; the second one consists of the pairs (x, η), where $x \in A$ and $\eta = (y, z)$, $y \in B$, $z \in C$; the third consists of the triples (x, y, z), where $x \in A$, $y \in B$, $z \in C$. Nevertheless, it is possible to see that the differences between these three sets are merely formal and we shall for brevity neglect them. Hence, for our purposes $(A \times B) \times C = A \times (B \times C) = A \times B \times C$. More generally, we will have $P \times Q = R$ whenever $P = A_1 \times A_2 \times \ldots \times A_m$, $Q = A_{m+1} \times A_{m+2} \times \ldots \times A_{m+n}$, $R = A_1 \times A_2 \times \ldots \times A_{m+n}$. On the other hand, we shall distinguish $A \times B$ from $B \times A$.

*) Topologie I, p. 7.

2.2. Let A and B be two given sets. Let f be a subset of the set $A \times B$ such that for every $x \in A$ there is just one $y \in B$ with $(x, y) \in f$. Then we say that f is a *mapping* of the set A *into* the set B. Besides the term mapping, the terms *transformation, operation, correspondence* are used; there is also the term *function*, which we use, however, in a narrower sense (see 2.3). If f is a mapping of a set A into a set B, then every element $x \in A$ determines uniquely the element $y \in B$ with $(x, y) \in f$; we write, as a rule, $y = f(x)$ and say that y is the *image* of the element x (under the mapping f).

If f_1 is a mapping of a set A_1 into a set B_1 and if f_2 is a mapping of a set A_2 into a set B_2 then $f_1 = f_2$ if and only if [1] $A_1 = A_2$; [2] $x \in A_1$ implies $f_1(x) = f_2(x)$.

If f is a mapping of a set A into a set B and if $y \in B$, there need not exist any element $x \in A$ with $f(x) = y$; there may be more, possibly an infinite number of such elements. If C is the set of all $y \in B$ such that there is at least one $x \in A$ with image y, we say that f is a mapping of the set A *onto* the set C.

2.3. Throughout the whole book, the symbol $\mathbf{R}$ denotes the set of all real numbers augmented by two elements: $\infty = +\infty$ and $-\infty$. A mapping of an arbitrary set A into the set $\mathbf{R}$ is said to be a *function* (on A). The set A is called the *domain* of the function f. The image $f(x)$ of an arbitrary $x \in A$ under the function f is called the *value* of the function f at the element x. If neither ∞ nor $-\infty$ appear among the values of a function f, we say that f is a *finite function*. If, moreover, there is a real number c such that $|f(x)| \leqq c$ for every $x \in A$, we say that f is a *bounded function*.

Let P be a given set. With every set $A \subset P$ we may associate a function χ_A on P, for which

$$\chi_A(x) = 1 \quad \text{for} \quad x \in A\,, \quad \chi_A(x) = 0 \quad \text{for} \quad x \in P - A\,.$$

The function χ_A is called the *characteristic function* of the set A (with domain P).

2.4. If f is a mapping of a set A into a set B and if there is given a set $M \subset A$, we define the *partial mapping* f_M of the set M into the set B by the formula

$$f_M = (M \times B) \cap f;$$

thus if $x \in M$, then $f_M(x) = f(x)$, while for $x \in A - M$ the symbol $f_M(x)$ is meaningless. If $B = \mathbf{R}$, we speak about *partial function.*

2.5. Let f be a mapping of a set A into a set B; let $\mathfrak{A}$ be the system of all subsets of A, and similarly let $\mathfrak{B}$ be the system of all subsets of B. The mapping f determines a mapping F of the system $\mathfrak{A}$ into the system $\mathfrak{B}$, and also a mapping φ of the system $\mathfrak{B}$ into the system $\mathfrak{A}$. These two new mappings are defined as follows: [1] if $M \in \mathfrak{A}$, then $F(M)$ is the set of all the images under f of all $x \in M$; [2] if $N \in \mathfrak{B}$, then $\varphi(N)$ is the set of all $x \in A$ such that $f(x) \in N$.

The mapping φ will be denoted by f_{-1}; it ts also denoted f^{-1}, The mapping F will be, without fear of misunderstanding, denoted simply by f. Thus, for $M \subset A$

$$f(M) = \underset{y}{\mathrm{E}}[y = f(x), x \in M]$$

and for $N \subset B$

$$f_{-1}(N) = \underset{x}{\mathrm{E}}[f(x) \in N] .$$

2.6. Let f be a mapping of a set A into a set B. We say that the mapping f is *one-to-one*, if

$$x \in A,\ y \in A,\ x \neq y \quad \text{imply} \quad f(x) \neq f(y) .$$

Let $C = f(A)$. If $y \in C$, there is only one $x \in A$ with $f(x) = y$; if we write $x = g(y)$, then g is evidently a one-to-one mapping of the set C onto the set A. We say that the mapping g is the *inverse mapping* to the mapping f. Thus, this notion is defined only for a *one-to-one* mapping f. An inverse mapping g to a one-to-one mapping f will be denoted by f_{-1} (it is also denoted f^{-1}); the reader will easily determine that it is not in contradiction with the symbol f_{-1} introduced in 2.5. Obviously $(f_{-1})_{-1} = f$.

Exercises

2.1. $(A \cup B)\times(C \cup D) = (A\times C) \cup (A\times D) \cup (B\times C) \cup (B\times D)$.
2.2. $A \cap C\times B \cap D = (A\times B) \cap (C\times D)$.
2.3. $(A - B)\times C = (A\times C) - (B\times C)$.
2.4. $A \subset B,\ C \subset D \Rightarrow (A\times C) \subset (B\times D)$. If $A \neq \emptyset \neq C$, we may write $\Leftrightarrow$ instead of $\Rightarrow$.
2.5. $(P\times Q) - (A\times B) = (P - A)\times Q \cup P\times(Q - B)$.
2.6. $A \subset P,\ B \subset Q \Rightarrow A\times B = (A\times Q) \cap (P\times B)$.

In exercises 2.7—2.18, f is a mapping of a set A into a set B.

2.7. $M_1 \subset M_2 \subset A \Rightarrow f(M_1) \subset f(M_2)$.
2.8. $M(z) \subset A \Rightarrow f[\bigcup_z M(z)] = \bigcup_z f[M(z)]$.
2.9. $M_1 \subset A,\ M_2 \subset A \Rightarrow f(M_1) - f(M_2) \subset f(M_1 - M_2)$.
2.10. $M(z) \subset A \Rightarrow f[\bigcap_z M(z)] = \bigcap_z f[M(z)]$.
2.11. $N_1 \subset N_2 \subset B \Rightarrow f_{-1}(N_1) \subset f_{-1}(N_2)$.
2.12. $N(z) \subset B \Rightarrow f_{-1}[\bigcap_z N(z)] = \bigcap_z f_{-1}[N(z)]$.
2.13*. $N_1 \subset B,\ N_2 \subset B \Rightarrow f_{-1}(N_1 - N_2) = f_{-1}(N_1) - f_{-1}(N_2)$.
2.14. $N(z) \subset B \Rightarrow f_{-1}[\bigcap_z N(z)] = \bigcap_z f_{-1}(N(z))$.
2.15. If the mapping f is one-to-one, on the left-hand side of 2.9 and 2.10 we may write $=$ instead of $\subset$.
2.16. If $N \subset B$, then $f_{-1}(N) = \emptyset$ implies $N \cap f(A) = \emptyset$.
2.17. If $N \subset B$, then $f(f_{-1}(N)) = N \cap f(A)$.
2.18. If $M \subset A$, $N \subset B$, then $(f_M)_{-1}(N) = M \cap f_{-1}(N)$.

In exercies 2.19—2.24 χ_A is the characteristic function of $A \supset P$ (P is the domain of the function χ_A); similarly χ_B for $B \supset P$ etc. The letter x denotes an arbitrary element of the set P.

2.19. $\chi_{A \cap B}(x) = \chi_A(x) . \chi_B(x) = \min [\chi_A(x), \chi_B(x)]$. *)
2.20. $\chi_{A \cup B}(x) = \max [\chi_A(x), \chi_B(x)]$.
2.21. If $A \cap B = \emptyset$, then $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$.
2.22. $\chi_{A-B}(x) = \max [0, \chi_A(x) - \chi_B(x)]$; if $B \subset A$, then $\chi_{A-B}(x) = \chi_A(x) - \chi_B(x)$.
2.23. If $A_n \subset A_{n+1}$, $A = \bigcup_{n=1}^{\infty} A_n$, then $\chi_A(x) = \lim \chi_{A_n}(x)$.
2.24. If $A_n \supset A_{n+1}$, $A = \bigcap_{n=1}^{\infty} A_n$, then $\chi_A(x) = \lim \chi_{A_n}(x)$.
2.25. $\chi_{A \cup B}(x) = 1 - [1 - \chi_A(x)] . [1 - \chi_B(x)]$.

§ 3. Countable sets

3.1. In this paragraph **N** denotes the set of all natural numbers $1, 2, 3, \dots$.

A mapping of the set **N** (into an arbitrary set) is called a *sequence*. The image of a natural number n in a sequence is denoted, as a rule, by a_n (or b_n, α_n, A_n, etc.) and is termed the n-th *term* of the sequence. The sequence is then denoted by $\{a_n\}$ or $\{a_n\}_1^{\infty}$ or $\{a_n\}_{n=1}^{\infty}$. A sequence a_n is *one-to-one* (in the sense of section 2.6), if: $m \in \mathbf{N}$, $n \in \mathbf{N}$, $m \neq n$ imply $a_m \neq a_n$. If the terms of a sequence $\{A_n\}$ are sets, we say that the sequence is *disjoint*, whenever: $m \in \mathbf{N}$, $n \in \mathbf{N}$, $m \neq n$ imply $A_m \cap A_n \neq \emptyset$.

The sequences just defined are sometimes called more precisely *infinite sequences*. By a *finite sequence* we understand a mapping of the set of all the natural numbers less than or equal to p, where p is a given natural number; notation $\{a_n\}_1^p$ etc.

If $\{i_n\}$ is an increasing (i.e. $i_n < i_{n+1}$) sequence of natural numbers (i.e. $i_n \in \mathbf{N}$) and if $\{a_n\}_{n=1}^{\infty}$ is an arbitrary sequence, we say that the sequence $\{a_{i_n}\}_{n=1}^{\infty}$ is a *subsequence* of the sequence $\{a_n\}_{n=1}^{\infty}$. Obviously a subsequence of a subsequence of a sequence $\{a_n\}$ is again a subsequence of $\{a_n\}$.

To define a sequence, it is sufficient: [1] to define the first term a_1, [2] for every n to define a_n, by using the terms $a_1, a_2, \dots, a_{n-1}$. Such a definition of a sequence is said to be *recursive*.

3.2. A set A is said to be *finite*, if it has a finite number of elements; otherwise, it is said to be *infinite*. The void set $\emptyset$ is taken as finite.

Every finite set is termed *countable*. An *infinite* set A is said to be *countable* if there exists a *one-to-one* mapping of the set **N** onto the set A; i.e. if there is a sequence $\{a_n\}_1^{\infty}$ such that $a_n \in A$ and such that, for every $a \in A$ there is a unique index n such that $a_n = a$. A set A is said to be *uncountable* if it is not countable. Many authors reserve the term countable for infinite countable sets only.

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*) The sense of the sign min (and similarly for max in the following) is obvious. Also, see section 4.10.

3.3. 3.3.1. *Every subset C of the set* $\mathbf{N}$ *is countable.*

This is evident for finite C. If C is infinite, there is a one-to-one *increasing* sequence $\{i_n\}$ such that C consists exactly of its terms.

3.3.2. *Every subset B of a countable set A is countable.*

This is evident for a finite set B. If B is infinite, the set A is also infinite, hence there is a one-to-one sequence $\{a_n\}$ such that A consists exactly of its terms. Since B is infinite, the set C of the numbers n with $a_n \in B$ is infinite and hence, by 3.3.1, there is a subsequence $\{a_{i_n}\}$ of the sequence $\{a_n\}$ (hence, one-to-one) such that its terms form exactly the set B.

3.4. 3.4.1. *Let A be a countable set. Let there exist a mapping f of the set A onto a set B. Then B is a countable set.*

This is evident for a finite set B. If B is infinite, then let us choose, for every $y \in A$, exactly one $x \in A$ with $f(x) = y$. Let C be the set of the chosen $x \in A$. We have $C \subset A$ and, hence, the set C is countable by 3.3.2. Evidently $\varphi = f_C$ (see 2.4) is a one-to-one mapping of the set C onto B. Since B is infinite and since φ is one-to-one, the set C is also infinite. Since C is countable, there is a one-to-one sequence $\{a_n\}$ such that its terms form exactly the set C. Then $\{\varphi(a_n)\}$ is a one-to-one sequence such that its terms form exactly the set B.

If we put $A = N$ in 3.4.1, we get

3.4.2. *The set of all terms of an arbitrary sequence is countable.*

3.5. 3.5.1. $\mathbf{N} \times \mathbf{N}$ *is an infinite countable set.*

Proof: For $(m, n) \in \mathbf{N} \times \mathbf{N}$ put $f(m, n) = 2^{m+n} + m$. Since $0 < m < 2^m < 2^{m+n}$, we have $2^{m+n} < f(m, n) < 2^{m+n+1}$, so that a value $f(m, n)$ determines the values $m + n$, m (and also n). Hence, f is a one-to-one mapping of the set $\mathbf{N} \times \mathbf{N}$ into the set $\mathbf{N}$. If $C = f(\mathbf{N} \times \mathbf{N})$, then f_{-1} is a (one-to-one) mapping of the (by 3.3.1) countable set C onto the set $\mathbf{N} \times \mathbf{N}$, and hence the set $\mathbf{N} \times \mathbf{N}$ is countable by 3.4.1. Of course, the set $\mathbf{N} \times \mathbf{N}$ is infinite.

If we assign the number m/n to every $(m, n) \in \mathbf{N} \times \mathbf{N}$, we obtain the following result:

3.5.2. *The set of all positive rational numbers is countable.*

3.6. *Let* $\mathbf{C} \neq \emptyset$ *be a countable set. Let, for every* $z \in \mathbf{C}$, $A(z)$ *be a countable set. Then* $\bigcup_{z \in \mathbf{C}} A(z)$ *is countable.*

Proof: If $\bigcup A(z) = \emptyset$, this is evident. Hence, let there exist an $\alpha \in \bigcup A(z)$ (we choose α arbitrarily, but fixed). Since $\mathbf{C} \neq \emptyset$ is a countable set, there is a one-to-one (finite or infinite) sequence $\{c_m\}_{m=1}^{p}$ ($p \in \mathbf{N}$) or $\{c_m\}_{m=1}^{\infty}$ such that $\mathbf{C}$ consists exactly of all the terms of the sequence. If the set $A(c_m)$ is infinite, there is a one-to-one infinite sequence $\{a_{mn}\}_{n=1}^{\infty}$ consisting exactly of the elements of $A(c_m)$; if the set $A(c_m)$ is finite, there exists a sequence (not one-to-one this time) $\{a_{mn}\}_{n=1}^{\infty}$, consisting exactly of the elements of the set $A(c_m) \cup (\alpha)$. If the set $\mathbf{C}$ is finite and if $m > p$, put $a_{mn} = \alpha$. For $(m, n) \in \mathbf{N} \times \mathbf{N}$ put $f(m, n) = a_{mn}$. Then f is a mapping of the countable (by 3.5.1) set $\mathbf{N} \times \mathbf{N}$ onto the set $\bigcup_{z \in \mathbf{C}} A(z)$, and hence this last set is countable by 3.4.1.

3.7. *Uncountable sets exist.* For:

3.7.1. *Countable sets of real numbers contain no intervals.*

Proof: Let there exist, on the contrary, real numbers a and b such that $a < b$, and also a sequence $\{c_n\}$ of real numbers containing every real number x such that $a < x < b$. Let us determine an index ν_1 such that: [1] $a < c_{\nu_1} < b$, [2] ν_1 is the least index with this property. Put $u_1 = c_{\nu_1}$. Let us determine an index μ_1 such that: [1] $a < u_1 < c_{\mu_1} < b$, [2] μ_1 is the least index with this property. Put $v_1 = c_{\mu_1}$. We proceed recursively to construct sequences $\{u_n\}_1^{\infty}$, $\{v_n\}_1^{\infty}$ so that, for every n: $a < u_n < v_n < b$ (as was the case for $n = 1$). If, for some $p > 1$, all the members u_n and v_n have been constructed for all $n < p$, we determine the index ν_p for which [1] $a < u_{p-1} < c_{\nu_p} < v_{p-1} < b$, [2] ν_p is the least index with this property; put $u_p = c_{\nu_p}$. Further, let us determine an index μ_p for which: [1] $a < u_p < c_{\mu_p} < v_{p-1} < b$, [2] μ_p is the least index with this property; put $v_p = c_{\mu_p}$. Then, for every n, $a < u_n < u_{n+1} < v_{n+1} < v_n$, and hence $\{u_n\}$ is an increasing bounded sequence of real numbers and hence, by a well-known theorem from the theory of real numbers, there exists $\lim u_n = \alpha$. We have $a < u_1 \leqq \alpha \leqq v_1 < b$. Hence, there is an index k with $\alpha = c_k$. For every n, $u_n < c_k < v_n$. Since ν_n and μ_n were always the least indices, we have $k > \nu_n$ for every n, which is impossible, since evidently $\nu_n < \mu_n < \nu_{n+1}$ and hence $\lim \nu_n = \infty$.

Exercises

3.1.* The set of all rational numbers is countable.

3.2. The set of all irrational numbers contained in an interval is uncountable.

3.3. If A and B are countable sets, then $A \times B$ is countable.

3.4. The system of all finite subsets of a countable set is countable.

3.5. The set of all numbers of the form $\sum_{n \in A} 2^{-n}$, where A varies over all the non-void subsets of $\mathbf{N}$, coincides with the interval $\mathrm{E}_t[0 < t \leqq 1]$.

3.6. The system of all subsets of the set $\mathbf{N}$ is uncountable.
3.7. Every infinite set contains an infinite countable subset.
3.8. The system of all subsets of an infinite set is uncountable.
3.9. The system of all infinite subsets of an infinite set is uncountable.
3.10. The system of all polynomials $\sum_{k=0}^{p} a_k x^k$ with integral coefficients $a_k = 0, \pm 1, \pm 2, \ldots, a_p \neq 0$, is countable.
3.11. The set of all real algebraic numbers is countable.
3.12. Real transcendental numbers exist.
3.13. The set of all transcendental numbers in a given interval is uncountable.
3.14.* Let A be a countable set such that $0 \in A$. Let $\mathfrak{A}$ be the system of all sequences $\{a_n\}_1^\infty$ such that: [1] $a_n \in A$ for every n, [2] there is an index p such that $a_n = 0$ for every $n > p$. Then the system $\mathfrak{A}$ is countable.

§ 4. Ordered sets

4.1. *To order* a set P means to give a rule by which we decide whether an element $a \in P$ *precedes* an element $b \in P$ or not; such a rule must satisfy the following three conditions:

[1] *if a precedes b, then b does not precede a;*
[2] *if neither a precedes b, nor b precedes a, then $a = b$;*
[3] *if a precedes b and b precedes c, then a precedes c.*

By [1], a never precedes a. If a precedes b, we say that *b follows a.* If either simultaneously a precedes b and b precedes c, or simultaneously a follows b and b follows c, we say that b is *between* a and c (or between c and a). We say that $a \in P$ is the *first* element, if no $x \in P$ precedes a; we say that $a \in P$ is the *last* element, if no $x \in P$ follows a. By [2] there is at most one first element and at most one last element.

A set P may certainly be ordered in various ways. If it is ordered by a given rule, we obtain a new ordering by stating that in the new sense, a precedes b if and only if a followed b in the former one. The new ordering is called the *inverse ordering* to the previous one. We say also that the two orderings are *mutually inverse.*

If a set P is ordered by some rule, then the rule also orders an arbitrary subset $A \subset P$. If we speak about an ordering of a subset A of an ordered set P, we mean, of course, the ordering of A determined by the ordering of the set P.

The symbol $\mathbf{E}_1$ denotes throughout the present book the set of all real numbers. $\mathbf{R}$ denotes, as stated above (in section 2.3), the set $\mathbf{E}_1$ augmented by two elements, one of which is denoted by $+\infty$ or simply ∞ and the second by $-\infty$. If $x \in \mathbf{R}$, $y \in \mathbf{R}$, then the assertion "x precedes y" means $x < y$, where (throughout all the book) we put:

$$-\infty < \infty\,, \quad -\infty < c\,, \quad c < \infty$$

for every $c \in \mathbf{E}_1$. This defines the so called *natural ordering* of the set $\mathbf{R}$, which determines the *natural ordering* of each of its subsets. $-\infty$ is the first, $+\infty$ the last element of the set **R** in its natural ordering; this contrasts with the naturally ordered $\mathbf{E}_1$, which has neither first nor last element.

4.2. Let P and Q be two ordered sets. We say that they are *similarly ordered* or that their given orderings are *similar**), if there exists a mapping f of the set P onto the set Q such that for $a \in P$, $b \in P$

$$a \text{ precedes } b \Rightarrow f(a) \text{ precedes } f(b).$$

In this sense we also say that f is a *similar mapping* of the set P onto the set Q. If $a \in P$, $b \in P$, $a \neq b$, then either a precedes b or b precedes a; hence either $f(a)$ precedes $f(b)$ or $f(b)$ precedes $f(a)$ and hence $f(a) \neq f(b)$. Thus the mapping f is one-to-one, so that there exists an inverse mapping f_{-1} of the set Q onto the set P. The reader may prove easily that f_{-1} is a *similar* mapping of the set Q onto the set P.

4.3. Let P be an ordered set. We say that P is *well ordered* if for every A, $0 \neq A \subset P$, there is a first element $x \in A$, i.e. there is an element $x \in A$ such that

$$y \in A,\ y \neq x \Rightarrow x \text{ precedes } y.$$

The natural ordering of the set of all natural numbers $1, 2, \ldots$, or of any of its subsets, is a well ordering. The theory of well ordered sets has numerous applications, many of them following from the famous Zermelo theorem, asserting that every set may be well ordered. Nevertheless, we shall not occupy ourselves with this theory. [An introduction to the theory of well ordered sets may be found, e. g., in the book A. Fraenkel, Abstract Set Theory, Amsterdam, 1958. Ed.]

4.4. If P is an ordered set, $a \in P$, $b \in P$, a precedes b and if there is no $x \in P$ between a and b, we say that *a immediately precedes b* or that *b immediately follows a*.

Remark: If P is an ordered set, $a \in P$, $b \in P$, a precedes b and the number of points $x \in P$ between a and b is finite ($\geqq 0$), then there exists an element x which immediately follows a, and an element z which immediately precedes b.

Let us, e.g., prove the first statement (the second one may be proved similarly, or it may be reduced to the first by using the inverse ordering): Let $x \in P$ following a be such that between a and x there is the *least possible* number of elements. Since b follows a and there is only a finite number of elements between a and b, such an x exists. If x does not immediately follow a, there is a $y \in P$ between a and x. Every element which is between a and y is also between a and x, but between a and y there are less elements than between a and x. This is a contradiction, so that x immediately follows a.

*) The term commonly used today is *isomorphic* (Ed.).

The natural ordering of the set of all integers 0, ± 1, $\pm 2, \ldots$ and the natural ordering of each its subset P have the property that for every pair $a \in P$, $b \in P$ there is only a finite number of elements between a and b. Conversely:

4.4.1. *Let a set $P \neq \emptyset$ be ordered in such a manner that whenever $a \in P$, $b \in P$, there is only a finite number of elements $x \in P$ between a and b. Let us distinguish the following four cases:* [1] *P has both first and last elements,* [2] *P has a first element and no last element,* [3] *P has no first element and has a last element,* [4] *P has neither last nor first element. Then the given ordering of the set P is similar to the natural ordering of:* [1] *the set of natural numbers less than or equal to p, where p is the number of the elements of the set P,* [2] *the set of all natural numbers,* [3] *the set of all negative integers,* [4] *the set of all integers.**)

Proof: Let us begin with the cases in which the first element exists and denote it by a_1. If a_n is defined for some n and if a_n is not the last element, then there is an element b following a_n. Since there is a finite number of elements between a_n and b by the previous remark there exists an a_{n+1} following immediately a_n. There are two possibilities: either (case α) we obtain a finite sequence $\{a_n\}_1^p$ such that a_p is the last element of the set P, or (case β) we obtain an infinite sequence $\{a_n\}_1^\infty$. It is easy to show that, in both cases, the sequence $\{a_n\}$ is one-to-one. Let us prove that in both cases the set of all terms of the sequence $\{a_n\}$ is equal to the whole set P. If we assume the contrary, there exists an $x \in P$ distinct from every a_n. Since a_1 is the first element, x follows a_1. The element x cannot follow every a_n, in the case α because a_p is the last element, in the case β because there is only a finite number of elements between a_1 and x. Hence, there is a term a_n of the sequence $\{a_n\}$ ($1 \leqq n < p$ in the case α) such that x follows a_n but x does not follow a_{n+1}; hence x is between a_n and a_{n+1}. This is a contradiction, since a_{n+1} immediately follows a_n. Thus, P is exactly the set of all terms of the sequence $\{a_n\}$. Putting $f(a_n) = n$ we obtain, as one sees easily, a similar mapping of the set P onto the set of all natural numbers less then or equal to p (case α) or onto the set of all natural numbers (case β). We also see that in the case β the set P has no last element.

Let us turn to the case where P has a last element. This case may be reduced. with the aid of the inverse ordering, to the previous one. Thus, if P has no first and has a last element, we construct a sequence $\{b_n\}_1^\infty$ such that on putting $f(b_n) = -n$ we obtain a similar mapping of the set P onto the set of all negative integers.

There remains the case where P has neither first nor last element. Choose arbitrarily $a_0 \in P$. Define $P_1 \subset P$ and $P_2 \subset P$ as follows:

$$P_1 = \mathop{\mathrm{E}}_x[a_0 \text{ precedes } x], \quad P_2 = \mathop{\mathrm{E}}_x[a_0 \text{ follows } x].$$

Since a_0 is neither first nor last element, $P_1 \neq \emptyset \neq P_2$. Moreover,

$$P = (a_0) \cup P_1 \cup P_2$$

*) Case [1] obtains, of course, if and only if the set P is finite.

with disjoint summands. Both the sets P_1 and P_2 have the property that between any two of their elements there is only a finite number of elements. Moreover, we see easily that P_1 has a first and has no last element, whereas P_2 has no first and has a last element. Hence, there are two sequences $\{a_n\}_1^\infty$ and $\{a_{-n}\}_1^\infty$ such that P_1 is the set of all terms of the sequence $\{a_n\}_1^\infty$, that P_2 is the set of all terms of the sequence $\{a_{-n}\}_1^\infty$, and such that

$$1 \leqq m < n \Rightarrow a_m \text{ precedes } a_n,\ a_{-n} \text{ precedes } a_{-m}.$$

Putting $f(a_n) = n$ for $n = 0, \pm 1, \pm 2, \ldots$, we obtain a similar mapping of the set P onto the set of all integers.

4.5. Let P be an ordered set. We say that P is *densely ordered*, or, that the given ordering is *dense*, if P contains at least two distinct elements and if there is no pair $a \in P$, $b \in P$ such that a immediately precedes b. *Every densely ordered set P is infinite.* Moreover, by the remark in section 4.4, *if $a \in P$, $b \in P$, $a \neq b$, the set of all $x \in P$ between a and b is infinite.*

The natural ordering of the set of all rational numbers is dense. Let us recollect (see ex. 3.1) that this set is countable.

4.6. 4.6.1. *Let P be a countable ordered set. Let H be a densely ordered set. Then there is a set $Q \subset H$ such that P and Q are similarly ordered.* (The ordering of the set Q is, of course, determined by the given ordering of the set $H \supset Q$ by 4.1.)

Proof: If $P = \emptyset$, it suffices to put $Q = \emptyset$. Hence, let $P \neq \emptyset$. P is either finite (case α) or infinite (case β). In the case α there exists a finite one-to-one sequence $\{a_n\}_1^p$, in the case β an infinite one-to-one sequence $\{a_n\}_1^\infty$ such that, in both cases, P is exactly the set of all terms of the sequence $\{a_n\}$. Since H is densely ordered, it is infinite, and hence we may choose a $b_1 \in H$ such that b_1 is neither the first nor the last element of the set H. (We do not at all assert that there is a first or last element in H!) For a given $q = 1, 2, 3, \ldots$ (in the case α let $q < p$), let there be constructed elements $b_n \in H$ ($1 \leqq n \leqq q$) in such a way that none of them is first nor last in H, and that, for $1 \leqq m \leqq q$, $1 \leqq n \leqq q$,

$$a_m \text{ precedes } a_n \Leftrightarrow b_m \text{ precedes } b_n. \tag{1}$$

(This is satisfied for $q = 1$.) We shall prove that it is possible to choose an element $b_{q+1} \in H$ which is neither first nor last in H, such that (1) holds for $1 \leqq m \leqq q + 1$, $1 \leqq n \leqq q + 1$. Let us distinguish three cases: [1] Let a_{q+1} precede a_n for $1 \leqq n \leqq q$. Evidently there is an index m ($1 \leqq m \leqq q$) such that

$$1 \leqq n \leqq q, \quad n \neq m \quad \text{if and only if} \quad a_m \text{ precedes } a_n.$$

Since b_m is not the first element in H and since H is densely ordered, there is a $b_{q+1} \in H$ such that b_{q+1} is not first in H and precedes b_m. It is easy to see that b_{q+1} is not the last element in H and that (1) holds for $1 \leqq m \leqq q + 1$, $1 \leqq n \leqq q + 1$. [2] Let a_{q+1} follow a_n for $1 \leqq n \leqq q$. We construct a b_{q+1} in a way similar to that in the previous case. [3] Let there be at least one index n such that $1 \leqq n \leqq q$ and that a_{q+1} precedes a_n, and let there also be at least one index n such that $1 \leqq n \leqq q$ and that a_{q+1} follows a_n. Evidently, there is an index h ($1 \leqq h \leqq q$) such that a_{q+1} precedes a_h and that

$$1 \leqq n \leqq q, \quad n \neq h, \quad a_{q+1} \text{ precedes } a_n \quad \Rightarrow \quad a_h \text{ precedes } a_n.$$

Similarly, there is an index k ($1 \leqq k \leqq q$) such that a_{q+1} follows a_k and that

$$1 \leqq n \leqq q, \quad n \neq k, \quad a_{q+1} \text{ follows } a_n \quad \Rightarrow \quad a_k \text{ follows } a_n.$$

Since H is densely ordered, there is a $b_{q+1} \in H$ between b_h and b_k; it is easy to see that b_{q+1} is neither first nor last in H and that (1) holds for $1 \leqq m \leqq q + 1$, $1 \leqq n \leqq q + 1$.

Proceeding in this way, we construct in the case α a finite sequence $\{b_n\}_1^p$ and in the case β an infinite sequence $\{b_n\}_1^\infty$ such that, putting $f(a_n) = b_n$, we obtain in both cases a similar mapping f of the set P onto the set $Q \subset H$ of all terms of the sequence $\{b_n\}$.

Putting the naturally ordered set of rational numbers for H in 4.6.1, we obtain the following theorem:

4.6.2. *Let P be a countable ordered set. There exists a set Q such that:* [1] *the elements of the set Q are rational numbers*; [2] *the given ordering of the set P is similar to the natural ordering of Q.*

4.7. 4.7.1. *Let P and Q be densely ordered sets without first and last elements. Then P and Q are similarly ordered.*

Proof: As P and Q are densely ordered, they are infinite. Thus, there are one-to-one sequences $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ such that the sets of all their terms are P and Q respectively. We shall construct recursively two new one-to-one sequences $\{u_n\}$ and $\{v_n\}$ as follows: Let $u_1 = a_1$, $v_1 = b_1$. For a certain $q = 1, 2, 3, \ldots$ let there be constructed terms $u_n \in P$ and $v_n \in Q$ ($1 \leqq n \leqq q$) such that, for $1 \leqq m \leqq q$, $1 \leqq n \leqq q$,

$$u_m \text{ precedes } u_n \text{ if and only if } v_m \text{ precedes } v_n. \tag{1}$$

Let us distinguish two cases. *First,* if q is odd, we choose $u_{q+1} = a_h$ where h is the *least* index such that $a_h \neq u_n$ for $1 \leqq n \leqq q$; then we put $v_{q+1} = b_k$, where k is chosen so that (1) holds for $1 \leqq m \leqq q$, $1 \leqq n \leqq q$. That such a b_k exists may be deduced in the same way as in the analogous consideration in the preceding proof. *Secondly,* if q is even, we choose $v_{q+1} = b_h$, where h is the *least* index such that $b_h \neq v_n$ for $1 \leqq n \leqq q$; then we put $u_{q+1} = a_k$, where k is chosen so that (1) holds for $1 \leqq m \leqq q + 1$, $1 \leqq n \leqq q + 1$. In this way we obtain two one-to-one sequences $\{u_n\}_1^\infty$ and $\{v_n\}_1^\infty$. Since the index h was always chosen the least possible, the set of all terms of the sequence $\{u_n\}$ is the set P and similarly for $\{v_n\}$ and Q. The relation (1) holds for all natural numbers m and n; consequently, putting $f(u_n) = v_n$, we obtain a similar mapping f of the set P onto the set Q.

If we put the naturally ordered set of all rational numbers for Q in 4.7.1, we obtain the theorem:

4.7.2. *Let P be a countable densely ordered set without the first and last elements. Then P is ordered similarly to the naturally ordered set of all rational numbers.*

4.8. Let P be an ordered set. We define a *cut* of the (ordered) set P to be any couple $\alpha = (A_1, A_2)$ where $A_1 \cup A_2 = P$ and

$$x_1 \text{ precedes } x_2 \text{ whenever } x_1 \in A_1,\ x_2 \in A_2. \tag{1}$$

Notice that condition (1) implies $A_1 \cap A_2 = \emptyset$. We call the set A_1 the *lower class* of the cut and the set A_2 the *upper class* of the cut.

An important case of a cut is the following. We choose an $a \in P$ and define A_1 and A_2 as follows:

$$x \text{ precedes } a \Rightarrow x \in A_1, \qquad x \text{ follows } a \Rightarrow x \in A_2,$$

while the point a will be included in either A_1 or A_2, certainly in only one of them. In either case, (A_1, A_2) is a cut of the set P; in both cases we say that (A_1, A_2) is a cut *generated* by the element a. If $a \in A_1$, the element a is the last element in the lower class; if $a \in A_2$, the element a is the first element in the upper class. Conversely, if $\alpha = (A_1, A_2)$ is a cut such that there is a last element in A_1, then α is generated by this element, and if $\alpha = (A_1, A_2)$ is a cut such that there is a first element in A_2, then α is generated by this element.

A cut $\alpha = (A_1, A_2)$ is called a *jump*, if there exist both a last element of the lower class a_1 and a first element of the upper class a_2; the cut α is then generated by both a_1 and a_2. Conversely, a cut which may be generated by two distinct elements is a jump. If there were an element $x \in P$ between a_1 and a_2, it could be neither in A_1 nor in A_2; thus, a_1 precedes a_2 immediately. Therefore, *a densely ordered set has no jumps.*

A cut $\alpha = (A_1, A_2)$ is said to be a *gap*, if $A_1 \neq \emptyset \neq A_2$ and if there is neither a last element of the lower class, nor a first element of the upper class. A gap can be generated by no $a \in P$.

The couple $\alpha_1 = (P, \emptyset)$ is a cut of the set P. If there exists a last element of the set P, α_1 is generated by it; if there is none, α_1 can be generated by no $a \in P$. The couple $\alpha_2 = (\emptyset, P)$ is a cut of the set P. If there exists a first element of the set P, α_2 is generated by it; if there is none, α_2 can be generated by no $a \in P$.

If an ordered set has both first and last element and if it has no gaps, then each of its cuts can be generated by an $a \in P$. If, moreover, P is densely ordered, then every cut is generated by a unique $a \in P$.

4.9. Let M be the set of all *rational* numbers. If α is an *irrational* number and if we put

$$A_1 = \mathop{\mathrm{E}}_x [x \in M,\ x < \alpha], \qquad A_2 = \mathop{\mathrm{E}}_x [x \in M,\ x > \alpha],$$

then (A_1, A_2) is a gap of the naturally ordered set M. It is said to be (in a somewhat different sense than in 4.8, since α is not an element of M) *generated* by the irrational number α. Distinct irrational numbers generate distinct gaps of the set M. It is well known from the theory of irrational numbers that, conversely, every gap of the set M is generated by a unique irrational number; in the Dedekind theory, the irrational numbers are *defined* as the gaps of the set M. The naturally ordered set $\mathbf{E}_1$ of all real numbers (both rational and irrational) has then no gaps.*) These facts form, as is well-known, the fundament of an exact construction of the whole of mathematical analysis. We shall deduce here their abstract basis.

Let P be an ordered set and let Q be the set of all its gaps. For clarity, we shall denote the elements of P by lower- case Roman letters and the elements of Q, i.e. the gaps of P, by lower- case Greek letters, putting $\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$ etc. Evidently $P \cap Q = \emptyset$. We know what is meant by "a precedes b" ($a \in P$, $b \in P$). Further, we say that [1] a precedes α [$a \in P$, $\alpha = (A_1, A_2) \in Q$] if $a \in A_1$; [2] α precedes a if $a \in A_2$; [3] α precedes β [$\alpha = (A_1, A_2)$, $\beta = (B_1, B_2)$] if $A_1 \subset B_1 \neq A_1$. We must prove three statements in order to show that we have defined an ordering of the set $P \cup Q$ (see 4.1):

I. Given two elements of $P \cup Q$, the first of them cannot precede the second one, if the second one precedes the first one. We know it is never the case that simultaneously a precedes b and b precedes a. If a precedes α and α precedes a, we have $a \in A_1$, $a \in A_2$, whilst $A_1 \cap A_2 = \emptyset$. If α precedes β and β precedes α, then $A_1 \subset B_1 \neq A_1$, $B_1 \subset A_1 \neq B_1$, which is also impossible.

II. If one element of $P \cup Q$ does not precede another one and the latter does not precede the former, then these elements are equal. First, we know that if neither a precedes b, nor b precedes a, then $a = b$. Secondly, let us prove that the assumption that a does not precede α and α does not precede a leads to a contradiction; really, such an assumption implies that neither $a \in A_1$ nor $a \in A_2$ while $a \in P = A_1 \cup A_2$. Thirdly, let neither α precede β nor β precede α; we have to show that $\alpha = \beta$. Since $A_2 = P - A_1$, $B_2 = P - B_1$, it suffices to show that $A_1 = B_1$. Let $A_1 \neq B_1$. As α does not precede β, A_1 is not a subset of B_1 and hence there is an $a \in A_1 - B_1$;

*) Certainly, the set $\mathbf{R}$ consisting of the elements of $\mathbf{E}_1$ and of the first element $-\infty$ and the last element $+\infty$ also has no gaps.

similarly, there is a $b \in B_1 - A_1$, since β does not precede α. We have $a \in A_1 - B_1 = A_1 \cap (P - B_1) = A_1 \cap B_2$ and similarly $b \in A_2 \cap B_1$. As $a \in A_1$, $b \in A_2$, a precedes b; on the other hand, $a \in B_2$, $b \in B_1$ and hence b precedes a. This is a contradiction.

III. Let some three elements of $P \cup Q$ have the property that the first one precedes the second one, and the second one precedes the third one. We have to prove that the first one precedes the third one. We must investigate eight cases: [1] a precedes b, b precedes c: We know that then a precedes c. [2] α precedes b, b precedes c: Since α precedes b, we have $b \in A_2$. If there were $c \in A_1$, c would precede b, since (A_1, A_2) is a cut. Thus, $c \in A_2$, i.e. α precedes c. [3] a precedes β, β precedes c: Then $a \in B_1$, $c \in B_2$ and hence a precedes c. [4] a precedes b, b precedes γ: Since b precedes γ, we have $b \in C_1$. If $a \in C_2$, then a follows b; thus, $a \in C_1$, i.e. a precedes γ. [5] α precedes β, β precedes c: Since β precedes c, we have $c \in B_2$. Since α precedes β, $A_1 \subset B_1$ and hence $A_2 = P - A_1 \supset P - B_1 = B_2$, and hence $c \in A_2$, i.e. α precedes c. [6] α precedes b, b precedes γ: Hence, $b \in A_2 \cap C_1$. If α did not precede γ, we either would have (see II above) $\alpha = \gamma$ or γ would precede α, so that $C_1 \subset A_1$ and hence $b \in A_2 \cap C_1 \subset A_1 \cap A_2 \neq \emptyset$; this is a contradiction. [7] a precedes β, β precedes γ: We have $a \in B_1 \subset C_1$, hence $a \in C_1$, i.e. a precedes γ. [8] α precedes β, β precedes γ: We have $A_1 \subset B_1 \subset C_1 \neq B_1$ and hence $A_1 \subset C_1 \neq A_1$, i.e. α precedes γ.

Thus, we have really constructed an ordering of the set $P \cup Q$ which determines the previously given ordering of the set P. This ordering has the following four properties:

[1] *Every $\alpha \in Q$ follows some $a \in P$, namely every $a \in A_1$.*

[2] *Every $\alpha \in Q$ precedes some $a \in P$, namely every $a \in A_2$.*

[3] *If $\alpha \in Q$, $\beta \in Q$ and $\alpha \neq \beta$, then there is always some $a \in P$ between α and β.* Let α precede β. Then $A_1 \subset B_1 \neq A_1$ and hence $\emptyset \neq B_1 - A_1 = A_2 \cap B_1$. Choosing an $a \in A_2 \cap B_1$ we see that a follows α and precedes β, i.e. a is between α and β.

[4] *If $a \in P$, $\alpha \in Q$, then there is always some $b \in P$ between a and α.* Let, e.g., a precede α, hence $a \in A_1$. Since $\alpha = (A_1, A_2)$ is a gap in P, there is no last element in A_1, and hence there is a $b \in A_1$ following a. As $b \in A_1$, b precedes α. As, moreover, b follows a, b is between a and α.

The ordering of the set $P \cup Q$ just constructed has no gaps.

Proof: Let $(\mathfrak{A}_1, \mathfrak{A}_2)$ be a cut of the set $P \cup Q$ and let $\mathfrak{A}_1 \neq \emptyset \neq \mathfrak{A}_2$. We have to prove that either there is a last element of the class $\mathfrak{A}_1$, or there is a first element of the class $\mathfrak{A}_2$. Put $A_1 = P \cap \mathfrak{A}_1$, $A_2 = P \cap \mathfrak{A}_2$. Obviously $\alpha = (A_1, A_2)$ is a cut of the set P. As $\mathfrak{A}_1 \neq \emptyset$, there exists a $\beta \in \mathfrak{A}_1$. If $\beta \in P$ we have $\beta \in A_1$; if $\beta \in Q$ we know that there is a $b \in P$ preceding β. Since b precedes β and $\beta \in \mathfrak{A}_1$, b is not an element of $\mathfrak{A}_2$. Thus, $b \in \mathfrak{A}_1$; as $b \in P$, we have $b \in A_1$. Thus, in both cases $A_1 \neq \emptyset$. Similarly we may prove that $A_2 \neq \emptyset$. Let us distinguish three cases:

First, let there be a last element b in the set A_1. Let us assume that b is not the last element of the set $\mathfrak{A}_1$. Then there is a $\beta \in \mathfrak{A}_1$ following b. As $P \cap \mathfrak{A}_1 = A_1$ and as β follows the last element of the set A_1, β is not in P and hence $\beta \in Q$. As b precedes β, there is, by the property [4] above, a $c \in P$ which is between b and β, i.e. which follows b and precedes β. Since c lies in P and follows the last element of the set A_1, c is not in A_1. Since c precedes $\beta \in \mathfrak{A}_1$, c does not belong to $\mathfrak{A}_2$. Therefore $c \in \mathfrak{A}_1$, and it follows $c \in P \cap \mathfrak{A}_1$; i.e., we have $c \in A_1$. This is a contradiction. Thus, b is the last element of the set $\mathfrak{A}_1$.

Secondly, let there be a first element b of the set A_2. We may show that b is the first element of the set $\mathfrak{A}_2$ in the same way as we did in the first case.

Thirdly, let there be neither a last element in A_1, nor a first element in A_2. As $A_1 \neq \emptyset \neq A_2$, $\alpha = (A_1, A_2)$ is a gap of the set P and hence $\alpha \in Q$. Let $\beta \in P \cup Q$ precede α. If $\beta \in P$, we have $\beta \in A_1$ and hence $\beta \in \mathfrak{A}_1$. If $\beta \in Q$, then, by the property [3] above, there is an $a \in P$ between α and β, i.e., a follows β and precedes α. Since $a \in P$ precedes α, we have $a \in A_1$, and hence $a \in \mathfrak{A}_1$. Since β precedes a, β is not in $\mathfrak{A}_2$, and hence $\beta \in \mathfrak{A}_1$. Thus, for $\beta \in P \cup Q$ it holds that

$$\beta \text{ precedes } \alpha \Rightarrow \beta \in \mathfrak{A}_1 .$$

Similarly we may prove that

$$\beta \text{ follows } \alpha \Rightarrow \beta \in \mathfrak{A}_2 .$$

As $\mathfrak{A}_1 \cup \mathfrak{A}_2 = P \cup Q$, we have either $\alpha \in \mathfrak{A}_1$ or $\alpha \in \mathfrak{A}_2$. In the first case, α is the last element in $\mathfrak{A}_1$; in the second one, α is the first element in $\mathfrak{A}_2$.

4.10. Let us finish this section recollecting some well known consequences of the fact that the natural ordering of the set $\mathbf{R}$ has neither jumps nor gaps.

Let an arbitrary set $M \subset \mathbf{R}$ be given. Denote by A_2 the set of all $x \in \mathbf{R}$ such that

$$y \in M \Rightarrow y < x,$$

and put $A_1 = \mathbf{R} - A_2$. We see easily that (A_1, A_2) is a cut of the naturally ordered set $\mathbf{R}$. By the remark at the end of 4.8 there is exactly one $\alpha \in \mathbf{R}$ generating the cut (A_1, A_2). It is easy to prove that the number α is characterized by the following two properties:

[1] if $\beta \in \mathbf{R}$ and $\beta < \alpha$, there is a $y \in M$ such that $y \geqq \beta$;

[2] $\beta \in \mathbf{R}$, $\beta > \alpha$, $y \in M \Rightarrow y < \beta$.

Following Hausdorff, this number α is called the *supremum* of the set M and it is denoted by

$$\sup M.^{*)}$$

*) If f is a function on P, then $\sup f(P)$ is also denoted by $\sup_{x \in P} f(x)$; similarly for infimum.

Instead of the term supremum, one uses sometimes the term *least upper bound.* If $\alpha \in M$, then α is evidently the greatest number contained in the set M; it is then called the *maximum* of the set M and sometimes denoted by

$$\max M.^{*)}$$

If $\alpha \in \mathbf{R} - M$, the maximum of the set M does not exist.

Similarly, for every $M \subset \mathbf{R}$ there exists a number $\alpha' \in \mathbf{R}$ characterized by the properties:

[1′] if $\beta \in \mathbf{R}$ and $\beta > \alpha'$, there is a $y \in M$ such that $y \leqq \beta$;
[2′] $\beta \in \mathbf{R}$, $\beta < \alpha'$, $y \in M \Rightarrow y > \beta$.

This number α' is called the *infimum* of the set M and is denoted by

$$\inf M.$$

Instead of the term infimum, one uses sometimes the term *greatest lower bound.* If $\alpha' \in M$, then α' is termed the *minimum* of the set M and is sometimes denoted by

$$\min M.$$

According to our definition we have

$$\sup \emptyset = -\infty, \qquad \inf \emptyset = \infty.$$

On the other hand,

$$\text{if} \quad M \neq \emptyset, \quad \text{then} \quad \sup M \geqq \inf M.$$

Suppose, on the contrary, that $\alpha < \alpha'$. Choose a $y \in M$. By [2] we have $y < \alpha'$; putting $\beta = y$ in [2′] we obtain the contradiction $y > y$.

A set M is said to be *bounded*, if $\sup M < \infty$ and $\inf M > -\infty$; otherwise it is said to be *unbounded*. We see easily that M is bounded if and only if there is a $c \in \mathbf{E}_1$ such that

$$-c < x < c \quad \text{for every} \quad x \in M.$$

If f is a function on P, then $f(P)$ is a bounded set in the sense just defined if and only if f is bounded in the sense of 2.3.

Exercises

4.1. A finite set with n elements has $n!$ orderings (this is true also for $n = 0$, since $0! = 1$, and for $n = 1$).

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*) $\max(a, b)$ denotes the maximum of the set consisting of two points $a \in \mathbf{R}$ and $b \in \mathbf{R}$; similarly for minimum.

4.2. The set of all orderings of the set of all natural numbers is uncountable.

4.3. Let P and Q be two ordered sets. Then their cartesian product may be ordered as follows: (x_1, y_1) precedes (x_2, y_2) if and only if either x_1 precedes x_2, or simultaneously $x_1 = x_2$ and y_1 precedes y_2.

4.4. For every element z of an ordered set $\mathbf{C} \neq \emptyset$ let there be an ordered set $A(z)$ and let the union $\bigcup_{x \in \mathbf{C}} A(z)$ be disjoint. Then the set $M = \bigcup_{x \in \mathbf{C}} A(z)$ may be ordered as follows: If $x_1 \in M$, $x_2 \in M$, there exist elements $z_1 \in \mathbf{C}$, $z_2 \in \mathbf{C}$ with $x_1 \in A(z_1)$, $x_2 \in A(z_2)$; x_1 precedes x_2 if and only if either z_1 precedes z_2, or simultaneously $z_1 = z_2$ and x_1 precedes x_2 in $A(z_1)$. If the given orderings of $\mathbf{C}$ and of all the sets $A(z)$ are well orderings, we obtain a well ordering of the set M.

4.5. The natural ordering of the set of all decadic rationals, or of the set of all rational numbers x with $\alpha < x < \beta$ ($\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\alpha < \beta$), or of the set of all algebraic numbers, is similar to the natural ordering of all rational numbers.

4.6. Let P be a densely ordered countable set. Then P is ordered similarly with the naturally ordered set of all rational numbers contained in the interval: [1] $\underset{x}{\mathrm{E}}[0 < x < 1]$, if P has neither a first nor a last element, [2] $\underset{x}{\mathrm{E}}[0 \leqq x < 1]$, if P has the first but has no last element, [3] $\underset{x}{\mathrm{E}}[0 < x \leqq 1]$, if P has no first, but has a last element, [4] $\underset{x}{\mathrm{E}}[0 \leqq x \leqq 1]$, if P has both the first and last elements.

4.7. The condition (1) in the definition of a cut at the beginning of 4.8. may be replaced by either of the following conditions:

x_1 precedes x_2, $x_2 \in A_1 \Rightarrow x_1 \in A_1$,
x_1 precedes x_2, $x_1 \in A_2 \Rightarrow x_2 \in A_2$.

4.8. Let (A_1, A_2) be a cut of an ordered set P. Then (A_2, A_1) is a cut of the inversely ordered set P. If (A_1, A_2) is a jump or a gap, then (A_2, A_1) is also a jump or a gap respectively.

4.9. Let P and P' be ordered sets; let f be a similar mapping of P onto P'. Let Q and Q' be the sets of all gaps of the sets P and P' respectively. Let us order the sets $P \cup Q$ and $P' \cup Q'$ as in 4.9. Then there exists exactly one similar mapping φ of $P \cup Q$ onto $P' \cup Q'$ such that $\varphi_P = f$ (in the sense of 2.4).

4.10. Let Q be a densely ordered set which has neither first nor last element and which has no gaps. Let $P \subset Q$ be a countable set. For every $x \in Q - P$ and $y \in Q - P$, $x \neq y$, let there eb an $a \in P$ between x and y. Then there exists a similar mapping f of the set Q onto the (naturally ordered) set $\mathbf{E}_1$ such that $f(P)$ is the set of all rational numbers.

4.11. Let P be a given set containing at least two elements. Let there be given a set $\mathbf{M} \subset P \times P \times P$ such that

[1] $(a, c, b) \in \mathbf{M} \Rightarrow a \neq b$;
[2] $(a, c, b) \in \mathbf{M} \Rightarrow (b, c, a) \in \mathbf{M}$;
[3] $(a, c, b) \in \mathbf{M} \Rightarrow a \neq c$;
[4] $(a, d, c) \in \mathbf{M}$, $(a, c, b) \in \mathbf{M} \Rightarrow (a, d, b) \in \mathbf{M}$;
[5] $(a, d, c) \in \mathbf{M}$, $(d, c, b) \in \mathbf{M} \Rightarrow (a, c, b) \in \mathbf{M}$;
[6] $(a, x, b) \in \mathbf{M}$, $(a, y, b) \in \mathbf{M}$, $x \neq y \Rightarrow$ either $(a, x, y) \in \mathbf{M}$ or $(a, y, x) \in \mathbf{M}$;
[7] $(a, c, x) \in \mathbf{M}$, $(a, c, y) \in \mathbf{M}$, $x \neq y \Rightarrow$ either $(c, x, y) \in \mathbf{M}$ or $(c, y, x) \in \mathbf{M}$;
[8] $a \neq b \neq c \neq a \Rightarrow$ either $(a, b, c) \in \mathbf{M}$ or $(b, c, a) \in \mathbf{M}$ or $(c, a, b) \in \mathbf{M}$.

Then there are exactly two orderings of the set P such that c is between a and b if and only if $(a, c, b) \in \mathbf{M}$. These orderings are mutually inverse.

§ 5. Cyclically ordered sets*)

5.1. We may define the ordering of a set P as a subset $\mathbf{U}$ of $P \times P$ satisfying certain conditions, namely, as the set $\underset{(x,y)}{\mathrm{E}}[x \text{ precedes } y]$. Similarly, we define the *cyclical ordering* of a set P as a subset $\mathbf{C}$ of the set $P \times P \times P$ satisfying the following four conditions:

[1] $(a, b, c) \in \mathbf{C} \Rightarrow (b, c, a) \in \mathbf{C}$;

[2] $(a, b, c) \in \mathbf{C}$ and $(b, a, c) \in \mathbf{C}$ never hold simultaneously;

[3] if neither $(a, b, c) \in \mathbf{C}$ nor $(b, a, c) \in \mathbf{C}$, then some two of the elements a, b, c are equal;

[4] $(a, b, c) \in \mathbf{C}$, $(a, c, d) \in \mathbf{C} \Rightarrow (a, b, d) \in \mathbf{C}$.

[1] and [2] yield:

[5] if $(a, b, c) \in \mathbf{C}$, then $(b, c, a) \in \mathbf{C}$ and $(c, a, b) \in \mathbf{C}$, while we have neither $(b, a, c) \in \mathbf{C}$ nor $(c, b, a) \in \mathbf{C}$ nor $(a, c, b) \in \mathbf{C}$.

[5] yields:

[6] $(a, b, c) \in \mathbf{C} \Rightarrow a \neq b \neq c \neq a$.

[3] and [5] yield:

[7] if $a_1 \neq a_2 \neq a_3 \neq a_1$, then $(a_{i_1}, a_{i_2}, a_{i_3}) \in \mathbf{C}$ holds for exactly three of the six permutations i_1, i_2, i_3 of the indices 1, 2, 3.

Finally, the following analogy of condition [4] holds: [8] $(a, b, d) \in \mathbf{C}$, $(b, c, d) \in \mathbf{C} \Rightarrow (a, c, d) \in \mathbf{C}$. In fact, by [5] we have $(d, a, b) \in \mathbf{C}$, $(d, b, c) \in \mathbf{C}$ and hence, by [4], $(d, a, c) \in \mathbf{C}$ and hence $(a, c, d) \in \mathbf{C}$ by [1].

5.2. Let P be a cyclically ordered set and let $a \in P$. If $x \in P - (a)$, $y \in P - (a)$, we say that x precedes y if and only if $(a, x, y) \in \mathbf{C}$.

To prove that we have defined an ordering, we must find out whether the three conditions stated at the beginning of 4.1 are satisfied.

I. Let x precede y. Then $(a, x, y) \in \mathbf{C}$. By [5] (a, y, x) does not belong to $\mathbf{C}$, i.e., y does not precede x.

II. Let neither x precede y, nor y precede x; then neither $(a, x, y) \in \mathbf{C}$, nor $(a, y, x) \in \mathbf{C}$ and hence, by [3] and [5], some two of the elements a, x, y are equal. Since $x \in P - (a)$, $y \in P - (a)$, we have $x = y$.

III. Let x precede y and y precede z. Then $(a, x, y) \in \mathbf{C}$, $(a, y, z) \in \mathbf{C}$ and hence, by [4], $(a, x, z) \in \mathbf{C}$, i.e. x precedes z.

Thus, we have actually defined an ordering of the set $P - (a)$. This ordering will be denoted by $\mathbf{U}(a)$, or, more precisely, by $\mathbf{U}_{\mathbf{C}}(a)$.

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*) The beginner is recommended to omit this section for the time being. The results will not be used until the last chapters of this book.

Conversely:

5.2.1. *Let $a \in P$ and let there be given an ordering of the set $P - (a)$. Then there is exactly one cyclical ordering $\mathbf{C}$ of the set P such that the given ordering of the set $P - (a)$ coincides with $\mathbf{U_C}(a)$.*

Proof: I. If such a cyclical ordering $\mathbf{C}$ exists, then, by [1], [6] and by the definition of the ordering $\mathbf{U_C}(a)$

$$(a, x, y) \in \mathbf{C} \Leftrightarrow (x, y, a) \in \mathbf{C} \Leftrightarrow (y, a, x) \in \mathbf{C} \Leftrightarrow x \in P - (a),\ y \in P - (a),\ x \text{ precedes } y.$$

Now, let x, y and z be three elements of the set $P - (a)$ such that x precedes z, i.e. $(a, x, z) \in \mathbf{C}$. If $(x, y, z) \in \mathbf{C}$, then, by [8], $(a, y, z) \in \mathbf{C}$, i.e. y precedes z. We have $(x, y, z) \in \mathbf{C}$ and, by [1], $(x, z, a) \in \mathbf{C}$, so that, by [4], $(x, y, a) \in \mathbf{C}$, and hence, by [5], $(a, x, y) \in \mathbf{C}$, i.e. x precedes y. Thus, y is between x and z. On the other hand, let y be between x and z. Since x precedes z, y follows x and precedes z, i.e. $(a, x, y) \in \mathbf{C}$, $(a, y, z) \in \mathbf{C}$. Therefore, by [5], $(y, z, a) \in \mathbf{C}$, $(y, a, x) \in \mathbf{C}$, hence, by [4], $(y, z, x) \in \mathbf{C}$ and hence, by [5], $(x, y, z) \in \mathbf{C}$.

Finally, let x, y and z be three elements of the set $P - (a)$ such that x follows z. Then z precedes x, so that

$$(z, y, x) \in \mathbf{C} \quad \text{if and only if } y \quad \text{is between } x \text{ and } z.$$

On the other hand, by [5], [6] and [7], $(x, y, z) \in \mathbf{C}$ if and only if $x \neq y \neq z \neq x$ and (z, y, x) is not an element of $\mathbf{C}$.

Thus, if x, y, z are elements of $P - (a)$, then $(x, y, z) \in \mathbf{C}$ if and only if either

$$x \text{ precedes } y, \quad y \text{ precedes } z,$$

or

$$y \text{ precedes } z, \quad z \text{ precedes } x,$$

or

$$z \text{ precedes } x, \quad x \text{ precedes } y.$$

Thus, the set $\mathbf{C}$ is fully determined by the given ordering of $P - (a)$,

II. It remains to show that $\mathbf{C}$ is a cyclical ordering of the set P, i.e. that the conditions [1]–[4] are satisfied. In fact, the given ordering of the set $P - (a)$ coincides with $\mathbf{U_C}(a)$ by the construction of $\mathbf{C}$.

First, let $(x, y, z) \in \mathbf{C}$. We see easily that $(y, z, x) \in \mathbf{C}$.

Secondly, we see easily that we never have both $(x, y, z) \in \mathbf{C}$ and $(y, z, x) \in \mathbf{C}$.

Thirdly, let neither $(x, y, z) \in \mathbf{C}$ nor $(y, x, z) \in \mathbf{C}$. Investigating individually the cases $x = a$, $y = a$, $z = a$, $x \neq a \neq y \neq a \neq z$ we find out that we never have $x \neq y \neq z \neq x$.

Fourthly, let $(x, y, z) \in \mathbf{C}$, $(x, z, u) \in \mathbf{C}$. Investigating individually the cases $x = a$, $y = a$, $z = a$, $u = a$, $x \neq a \neq y \neq a \neq z \neq a \neq u$, we find that we always have $(x, y, u) \in \mathbf{C}$.

5.3. Let P be a cyclically ordered set and let $a \in P$, $b \in P$, $a \neq b$. Denote by $J(a, b)$ [more exactly, by $J_{\mathbf{C}}(a, b)$] the set $\underset{x}{\mathrm{E}}\,[(a, x, b) \in \mathbf{C}]$ and call this set an *interval* of the (cyclically ordered) set P, with the *beginning* a and the *end* b. If $x \in J(a, b)$, $y \in J(a, b)$, then $a \neq x \neq b$, $a \neq y \neq b$, so that the elements x and y are in both sets $P - (a)$ and $P - (b)$. If x precedes y with respect to the ordering $\mathbf{U}(a)$ of the set $P - (a)$, we have $(a, x, y) \in \mathbf{C}$; since also $(a, y, b) \in \mathbf{C}$, we have, by [5], $(y, b, a) \in \mathbf{C}$, $(y, a, x) \in \mathbf{C}$, and hence, by [4], $(y, b, x) \in \mathbf{C}$. Thus, by [1], $(b, x, y) \in \mathbf{C}$, so that x precedes y with respect to the ordering $\mathbf{U}(b)$ of the set $P - (b)$. Conversely, if x precedes y with respect to the ordering $\mathbf{U}(b)$ of the set $P - (b)$, we have $(b, x, y) \in \mathbf{C}$; since also $(a, x, b) \in \mathbf{C}$, we have, by [5], $(y, b, x) \in \mathbf{C}$, $(b, a, x) \in \mathbf{C}$. Therefore, by [8], $(y, a, x) \in \mathbf{C}$, and hence, by [1], $(a, x, y) \in \mathbf{C}$, i.e. x precedes y with respect to the ordering $\mathbf{U}(a)$ of $P - (a)$.

Thus, the orderings of $J(a, b)$ determined by the orderings $\mathbf{U}(a)$ of $P - (a)$ and $\mathbf{U}(b)$ of $P - (b)$ coincide. The ordering thus obtained is denoted by $\mathbf{U}(a, b)$, more precisely, by $\mathbf{U}_{\mathbf{C}}(a, b)$.

Evidently

$$P = (a) \cup (b) \cup J(a, b) \cup J(b, a)$$

with disjoint summands. On the other hand:

5.3.1. *Let*

$$P = (a) \cup (b) \cup A \cup B$$

with disjoint summands. Let the sets A and B be ordered. Then there is exactly one cyclical ordering $\mathbf{C}$ of the set P such that: [1] $A = J(a, b)$, $B = J(b, a)$; [2] *the given orderings of the sets A and B coincide with $\mathbf{U}_{\mathbf{C}}(a, b)$ and $\mathbf{U}_{\mathbf{C}}(b, a)$ respectively.*

Proof: I. Let the required cyclical ordering exist. It determines an ordering $\mathbf{U}(a)$ of the set $P - (a) = (b) \cup A \cup B$. If $x \in A$, $y \in A$ or $x \in B$, $y \in B$, then x precedes y with respect to the ordering $\mathbf{U}(a)$ if and only if x precedes y with respect to the given ordering of the set A, B respectively. If $x \in A$, then $(a, x, b) \in \mathbf{C}$, so that x precedes b with respect to the ordering $\mathbf{U}(a)$. If $y \in B$, then $(b, y, a) \in \mathbf{C}$ and hence, by [5], $(a, b, y) \in \mathbf{C}$, hence b precedes y with respect to the ordering $\mathbf{U}(a)$. Finally, if $x \in A$, $y \in B$, we have $(a, x, b) \in \mathbf{C}$, $(b, y, a) \in \mathbf{C}$, hence (by [1]) $(a, x, b) \in \mathbf{C}$, $(a, b, y) \in \mathbf{C}$, hence (by [4]) $(a, x, y) \in \mathbf{C}$ and hence x precedes y with respect to the ordering $\mathbf{U}(a)$. Thus, the ordering $\mathbf{U}(a)$ of the set $P - (a)$ is uniquely determined.

II. By 5.2.1 there is exactly one cyclical ordering $\mathbf{C}$ of the set P such that the ordering just constructed of the set $P - (a)$ coincide with $\mathbf{U}_{\mathbf{C}}(a)$. If $x \in J(a, b)$, then $(a, x, b) \in \mathbf{C}$; thus, $x \in P - (a)$ and x precedes b with respect to $\mathbf{U}(a)$ and hence $x \in A$. Conversely, if $x \in A$, we have $x \in P - (a)$ and x precedes b with respect to $\mathbf{U}(a)$, hence $(a, x, b) \in \mathbf{C}$, i.e. $x \in J(a, b)$. Thus, $A = J(a, b)$ and we may prove similarly that $B = J(b, a)$. The given ordering of the set A is a part of the ordering

$\mathbf{U}(a)$ of the set $P - (a) \supset A$, hence, it coincides with the ordering $\mathbf{U}(a, b)$. Similarly, the given ordering of the set B coincides with $\mathbf{U}(b, a)$.

5.4. Let $\mathbf{C}$ be a cyclical ordering of a set P. Define $\mathbf{C}^* \subset P \times P \times P$ as follows:

$$(a, b, c) \in \mathbf{C}^* \Leftrightarrow (c, b, a) \in \mathbf{C}.$$

Then $\mathbf{C}^*$ is a cyclical ordering of the set P.

We must show that the four conditions stated at the beginning of section 5.1 are satisfied.

I. Let $(a, b, c) \in \mathbf{C}^*$. Then $(c, b, a) \in \mathbf{C}$, so that, by [5], $(a, c, b) \in \mathbf{C}$ and hence $(b, c, a) \in \mathbf{C}^*$.

II. If we have simultaneously $(a, b, c) \in \mathbf{C}^*$ and $(b, a, c) \in \mathbf{C}^*$, we have $(c, b, a) \in \mathbf{C}$ and $(c, a, b) \in \mathbf{C}$, which is, by [5], impossible.

III. If neither $(a, b, c) \in \mathbf{C}^*$ nor $(b, a, c) \in \mathbf{C}^*$, then we have neither $(c, b, a) \in \mathbf{C}$, nor $(c, a, b) \in \mathbf{C}$. Thus, by [5], we have neither $(b, a, c) \in \mathbf{C}$ nor $(a, b, c) \in \mathbf{C}$, so that, by [3], some two of the elements a, b, c are equal.

IV. If $(a, b, c) \in \mathbf{C}^*$, $(a, c, d) \in \mathbf{C}^*$, then $(d, c, a) \in \mathbf{C}$, $(c, b, a) \in \mathbf{C}$, so that, by [8], $(d, b, a) \in \mathbf{C}$, i.e. $(a, b, d) \in \mathbf{C}^*$.

The cyclical ordering $\mathbf{C}^*$ is termed the *inverse* cyclical ordering to $\mathbf{C}$. Of course, conversely, $\mathbf{C}$ is inverse to $\mathbf{C}^*$. We also say that $\mathbf{C}$ and $\mathbf{C}^*$ are mutually inverse.

If $a \in P$, $b \in P$, $a \neq b$, we have evidently

$$J_{\mathbf{C}^*}(a, b) = J_{\mathbf{C}}(b, a).$$

5.5. Let P be a cyclically ordered set. If $a \in P$, $b \in P$, $a \neq b$, then

$$J(a, b) \cup J(b, a) = P - [(a) \cup (b)], \qquad J(a, b) \cap J(b, a) = \emptyset.$$

5.5.1. *If $c \in J(a, b)$, then*

$$J(a, b) = (c) \cup J(a, c) \cup J(c, b)$$

with disjoint summands.

Proof: Let $x \in J(a, b)$, $x \neq c$. Since $c \in P - (a)$, $x \in P - (a)$, exactly one of the two following statements holds: "x precedes c" or "c precedes x", with respect to the ordering $\mathbf{U}(a)$ of the set $P - (a)$. First, if x precedes c, we have $(a, x, c) \in \mathbf{C}$, i.e. $x \in J(a, c)$. Secondly, if c precedes x, we have $(a, c, x) \in \mathbf{C}$; as $x \in J(a, b)$, we have $(a, x, b) \in \mathbf{C}$. By [5], $(x, b, a) \in \mathbf{C}$, $(x, a, c) \in \mathbf{C}$ and hence, by [4], $(x, b, c) \in \mathbf{C}$; thus, by [5], $(c, x, b) \in \mathbf{C}$, i.e. $x \in J(c, b)$. We have proved that

$$J(a, b) = (c) \cup [J(a, b) \cap J(a, c)] \cup [J(a, b) \cap J(c, b)]$$

with disjoint summands.*) It remains to prove that $J(a, c) \cup J(c, b) \subset J(a, b)$. First, if $x \in J(a, c)$, we have $(a, x, c) \in \mathbf{C}$. Since $c \in J(a, b)$, we have $(a, c, b) \in \mathbf{C}$; thus, by [4], $(a, x, b) \in \mathbf{C}$, i.e. $x \in J(a, b)$. Secondly, if $x \in J(c, b)$, we have $(c, x, b) \in \mathbf{C}$. Since $(a, c, b) \in \mathbf{C}$, $(c, x, b) \in \mathbf{C}$, we have, by [8], $(a, x, b) \in \mathbf{C}$, i.e. $x \in J(a, b)$.

On the other hand:

5.5.2. *Let a set P have at least three distinct elements. With every pair (a, b) of distinct elements ot the set P let there be associated two subsets A and B of the set P such that*

$$A \cup B = P - [(a) \cup (b)], \quad A \cap B = 0.$$

Let the subsets associated with (b, a) be the same as those associated with (a, b). If A and B are associated with a pair (a, b) and if $c \in A$ (and hence $a \neq c \neq b$), then let one of the two subsets associated with the pair (a, c) (denote it by C_1) and one of the two subsets associated with the pair (c, b) (denote it be C_2) be such that

$$A = (c) \cup C_1 \cup C_2$$

with disjoint summands. Then there are exactly two cyclical orderings $\mathbf{C}$ of the set P such that for every pair (a, b) $(a \neq b)$ the associated subsets coincide with $J(a, b)$, $J(b, a)$. These two cyclical orderings are mutually inverse.

Proof: I. Choose a fixed $a \in P$ and a $b \in P$ and denote by A and B the two subsets associated with (a, b). It suffices to show that there is exactly one cyclical ordering $\mathbf{C}$ of the set P satisfying the conditions above such that

$$A = J_{\mathbf{C}}(a, b). \tag{1}$$

It then follows that there exists exactly one cyclical ordering $\mathbf{C}'$ of the set P satisfying the conditions above such that

$$A = J_{\mathbf{C}'}(b, a). \tag{2}$$

Moreover, every cyclical ordering of the set P satisfying the conditions above satisfies exactly one of the conditions (1), (2). Further, if $\mathbf{C}$ satisfies the conditions above and condition (1), the inverse cyclical ordering $\mathbf{C}^*$ satisfies the conditions above and condition (2). Thus, $\mathbf{C}' = \mathbf{C}^*$.

II. If $x \in A$, then one of the two sets associated with the pair (a, x) — denote it by $F_1(x)$ — and one of the two sets associated with the pair (b, x) — denote it by $F_2(x)$ — are such that

$$A = (x) \cup F_1(x) \cup F_2(x) \text{ with disjoint summands.} \tag{3}$$

*) Actually, if there were an x with $x \in J(a, c) \cap J(c, b)$, we would obtain, by [1] and [8], $(a, x, c) \in \mathbf{C}$, $(c, x, b) \in \mathbf{C}$, hence $(a, x, c) \in \mathbf{C}$ and $(x, b, c) \in \mathbf{C}$, hence $(a, b, c) \in \mathbf{C}$, hence $(c, a, b) \in \mathbf{C}$. This would be, by [2], in contradiction with $c \in J(a, b)$, i.e. with $(a, c, b) \in \mathbf{C}$.

The second one of the two sets associated with the pair (a, x) is then

$$G_1(x) = P - [(a) \cup (x) \cup F_1(x)];$$

since $b \in P - A$, we have, by (3), $b \in G_1(x)$. The second one of the two sets associated with the pair (b, x) is

$$G_2(x) = P - [(b) \cup (x) \cup F_2(x)];$$

as $a \in P - A$, we have, by (3), $a \in G_2(x)$. Since neither a nor b belongs to the set A, the relation (3) loses validity, if we replace $F_1(x)$ by $G_1(x)$ and $F_2(x)$ by $G_2(x)$. Thus, the sets $F_1(x)$ and $F_2(x)$ are uniquely determined.

If $x \in B$, then one of the two sets associated with the pair (a, x)—denote it by $F_1(x)$—and one of the two sets associated with the pair (b, x)—denote it by $F_2(x)$—are such that

$$B = (x) \cup F_1(x) \cup F_2(x) \text{ with disjoint summands.} \tag{4}$$

Again, the sets $F_1(x)$ and $F_2(x)$ are uniquely determined.

III. Let $x \in A$, $y \in A$, $x \neq y$. By (3), there occurs exactly one of the cases $y \in F_1(x)$, $y \in F_2(x)$. *First*, let $y \in F_1(x)$. By the assumption we obtain

$$F_1(x) = (y) \cup H_1(y) \cup H_2(y),$$

where $H_1(y)$ is equal to either $F_1(y)$ or $G_1(y)$, and $H_2(y)$ is one of the two sets associated with the pair (x, y). Since $b \in G_1(y)$, $b \in P - F_1(x)$, we have $H_1(y) = F_1(y)$, and hence

$$x \in A, \quad y \in F_1(x) \Rightarrow (y) \cup F_1(y) \subset F_1(x). \tag{5}$$

Secondly, let $y \in F_2(x)$. By the assumption we obtain

$$F_2(x) = (y) \cup K_1(y) \cup K_2(y),$$

where $K_1(y)$ is one of the two sets associated with the pair (x, y) and $K_2(y)$ is equal to either $F_2(y)$ or $G_2(y)$. Since $a \in G_2(y)$, $a \in P - F_2(x)$, we have $K_2(y) = F_2(y)$ and hence $F_2(x) \supset (y) \cup F_2(y)$. By the relation (3), which is valid for both x and y, we obtain that

$$x \in A, \quad y \in F_2(x) \Rightarrow (x) \cup F_1(x) \subset F_1(y). \tag{6}$$

It follows from relations (3), (5), (6) that, for $x \in A$, $y \in A$, exactly one of the following relations holds: $x = y$, $(x) \cup F_1(x) \subset F_1(y)$, $(y) \cup F_1(y) \subset F_1(x)$. Let "$x$ precedes y" mean that $(x) \cup F_1(x) \subset F_1(y)$. We see easily that in this way we have defined an ordering of the set A, which will be denoted by V_1. Similarly we prove that there is an ordering V_2 of the set B in which x precedes y if and only if $x \neq y$ and $F_1(x) \supset F_1(y)$.

IV. Let the required cyclical ordering **C** of the set P exist. We have $A = J(a, b)$, so that **C** determines (see section 5.3) an ordering **U**(a, b) of the set A. If $x \in A$, we have either $F_1(x) = J(a, x)$ or $G_1(x) = J(a, x)$. Since $b \in G_1(x)$, we would have in the second case $b \in J(a, x)$, i.e. $(a, b, x) \in \mathbf{C}$; this is impossible by [5], since $x \in J(a, b)$, i.e. $(a, x, b) \in \mathbf{C}$. Thus, $x \in A$ implies $F_1(x) = J(a, x)$. Let $x \in A$, $y \in A$. *First*, if x precedes y with respect to the ordering V_1, then $(x) \cup F_1(x) \subset F_1(y)$, hence, $x \in F_1(y) = J(a, y)$, hence $(a, x, y) \in \mathbf{C}$ and consequently x precedes y with respect to the ordering $\mathbf{U}(a, b)$. *Secondly*, if x precedes y with respect to the ordering $\mathbf{U}(a, b)$, we have $(a, x, y) \in \mathbf{C}$ and hence $x \in J(a, y) = F_1(y)$, so that, by [5], $(x) \cup \cup F_1(x) \subset F_1(y)$; thus, x precedes y with respect to the ordering V_1. Thus, the orderings V_1 and $\mathbf{U}(a, b)$ of the set $A = J(a, b)$ coincide. Similarly it can be proved that the orderings V_2 and $\mathbf{U}(b, a)$ of the set $B = J(b, a)$ coincide. Thus, by 5.3, the cyclical ordering $\mathbf{C}$ is uniquely determined.

V. It remains to be shown that the cyclical ordering $\mathbf{C}$ of the set $P = (a) \cup (b) \cup \cup A \cup B$, determined (by 5.3.1) by the conditions $A = J(a, b)$, $B = J(b, a)$, $V_1 = = \mathbf{U}(a, b)$, $V_2 = \mathbf{U}(b, a)$, has the property that for every pair (x, y), where $x \in P$, $y \in P$, $x \neq y$, the two sets associated with the pair (x, y) are $J(x, y)$ and $J(y, x)$. *First*, this is evident for the pair (a, b). *Secondly*, let us investigate a pair (a, x) with $x \in A$. $J(a, x)$ is the set of all $y \in J(a, b)$ for which y precedes x with respect to the ordering $\mathbf{U}(a, b)$. $F_1(x)$ is the set of all $y \in A$ for which y precedes x with respect to the ordering V_1. Since $A = J(a, b)$, $V_1 = \mathbf{U}(a, b)$, we have $J(a, x) = F_1(x)$. As $P = (a) \cup (x) \cup J(a, x) \cup J(x, a) = (a) \cup (x) \cup F_1(x) \cup G_1(x)$ with disjoint summands, we have $J(x, a) = G_1(x)$. The *third* case of a pair (b, x) with $x \in A$, and similarly, the *fourth* and *fifth* cases of pairs (a, x), (b, x), respectively, with $x \in B$, may be treated in the same way as was the second. Let us investigate the *sixth* case of a pair (x, y) with $x \in A$, $y \in B$. Let C_1 and C_2 be the two sets associated with the pair (x, y). As $P = (x) \cup (y) \cup C_1 \cup C_2$ with disjoint summands, we may assume that $a \in C_1$. By the assumption, $C_1 = (a) \cup H \cup K$ with disjoint summands, where H is one of the two sets associated with the pair (a, x) and K is one of the two sets associated with the pair (a, y). We have either $H = F_1(x)$ or $H = G_1(x)$, and either $K = F_1(y)$ or $K = G_1(y)$. As $y \in B \subset G_1(x)$, $x \in A \subset G_1(y)$ and as neither x nor y is contained in C_1, we obtain $H = F_1(x)$, $K = F_1(y)$ and hence $C_1 = (a) \cup F_1(x) \cup \cup F_1(y) = (a) \cup J(a, x) \cup J(y, a)$. Since $x \in A = J(a, b)$, $y \in B = J(b, a)$, we have $(a, x, b) \in \mathbf{C}$, $(b, y, a) \in \mathbf{C}$ and hence, by [5], $(a, b, y) \in \mathbf{C}$, so that, by [4], $(a, x, y) \in \mathbf{C}$. Thus, by [5], $(y, a, x) \in \mathbf{C}$, hence $a \in J(y, x)$, so that $(a) \cup J(y, a) \cup J(a, x) = J(x, y)$ and hence $C_1 = J(x, y)$. Since $P = (x) \cup (y) \cup C_1 \cup C_2 = (x) \cup (y) \cup J(x, y) \cup \cup J(y, x)$ with disjoint summands, we have $C_2 = J(y, x)$. There remains the case of a pair (x, y) with either $(x) \cup (y) \subset A$ or $(x) \cup (y) \subset B$ to be discussed. Let, e.g., $(x) \cup (y) \subset A$; similarly as in III, let us distinguish two subcases: $y \in F_1(x)$ and $y \in F_2(x)$. If $y \in F_1(x)$, we saw in III that $F_1(x) = (y) \cup F_1(y) \cup H_2(y)$ with disjoint summands, where $H_2(y)$ is one of the two sets associated with (x, y). We have $F_1(x) = = J(a, x)$, $F_1(y) = J(a, y)$, $y \in F_1(x)$, hence $J(a, x) = (y) \cup J(a, y) \cup H_2(y)$ with disjoint summands; thus, by 5.5.1, $H_2(y) = J(y, x)$. The second set associated with the pair (x, y) is, of course, $P - [(x) \cup (y) \cup H_2(y)] = P - [(x) \cup (y) \cup J(y, x)] =$

$= J(x, y)$. If $y \in F_2(x)$, then, by III, $F_2(x) = (y) \cup F_2(y) \cup K_1(y)$, where $K_1(y)$ is one from the two sets associated with the pair (x, y). We have $F_2(x) = J(x, b)$, $F_2(y) = J(y, b)$, $y \in F_2(x)$, so that $J(x, b) = (y) \cup J(y, b) \cup K_1(y)$ with disjoint summands; hence, by 5.5.1, $K_1(y) = J(x, y)$. The second set associated with the pair (x, y) is, of course, $P - [(x) \cup (y) \cup K_1(y)] = P - [(x) \cup (y) \cup J(x, y)] = J(y, x)$.

Exercises

5.1. A finite set with $n > 0$ elements has $(n - 1)!$ cyclical orderings. All these cyclical orderings are mutually similar, if we define the similarity of cyclical orderings analogously as we did with orderings in 4.2.

5.2. The set of all cyclical orderings of the set of all natural numbers is uncountable.

5.3. Let $\mathbf{C}$ be a cyclical ordering of a set P. Let us define, for $n = 3, 4, 5, \ldots$, subsets $\mathbf{C}_n$ of $P \times P \times \ldots \times P$ (with n factors) as follows: [1] $\mathbf{C}_3 = \mathbf{C}$, [2] $\mathbf{C}_{n+1}$ is the set of all $(a_1, a_2, \ldots, a_{n+1})$ such that $(a_1, a_2, \ldots, a_n) \in \mathbf{C}_n$ and $(a_1, a_n, a_{n+1}) \in \mathbf{C}$. If $(a_1, a_2, \ldots, a_n) \in \mathbf{C}_n$ and if $1 \leqq i < j < k \leqq n$, we have $(a_i, a_j, a_k) \in \mathbf{C}$.

5.4. Let P be a cyclically ordered set. Let $J(a, b) \neq \emptyset$ for every $a \in P$, $b \in P$, $a \neq b$. Then all the sets $J(a, b)$ are infinite.

5.5. Let K be the set of all complex numbers $x + iy$ with $x^2 + y^2 = 1$. Denote by $Im(x + iy)$ the term y. If $\alpha \in K$, $\beta \in K$, $\gamma \in K$ then let $(\alpha, \beta, \gamma) \in \mathbf{C}$ mean that $\beta \neq \gamma$ and $Im \dfrac{\beta - \alpha}{\gamma - \beta} > 0$. Then $\mathbf{C}$ is a cyclical ordering of the set K.