Topological spaces

Appendix

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APPENDIX

41. COMPACTNESS AND COMPLETENESS

A uniform space $\mathscr{P}$ is said to be complete if the following condition is fulfilled: if f is a uniformly continuous mapping of a dense subspace $\mathscr{Q}$ of a uniform space $\mathscr{R}$ into $\mathscr{P}$, then f is the restriction of a uniformly continuous mapping of $\mathscr{R}$ into $\mathscr{P}$; stated in other words, every uniformly continuous mapping into $\mathscr{P}$ of any dense subspace of any uniform space $\mathscr{R}$ has a uniformly continuous domain-extension to $\mathscr{R}$. It turns out that a uniform space $\langle P, \mathscr{U} \rangle$ is complete if and only if the following condition is fulfilled: if $\mathscr{X}$ is a proper filter on P containing "arbitrarily small sets", i.e. each $U \in \mathscr{U}$ contains a set $X \times X$ with X in $\mathscr{X}$, then the intersection of closures of sets of $\mathscr{X}$ is non-void.

In a totally bounded uniform space every proper filter is contained in a proper filter containing arbitrarily small sets, and therefore in a complete totally bounded uniform space the intersection of closures of sets of any proper filter is non-void. Closure spaces possessing the last property are said to be compact. It should be remarked that compact spaces were considered in the exercises to Sections 17 and 27, and in 29 B and 31 D.

In subsection A we shall describe the fundamental properties of complete uniform spaces and compact closure spaces. It may be pointed out that we shall define complete uniform spaces by the condition on filters containing arbitrarily small sets, and the equivalence with the extension property, mentioned above, will be proved in 41 B. The most profound result states that the class of all complete uniform spaces as well as the class of all compact spaces are completely productive.

In subsection B we shall prove that every uniform space admits an identity embedding into a complete uniform space, and that every uniformly continuous mapping of a dense subspace of a uniform space $\mathscr{P}$ into a complete uniform space has a uniformly continuous domain-extension on $\mathscr{P}$. Particular attention is given to topological groups whose two-sided uniformity is complete.

In subsection C we shall be concerned with the development of the properties of compact spaces in the class of all topological spaces and the class of all uniformizable spaces. Various formulations of the Stone-Weierstrass Theorem for uniformizable spaces (41 C.15) are given.

In subsection D the theory of compactifications of uniformizable spaces is studied; the Čech-Stone compactification is introduced, and the Čech-Stone Theorem is proved. The exposition is based on the theory of completion of uniform spaces. In subsection E the main results of D are reproved by means of the structure spaces of algebras of bounded continuous functions.

A. GENERALITIES

In the exercises to sections 15 and 16 the concepts of a cluster point and a limit point of a proper filter of sets on a space were introduced and studied. For convenience we shall recall the definitions and properties which will be needed.

41 A.1. Definition. Let $\langle P, u\rangle$ be a closure space, $\mathscr{X}$ be a proper filter on P and x be an element of P. We shall say that x *is a cluster point of* $\mathscr{X}$ *in* $\langle P, u\rangle$ if x belongs to $\bigcap\{uX \mid X \in \mathscr{X}\}$, i.e., if each neighborhood of x intersects each $X \in \mathscr{X}$. We shall say that x *is a limit point of* $\mathscr{X}$ *in* $\langle P, u\rangle$ or that $\mathscr{X}$ *converges to* x *in* $\langle P, u\rangle$ if each neighborhood of x contains an element of $\mathscr{X}$, i.e. if each neighborhood of x belongs to $\mathscr{X}$. If $\mathscr{Y}$ is a centered collection of subsets of P (i.e. a collection with the finite intersection property, or a sub-base for a proper filter on P) then we shall say that x *is a cluster point* or *a limit point of* $\mathscr{Y}$ *in* $\langle P, u\rangle$ if x is, respectively, a cluster point or a limit point of the smallest filter on P containing $\mathscr{Y}$, i.e. of the filter generated by $\mathscr{Y}$.

41 A.2. Remarks. (a) An ultrafilter on a closure space converges to each of its cluster points. (b) If $\langle P, u\rangle$ is a non-void closure space, then each point of P is a cluster point of the smallest filter (P) on P, but (P) converges to each point of P if and only if $\langle P, u\rangle$ is an accrete space. (c) If $\langle P, u\rangle$ is a closure space, then each point is a cluster point of the filter (P) but x is a cluster point of the set P if and only if $x \in u(P - (x))$. (d) Let f be a mapping of a closure space $\mathscr{P}$ into a closure space $\mathscr{Q}$ and let $\mathscr{U}$ be the neighborhood system of a point x of $\mathscr{P}$. Then $f[[\mathscr{U}]]$ $(= \mathsf{E}\{f[U] \mid U \in \mathscr{U}\})$ is a base of a proper filter $\mathscr{V}$ on $\mathscr{Q}$ and f is continuous at x if and only if $\mathscr{V}$ converges to fx in $\mathscr{Q}$. (e) A mapping f of a closure space $\mathscr{P}$ into another one $\mathscr{Q}$ is continuous if and only if the following condition is fulfilled: if x is a cluster (limit) point of a proper filter $\mathscr{X}$ on $\mathscr{P}$, then fx is a cluster (limit) point of $f[[\mathscr{X}]]$.

Now we proceed to the subject proper of this subsection.

41 A.3. Definition. A *closure space* $\mathscr{P}$ is said to be *compact* if every proper filter of sets on $\mathscr{P}$ has a cluster point in $\mathscr{P}$. A *Cauchy filter on a uniform space* $\langle P, \mathscr{U}\rangle$ is a proper filter of sets $\mathscr{X}$ on P such that each element U of $\mathscr{U}$ contains a set $X \times X$ with X in $\mathscr{X}$. A *uniform space* $\mathscr{P}$ is said to be *complete* if each Cauchy filter $\mathscr{X}$ on $\mathscr{P}$ has a cluster point in $\mathscr{P}$.

41 A.4. Examples. (a) Evidently every compact uniform space is complete. On the other hand a complete uniform space need not be compact; e.g. every uniformly discrete uniform space $\langle P, \mathscr{U}\rangle$ is complete but no infinite discrete space is compact. Indeed, if $\mathscr{X}$ is a Cauchy filter on $\langle P, \mathscr{U}\rangle$ then $X \times X$ is contained in the diagonal J_P of $P \times P$ for some $X \in \mathscr{X}$ because $\mathsf{J}_P \in \mathscr{U}$; but clearly $X \times X \subset \mathsf{J}$ implies that X has at most one element, and hence $X = (x)$ for some $x \in P$ because $X \neq \emptyset$. Thus $\mathscr{X}$ is the fixed filter with base (x) and hence $\bigcap \mathscr{X} = (x)$. If $\langle P, u \rangle$ is an infinite discrete space, then there exists a free filter $\mathscr{X}$ on P, and $\mathscr{X}$ has no cluster point because $uX = X$ for each $X \subset P$ and hence $\bigcap\{uX \mid X \in \mathscr{X}\} = \bigcap \mathscr{X} = \emptyset$. Relations between complete uniform spaces and compact spaces will be described more fully in 41 A.8. (b) Every finite closure space is compact because there is no free proper filter on a finite set. (c) A monotone ordered space $\langle P, \leqq, u \rangle$ is compact if and only if $\langle P, \leqq \rangle$ is order-complete. First suppose that $\langle P, \leqq \rangle$ is order-complete and $\mathscr{X}$ is a proper filter on P. Let x be the infimum of all $y \in P$ such that y is an upper bound of an $X \in \mathscr{X}$. It is easily seen that x is a cluster point of $\mathscr{X}$ in $\langle P, u \rangle$. Conversely, suppose that $\langle P, \leqq \rangle$ is not order-complete and let X be a non-void subset such that $\sup X$ or $\inf X$ does not exist. If $\sup X$ does not exist then the collection of all $X \cap [\![x, \rightarrow]\!]$, $x \in X$, is a filter base for a proper filter on $\langle P, u \rangle$ which has no cluster point; in fact, if y is an upper bound of X, then $P - X$ is a neighborhood of y which do not intersect X, and if y is not an upper bound of X, then we can choose an $x \in X \cap]\!] y, \rightarrow]\!]$, and clearly $[\![\leftarrow, x [\![$ is a neighborhood of y which does not intersect $X \cap [\![x, \rightarrow]\!]$. (d) Every closed bounded interval of reals is compact but R is not compact (by (c)). (e) If a closure space $\langle P, u \rangle$ is separated then $\langle P, u \rangle$ is compact if and only if u is a coarse separated closure (by 31 D.8). (f) Many examples of non-complete uniform spaces will be given later.

It is interesting to note that a complete uniform space can be characterized without any reference to the induced closure.

41 A.5. Theorem. *A uniform space $\langle P, \mathscr{U} \rangle$ is complete if and only if $\bigcap\{U[X] \mid U \in \mathscr{U}, X \in \mathscr{X}\} \neq \emptyset$ for each Cauchy filter $\mathscr{X}$ on $\langle P, \mathscr{U} \rangle$.*

Proof. If u is the closure induced by $\mathscr{U}$, then $uX = \bigcap\{U[X] \mid U \in \mathscr{U}\}$ for each $X \in \mathscr{X}$ (23 B.5), and hence $\bigcap\{U[X] \mid X \in \mathscr{X}, U \in \mathscr{U}\} = \bigcap\{uX \mid X \in \mathscr{X}\}$.

The next theorem asserts that, to prove the compactness of a closure space or the completeness of a uniform space, it is sufficient to show that every ultrafilter or Cauchy ultrafilter, respectively, has a cluster point.

41 A.6. Theorem. *A closure space $\mathscr{P}$ is compact if and only if every ultrafilter on $\mathscr{P}$ has a cluster point. A uniform space $\mathscr{P}$ is complete if and only if every Cauchy ultrafilter on $\mathscr{P}$ has a cluster point.*

Proof. "Only if" is evident because each (Cauchy) ultrafilter is a (Cauchy) proper filter. Conversely, suppose that $\mathscr{P}$ is a closure space (uniform space) such that every ultrafilter (Cauchy ultrafilter) has a cluster point, and let $\mathscr{X}$ be any proper filter (Cauchy filter) on $\mathscr{P}$. By 12 C.2 there exists an ultrafilter $\mathscr{Y}$ on $\mathscr{P}$ containing $\mathscr{X}$ (clearly $\mathscr{Y}$ is a Cauchy filter). By our assumption the ultrafilter $\mathscr{Y}$ has a cluster point, say x; since $\mathscr{Y} \supset \mathscr{X}$, x is a cluster point of $\mathscr{X}$.

It is to be noted that the preceding theorem is profound even though the topological part of the proof is almost evident (we made use of the rather profound result that every proper filter is contained in an ultrafilter).

Before proceeding we shall give a characterization of Cauchy filters which will often be needed.

41 A.7. Theorem. *Let $\langle P, \mathscr{U}\rangle$ be a uniform space. A proper filter $\mathscr{X}$ on $\langle P, \mathscr{U}\rangle$ is a Cauchy filter if and only if $\mathscr{X} \cap \mathbf{E}\{U[x] \mid x \in P\} \neq \emptyset$ for each U in $\mathscr{U}$ (i.e. for each U in $\mathscr{U}$ there exists an x in P such that $U[x] \in \mathscr{X}$).*

Proof. Let $\mathscr{X}$ be a Cauchy filter and $U \in \mathscr{U}$. There exists an X in $\mathscr{X}$ such that $X \times X \subset U$. If $x \in X$, then $U[x] \supset X$ and hence $U[x] \in \mathscr{X}$. Conversely, suppose that $\mathscr{X}$ fulfils the condition and U is any element of $\mathscr{U}$. Choose a symmetric element V of $\mathscr{U}$ so that $V \circ V \subset U$ and an x so that $X = V[x] \in \mathscr{X}$. Clearly $X \times X \subset U$, which shows that $\mathscr{X}$ is a Cauchy filter.

Corollary. *If $\mathscr{U}$ is the neighborhood system of a point of a uniform space $\mathscr{P}$, then $\mathscr{U}$ is a Cauchy filter on $\mathscr{P}$.*

Remark. Intuitively, a Cauchy filter on a uniform space $\langle P, \mathscr{U}\rangle$ is a proper filter on P containing "arbitrarily small sets" and this means that for each U in $\mathscr{U}$ there exists a "U-small set" in $\mathscr{X}$. In definition 41 A.3 the expression "X is U-small" was interpreted as $X \times X \subset U$, i.e. $U[x] \supset X$ for each $x \in X$, or stated in other words, every two points of X are U-related. According to the foregoing theorem the expression "X is U-small" can be interpreted as meaning that $X \subset U[x]$ for some $x \in P$ which is an essentially weaker requirement. It is easily seen that in a semi-uniform space, which is not uniform, these two definitions of "X is U-small" lead to different notions of a Cauchy filter, and it turns out that resulting notions of a complete semi-uniform space are indeed distinct.

We shall not study the completeness of semi-uniform spaces, since we are concerned with an examination of the extension of mappings, and for this purpose completeness of semi-uniform spaces is not needed. It may be in place to notice that if d is a semi-metric for a set P, r is a positive real and U is the set of all $\langle x, y\rangle$ with $d\langle x, y\rangle \leqq r$, then $X \times X \subset U$ means that the diameter of X is at most r, and $X \subset U[x]$ for some $x \in P$ means that X is contained in a closed r-sphere.

As a consequence of the foregoing two theorems we obtain one part of the following theorem which completely describes the relationship between compact uniform spaces and complete uniform spaces.

41 A.8. Theorem. *In order that a uniform space $\mathscr{P}$ be compact it is necessary and sufficient that $\mathscr{P}$ be complete and totally bounded. More precisely, if $\langle P, \mathscr{U}\rangle$ is a uniform space, then the closure induced by $\mathscr{U}$ is compact if and only if $\langle P, \mathscr{U}\rangle$ is a totally bounded complete uniform space.*

Proof. I. First suppose that $\langle P, \mathscr{U}\rangle$ is a complete totally bounded uniform space. To prove that the closure u is compact it is enough to show, by 41 A.6, that every ultrafilter $\mathscr{X}$ in the totally bounded uniform space $\langle P, \mathscr{U}\rangle$ is a Cauchy filter, and this follows from 41 A.7. In fact, if $U \in \mathscr{U}$, then $\{U[x] \mid x \in X\}$ is a cover of $\mathscr{P}$ for some finite subset X of P, and $\mathscr{X}$ being an ultrafilter, some $U[x]$, $x \in X$, must belong to $\mathscr{X}$

(by 12 C.8); consequently, by 41 A.7, $\mathscr{X}$ is a Cauchy filter on $\langle P, \mathscr{U}\rangle$. – II. Now suppose that the closure is compact. Evidently $\langle P, \mathscr{U}\rangle$ is complete. To prove that $\langle P, \mathscr{U}\rangle$ is totally bounded it is enough to show that each cover $\{U[x] \mid x \in P\}$, $U \in \mathscr{U}$, has a finite subcover, and this follows from the following theorem (remember that $\{U[x] \mid x \in P\}$ is an interior cover of $\langle P, \mathscr{U}\rangle$).

41 A.9. Theorem. *In order that a closure space $\langle P, u\rangle$ be compact it is necessary and sufficient that every interior cover $\mathscr{Y}$ of $\langle P, u\rangle$ have a finite subcover.*

Proof. Let $\mathscr{X}$ be a collection of subsets of P and let $\mathscr{Y}$ be the collection of sets of the form $P - X$, $X \in \mathscr{X}$. First observe that $\bigcap u[\mathscr{X}] = \emptyset$ if and only if $\bigcup \operatorname{int}[\mathscr{Y}] = \bigcup\{\operatorname{int}(P - X) \mid X \in \mathscr{X}\} = P$. Indeed, by de Morgan formula, $\bigcap u[\mathscr{X}] = P - \bigcup\{P - uX \mid X \in \mathscr{X}\} = P - \bigcup\{\operatorname{int}(P - X) \mid X \in \mathscr{X}\}$. Next, again by de Morgan formula, $\mathscr{X}$ is a proper filter sub-base, i.e. $\bigcap \mathscr{X}_1 \neq \emptyset$ for each finite subcollection, if and only if $\mathscr{Y}$ contains no finite subcover; in fact,

$$\bigcap \mathscr{X}_1 = P - \bigcup\{P - X \mid X \in \mathscr{X}_1\}.$$

Thus there exists an interior cover of $\langle P, u\rangle$ containing no finite subcover if and only if there exists a filter sub-base $\mathscr{X}$ on P such that $\bigcap u[\mathscr{X}] = \emptyset$. Thus, if $\langle P, u\rangle$ is not compact, then there exists a filter $\mathscr{X}$ on P such that $\bigcap u[\mathscr{X}] = \emptyset$, and then the collection $\mathscr{Y}$ consisting of complements of sets of $\mathscr{X}$ is an interior cover containing no finite subcover. Conversely, if $\mathscr{Y}$ is an interior cover of $\langle P, u\rangle$ containing no finite subcover, then the collection $\mathscr{X}$ consisting of complements of sets of $\mathscr{Y}$ is a proper filter sub-base of sets on P such that $\bigcap u[\mathscr{X}] = \emptyset$; if $\mathscr{X}_1$ is a proper filter containing $\mathscr{X}$, then $\bigcap u[\mathscr{X}_1] \subset \bigcap u[\mathscr{X}] = \emptyset$.

Corollary. *Every compact topological regular space and also every compact topological separated space is paracompact, hence normal, and thus uniformizable.*

It follows from 41 A.8 that no infinite totally bounded discrete uniform space is complete; in particular, the Čech uniformity of no infinite discrete space is complete.

41 A.10. Theorem. *Every closed subspace of a compact space is a compact space, and every closed subspace of a complete uniform space is a complete uniform space.*

Proof. Let $\langle Q, v\rangle$ be a closed subspace of a compact closure space $\langle P, u\rangle$ and let $\mathscr{X}$ be a proper filter on Q. Let us consider the smallest filter $\mathscr{Y}$ on P containing $\mathscr{X}$. Since $uQ = Q$, we have $vX = uX$ for each $X \subset Q$, and hence $\bigcap v[\mathscr{X}] = \bigcap u[\mathscr{X}]$. Next, $\mathscr{X}$ is a filter base for $\mathscr{Y}$ and therefore $\bigcap u[\mathscr{X}] = \bigcap u[\mathscr{Y}]$. However, $\langle P, u\rangle$ is compact and hence $\bigcap u[\mathscr{Y}] \neq \emptyset$; consequently also $\bigcap v[\mathscr{X}] \neq \emptyset$ which establishes the compactness of $\langle Q, v\rangle$. Now let $\langle Q, \mathscr{V}\rangle$ be a closed subspace of a complete uniform space $\langle P, \mathscr{U}\rangle$ and let $\mathscr{X}$ be a Cauchy filter in $\langle Q, \mathscr{V}\rangle$. The smallest filter $\mathscr{Y}$ on P containing $\mathscr{X}$ is evidently a Cauchy filter in $\langle P, \mathscr{U}\rangle$, and hence $\bigcap u[\mathscr{Y}] \neq \emptyset$ where u is the closure induced by $\mathscr{U}$. However, the closure v induced by $\mathscr{V}$ is a relativization of u and hence the argument used above yields $\bigcap v[\mathscr{X}] \neq \emptyset$; this establishes the completeness of $\langle Q, \mathscr{V}\rangle$.

Example. We know that every bounded closed interval of reals is compact. It follows from 41 A.10 that every closed bounded subspace of R is compact. This fact enables us to prove that the uniform space of reals is complete. If $\mathscr{X}$ is a Cauchy filter on R, then clearly $\mathscr{X}$ contains a bounded set Y and therefore $\overline{Y} \cap \bigcap\{\overline{X \cap Y} \mid X \in \mathscr{X}\} \neq \emptyset$ because $\overline{Y}$ is compact, in particular, $\bigcap\{\overline{X} \mid X \in \mathscr{X}\} \neq \emptyset$. Next, if follows from the second statement of 41 A.10 that every closed uniform subspace of R is complete.

The converse of 41 A.10 is not true, i.e. a compact subspace of a space need not be closed and a complete uniform subspace of a uniform space need not be closed. For example, if $\mathscr{P}$ is an accrete closure space (an accrete uniform space), then every subspace of $\mathscr{P}$ is an accrete closure space (an accrete uniform space), and hence each subspace of $\mathscr{P}$ is compact (complete), but no non-void proper subset of $|\mathscr{P}|$ is closed in $\mathscr{P}$. On the other hand the following important theorem holds.

41 A.11. Theorem. *If $\langle Q, v\rangle$ is a compact subspace of a separated closure space $\langle P, u\rangle$, then Q is closed in $\langle P, u\rangle$. If $\langle Q, \mathscr{V}\rangle$ is a complete uniform subspace of a separated uniform space $\langle P, \mathscr{U}\rangle$ then Q is closed in $\langle P, \mathscr{U}\rangle$.*

Proof. It may be noted that the first statement was proved in 31 D.10. Nevertheless, for the sake of completeness, we recall this proof. If $x \in uQ - Q$ and $\mathscr{X}$ is the neighborhood system at x, then $\bigcap u[\mathscr{X}] = (x)$ because $\langle P, u\rangle$ is separated, and hence $\bigcap v[\mathscr{Y}] = \emptyset$, where $\mathscr{Y} = [\mathscr{X}] \cap Q$, and clearly $\mathscr{Y}$ is a proper filter on Q because $X \cap Q \neq \emptyset$ for each $X \in \mathscr{X}$; thus $\langle Q, v\rangle$ is not compact. Now let $\langle Q, \mathscr{V}\rangle$ be a subspace of a separated uniform space $\langle P, \mathscr{U}\rangle$ and let u and v be the closures induced by $\mathscr{U}$ and $\mathscr{V}$; we know that v is a relativization of u. We shall show that $\langle Q, \mathscr{V}\rangle$ is not complete if Q is not closed in $\langle P, u\rangle$. Suppose $x \in uQ - Q$ and let $\mathscr{X}$ be the neighborhood system at x in $\langle P, u\rangle$. By the corollary to 41 A.7, $\mathscr{X}$ is a Cauchy filter in $\langle P, \mathscr{U}\rangle$. As above, $\mathscr{Y} = [\mathscr{X}] \cap Q$ is a filter on Q such that $\bigcap v[\mathscr{Y}] = \emptyset$. Since $\mathscr{X}$ is a Cauchy filter in $\langle P, \mathscr{U}\rangle$, $\mathscr{Y}$ is a Cauchy filter in $\langle Q, \mathscr{V}\rangle$, and hence the uniform space $\langle Q, \mathscr{V}\rangle$ is not complete.

Example. By example following 41 A.10 each closed bounded subspace of R is compact. Conversely, if X is a compact subspace of R, then X is closed in R by the preceding theorem, and X is totally bounded (by 41 A.8) and hence bounded. Thus a subspace of R is compact if and only if X is closed and bounded.

Now we proceed to the most important result.

41 A.12. Theorem. *The class of all compact spaces is completely productive. The class of all complete uniform spaces is completely productive.*

The proof of both statements is based on the following lemma.

41 A.13. Lemma. *Let $\langle P, u\rangle$ be the product of a family $\{\langle P_a, u_a\rangle \mid a \in A\}$ of closure spaces. If $\mathscr{X}$ is an ultrafilter on P, then $\mathscr{X}_a = \mathbf{E}\{\mathrm{pr}_a[X] \mid X \in \mathscr{X}\}$ is a proper filter on P_a for each a and*

$$\bigcap u[\mathscr{X}] = \Pi\{\bigcap u_a[\mathscr{X}_a] \mid a \in A\};$$

in particular, if $\bigcap u_a[\mathscr{X}_a] \neq \emptyset$ for each a, then $\bigcap u[\mathscr{X}] \neq \emptyset$.

Proof. Clearly $\mathscr{X}_a$ are proper filters. Let $\mathscr{X}$ be an ultrafilter on P. If x is a cluster point of $\mathscr{X}$ in $\langle P, u\rangle$, then $\mathrm{pr}_a x$ is a cluster point of $\mathscr{X}_a$ in $\langle P_a, u_a\rangle$ for each a by 41 A.2 (e) because of the continuity of projections. Thus $\bigcap u[\mathscr{X}] \subset \Pi\{\bigcap u_a[\mathscr{X}_a]\}$. To prove the inverse inclusion, suppose that x is a point of $\langle P, u\rangle$ such that $\mathrm{pr}_a x$ is a cluster point of $\mathscr{X}_a$ in $\langle P_a, u_a\rangle$ for each a. We must show that x is a cluster point of $\mathscr{X}$ in $\langle P, u\rangle$. It is enough to prove that each canonical neighborhood of x belongs to $\mathscr{X}$; since $\mathscr{X}$ is a filter, it suffices to show that if $a \in A$ and U is a neighborhood of $\mathrm{pr}_a x$, then $V = P \cap \mathrm{pr}_a^{-1}[U]$ belongs to $\mathscr{X}$. Since $U \cap \mathrm{pr}_a X \neq \emptyset$ for each $X \in \mathscr{X}$, we have $V \cap X \neq \emptyset$ for each $X \in \mathscr{X}$ (notice that V is the inverse image of U under $\mathrm{pr}_a \cap (P \times P_a)$ and $X \subset P$), and this implies $V \in \mathscr{X}$ because $\mathscr{X}$ is an ultrafilter.

Proof of 41 A.12. Let $\langle P, u\rangle$ be the product of a family $\{\langle P_a, u_a\rangle \mid a \in A\}$ of compact spaces. To prove that $\langle P, u\rangle$ is compact, by 41 A.6 it is sufficient to show that $\bigcap u[\mathscr{X}] \neq \emptyset$ for each ultrafilter $\mathscr{X}$ on P; but this follows from the foregoing lemma. Now let $\langle P, \mathscr{U}\rangle$ be the product of a family $\{\langle P_a, \mathscr{U}_a\rangle \mid a \in A\}$ of complete uniform spaces, and let u be the closure induced by $\mathscr{U}$ and u_a be the closures induced by $\mathscr{U}_a$ for each a. By 23 D.11, $\langle P, u\rangle$ is the product of the family $\{\langle P_a, u_a\rangle\}$. By 41 A.6 to prove that $\langle P, \mathscr{U}\rangle$ is complete it is enough to show that $\bigcap u[\mathscr{X}] \neq \emptyset$ for each Cauchy ultrafilter $\mathscr{X}$ in $\langle P, \mathscr{U}\rangle$, and this will follow from Lemma 41 A.13 if we show that, with the notation of 41 A.13, each $\mathscr{X}_a$ is a Cauchy filter in $\langle P_a, \mathscr{U}_a\rangle$; but this is a result of the proposition which follows because the projections of a product of semi-uniform spaces into the coordinate spaces are uniformly continuous.

It should be noted that another proof of the first assertion of 41 A.12 was given in 29 B.5.

41 A.14. *If f is a uniformly continuous mapping of a uniform space $\mathscr{P}$ into a uniform space $\mathscr{Q}$ and if $\mathscr{X}$ is a Cauchy filter on $\mathscr{P}$, then $f[[\mathscr{X}]] = \mathbf{E}\{f[X] \mid X \in \mathscr{X}\}$ is a filter base for a Cauchy filter on $\mathscr{Q}$.*

Proof. Let $\mathscr{Y}$ be the filter generated by the collection of all $f[X]$, $X \in \mathscr{X}$. Clearly $\mathscr{Y}$ is a proper filter on $\mathscr{Q}$. We shall prove that $\mathscr{Y}$ contains arbitrarily small sets. If V is an element of the uniform structure of $\mathscr{Q}$, then $U = (f \times f)^{-1}[V]$ is an element of the uniform structure of $\mathscr{P}$ (because of the uniform continuity of f), and we can choose an X in $\mathscr{X}$ so that $X \times X \subset U$; clearly $f[X] \in \mathscr{Y}$ and $f[X] \times f[X] \subset V$.

Up to now the theory of compact spaces and the theory of complete uniform spaces have been parallel. Now we have reached a point where the two theories diverge; compare 41 A.15 and 41 A.16 which follow.

41 A.15. *Any image under a continuous mapping of a compact space is compact, more precisely, if f is a surjective continuous mapping and* $\mathbf{D}^*f$ *is compact, then* $\mathbf{E}^*f$ *is compact.*

Proof. Let f be a mapping of a compact closure space $\langle P, u\rangle$ onto a closure space $\langle Q, v\rangle$ and let $\mathscr{X}$ be a proper filter on Q. If $\mathscr{Y}$ is the collection of all the sets of the form $f^{-1}[X]$, $X \in \mathscr{X}$, then $\mathscr{Y}$ is a filter base of sets on P and hence $\bigcap u[\mathscr{Y}] \neq \emptyset$. However $vf[Y] \supset f[uY]$ for each $Y \subset P$ (because of the continuity of f) and therefore $f[\bigcap u[\mathscr{Y}]] \subset \bigcap v[\mathscr{X}]$ and hence the latter set is non-void; this establishes the compactness of $\langle Q, v \rangle$. (One can also use 41 A.2 (e): if x is a cluster point of $\mathscr{Y}$, then fx is a cluster point of $\mathscr{X}$.)

41 A.16. Every uniform space is the image under a bijective uniformly continuous mapping of a complete uniform space, namely a uniform space $\langle P, \mathscr{U} \rangle$ is the range carrier of the uniformly continuous mapping $J : \langle P, \mathscr{V} \rangle \to \langle P, \mathscr{U} \rangle$ where $\mathscr{V}$ is the uniformly finest uniformity for P. On the other hand: *If $\mathscr{U}$ and $\mathscr{V}$ are topologically equivalent uniformities for a set P, $\mathscr{U} \subset \mathscr{V}$ and $\mathscr{U}$ is complete, then $\mathscr{V}$ is complete.* Indeed, if $\mathscr{X}$ is a Cauchy filter on $\langle P, \mathscr{V} \rangle$, then $\mathscr{X}$ is a Cauchy filter on $\langle P, \mathscr{U} \rangle$. – It is to be noted that the assumption $\mathscr{U} \subset \mathscr{V}$ is essential, e.g. consider the uniformly finest uniformity $\mathscr{U}$ for an infinite set P and the uniformly finest totally bounded uniformity $\mathscr{V}$ for P.

In conclusion we shall describe compactness and completeness by means of convergent nets. We begin with a definition.

41 A.17. Definition. A *Cauchy net* in a uniform space $\langle P, \mathscr{U} \rangle$ is a net N in $\langle P, \mathscr{U} \rangle$ such that for each U in $\mathscr{U}$ there exists a residual set A of indices such that $(N \times N)[A \times A] \subset U$, i.e. $a \in A, b \in A \Rightarrow \langle N_a, N_b \rangle \in U$.

41 A.18. Theorem. *A closure space $\mathscr{P}$ is compact if and only if every net in $\mathscr{P}$ has an accumulation point. A uniform space $\mathscr{P}$ is complete if and only if every Cauchy net in $\mathscr{P}$ has an accumulation point.*

Proof. I. If $\mathscr{P}$ is not compact then there exists a proper filter $\mathscr{X}$ on $\mathscr{P}$ which has no cluster point. Let N be a single-valued relation such that $\mathbf{D}N = \mathscr{X}$ and $NX \in X$ for each $X \in \mathscr{X}$. Clearly $\langle N, \supset \rangle$ is a net in $\mathscr{P}$ which has no accumulation point. If a uniform space $\mathscr{P}$ is not complete, then there exists a Cauchy filter $\mathscr{X}$ without cluster points; then the net constructed above is a Cauchy net.

II. If N is a net in a closure space $\mathscr{P}$ and $\mathscr{X}$ is the collection of all subsets X of $|\mathscr{P}|$ such that N is eventually in X, then $\mathscr{X}$ is a proper filter on $\mathscr{P}$ (remember that $N \neq \emptyset$), and each cluster point of $\mathscr{X}$ is an accumulation point of N (15 B.3). Thus if $\mathscr{P}$ is compact then N has a cluster point. Now if N is a Cauchy net in a uniform space $\mathscr{P}$, then the filter $\mathscr{X}$ constructed above is a Cauchy filter. As a consequence, if $\mathscr{P}$ is complete, then N has a cluster point.

For the sake of completeness we shall prove the following result:

41 A.19. *Every convergent net in a uniform space is a Cauchy net. In a uniform space every Cauchy net converges to each of its accumulation points.*

Proof. Let $\langle P, \mathscr{U} \rangle$ be a uniform space. Suppose that N converges to x in $\langle P, \mathscr{U} \rangle$ and $U \in \mathscr{U}$. Choose symmetric V in $\mathscr{U}$ so that $V \circ V \subset U$ and a residual set A of indexes such that $N[A] \subset V[x]$. Clearly, $a \in A$, $b \in A \Rightarrow \langle N_a, N_b \rangle \in V \circ V \subset U$. Thus N is a Cauchy net. Now let x be an accumulation point of a Cauchy net N and G be a neighborhood of x. Choose a symmetric U in $\mathscr{U}$ so that $(U \circ U)[x] \subset G$ and then a residual set A of indices so that $(N \times N)[A \times A] \subset U$. Since x is an accumulation point of N, there exists an $a \in A$ with $N_a \in U[x]$. If $b \in A$, then $\langle N_b, N_a \rangle \in U$, $\langle N_a, x \rangle \in U$, hence $\langle N_b, x \rangle \in U \circ U$ and therefore $N_b \in (U \circ U)[x] \subset G$. Thus $b \in A$ implies $N_b \in G$.

Remark. It turns out that the class of all Cauchy nets on a uniform space $\mathscr{P}$ does not determine $\mathscr{P}$. This follows from 41 A.19 and the fact that there exist two distinct topologically equivalent complete uniformities (see 41 ex. 4).

41 A.20. *Every Cauchy filter converges to each of its cluster points.*

Proof. Let x be a cluster point of a Cauchy filter $\mathscr{X}$ on a uniform space $\langle P, \mathscr{U} \rangle$ and G be a neighborhood of x. We must find an X in $\mathscr{X}$ such that $X \subset G$. Choose a symmetric U in $\mathscr{U}$ such that $(U \circ U)[x] \subset G$ and then an $X \in \mathscr{X}$ such that $X \times \times X \subset U$. We shall prove that $X \subset G$. Since x belongs to the closure of X, the neighborhood $U[x]$ of x intersects X, and $X \times X$ being contained in U, we obtain $U[X \cap U[x]] \supset X$. However, $U[X \cap U[x]] \subset U[U[x]] = (U \circ U)[x] \subset G$.

B. COMPLETIONS

Recall that in the introduction to the present section a complete uniform space was defined as a uniform space $\mathscr{P}$ such that uniformly continuous mappings into $\mathscr{P}$ have certain uniformly continuous domain-extensions. In 41 A we adopted another definition. Now we shall prove that the two definitions are equivalent.

41 B.1. Theorem. *A uniform space $\mathscr{P}$ is complete if and only if the following condition is fulfilled:*

If f is a uniformly continuous mapping of a dense subspace of a uniform space $\mathscr{Q}$ into $\mathscr{P}$, then f is the restriction of a uniformly continuous mapping of $\mathscr{Q}$ into $\mathscr{P}$.

Remark. The condition can be restated as follows: every uniformly continuous mapping into $\mathscr{P}$ from a dense subspace of a uniform space $\mathscr{Q}$ has a uniformly continuous domain-extension to $\mathscr{Q}$.

Proof. I. Suppose that $\mathscr{P}$ is complete and f is a uniformly continuous mapping of a dense subspace $\mathscr{R}$ of a uniform space $\mathscr{Q}$ into $\mathscr{P}$. If $\mathscr{U}_x$ is the neighborhood system at a point $x \in |\mathscr{Q}| - |\mathscr{R}|$, then clearly $\mathscr{V}_x = |\mathscr{R}| \cap [\mathscr{U}_x]$ is a Cauchy filter on $\mathscr{R}$ (x belongs to the closure of $|\mathscr{R}|$ in $\mathscr{Q}$ and therefore $\emptyset \notin \mathscr{V}_x$; clearly $\mathscr{V}_x$ contains arbitrarily small sets), and, by 41 A.14, $f[[\mathscr{V}_x]]$ (i.e. $\mathbf{E}\{f[V] \mid V \in \mathscr{V}_x\}$) is a base for a Cauchy filter $\mathscr{W}_x$ on $\mathscr{P}$. Since $\mathscr{P}$ is a complete uniform space, the Cauchy filter has at least one cluster point, and hence, by 41 A.19, $\mathscr{W}_x$ converges to at least one point. As a consequence, there exists a domain-extension F of f to $\mathscr{Q}$ such that Fx is a limit point of $\mathscr{W}_x$ in $\mathscr{P}$ for each x in $|\mathscr{Q}| - |\mathscr{R}|$. Since f is uniformly continuous, to prove that F is uniformly continuous it is sufficient to show that, for each $x \in |\mathscr{Q}| - |\mathscr{R}|$, the domain-restriction F_x of F to the subspace $|\mathscr{R}| \cup (x)$ of $\mathscr{Q}$ is continuous at x (by 27 B.15); but this is obvious because $\mathscr{W}_x$ converges to Fx and hence each neighborhood G of Fx contains a $f[U \cap |\mathscr{R}|]$ $(= F_x[U \cap |\mathscr{R}|])$, $U \in \mathscr{U}_x$, and hence $F_x[U \cap \cap (|\mathscr{R}| \cup x)]$; but $U \cap (|\mathscr{R}| \cup (x))$ is a neighborhood of x in the subspace $|\mathscr{R}| \cup (x)$ of $\mathscr{Q}$.

II. Now suppose that $\mathscr{P} = \langle P, \mathscr{U} \rangle$ is not complete. We shall construct $\mathscr{Q}$ such that $\mathscr{P}$ is a dense subspace of $\mathscr{Q}$ and $f = \mathrm{J} : \mathscr{P} \to \mathscr{P}$ has no continuous domain-extension to $\mathscr{Q}$. There exists a Cauchy filter $\mathscr{X}$ on $\mathscr{P}$ which has no cluster point. Let Q be the set consisting of $\mathscr{X}$ and of all elements of P*). For each symmetric U in $\mathscr{U}$ let U^* consist of all elements of U, the element $\langle \mathscr{X}, \mathscr{X} \rangle$ and all the elements $\langle x, \mathscr{X} \rangle$ and $\langle \mathscr{X}, x \rangle$ such that $U[x] \in \mathscr{X}$; thus $U^* \cap (P \times P) = U$. Evidently each U^* is a symmetric vicinity of the diagonal of $Q \times Q$, and we shall prove that

(*) $V \circ V \subset U$ implies $V^* \circ V^* \subset U^*$,

where both V and U are assumed to be symmetric. Suppose that $x, y, z \in P$. If $\langle x, y \rangle \in V^*$, $\langle y, z \rangle \in V^*$, then $\langle x, y \rangle \in V$, $\langle y, z \rangle \in V$, and hence $\langle x, z \rangle \in V \circ V \subset \subset U \subset U^*$. If $\langle x, \mathscr{X} \rangle \in V^*$ and $\langle \mathscr{X}, y \rangle \in V^*$, then $V[x] \in \mathscr{X}$, $V[y] \in \mathscr{X}$; therefore $V[x] \cap V[y] \neq \emptyset$ and hence $x \in V \circ V[y]$, i.e. $\langle x, y \rangle \in V \circ V \subset U \subset U^*$. If $\langle \mathscr{X}, x \rangle \in \in V^*$, $\langle x, y \rangle \in V^*$, then $V[x] \in \mathscr{X}$, $\langle x, y \rangle \in V$ and thus $U[y] \supset V \circ V[y] \supset V[x]$, and hence $U[y] \in \mathscr{X}$, i.e. $\langle \mathscr{X}, y \rangle \in U^*$; this completes the proof of (*). Now it is clear that the collection of all U^*, U varying over all symmetric elements of $\mathscr{U}$, is a base for a uniformity $\mathscr{V}$ for Q, and clearly $\mathscr{U}$ is the relativization of $\mathscr{V}$ (remember that $U^* \cap Q \times Q = U$). Next, let us consider the filter $\mathscr{Y}$ on $\mathscr{Q} = \langle Q, \mathscr{V} \rangle$ generated by $\mathscr{X}$ (i.e. the smallest filter on $\mathscr{Q}$ containing $\mathscr{X}$). Since $\mathscr{X}$ is a base for $\mathscr{Y}$, $\mathscr{X}$ is a Cauchy filter on $\mathscr{P}$ and $\mathscr{P}$ is a subspace of $\mathscr{Q}$, $\mathscr{Y}$ is a Cauchy filter on $\mathscr{Q}$. Evidently $\mathscr{Y}$ converges to $\mathscr{X}$ in $\mathscr{Q}$ $(X \in \mathscr{X}, U \in \mathscr{U}, X \times X \subset U \Rightarrow X \subset U^*[\mathscr{X}])$, and hence $\mathscr{P}$ is dense in $\mathscr{Q}$. Now let f be the identity mapping of $\mathscr{P}$ onto $\mathscr{P}$. Suppose that F is a uniformly continuous mapping of $\mathscr{Q}$ into $\mathscr{P}$ whose domain-restriction to $\mathscr{P}$ is f. We shall show that this assumption leads to a contradiction. Since $\mathscr{Y}$ converges to $\mathscr{X}$, the smallest filter $\mathscr{Z}$ on $\mathscr{P}$ containing all $F[Y]$, $Y \in \mathscr{Y}$, converges to $F\mathscr{X}$. But clearly $\mathscr{Z} = \mathscr{X}$ (in fact, if $X \in \mathscr{X}$, then $F[X] = f[X] = X$, and since $\mathscr{X}$ is a base for $\mathscr{Y}$, $\mathscr{X}$ is a base for $\mathscr{Z}$; but $\mathscr{X}$ is a filter on $\mathscr{P}$) and $\mathscr{X}$ has no cluster point in $\mathscr{P}$ by our assumption.

In the second part of this proof we constructed a uniform space containing a given uniform space $\mathscr{P}$ as a dense subspace and such that the given Cauchy filter $\mathscr{X}$ on $\mathscr{P}$ is convergent in $\mathscr{Q}$. If we take the set ξ of all Cauchy filters on $\mathscr{P}$ which have no cluster points, then the set $Q = |\mathscr{P}| \cup \xi$ can be endowed with a uniformity $\mathscr{V}$ such that $\mathscr{P}$ is a dense subspace of $\mathscr{Q} = \langle Q, \mathscr{V} \rangle$ and $\mathscr{X}$ converges to $\mathscr{X}$ in $\mathscr{Q}$ for each $\mathscr{X}$ in ξ. It will follow that $\mathscr{Q}$ is complete (see the proof of 41 B.5). If $\mathscr{Y}$ is a neighborhood system of $\mathscr{X}$ in $\mathscr{Q}$, then $|\mathscr{P}| \cap [\mathscr{Y}] \subset \mathscr{X}$ but in general $|\mathscr{P}| \cap [\mathscr{Y}] \neq \mathscr{X}$. It turns out that we can take an appropriate subset of ξ instead of ξ such that the resulting space $\mathscr{Q}$ is complete, and $|\mathscr{P}| \cap [\mathscr{Y}] = \mathscr{X}$ for each $\mathscr{X}$. The choice of ξ will be clear from two propositions which follow.

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*) For convenience we assume that $\mathscr{X} \notin P$; see 35 F.1.

41 B.2. *Let $\langle P, \mathscr{U}\rangle$ be a uniform space. If $\mathscr{X}$ is a Cauchy filter on $\langle P, \mathscr{U}\rangle$ then the collection $\mathsf{E}\{U[X] \mid U \in \mathscr{U}, X \in \mathscr{X}\}$ is a base for a Cauchy filter contained in $\mathscr{X}$ which will be denoted by $m(\mathscr{X})$. If $\mathscr{X}$ is a Cauchy filter, then $m(m(\mathscr{X})) = m(\mathscr{X})$ and each of the following three conditions is necessary and sufficient for $\mathscr{X} = m(\mathscr{X})$:*

(a) *each $X \in \mathscr{X}$ is a proximal neighborhood of a set of $\mathscr{X}$;*

(b) *if $\mathscr{Y}$ is a Cauchy filter, then either $\mathscr{Y} \supset \mathscr{X}$ or $X \cap Y = \emptyset$ for some $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$;*

(c) *$\mathscr{X}$ is a minimal Cauchy filter, i.e. if $\mathscr{Y} \subset \mathscr{X}$* and *$\mathscr{Y}$ is a Cauchy filter, then $\mathscr{Y} = \mathscr{X}$.*

Proof. I. Let $\mathscr{X}$ be a Cauchy filter on $\langle P, \mathscr{U}\rangle$ and let $\mathscr{Z}$ be the collection consisting of all $U[X]$, $U \in \mathscr{U}$, $X \in \mathscr{X}$. If $Z_i = U_i[X_i]$, $i = 1, 2$, where $U_i \in \mathscr{U}$ and $X_i \in \mathscr{X}$, then $X = (X_1 \cap X_2) \in \mathscr{X}$, $U = (U_1 \cap U_2) \in \mathscr{U}$ and hence $Z = U[X] \in \mathscr{Z}$; clearly $\emptyset \neq Z \subset Z_1 \cap Z_2$. Thus $\mathscr{Z}$ is a base for some filter $m(\mathscr{X})$ on $\langle P, \mathscr{U}\rangle$. We shall show that $m(\mathscr{X})$ is a Cauchy filter. According to 41 A.7 it is sufficient to show that for each U in $\mathscr{U}$ there exists an x in P such that $U[x] \in m(\mathscr{X})$. Given U in $\mathscr{U}$, choose a symmetric, V in $\mathscr{U}$ so that $V \circ V \subset U$ and an $x \in P$ so that $V[x] \in \mathscr{X}$ (this is possible by 41 A.7). We have $U[x] \supset V[V[x]] \in \mathscr{Z}$ and hence $U[x] \in m(\mathscr{X})$. – II. Evidently $\mathscr{X} \supset m(\mathscr{X})$, and to prove that $m(m(\mathscr{X})) = m(\mathscr{X})$ it is enough to show that $m(m(\mathscr{X})) \supset m(\mathscr{X}))$. Let $Y \in m(\mathscr{X})$ and choose an X in $\mathscr{X}$ and U in $\mathscr{U}$ so that $U[X] \subset Y$, and a symmetric V in $\mathscr{U}$ so that $V \circ V \subset U$. We have $V[X] \in m(\mathscr{X})$ and hence $V[V[X]] \in m(m(\mathscr{X}))$; however $V[V[X]] = V \circ V[X] \subset U[X] \subset Y$ and hence $Y \in m(m(\mathscr{X}))$. – III. Suppose that $\mathscr{X}$ is a Cauchy filter. If $\mathscr{X} = m(\mathscr{X})$ and $X \in \mathscr{X}$, then $X \supset U[Y]$ for some U in $\mathscr{U}$ and Y in $\mathscr{X}$ and hence X is a proximal neighborhood of Y. Thus condition (a) is necessary. Next we shall prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that (c) implies $\mathscr{X} = m(\mathscr{X})$. Evidently (b) $\Rightarrow$ (c), and if (c) is fulfilled then $\mathscr{X} = m(\mathscr{X})$ because $m(\mathscr{X})$ is a Cauchy filter contained in $\mathscr{X}$. It remains to show that (a) implies (b). Suppose that $\mathscr{Y}$ is a Cauchy filter such that $X \cap Y \neq \emptyset$ for each $X \in \mathscr{X}$, $Y \in \mathscr{Y}$; we must prove $\mathscr{X} \subset \mathscr{Y}$. Let $X \in \mathscr{X}$; to prove $X \in \mathscr{Y}$ it is sufficient to find a Y in $\mathscr{Y}$ such that $Y \subset X$. Choose an X' in $\mathscr{X}$ so that X is a proximal neighborhood of X', i.e. $U[X'] \subset X$ for some U in $\mathscr{U}$, and then a set Y in $\mathscr{Y}$ so that $Y \times Y \subset U$. Since $X' \cap Y \neq \emptyset$ we obtain that $Y \subset U[X' \cap Y] \subset U[X'] \subset X$.

41 B.3. *Let $\langle P, \mathscr{U}\rangle$ be a uniform space. The neighborhood system of each point of $\langle P, \mathscr{U}\rangle$ is a minimal Cauchy filter. If x is a cluster point of a Cauchy filter $\mathscr{X}$, then $\mathscr{X}$ converges to x, and if $\mathscr{X}$ is also a minimal Cauchy filter, then $\mathscr{X}$ is the neighborhood system of x.*

Proof. I. The first statement follows from description 41 B.2 (a) of minimal Cauchy filters. Indeed, if $V \in \mathscr{U}$, $U \in \mathscr{U}$ and $V \circ V \subset U$, then $U[x]$ is a proximal neighborhood of $V[x]$.

II. Let x be a cluster point of a Cauchy filter $\mathscr{X}$ and let $\mathscr{Y}$ be the neighborhood system of x. By I, $\mathscr{Y}$ is a minimal Cauchy filter and clearly $X \cap Y \neq \emptyset$ for each $X \in \mathscr{X}$

and $Y \in \mathscr{Y}$. By 41 B.2 (condition (b)) we obtain $\mathscr{X} \supset \mathscr{Y}$, which implies that $\mathscr{X}$ converges to x; if $\mathscr{X}$ is a minimal Cauchy filter then the inclusion $\mathscr{X} \supset \mathscr{Y}$ implies $\mathscr{X} = \mathscr{Y}$.

Corollaries to 41 B.2 and 41 B.3. (a) *A uniform space $\mathscr{P}$ is complete if and only if the intersection of each minimal Cauchy filter on $\mathscr{P}$ is non-void.* (b) *Let $\mathscr{P}$ be a subspace of a uniform space $\mathscr{Q}$. If $\mathscr{X}$ is a minimal Cauchy filter on $\mathscr{Q}$ and if $X \cap |\mathscr{P}| \neq \emptyset$ for each $X \in \mathscr{X}$, then $[\mathscr{X}] \cap |\mathscr{P}|$ is a minimal Cauchy filter on $\mathscr{P}$; in particular, if x belongs to the closure of $|\mathscr{P}|$ in $\mathscr{Q}$ and if $\mathscr{X}$ is the neighborhood system of x in $\mathscr{Q}$, then $[\mathscr{X}] \cap |\mathscr{P}|$ is a minimal Cauchy filter on $\mathscr{P}$. If $\mathscr{P}$ is dense in $\mathscr{Q}$ and if $\mathscr{X}$ is a minimal Cauchy filter on $\mathscr{Q}$, then $[\mathscr{X}] \cap P$ is a minimal Cauchy filter on $\mathscr{P}$ (because $X \cap |\mathscr{P}| \neq \emptyset$ for each $X \in \mathscr{X}$).*

41 B.4. Definition. A *completion of a uniform space* $\mathscr{P}$ is a complete uniform space containing $\mathscr{P}$ as a dense subspace. A *completion-embedding* is a uniform embedding f such that $\mathbf{E}^*f$ is a completion of the subspace $\mathbf{E}f$ of $\mathbf{E}^*f$. A *completion* $\mathscr{Q}$ of a uniform space $\mathscr{P}$ is said to be *augmentation-separated* if $x \in |\mathscr{Q}| - |\mathscr{P}|$, $y \in |\mathscr{Q}|$, $x \neq y$ imply that x and y are separated, i.e. $\bigcap\{X \mid X \in \mathscr{X}\} = (x)$ whenever $x \in |\mathscr{Q}| - |\mathscr{P}|$ and $\mathscr{X}$ is the neighborhood system of x in $\mathscr{Q}$.

41 B.5. Theorem. *Every uniform space has an augmentation-separated completion.*

Proof. I. Let $\mathscr{P} = \langle P, \mathscr{U}\rangle$ be a uniform space and let ξ be a set of Cauchy filters on $\mathscr{P}$ without cluster points. Put*) $Q = P \cup \xi$. For each symmetric $U \in \mathscr{U}$ let U^* consist of all elements of U, of all pairs $\langle x, \mathscr{X}\rangle$ and $\langle \mathscr{X}, x\rangle$ such that $x \in P$, $\mathscr{X} \in \xi$ and $U[x] \in \mathscr{X}$, and of all pairs $\langle \mathscr{X}, \mathscr{Y}\rangle$ such that $\mathscr{X} \in \xi$, $\mathscr{Y} \in \xi$ and $X \times Y \subset U$ for some $X \in \mathscr{X}$, $Y \in \mathscr{Y}$. It is obvious that

(*) $U^* \cap (P \times P) = U, J_Q \subset U^* \subset Q \times Q, U^* \cap V^* = (U \cap V)^*, (U^*)^{-1} = U^*.$

We shall prove that

(**) $V \circ V \subset U \Rightarrow V^* \circ V^* \subset U^*.$

According to the second part of the proof of 41 B.1 it remains to show that $\langle \mathscr{X}, \mathscr{Y}\rangle \in V^*$, $\langle \mathscr{Y}, \mathscr{Z}\rangle \in V^*$ implies $\langle \mathscr{X}, \mathscr{Z}\rangle \in U^*$. Choose $X \in \mathscr{X}$, $Y \in \mathscr{Y}$ and $Z \in \mathscr{Z}$ such that $X \times Y \subset V$, $Y \times Z \subset V$. Since $(Y \times Z) \circ (X \times Y) = X \times Z$ we have $X \times Z \subset V \circ V \in U$ and hence $\langle \mathscr{X}, \mathscr{Z}\rangle \in U^*$. It follows from (*) and (**) that the set of all U^* is a base for a uniformity $\mathscr{V}$ for Q and $\mathscr{U}$ is a relativization of $\mathscr{V}$, i.e. $\langle P, \mathscr{U}\rangle$ is a subspace of the uniform space $\langle Q, \mathscr{V}\rangle$. We shall prove that

(***) if $\mathscr{X} \in \xi$ and $\mathscr{Y}$ is the neighborhood system of $\mathscr{X}$ in $\langle \mathscr{Q}, \mathscr{V}\rangle$, then $\mathscr{Z} = P \cap [\mathscr{Y}]$ is the minimal Cauchy filter contained in $\mathscr{X}$.

First we shall show that $\mathscr{X} \supset \mathscr{Z}$. If U is a symmetric element of $\mathscr{U}$, $X \in \mathscr{X}$, $X \times X \subset U$, then clearly $U^*[\mathscr{X}] \supset X$; since the sets of the form $U^*[\mathscr{X}]$ form a base for $\mathscr{Y}$, it follows that $\mathscr{X} \supset \mathscr{Z}$. The inclusion $\mathscr{X} \supset \mathscr{Z}$ implies that $Y \cap P \neq \emptyset$ for each $Y \in \mathscr{Y}$ and hence, by the corollary of 41 B.3, $\mathscr{Z}$ is a minimal Cauchy filter. It follows from

*) For convenience we assume that $\xi \cap P = \emptyset$; see 35 F.1.

(***) that $\langle P, \mathscr{U} \rangle$ is dense in $\langle Q, \mathscr{V} \rangle$ and the Cauchy filter base $\mathscr{X}$ converges to $\mathscr{X}$ in $\langle Q, \mathscr{V} \rangle$ for each $\mathscr{X} \in \xi$.

II. If every free minimal Cauchy filter on $\langle P, \mathscr{U} \rangle$ is contained in an $\mathscr{X} \in \xi$, then $\langle Q, \mathscr{V} \rangle$ is complete; in particular, if ξ consists of all free minimal Cauchy filters on $\langle P, \mathscr{U} \rangle$, then $\langle Q, \mathscr{V} \rangle$ is complete.

Proof. It is sufficient to show that every minimal Cauchy filter $\mathscr{Z}$ on $\langle Q, \mathscr{V} \rangle$ has a cluster point. Let $\mathscr{Z}$ be a minimal Cauchy filter on $\langle Q, \mathscr{V} \rangle$ without cluster points, i.e. $\mathscr{Z}$ is a free minimal Cauchy filter on $\langle Q, \mathscr{V} \rangle$. Since P is dense in $\langle Q, \mathscr{V} \rangle$, $\mathscr{Y} = [\mathscr{Z}] \cap P$ is a free minimal Cauchy filter on $\langle P, \mathscr{U} \rangle$, which is contained, by our assumption, in an $\mathscr{X} \in \xi$. Since the Cauchy filter base $\mathscr{X}$ converges to $\mathscr{X}$ in $\langle Q, \mathscr{V} \rangle$, $\mathscr{Z}$ also converges to $\mathscr{X}$.

III. Now let ξ be the set of all free minimal Cauchy filters on $\langle P, \mathscr{U} \rangle$ and let $\langle Q, \mathscr{V} \rangle$ be the space constructed in I. By I and II the space $\langle Q, \mathscr{V} \rangle$ is a completion of $\langle P, \mathscr{U} \rangle$. We shall prove that $\langle Q, \mathscr{V} \rangle$ is an augmentation-separated completion. Let $\mathscr{X} \in \xi$ and let $\mathscr{Y}$ be the neighborhood system of $\mathscr{X}$ in $\langle Q, \mathscr{V} \rangle$. We must show that $\bigcap \mathscr{Y} = (\mathscr{X})$. Since $[\mathscr{Y}] \cap P \subset \mathscr{X}$ by (***), we have $\bigcap \mathscr{Y} \subset \xi$. Let $\mathscr{X}_1 \in (\xi - \mathscr{X})$. Since $\mathscr{X}$ and $\mathscr{X}_1$ are distinct minimal Cauchy filters, we can choose $X \in \mathscr{X}$, $X_1 \in \mathscr{X}_1$ and a symmetric $U \in \mathscr{U}$ such that $U[X] \cap U[X_1] = \emptyset$. It is easily seen that $\mathscr{X}_1 \notin$ $\notin U^*[(\mathscr{X})]$. The proof is complete.

41 B.6. *A completion $\mathscr{Q}$ of a uniform space $\mathscr{P}$ is augmentation-separated if and only if no proper subspace $\mathscr{R}$ of $\mathscr{Q}$, $|\mathscr{R}| \supset |\mathscr{P}|$, is complete. In particular, a uniform space $\mathscr{P}$ is complete if and only if $\mathscr{P}$ is the only augmentation-separated completion of $\mathscr{P}$.*

Proof. I. Let $\mathscr{Q}$ be an augmentation-separated completion of $\mathscr{P}$ and $\mathscr{R}$ be a proper subspace of $\mathscr{Q}$, $|\mathscr{P}| \subset |\mathscr{R}|$. If $\mathscr{X}$ is the neighborhood system of a point $x \in |\mathscr{Q}| - |\mathscr{R}|$, then $\bigcap \mathscr{X} = (x)$, and hence $|\mathscr{R}| \cap \mathscr{X}$ is a Cauchy filter on $\mathscr{R}$ which has no cluster point (notice e.g. that $|\mathscr{R}| \cap \mathscr{X}$ is a free minimal Cauchy filter on $\mathscr{R}$). – II. If a completion $\mathscr{Q}$ of $\mathscr{P}$ is not augmentation-separated then there exists a point $x \in |\mathscr{Q}| - |\mathscr{P}|$ such that $\bigcap \mathscr{X} \neq (x)$, where $\mathscr{X}$ is the neighborhood system of x in $\mathscr{Q}$. If $y \in \bigcap \mathscr{X} -$ $- (x)$, then a Cauchy filter converges to y if (and only if) it converges to x. It follows that the subspace $|\mathscr{Q}| - (x)$ of $\mathscr{Q}$ is complete. – III. The second statement is an immediate consequence of the first.

41 B.7. Theorem. *If $\mathscr{P}$ is a uniform space, $\mathscr{Q}$ is an augmentation-separated completion of $\mathscr{P}$ and $\mathscr{R}$ is a completion of $\mathscr{P}$, then there exists a uniformly continuous mapping f of $\mathscr{R}$ into $\mathscr{Q}$ such that $f : \mathscr{P} \to \mathscr{P}$ is the identity mapping. The mapping f is a surjective projective generating mapping.*

Corollary. *An augmentation-separated completion of a uniform space $\mathscr{P}$ is uniquely determined up to a uniform homeomorphism which is an identity on $\mathscr{P}$.*

Proof. I. By 41 B.1 the mapping $J : \mathscr{P} \to \mathscr{Q}$ has a uniformly continuous domain-extension f to $\mathscr{R}$. If $x \in |\mathscr{Q}| - |\mathscr{P}|$ and if $\mathscr{X}$ is the neighborhood system of x in $\mathscr{Q}$, then $[\mathscr{X}] \cap |\mathscr{P}|$ is a Cauchy filter on $\mathscr{P}$ which has no cluster point in $\mathscr{P}$. Since $\mathscr{R}$ is complete, the Cauchy filter base $[\mathscr{X}] \cap |\mathscr{P}|$ converges to a point $y \in \mathscr{R}$, and f being continuous, $fy = x$. Thus f is surjective. The fact that f is a projective generating mapping follows from 37 ex. 2.

II. Let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be two augmentation-separated completions of $\mathscr{P}$ and let f_1 (f_2) be a uniformly continuous mapping of $\mathscr{Q}_1$ onto $\mathscr{Q}_2$ ($\mathscr{Q}_2$ onto $\mathscr{Q}_1$) such that $f_1 : \mathscr{P} \to \mathscr{P}$ $(f_2 : \mathscr{P} \to \mathscr{P})$ is the identity mapping of $\mathscr{P}$. Clearly $f = f_2 \circ f_1$ is a uniformly continuous mapping of $\mathscr{Q}_1$ onto $\mathscr{Q}_1$ and $f : \mathscr{P} \to \mathscr{P}$ is the identity mapping. We must show that f is the identity mapping of $\mathscr{Q}_1$ onto $\mathscr{Q}_1$. If $x \in |\mathscr{Q}_1| - |\mathscr{P}|$ and $y \neq x$, $y \in |\mathscr{Q}_1|$, then there exists a neighborhood U of x and V of y such that $U \cap V = \emptyset$. Since x belongs to the closure of $U \cap |\mathscr{P}|$, fx belongs to the closure of $f[U \cap |\mathscr{P}|] = U \cap |\mathscr{P}|$, and hence $fx \neq y$. Thus $fx \neq y$ for each $y \in |\mathscr{Q}_1|$, $y \neq x$, i.e. $fx = x$.

Remark. If $\mathscr{P}$ is separated then the second part of the proof follows from the uniqueness theorem (27 A.8). Indeed, $\mathscr{Q}$ is then separated and the identity mapping f of $\mathscr{Q}_1$ onto $\mathscr{Q}_1$ is the unique mapping of $\mathscr{Q}_1$ into $\mathscr{Q}_1$ such that $f : \mathscr{P} \to \mathscr{P}$ is the identity mapping.

41 B.8. Theorem. *If $\mathscr{Q}$ is a completion of a uniform space $\mathscr{P}$ and $\mathscr{R}$ is a subspace of $\mathscr{P}$, then the closure of $|\mathscr{R}|$ in $\mathscr{Q}$ is a completion of $\mathscr{R}$. If $\mathscr{P}$ is the product of a family $\{\mathscr{P}_a\}$ of uniform spaces and $\mathscr{Q}_a$ is a completion of $\mathscr{P}_a$ for each a, then the product $\mathscr{Q}$ of $\{\mathscr{Q}_a\}$ is a completion of $\mathscr{P}$.*

Proof. The first statement follows from the fact that a closed subspace of a complete uniform space is complete. To prove the second statement only recall that $\mathscr{P}$ is dense in $\mathscr{Q}$ (17 C.2) and $\mathscr{Q}$ is complete (41 A.12).

41 B.9. *Let $\mathscr{Q}$ be a completion of a uniform space $\mathscr{P}$. The space $\mathscr{P}$ is pseudometrizable if and only if $\mathscr{Q}$ is pseudometrizable; $\mathscr{P}$ is totally bounded if and only if $\mathscr{Q}$ is totally bounded.*

Proof. Since $\mathscr{P}$ is dense in $\mathscr{Q}$ and $\mathscr{Q}$ is a uniform space, the statements follow from ex. 1, 2.

Corollary. *Every completion of a totally bounded uniform space is compact.*

Proof. By 41 A.8 every complete totally bounded uniform space is compact.

41 B.10. *Let $\mathscr{Q}$ be an augmentation-separated completion of a uniform space $\mathscr{P}$. If $\mathscr{P}$ is separated, then $\mathscr{Q}$ is separated. If $\mathscr{P}$ is metrizable, then $\mathscr{Q}$ is metrizable.*

Proof. The first statement is obvious and the second follows from 41 B.9 and the fact that a separated pseudometrizable space is metrizable.

The remaining part of this subsection is devoted to topological groups. Let $\mathscr{G} = \langle G, \cdot, u \rangle$ be a topological group, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{U}$ be the right uniformity, the left uniformity and the two-sided uniformity of $\mathscr{G}$. Recall that if $\mathcal{O}$ is the neighborhood system of the unit, then the set of all $O_R = \mathbf{E}\{\langle x, y \rangle \mid x \,.\, y^{-1} \in O\}$, $O \in \mathcal{O}$, is a base for $\mathscr{R}$, the set of all $O_L = \mathbf{E}\{\langle x, y \rangle \mid x^{-1} \,.\, y \in O\}$, $O \in \mathcal{O}$, is a base for $\mathscr{L}$, and

$\mathscr{R} \cup \mathscr{L}$ is a sub-base for $\mathscr{U}$. Thus $X \times X \subset O_R$ if and only if $x \,.\, y^{-1} \in O$ for each $x \in X$, $y \in X$, i.e. $X \,.\, X^{-1} \subset O$, and $X \times X \subset O_L$ if and only if $X^{-1} \,.\, X \subset O$. Clearly

41 B.11. *If $\mathscr{X}$ is a proper filter on a topological group $\mathscr{G}$, then*

(a) *$\mathscr{X}$ is a Cauchy filter with respect to the two-sided uniformity of $\mathscr{G}$ if and only if $\mathscr{X}$ is a Cauchy filter with respect to both the right and left uniformities of $\mathscr{G}$.*

(b) *$\mathscr{X}$ is a Cauchy filter with respect to the right uniformity of $\mathscr{G}$ if and only if the filter base $\mathsf{E}\{[X] \,.\, [X^{-1}] \mid X \in \mathscr{X}\}$ converges to the unit in $\mathscr{G}$.*

(c) *$\mathscr{X}$ is a Cauchy filter with respect to the left uniformity of $\mathscr{G}$ if and only if the filter base $\mathsf{E}\{[X^{-1}] \,.\, [X] \mid X \in \mathscr{X}\}$ converges to the unit in $\mathscr{G}$.*

41 B.12. *If the right uniformity or the left uniformity of a topological group $\mathscr{G}$ is complete, then all three uniformities of $\mathscr{G}$ are complete.*

Proof. $\{x \to x^{-1}\} : \langle |\mathscr{G}|, \mathscr{R}\rangle \to \langle |\mathscr{G}|, \mathscr{L}\rangle$ is a uniform homeomorphism, and therefore the right uniformity is complete if and only if the left uniformity is complete. The two-sided uniformity is uniformly finer than the right uniformity and topologically equivalent with the right uniformity, and therefore, by 41 A.16, if the right uniformity is complete then the two-sided uniformity is also complete.

41 B.13. Example. If $\mathscr{G}$ is a subgroup of a topological group $\mathscr{H}$, then the left uniformity, right uniformity and the two-sided uniformity of $\mathscr{G}$ are relativizations of the corresponding uniformities of $\mathscr{H}$. A natural problem arises: given $\mathscr{G}$, under what conditions does there exist a $\mathscr{H}$ such that $\mathscr{G}$ is dense in $\mathscr{H}$ and the right (equivalently, left) uniformity or the two-sided uniformity of $\mathscr{H}$ is complete. We restrict our attention to separated $\mathscr{G}$ and $\mathscr{H}$. We shall prove that $\mathscr{H}$ can be chosen so that the two-sided uniformity of $\mathscr{H}$ is complete. On the other hand it is clear that if the two-sided uniformity of $\mathscr{G}$ is complete, then $\mathscr{G}$ is closed in any separated group (by 41 A.11) and therefore, if in addition the right uniformity of $\mathscr{G}$ is not complete, then $\mathscr{G}$ is a subgroup of no separated topological group whose right uniformity is complete (because any closed subspace of a complete uniform space is complete).

We begin with a very simple particular case which includes commutative groups. First define a topological group $\langle G, \sigma, u\rangle$ to be uniformly continuous if the mapping $\sigma : \langle G, \mathscr{U}\rangle \times \langle G, \mathscr{U}\rangle \to \langle G, \mathscr{U}\rangle$ is uniformly continuous where $\mathscr{U}$ is the two-sided uniformity.

41 B.14. Theorem. *Every uniformly continuous separated topological group $\mathscr{G}$ is a dense subgroup of a uniformly continuous separated topological group $\mathscr{H}$ whose two-sided uniformity is complete. More precisely, if $\mathscr{U}$ is the two-sided uniformity of a uniformly continuous separated topological group $\mathscr{G} = \langle G, \sigma, u\rangle$ and if $\langle H, \mathscr{V}\rangle$ is an augmentation-separated completion of $\langle G, \mathscr{U}\rangle$, then the uniformly continuous extension $\mu : \langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle \to \langle H, \mathscr{V}\rangle$ of $\sigma : \langle G, \mathscr{U}\rangle \times \langle G, \mathscr{U}\rangle \to \langle G, \mathscr{U}\rangle$ defines a group structure on H, the closure v induced by $\mathscr{V}$ is compatible for $\langle H, \mu\rangle$ and $\mathscr{V}$ is the two-sided uniformity of $\mathscr{H} = \langle H, \mu, v\rangle$.*

Remark. If $\mathscr{G}$ is commutative, then $\mathscr{H}$ is commutative.

Proof. I. Let, $\langle H, \mathscr{V}\rangle$ be an augmentation-separated completion of $\langle G, \mathscr{U}\rangle$. Since the mapping σ of $\langle G, \mathscr{U}\rangle \times \langle G, \mathscr{U}\rangle$ into $\langle G, \mathscr{U}\rangle$ and hence into $\langle H, \mathscr{V}\rangle$ is uniformly continuous and $\langle H, \mathscr{V}\rangle$ is complete and separated, by 41 B.1 and 41 B.8 there exists a unique uniformly continuous extension of σ to a mapping μ of $\langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle$ into $\langle H, \mathscr{V}\rangle$. Let f be the inversion of $\langle G, \sigma, u\rangle$; $f : \langle G, \mathscr{U}\rangle \to \langle G, \mathscr{U}\rangle$ is uniformly continuous and therefore, by 41 B.1, there exists an extension g of f to a uniformly continuous mapping of $\langle H, \mathscr{V}\rangle$ into $\langle H, \mathscr{V}\rangle$. Since f is a uniform homeomorphism, g is also a uniform homeomorphism. It is easily seen that μ is a group structure on H and g is the inversion of $\langle H, \mu\rangle$; the proof is based on the unicity theorem (27 A.8) for continuous mappings into a separated closure space. We shall prove transitivity; the remainder is left to the reader. The mappings $\{\langle x, y, z\rangle \to x\mu(y\mu z)\}$ and $\{\langle x, y, z\rangle \to (x\mu y)\mu z\}$ of $\langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle$ into $\langle H, \mathscr{V}\rangle$ are continuous and coincide on a dense subspace, namely on $\langle G, \mathscr{U}\rangle \times \langle G, \mathscr{U}\rangle \times \langle G, \mathscr{U}\rangle$, and therefore, by the unicity theorem, they coincide. — II. Let v be the closure induced by $\mathscr{V}$. Clearly u is a relativization of v. Since μ is uniformly continuous and the inversion of μ is uniformly continuous (of course, under $\mathscr{V}$), $\langle \mu, v\rangle$ is continuous and its inversion is also continuous. Thus $\mathscr{H} = \langle H, \mu, v\rangle$ is a topological group and $\langle G, \sigma, u\rangle$ is a subgroup of $\mathscr{H}$. If $\mathscr{W}$ is the two-sided uniformity of $\mathscr{H}$, then $\mathscr{U}$ is a relativization of $\mathscr{W}$, but $\mathscr{U}$ is also a relativization of $\mathscr{V}$, $\mathscr{W}$ and $\mathscr{V}$ are topologically equivalent and G is dense in $\mathscr{H}$. In consequence, $\mathscr{V} = \mathscr{W}$ by 27 B.13. Since $\mathscr{V}$ is complete, $\mathscr{W}$ is complete. — III. The remark is proved in the same way as e.g. the transitivity of μ was proved above.

If the multiplication is not uniformly continuous then we cannot use the extension theorem 41 B.1 and we must prove the continuity directly; the main step is contained in the following lemma.

41 B.15. Lemma. *Let $\mathscr{R}$ be the right uniformity of a topological group $\mathscr{G} = \langle G, \cdot, u\rangle$. If $\mathscr{X}$ and $\mathscr{Y}$ are Cauchy filters on $\langle G, \mathscr{R}\rangle$ then $\mathscr{Z} = \mathbf{E}\{[X] . [Y] \mid X \in \mathscr{X}, Y \in \mathscr{Y}\}$ is a Cauchy filter base on $\langle G, \mathscr{R}\rangle$. A similar result holds for the left uniformity.*

Proof. Let O be a neighborhood of the unit element. We must find an $X \in \mathscr{X}$ and a $Y \in \mathscr{Y}$ so that $x, x_1 \in X$, $y, y_1 \in Y$ imply $x . y . (x_1 . y_1)^{-1} \in O$. If z is any element of G, then $x . y . (x_1 . y_1)^{-1} = x . y . y_1^{-1} . x_1^{-1} = (x . a^{-1}) . a . y . y_1^{-1} . a^{-1} . (a . x_1^{-1})$. Choose a neighborhood V of the unit such that $[V] . [V] . [V] \subset O$, an $X \in \mathscr{X}$ such that $[X] . [X^{-1}] \subset V$, an $a \in X$, a neighborhood U of the unit such that $[a . U] . a^{-1} \subset V$, and finally a Y in $\mathscr{Y}$ such that $[Y] . [Y^{-1}] \subset U$. Clearly $x . y . (x_1 . y_1)^{-1} \in O$ for each $x, x_1 \in X$ and $y, y_1 \in Y$.

Corollary. *Under the assumption of* 41 B.15, *if $\mathscr{X}$ and $\mathscr{Y}$ are Cauchy filters on $\langle G, \mathscr{U}\rangle$ then $\mathscr{Z}$ is a Cauchy filter base on the same space.* — Apply 41 B.11.

41 B.16. Theorem. *Every separated topological group $\mathscr{G}$ is a dense subgroup of a separated topological group $\mathscr{H}$ whose two-sided uniformity is complete.*

Proof. Let $\mathscr{U}$ be the two-sided uniformity of $\mathscr{G} = \langle G, \sigma, u\rangle$ and let $\langle H, \mathscr{V}\rangle$ be an augmentation-separated completion of $\langle G, \mathscr{U}\rangle$. The inversion f of $\langle G, \sigma\rangle$ is uniformly continuous under $\mathscr{U}$ and therefore, by 41 B.1, f has a uniformly continuous extension g of $\langle H, \mathscr{V}\rangle$ into $\langle H, \mathscr{V}\rangle$; since f is a uniform homeomorphism, g is also a uniform homeomorphism. The multiplication σ need not be uniformly continuous and therefore 41 B.1 does not apply. It follows, however, that σ can be continuously (not necessarily uniformly) extended to a mapping of $\langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle$. The remainder of the proof follows the pattern of the proof of 41 B.15. If $\langle x, y\rangle \in H \times H$, $\mathscr{X}$ is the neighborhood system of x and $\mathscr{Y}$ is the neighborhood system of y and $\mathscr{X}_1 = G \cap [\mathscr{X}]$, $\mathscr{Y}_1 = G \cap [\mathscr{Y}]$, then $\mathscr{X}_1$ and $\mathscr{Y}_1$ are Cauchy filters on $\langle G, \mathscr{U}\rangle$, and hence, by the corollary of 41 B.15, $\mathscr{Z} = \mathbf{E}\{[X] . [Y] \mid X \in \mathscr{X}, Y \in \mathscr{Y}\}$ is a Cauchy filter base on $\langle G, \mathscr{U}\rangle$ which converges to a unique point $x\mu y$ in $\langle H, \mathscr{V}\rangle$. Clearly σ is a restriction of μ, and it follows from 27 B.10 that $\mu : \langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle \to \langle H, \mathscr{V}\rangle$ is continuous.

41 B.17. Theorem. *In order that a separated topological group $\mathscr{G} = \langle G, \sigma, u\rangle$ be a dense subgroup of a separated topological group $\mathscr{H} = \langle H, \mu, v\rangle$ whose right uniformity is complete, it is necessary and sufficient that the inversion of $\mathscr{G}$ carry Cauchy filters under the right uniformity into Cauchy filters under the right uniformity, i.e. that the Cauchy filters under the right and left uniformities coincide.*

Proof. I. Suppose that such an $\mathscr{H}$ exists. Since the right uniformity of $\mathscr{H}$ is complete, the left uniformity is also complete, and since these uniformities are topologically equivalent, they have the same Cauchy filters. Since the right and the left uniformities of $\mathscr{G}$ are relativizations of the corresponding uniformities of $\mathscr{H}$, the same holds for the left and the right uniformities of $\mathscr{G}$. – II. Let $\mathscr{R}$ be the right uniformity of $\mathscr{G}$ and let $\langle H, \mathscr{V}\rangle$ be an augmentation-separated completion of $\langle G, \mathscr{R}\rangle$. Using Lemma 41 B.15 instead of Corollary of 41 B.15 the procedure of the proof of the preceding theorem makes it possible to obtain a continuous domain-extension of $\sigma : \langle G, u\rangle \times \langle G, u\rangle \to \langle H, \mathscr{V}\rangle$ to $\langle H, \mathscr{V}\rangle \times \langle H, \mathscr{V}\rangle$. Using the condition and the same procedure, we obtain an extension of the inversion of $\mathscr{G}$ to a continuous mapping of $\langle H, \mathscr{V}\rangle$ into $\langle H, \mathscr{V}\rangle$. The remainder is clear.

Completions of topological rings, fields, modules and algebras are considered in the exercises. It may be in place to point out that the only difficult step is the extension of the multiplication of a topological ring to a completion of the underlying additive topological group.

C. SPECIAL PROPERTIES OF COMPACT SPACES

In the present subsection, some special properties of compact spaces are developed further. Concerning the contents of the subsection, the main result is Theorem 41 C.4. Then we shall prove the following two important properties of compact spaces: a continuous mapping of a compact space into a separated space is inversely upper semi-continuous, and a mapping f of a space into a compact space such that the graph of f is closed in the product space is continuous. The concluding part is devoted to an investigation of compact uniformizable spaces. The main characterizations of compactness in the class of all uniformizable spaces are summarized in Theorem 41 C.23. Special attention is given to algebras of continuous functions, in particular, various formulations of the so-called Stone-Weierstrass Theorem are presented. We shall prove that a uniformity (uniformizable proximity) is fine around each compact subspace; thus these subspaces behave similarly as finite subspaces. We invite the reader to give a special attention to 41 C.14 where, under some additional assumptions, the following characterization of compactness of a space $\mathscr{P}$ is given: if a family $\{f_a\}$ of continuous mappings of $\mathscr{P}$ "distinguishes" separated points of $\mathscr{P}$, then $\{f_a\}$ projectively generates $\mathscr{P}$.

For brevity we shall write $\overline{\mathscr{X}}^{\mathscr{P}}$ or $\overline{\mathscr{X}}$ to denote the collection of all $\overline{X}^{\mathscr{P}}$, $X \in \mathscr{X}$.

41 C.1. Definition. A *complete accumulation point* of a subset X of a closure space $\mathscr{P}$ is a point x of $\mathscr{P}$ such that card $(U \cap X) =$ card $X \geqq \aleph_0$ for each neighborhood U of x.

Obviously, every complete accumulation point is an accumulation point.

41 C.2. Theorem. *In a compact space $\mathscr{P}$,*

(a) *every infinite subset of $\mathscr{P}$ has a complete accumulation point.*

If a closure space $\mathscr{P}$ possesses property (a), *then*

(b) *for each non-void monotone collection $\mathscr{X}$ of non-void subsets of $\mathscr{P}$ the intersection of $\overline{\mathscr{X}}$ is non-void.*

If $\mathscr{P}$ is a topological space, then (b) *implies the compactness of $\mathscr{P}$. In particular, for topological spaces each of the preceding conditions* (a) *and* (b) *is necessary and sufficient for $\mathscr{P}$ to be a compact space.*

Proof. Let X be an infinite subset of a compact space. If X possesses no complete accumulation point, then there exists a family $\{U_x \mid x \in \mathscr{P}\}$ such that U_x is a neighborhood of x and card $(U_x \cap X) <$ card X. Since $\mathscr{P}$ is compact, some finite subfamily $\{U_x \mid x \in F\}$ covers $\mathscr{P}$. As a consequence $X = \bigcup\{U_x \cap X \mid x \in F\}$, and hence

(*) card $X \leqq \Sigma\{\text{card}\,(U_x \cap X) \mid x \in F\} = \mathfrak{m}$.

Since F is finite and card $(U_x \cap X)$ is less than the cardinal of X, $\mathfrak{m}$ is less than the cardinal of X, which contradicts (*).

Now we shall prove that in any space (a) implies (b). Suppose (a). Let $\mathscr{X}$ be a monotone non-void collection of non-void subsets of $\mathscr{P}$. If $\mathscr{X}$ contains the smallest element X (relative to the order $\subset$ on exp P), then $\emptyset \neq X = \bigcap\mathscr{X} \subset \bigcap\overline{\mathscr{X}}$. Suppose $\mathscr{X}$ contains no smallest element. By 11 A.18 there exists a cofinal subset $\mathscr{Y}$ of $\mathscr{X}$ which is minimally well-ordered by $\supset$, i.e. $\mathscr{Y}$ is well-ordered and the cardinal of any down bounded subcollection of $\mathscr{Y}$ is less than the cardinal of $\mathscr{Y}$. For each Y in $\mathscr{Y}$ let us choose a point $x(Y)$ in the difference of Y and the successor of Y. If $Y_1, Y_2 \in \mathscr{Y}$ and $Y_1 \neq Y_2$, then $x(Y_1) \neq x(Y_2)$, because $Y_1 \subset Y_2$ or $Y_2 \subset Y_1$ and in the first case $x(Y_2) \notin Y_1$ and in the second $x(Y_1) \notin Y_2$. Thus the cardinal of the set X of all $x(Y)$, $Y \in \mathscr{Y}$, and that of $\mathscr{Y}$ are equal. Let x be a complete accumulation point of X; it will be shown that $x \in \bigcap \overline{\mathscr{Y}}$. If $x \notin \overline{Y}_0$ for some Y_0 in Y, then $|\mathscr{P}| - Y_0$ is a neighborhood of x. If $Y \in \mathscr{Y}$, $Y \subset Y_0$, then $x(Y) \notin |\mathscr{P}| - Y_0$, and consequently the cardinal of $(|\mathscr{P}| - Y_0) \cap X$ is less than that of X, since $\mathscr{Y}$ is minimally well-ordered; this contradicts our assumption that x is a complete accumulation point of X. Thus $x \in \bigcap \overline{\mathscr{Y}}$ and hence $\bigcap \overline{\mathscr{Y}} \neq \emptyset$. Since $\mathscr{Y}$ is cofinal in $\mathscr{X}$, i.e. each $X \in \mathscr{X}$ contains a $Y \in \mathscr{Y}$, we have $\bigcap \overline{\mathscr{Y}} \subset \bigcap \overline{\mathscr{X}}$, and consequently $\bigcap \overline{\mathscr{X}} \neq \emptyset$; this completes the proof of the implication (a) $\Rightarrow$ (b).

It remains to show that a topological space $\mathscr{P}$ is compact provided that it has property (b). Suppose $\mathscr{P}$ is a topological space with property (b). Let $\mathscr{X}$ be a centered collection of subsets of $\mathscr{P}$ and let $\mathscr{Y}$ stand for $\overline{\mathscr{X}}$. Since $\mathscr{P}$ is a topological space, $\mathscr{Y}$ is a collection of closed subsets of $\mathscr{P}$ and obviously $\mathscr{Y}$ is centered. Now suppose $\bigcap \mathscr{Y} = \emptyset$. Then there exists the smallest cardinal $\mathfrak{m}$ such that the intersection of a subcollection $\mathscr{Y}_0$ of $\mathscr{Y}$ of cardinal $\mathfrak{m}$ is empty. Since $\mathscr{Y}$ is centered, the cardinal $\mathfrak{m}$ is infinite. There exists a minimal well-order $\succ$ on $\mathscr{Y}_0$. For each Y in $\mathscr{Y}_0$ let $Z(Y)$ be the intersection of all Y' preceding Y, and let $\mathscr{Z}$ be the collection of all $Z(Y)$, $Y \in \mathscr{Y}_0$. Obviously $\mathscr{Z}$ is monotone, and every set from $\mathscr{Z}$ is non-void, because $\mathscr{Y}_0$ is minimally well-ordered and the intersection of a subfamily of $\mathscr{Y}_0$ with cardinal less than $\mathfrak{m}$ is not empty. By (b) the intersection of $\overline{\mathscr{Z}}$ is non-void. But every set from $\mathscr{Y}$ and hence from $\mathscr{Z}$ is closed, and consequently $\mathscr{Z} = \overline{\mathscr{Z}}$. Thus the intersection of $\mathscr{Z}$ is non-void. Since every $Y \in \mathscr{Y}_0$ contains a $Z \in \mathscr{Z}$, namely $Z(Y)$, $\bigcap \mathscr{Y}_0 \supset \bigcap \mathscr{Z}$. Thus $\bigcap \mathscr{Y}_0$ is non-void, which contradicts our choice of $\mathscr{Y}_0$. The proof is complete.

Remark. It is to be noted that in general condition (a) does not imply compactness (see ex. 5 (b)), and (b) does not imply (a) (see ex. 5 (c)).

41 C.3. Theorem. *Each of the following two conditions is necessary and sufficient for a topological space $\mathscr{P}$ to be compact:*

(a) *the intersection of any centered family of closed subsets of $\mathscr{P}$ is non-void;*

(b) *every open cover of $\mathscr{P}$ contains a finite subcover.*

Proof. By the de Morgan formula the conditions (a) and (b) are equivalent. Indeed, an $\mathscr{X}$ is a centered collection of closed subsets with empty intersection if and only if the family $\mathscr{U}$ consisting of complements of all sets from $\mathscr{X}$ is an open cover of $\mathscr{P}$ containing no finite subcover. If $\mathscr{U}$ is a compact space (not necessarily topological), then (a) is obviously valid. Conversely, if $\mathscr{P}$ is a topological space satisfying (a) and $\mathscr{X}$ is a centered collection of subsets of $\mathscr{P}$, then $\overline{\mathscr{X}}$ is a centered collection of closed sets and hence $\bigcap \overline{\mathscr{X}} \neq \emptyset$ by (a).

Remark. Let $\mathscr{X}$ be a centered collection of closed subsets of a space $\mathscr{P}$. Since the property "to be centered" is of a finite character, by 4 C.5 there exists a maximal centered collection $\mathscr{Y} \supset \mathscr{X}$ of closed subsets of $\mathscr{P}$, i.e. a centered collection $\mathscr{Y}$ of closed subsets with the following property: if $\mathscr{Z} \supset \mathscr{Y}$ is a centered collection of closed subsets of $\mathscr{P}$, then $\mathscr{Z} = \mathscr{Y}$. Now it is obvious that the condition (a) in 41 C.3 can be replaced by the following formally weaker condition:

The intersection of every maximal centered collection of closed subsets of $\mathscr{P}$ is non-void.

The first part of this subsection devoted to various characterizations of compactness is concluded by the following useful, interesting and non-trivial characterization 41 C.4 of compactness for topological spaces. Obviously a topological space $\mathscr{P}$ is compact provided that the intersection of any centered collection of sets of a certain closed base of $\mathscr{P}$ is non-void, or dually, if every cover consisting of elements of a certain open base for $\mathscr{P}$ contains a finite subcover. The following theorem asserts that the word "base" can replaced by "sub-base".

41 C.4. Theorem. *Each of the following conditions is necessary and sufficient for a topological space $\mathscr{P}$ to be compact:*

(a) *there exists a closed sub-base $\mathscr{F}$ for $\mathscr{P}$ such that the intersection of every centered subcollection of $\mathscr{F}$ is non-void;*

(b) *there exists an open sub-base $\mathscr{U}$ for $\mathscr{P}$ such that every open cover consisting of sets from $\mathscr{U}$ contains a finite subcover.*

Proof. The necessity of either condition is evident. By de Morgan formula an $\mathscr{F}$ is a closed sub-base of $\mathscr{P}$ with the property required by (a) if and only if the collection $\mathscr{U}$ of all $|\mathscr{P}| - F$, $F \in \mathscr{F}$, is an open sub-base of $\mathscr{P}$ possessing the property required by (b). Thus both conditions (a) and (b) are equivalent. It remains to show that, for example, (a) is sufficient. Let us suppose that $\mathscr{P}$ is not compact. By the remark following 41 C.3 there exists a maximal centered collection $\mathscr{X}$ of closed subsets of $\mathscr{P}$ such that $\bigcap \mathscr{X} = \emptyset$. Let $\mathscr{F}$ be a closed sub-base of $\mathscr{P}$. It will be shown that $\bigcap(\mathscr{F} \cap \mathscr{X}) = \emptyset$.

If $x \in |\mathscr{P}|$, then there exists an X in $\mathscr{X}$ with $x \notin X$. Since $\mathscr{F}$ is a closed sub-base there exists a finite sub-collection $\mathscr{F}_0$ of $\mathscr{F}$ such that the union F_0 of $\mathscr{F}_0$ contains X and does not contain x. Since $F_0 \supset X \in \mathscr{X}$, necessarily $F_0 \in \mathscr{X}$. Since $\mathscr{X}$ is a maximal centered collection and the union F_0 of the finite collection $\mathscr{F}_0$ belongs to $\mathscr{X}$, some $F \in \mathscr{F}_0$ belongs to $\mathscr{X}$. Thus $F \in \mathscr{F} \cap \mathscr{X}$. By our choice $x \notin F_0$ and consequently $x \notin F$, because $F \subset F_0$. The proof is complete.

Now we proceed to an investigation of the properties of compact spaces. In 41 A.12 we proved that the class of all compact spaces is completely productive and closed under continuous mappings, closed subspaces of compact spaces are compact and a compact subspace of a separated space is closed. Next, a topological separated or regular compact space is uniformizable (because it is paracompact).

41 C.5. Theorem. *A continuous mapping of a compact space into a separated closure space is inversely upper semi-continuous, in particular, it is a quotient mapping.*

Proof. Let f be a continuous mapping of a compact space $\langle P, u\rangle$ into a separated space $\langle Q, v\rangle$. Let $y \in \mathbf{E}f$ and G be a neighborhood of $f^{-1}[y]$ in $\langle P, u\rangle$. We must find a neighborhood U of y in $\langle Q, v\rangle$ such that $f^{-1}[U] \subset G$. For each $z \in Q - (y)$

let U_z be a neighborhood of y and V_z be a neighborhood of z such that $U_z \cap V_z = \emptyset$. Since f is continuous, the family $\{f^{-1}[V_z]\}$ interiorly covers $P - f^{-1}[y]$, and hence the collection consisting of G and of all $f^{-1}[V_z]$ is an interior cover of $\langle P, u \rangle$. Since $\langle P, u \rangle$ is compact, some finite subcollection covers P, and therefore $\bigcup\{f^{-1}[V_z] \mid z \in \in Z\} \supset P - G$ for some finite subset Z of $Q - (y)$. Put $U = \bigcap\{U_z \mid z \in Z\}$. Evidently U is a neighborhood of y and $f^{-1}[U] \subset G$.

Remark. If $\langle P, u \rangle$ is topological then it is enough to show that f is closed, and this follows from the facts that a closed subspace X of $\langle P, u \rangle$ is compact, its continuous image $f[X]$ is compact and hence, $\langle Q, v \rangle$ being separated, $f[X]$ is closed in $\langle Q, v \rangle$ and hence in $\mathbf{E}f$.

Corollary. *A one-to-one continuous mapping of a compact space into a separated space is an embedding.*

If $\mathscr{P}$ is a separated space and f is a continuous mapping of a space $\mathscr{Q}$ into $\mathscr{P}$, then the graph of f is closed in the product space $\mathscr{Q} \times \mathscr{P}$ (by 27 A.9), but the converse need not be true.

41 C.6. Theorem. *If f is a mapping of a closure space $\mathscr{Q}$ into a compact space $\mathscr{P}$ such that the graph of f is closed in the product space $\mathscr{Q} \times \mathscr{P}$, then f is continuous.*

Proof. Suppose that a net $\{x_a\}$ converges to x in $\mathscr{Q}$. We shall prove that $\{fx_{a_b}\}$ converges to fx for some subnet $\{x_{a_b}\}$ of $\{x_a\}$. Since $\mathscr{P}$ is compact the net $\{fx_a\}$ has an accumulation point y in $\mathscr{P}$ and hence some subnet $\{fx_{a_b}\}$ converges to y in $\mathscr{P}$; the subnet $\{x_{a_b}\}$ of $\{x_a\}$ converges to x. As a consequence the net $N = \{\langle x_{a_b}, fx_{a_b} \rangle\}$ converges to $\langle x, y \rangle$, and since N ranges in gr f and gr f is closed, $\langle x, y \rangle$ must also belong to gr f, i.e. $y = fx$.

It turns out that the property of compact spaces stated in the preceding theorem characterizes the compact spaces in the class of all separated spaces.

41 C.7. *If a separated space $\mathscr{P}$ is not compact, then there exists a mapping of a separated topological space $\mathscr{Q}$ into $\mathscr{P}$ such that the graph of f is closed in $\mathscr{Q} \times \mathscr{P}$ but f is not continuous.*

Proof. Let $\mathscr{X}$ be a filter on $\mathscr{P} = \langle P, u \rangle$ without cluster points, $Q = P \cup (\mathscr{X})$, and let v be the closure for Q such that each point of P is isolated in $\langle Q, v \rangle$ and $\{(\mathscr{X}) \cup X \mid X \in \mathscr{X}\}$ is the neighborhood system of $(\mathscr{X})$ in $\langle Q, v \rangle$. Let f be a mapping of $\langle Q, v \rangle$ into $\mathscr{P}$ such that $fx = x$ if $x \in P$. The mapping f is not continuous at $(\mathscr{X})$. In fact, choose an X in $\mathscr{X}$ such that $f\mathscr{X}$ does not belong to the closure of X; since $\mathscr{X} \in vX$, $f[X] = X$, f is not continuous at $\mathscr{X}$. On the other hand, it is easily seen that if $\mathscr{P}$ is separated, then gr $f \cap (P \times P)$ is closed in $\langle Q, v \rangle \times \mathscr{P}$ and hence gr f is also closed.

Now we proceed to semi-uniformities and proximities.

41 C.8. Theorem. *If $\langle P, u \rangle$ is a compact uniformizable space, then there exists exactly one uniformity inducing the closure u; this uniformity consists of all neighborhoods of the diagonal.*

We shall prove somewhat more:

41 C.9. *Suppose that the closure structure of a compact topological space $\mathscr{P}$ is induced by a semi-uniformity $\mathscr{U}$ such that the collection $\mathscr{V}$ of all closed (in $\mathscr{P} \times \mathscr{P}$) elements of $\mathscr{U}$ is a base for $\mathscr{U}$. Then $\mathscr{U}$ contains all open neighborhoods of the diagonal of $\mathscr{P} \times \mathscr{P}$.*

Proof of 41 C.8. Suppose that the closure structure of a compact space is induced by a uniformity $\mathscr{U}$. We know that each element of a uniformity is a neighborhood of the diagonal. On the other hand, the collection $\mathscr{V}$ of all closed elements of $\mathscr{U}$ is a base for $\mathscr{U}$ and therefore, by 41 C.9, $\mathscr{U}$ contains each neighborhood of the diagonal. Thus $\mathscr{U}$ is the set of all neighborhoods of the diagonal.

Proof of 41 C.9. Let U be any neighborhood of the diagonal and let G be the interior of U; thus G is an open neighborhood of the diagonal. Let us consider the collection $\mathscr{X}$ of all sets $V - G$, $V \in \mathscr{V}$. Since $\mathscr{V}$ is a base for $\mathscr{U}$ and $\mathscr{U}$ induces the closure structure of $\mathscr{P}$, we have $\bigcap \mathscr{X} = \emptyset$ (for each x in $|\mathscr{P}|$ there exists a V in $\mathscr{V}$ such that $V[x] \subset G[x]$). The elements of $\mathscr{X}$ are closed and the product space $\mathscr{P} \times \mathscr{P}$ is compact and therefore $\mathscr{X}$ is not centered, and hence $\emptyset \in \mathscr{X}$ because $\mathscr{X}$ is multiplicative. Thus $V \subset G$ for some V in $\mathscr{V}$, which shows that $G \in \mathscr{U}$ and hence $U \in \mathscr{U}$.

41 C.10. Corollary. *If $\mathscr{U}$ and $\mathscr{V}$ are topologically equivalent continuous uniformities for a compact space, then $\mathscr{U} = \mathscr{V}$. If p and q are topologically equivalent continuous uniformizable proximities for a compact space, then $p = q$; in particular, there exists exactly one uniformizable proximity inducing the closure structure of a given uniformizable compact space.*

Proof. A closure coarser than a compact closure is compact, and therefore the first statement follows immediately from 41 C.8. The statements concerning uniformizable proximities follow from the corresponding statements concerning uniformities (each uniformizable proximity is induced by a uniformity).

Remarks. (a) There exists a non-compact space the closure structure of which is induced by exactly one uniformity; e.g. the ordered space of all countable ordinals has this property (see 41 D.11).

(b) It need not be true that the semi-uniformity $\mathscr{U}$ of 41 C.9 consists of neighborhoods of the diagonal; e.g. let $\mathscr{P}$ be the ordered space $[\![0, 1]\!]$ of reals and let $\mathscr{W}$ be the set of all W_r, $2 > r > 0$, where W_r consists of all $\langle x, y\rangle$ in $P \times P$ such that either $|y - x| \leqq r \cdot x$ or $x = 0$, $|y| \leqq r$ or $y = 0$, $|x| \leqq r$. Clearly $\mathscr{W}$ is a base for a semi-uniformity $\mathscr{U}$ inducing the closure structure of $\mathscr{P}$ and each W_r is closed in $\mathscr{P} \times \mathscr{P}$.

In 41 C.8 and 41 C.10 uniformities and uniformizable proximities on compact spaces were considered. Now we shall exhibit a basic property of compact subsets of uniform spaces and uniformizable proximity spaces.

41 C.11. Theorem. *Each uniformity (uniformizable proximity) is fine around each compact subset. More precisely, if $\mathscr{U}$ is a uniformity for a set P, p is the proximity induced by $\mathscr{U}$, u is the closure induced by p and X is a compact subset of $\langle P, u\rangle$ (i.e., the corresponding subspace is compact), then $\mathscr{U}$ is fine around X*

(i.e. if G is a neighborhood of X then $U[X] \subset G$ for some U in $\mathscr{U}$) and p is fine around X (i.e. each neighborhood of X is a proximal neighborhood of X).

Proof. It is sufficient to prove the statement concerning $\mathscr{U}$. For each x in X let us choose an open element U_x in $\mathscr{U}$ such that $U_x \circ U_x[x] \subset G$. The open cover $\{U_x[x] \mid x \in X\}$ of X contains a finite subcover $\{U_x[x] \mid x \in F\}$, because the subspace X of $\langle P, u\rangle$ is compact. Put $U = \bigcap\{U_x \mid x \in F\}$. We have $U[X] \subset U[\bigcup\{U_x[x] \mid x \in F\}] \subset \bigcup\{U[U_x[x]] \mid x \in F\} \subset \bigcup\{U_x \circ U_x[x] \mid x \in F\} \subset G$.

The preceding result has the following important consequence.

41 C.12. Theorem. *Let $\mathscr{P}$ be the product of a family $\{\mathscr{P}_a \mid a \in A\}$ of uniformizable spaces and let G be a neighborhood of a set $X = \Pi\{X_a\}$ in $\mathscr{P}$. If each X_a is compact in $\mathscr{P}_a$, then there exists a canonical neighborhood $G' = \Pi\{G_a\}$ of X in $\mathscr{P}$ such that $G' \subset G$.*

Before presenting the proof we shall state the following particular case.

Corollary. *Let $\mathscr{P} = \mathscr{P}_1 \times \mathscr{P}_2$ and $X = X_1 \times X_2$, where $\mathscr{P}_i$ are uniformizable and X_i is compact in $\mathscr{P}_i$. If G is a neighborhood of X in $\mathscr{P}$, then there exists neighborhoods G_i of X_i in $\mathscr{P}_i$, $i = 1, 2$, such that $G_1 \times G_2 \subset G$.*

Proof of 41 C.12. For each a let $\mathscr{U}_a$ be a uniformity inducing the closure structure of $\mathscr{P}_a$ and let $\mathscr{U}$ be the product uniformity. Since X is compact in $\mathscr{P}$ and $\mathscr{U}$ induces the closure structure of $\mathscr{P}$, we can choose (by 41 C.11) a U in $\mathscr{U}$ such that $U[X] \subset G$. By Definition 23.D.10 of the product uniformity, there exists a finite subset F of A and a family $\{V_a \mid a \in A\}$ such that $V_a \in \mathscr{U}_a$ for each a, $a \in A - F$ implies $V_a = |\mathscr{P}_a|$, and the relational product V of $\{V_a\}$ is contained in U. Then

$$G \supset U[X] \supset V[X] = \Pi\{U_a[X_a]\},$$

which completes the proof.

Remark. It should be noted that similar results can be proved without uniformities or assuming that the space is uniformizable (see 41. ex. 7).

The concluding part is devoted to an investigation of algebras of bounded continuous functions in connection with compactness. Recall that algebras of bounded proximally continuous functions were studied in 25 D (main result: $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ is a closed subalgebra of unif $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$) and 25 E (main result: the Stone-Weierstrass Theorem for proximity spaces).

First we shall try to carry over the Stone-Weierstrass Theorem for proximity spaces to closure spaces. The Stone-Weierstrass Theorem states that, given a proximity space $\mathscr{P}$, the smallest algebra containing all constant functions on $\mathscr{P}$ and also a collection of bounded functions projectively generating $\mathscr{P}$ is dense in the normed algebra $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$ of all bounded proximally continuous functions on $\mathscr{P}$. Now let us consider a uniformizable closure space $\langle P, u\rangle$, a uniformizable proximity p inducing the closure u, and the set $\mathscr{A}$ of all $f : \langle P, u\rangle \to \mathsf{R}$, $f \in \mathbf{P}^*(\langle P, p\rangle, \mathsf{R})$. By 39 B.2 the collection $\mathscr{A}$ projectively generates $\langle P, u\rangle$. Clearly $\mathscr{A}$ is a closed subalgebra of $\mathbf{C}^*(\langle P, u\rangle, \mathsf{R})$ containing all constant functions, and $\mathscr{A} = \mathbf{C}^*(\langle P, u\rangle, \mathsf{R})$ if and only if p is the Čech proximity of $\langle P, u\rangle$. It follows that if the closure u is induced by a uniformizable proximity which is not the Čech proximity of $\langle P, u\rangle$, i.e. if u is induced by two distinct uniformizable proximities, then $\langle P, u\rangle$ is projectively generated by a proper closed subalgebra of $\mathbf{C}^*(\langle P, u\rangle, \mathsf{R})$ containing all constant functions. Conversely, assume that $\langle P, u\rangle$ is induced by exactly one uniformizable proximity p, which is necessarily the Čech proximity of $\langle P, u\rangle$, and let us consider any collection $\mathscr{F}$ of bounded functions projectively generating $\langle P, u\rangle$. The collection $\mathscr{F}' = \mathbf{E}\{f : P \to \mathsf{R} \mid f \in \mathscr{F}\}$ projectively generates a uniformizable proximity q for P. Again by 39 B.2 the proximity q induces the closure projectively generated by $\{f : P \to \mathsf{R} \mid f \in \mathscr{F}\}$, i.e. q induces u. Consequently $p = q$. By the Stone-Weierstrass Theorem 25 E.2, the smallest algebra $\mathscr{A}'$ containing all $f : \langle P, p\rangle \to \mathsf{R}, f \in \mathscr{F}$, and all constant functions on $\langle P, p\rangle$ is dense in $\mathbf{P}^*(\langle P, p\rangle, \mathsf{R})$. Let ϱ be the single-valued relation which assigns to each $f \in \mathbf{F}^*(\langle P, u\rangle, \mathsf{R})$ the function $f : \langle P, p\rangle \to \mathsf{R}$. Clearly $\varrho : \mathbf{F}^*(\langle P, u\rangle, \mathsf{R}) \to \mathbf{F}^*(\langle P, p\rangle, \mathsf{R})$ is a normed-algebra-isomorphism, and $\varrho[\mathscr{F}] = \mathscr{F}'$. Since p is the Čech proximity of $\langle P, u\rangle$ we have $\varrho[\mathbf{C}^*(\langle P, u\rangle, \mathsf{R})] = \mathbf{P}^*(\langle P, p\rangle, \mathsf{R})$. If $\mathscr{A}$ is the smallest subalgebra of $\mathbf{F}^*(\langle P, u\rangle, \mathsf{R})$ containing $\mathscr{F}$ and all constant functions, then $\varrho[\mathscr{A}] = \mathscr{A}'$ because $\varrho[\mathscr{F}] = \mathscr{F}'$. Finally, ϱ is a homeomorphism and therefore the image under ϱ of $\bar{\mathscr{A}}$ is the closure of $\mathscr{A}'$, i.e. $\mathbf{P}^*(\langle P, p\rangle, \mathsf{R})$. Thus $\bar{\mathscr{A}} = \mathbf{C}^*(\langle P, u\rangle, \mathsf{R})$. We have proved the following theorem.

41 C.13. Theorem. *The following two conditions on a uniformizable closure space $\mathscr{P}$ are equivalent:*

(a) *There exists a unique uniformizable proximity inducing the closure structure of $\mathscr{P}$.*

(b) *If $\mathscr{P}$ is projectively generated by a collection $\mathscr{F}$ of bounded functions, then the smallest subalgebra of $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ containing $\mathscr{F}$ and all constant functions is dense in $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$.*

For example, if $\mathscr{P}$ is a compact uniformizable space, then (a) is fulfilled (by 41 C.8) and therefore (b) is fulfilled (by 41 C.13). On the other hand a space with property (a) need not be compact. We want to strengthen condition (b) to obtain a characterization of compactness.

If a collection $\mathscr{F}$ of functions projectively generates a closure space $\langle P, u\rangle$ and $x \notin u(y)$, then $fx \neq fy$ for some f in $\mathscr{F}$. It is almost evident that the converse is not true. On the other hand, if $\langle P, u\rangle$ is compact and topological, then the converse does hold.

41 C.14. Lemma. *Suppose that $\mathscr{F}$ is a collection of continuous mappings of a space $\mathscr{P} = \langle P, u\rangle$ into separated closure spaces.* (a) *If $\mathscr{P}$ is a compact topological space and for each $x \notin u(y)$ there exists an f in $\mathscr{F}$ such that $fx \neq fy$, then $\mathscr{P}$ is projectively generated by the collection $\mathscr{F}$.* (b) *If $\mathscr{P}$ is a separated compact closure space and $\mathscr{F}$ distinguishes the points of $\mathscr{P}$ (i.e. for each $x \neq y$ there exists an f in $\mathscr{F}$ such that $fx \neq fy$), then $\mathscr{F}$ projectively generates $\mathscr{P}$.*

Proof. I. Under the assumptions of (a) let x be any point of $\mathcal{P}$, G be any open neighborhood of x in $\mathcal{P}$, and $\mathcal{X}$ be the smallest multiplicative collection of sets containing each set $f^{-1}[U] - G$ where $f \in \mathcal{F}$ and U is a neighborhood of fx in $\mathbf{E}^*f$. We must show that $\mathcal{X}$ contains the empty set. Because of compactness, on assuming the contrary we can choose a point y which belongs to each $uX, X \in \mathcal{X}$. Since G is open, $y \in \in |\mathcal{P}| - G$, hence $x \notin u(y)$ and therefore we can choose an f in $\mathcal{F}$ such that $fx \neq fy$. Since $\mathbf{E}^*f$ is separated we can choose a neighborhood U of fx in $\mathbf{E}^*f$ the closure of which does not contain fy; since f is continuous, we have $y \notin uf^{-1}[U]$, which contradicts our assumption that $y \in uX$ for each $X \in \mathcal{X}$. – II. Assuming in (b) that $\mathcal{P}$ is topological, the conclusion of (b) becomes an immediate consequence of (a). Although statement (b) will only be applied in this case, we shall prove (b) in the general setting. It should be noted that the same proof may be applied; since G need not be open, we only obtain $y \in |\mathcal{P}| - \operatorname{int} G$, which implies $x \neq y$, and P being separated, this yields $x \notin u(y)$. Nevertheless the following proof is simpler: consider the closure v projectively generated by the family $\{f : |\mathcal{P}| \to \mathbf{E}^*f \mid f \in \mathcal{F}\}$; v is coarser than u and, clearly, separated. It follows from 41 C.5 that $u = v$.

Now we are prepared to state the main theorem.

41 C.15. Theorem. *The following condition is necessary and sufficient for a uniformizable space $\mathcal{P} = \langle P, u \rangle$ to be compact:*

If $\mathcal{F}$ is a set of bounded continuous functions on $\mathcal{P}$ such that $x \notin u(y)$ implies $fx \neq fy$ for some f in $\mathcal{F}$, then the smallest subalgebra of $\mathbf{F}^*(\mathcal{P}, \mathrm{R})$ containing $\mathcal{F}$ and the constant function $\{x \to 1\} : \mathcal{P} \to \mathrm{R}$ is dense in $\mathbf{C}^*(\mathcal{P}, \mathrm{R})$.

As an immediate consequence of 41 C.14 we obtain the following characterization of compactness in the class of all separated uniformizable spaces.

41 C.16. Theorem. *The following condition is necessary and sufficient for a separated uniformizable space $\mathcal{P}$ to be compact:*

If $\mathcal{F}$ is a set of bounded continuous functions which distinguishes the points of $\mathcal{P}$ (i.e. $x \neq y$ implies that $fx \neq fy$ for some f in $\mathcal{F}$), then the smallest subalgebra of $\mathbf{F}^*(\mathcal{P}, \mathrm{R})$ containing $\mathcal{F}$ and the constant function $\{x \to 1\} : \mathcal{P} \to \mathrm{R}$ is dense in $\mathbf{C}^*(\mathcal{P}, \mathrm{R})$.

Indeed, if $\mathcal{P}$ is separated then the conditions of 41 C.15 and 41 C.16 are equivalent.

Proof of 41 C.15. I. Suppose that $\mathcal{P}$ is compact and the assumptions of the condition are fulfilled. By 41 C.14 (a), the collection $\mathcal{F}$ projectively generates $\mathcal{P}$. By 41 C.8, condition (a) of 41 C.13 is fulfilled (because of compactness of $\mathcal{P}$) and hence 41 C.13 (b) is fulfilled, i.e. the smallest subalgebra of $\mathbf{F}^*(\mathcal{P}, \mathrm{R})$ containing $\mathcal{F}$ and all constant functions is dense in $\mathbf{C}^*(\mathcal{P}, \mathrm{R})$.

II. Assuming that a uniformizable space $\mathcal{P}$ is not compact, we shall construct a subalgebra $\mathcal{F}$ of $\mathbf{C}^*(\mathcal{P}, \mathrm{R})$ containing all constant functions such that $\mathcal{F}$ is not dense in $\mathbf{C}^*(\mathcal{P}, \mathrm{R})$, but $x \notin u(y)$ implies that $fx \neq fy$ for some f in $\mathcal{F}$. Since $\mathcal{P}$ is not compact we can choose a proper filter $\mathcal{X}$ on $\mathcal{P}$ such that $\bigcap u[\mathcal{X}] = \emptyset$. Consider the collection $\mathcal{Z}$ of exact closed sets of $\mathcal{X}$. Since $\mathcal{P}$ is uniformizable we have $\bigcap \mathcal{Z} =$

$= \bigcap u[\mathscr{X}] = \emptyset$. Choose any point x of $\mathscr{P}$ and let us consider the set $\mathscr{F}$ of all bounded continuous functions on $\mathscr{P}$ such that $f[Z] = (fx)$ for some Z in $\mathscr{Z}$. We shall prove that $\mathscr{F}$ has the required properties. It is almost self-evident that $\mathscr{F}$ is an algebra ($\mathscr{Z}$ is multiplicative). If $x \notin u(y)$, then there exists a bounded continuous function g on $\mathscr{P}$ such that $gx = 0$ and $gy = 1$. Since $\bigcap \mathscr{Z} = \emptyset$, we can choose a Z in $\mathscr{Z}$ such that $y \notin Z$, and then we can choose a bounded continuous function h on $\mathscr{P}$ such that $Z = \mathbf{E}\{z \mid hz = 0\}$. If $f = h \,.\, g$, then $fx = 0$ and $f[Z] = (0)$, which shows that $f \in \mathscr{F}$, and clearly $fx \neq fy$. The general case is now obvious. But $\mathscr{F}$ is not dense in $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$. Indeed, choose a Z in $\mathscr{Z}$ such that $x \notin Z$, and then choose a bounded continuous function h such that $Z = \mathbf{E}\{z \mid hz = 0\}$. Thus $hx \neq 0$. If $f \in \mathscr{F}$ then $\|f - h\| \geqq \frac{1}{2}|hx|$. Indeed, if $|fx - hx| \leqq \frac{1}{2}|hx|$ then necessarily $|fz - hz| \geqq \frac{1}{2}|hx|$ for some z in Z. Thus $\mathscr{F}$ is not dense in $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$. The proof is complete.

From the proof of the preceding theorem one may, incidentally, obtain an interesting characterization of compactness in the class of all uniformizable spaces. Let us consider the closure v projectively generated by the family $\{f : |\mathscr{P}| \to \mathsf{R} \mid f \in \mathscr{F}\}$, where $\mathscr{F}$ is the algebra of the second part of the proof of 41 C.15. It is easily seen that v is strictly coarser than u. Indeed, $x \in vZ$ for each Z in $\mathscr{Z}$ but $x \notin Z = uZ$ for some Z in $\mathscr{Z}$. Thus we have proved: If $\langle P, u \rangle$ is a non-compact uniformizable space then there exists a uniformizable closure v strictly coarser than u such that $x \in v(y)$ implies that $x \in u(y)$ (and hence, $x \in u(y) \Leftrightarrow x \in v(y)$). On the other hand, if $\langle P, u \rangle$ is a compact uniformizable space and v is a uniformizable closure coarser than u such that $x \in v(y)$ implies $x \in u(y)$, then it follows from 41 C.14 (a) that the family $\{f : P \to \mathsf{R} \mid f \in \mathbf{C}^*(\langle P, v \rangle, \mathsf{R})\}$ projectively generates u, and hence $u = v$. Thus we have proved the following result.

41 C.17. Theorem. *A uniformizable closure u is compact if and only if there exists no uniformizable closure v strictly coarser than u such that $x \in u(y)$ if and only if $x \in v(y)$ (i.e. such that the quasi-discrete modifications of u and v coincide). In particular, a separated uniformizable closure u is compact if and only if there exists no separated uniformizable closure strictly coarser than u.*

Remark. If $\langle P, u \rangle$ is a uniformizable closure space, then the condition "if $x \notin u(y)$ then $fx \neq fy$ for some f in $\mathscr{F}$" can be stated as follows: $\mathscr{F}$ distinguishes separated points of $\langle P, u \rangle$.

Some of the preceding theorems stated that the smallest subalgebra containing a given set $\mathscr{F}$ and also all constant functions (i.e. the unit) is dense in the algebra of bounded proximally continuous functions or of bounded continuous functions. We want to describe the smallest closed algebra containing $\mathscr{F}$. The next simple lemma reduces this question to the corresponding preceding results. For convenience we review some of earlier definitions and results for real algebras, which will be needed.

An ideal in an algebra $\mathscr{A}$ is a linear subspace L of $\mathscr{A}$ such that $f \,.\, g \in L$ for each f in $\mathscr{A}$ and g in L. An ideal L is proper if $L \neq |\mathscr{A}|$. A maximal ideal is a proper ideal which is a proper subset of no proper ideal. If f belongs to a proper ideal, then the multiplicative inverse of f, denoted by $1/f$, does not exist; stated in other words, each proper ideal consists of (multiplicatively) non-invertible elements. If $\mathscr{A}$ has a unit, then each proper ideal is contained in a maximal ideal. If $\mathscr{A}$ has a unit then the following conditions on any subset L of $\mathscr{A}$ are equivalent:

(a) $L = \varphi^{-1}[(0)]$ for some non-zero homomorphism of $\mathscr{A}$ into R;

(b) L is a linear subspace of $\mathscr{A}$ and each element of $\mathscr{A}$ can be uniquely written as $r \cdot 1 + f, f \in L, r \in$ R;

(c) L is a maximal ideal in $\mathscr{A}$.

Remember that $\varphi g = r$ if $g = r \cdot 1 + f$ where φ is of (a) and $r \cdot 1 + f$ is the decomposition of g as stated in (b). Now let $\mathscr{A}$ be a topological algebra. The closure of an ideal is an ideal. It follows that a maximal ideal is closed or dense, and hence if $\mathscr{A}$ has a unit and the set of all non-invertible elements is closed (this is the case when $\mathscr{A} =$ $=$ **C***($\mathscr{P}$, R); indeed, f is invertible if and only if $|f| \geqq r > 0$), then each maximal ideal is closed. The constant function $\{x \to 1\} : \mathscr{P} \to$ R is the unit of **F***($\mathscr{P}$, R) for each non-void struct $\mathscr{P}$.

41 C.18. Lemma. *Let $\mathscr{A}$ be a topological algebra with unit and let $\mathscr{F} \subset |\mathscr{A}|$. If $\mathscr{A}_1$ is the smallest closed subalgebra containing $\mathscr{F}$ and the unit and $\mathscr{A}_2$ is the smallest closed subalgebra containing $\mathscr{F}$, then either $\mathscr{A}_2 = \mathscr{A}_1$ or $\mathscr{A}_2$ is a maximal ideal in $\mathscr{A}_1$ (which is necessarily closed).*

Proof. Assuming that $\mathscr{A}_2 \neq \mathscr{A}_1$, it is sufficient to prove that each element of $\mathscr{A}_1$ can be written uniquely in the form $r \cdot 1 + f, f \in \mathscr{A}_2$. If $r \cdot 1 + f = s \cdot 1 + g$, where $f \in \mathscr{A}_2$, $g \in \mathscr{A}_2$, then $(r - s) \cdot 1 = (g - f) \in \mathscr{A}_2$, and hence $r - s = 0$ because $1 \notin \mathscr{A}_2$; consequently $r = s$ and $g = f$, which establishes uniqueness. To prove the existence let us consider the set $\mathscr{A}_3$ of all $r \cdot 1 + f$, $r \in$ R, $f \in \mathscr{A}_2$. We must prove $\mathscr{A}_1 = \mathscr{A}_3$. Clearly $\mathscr{A}_3$ is a subalgebra of $\mathscr{A}$ containing $\mathscr{F}$ and 1, and $\mathscr{A}_3 \subset \mathscr{A}_1$. Thus it is sufficient to show that $\mathscr{A}_3$ is closed. Suppose that a net $\{r_a \cdot 1 + f_a\}$ in $\mathscr{A}_3$ (where $f_a \in \mathscr{A}_2$ and $r_a \in$ R) converges to a $g \in \mathscr{A}_1$. We must show that $g = r \cdot 1 + f$ with f in $\mathscr{A}_2$ and r in R. If we show that the net $\{r_a \cdot 1\}$ converges to some $r \cdot 1$, $r \in$ R, then the net $\{f_a\}$ will converge to $g - r \cdot 1$ because $f_a = (r_a \cdot 1 + f_a) - r_a \cdot 1$, and $\mathscr{A}_2$ being closed, $g - r \cdot 1 \in \mathscr{A}_2$, i.e. $g - r \cdot 1 = f$ for some f in $\mathscr{A}_2$, and hence $g = r \cdot 1 + f$. To prove that $\{r_a \cdot 1\}$ is convergent it suffices to show that $\{r_a \cdot 1\}$ is a Cauchy net. Consider the mapping φ of the product algebra R $\times \mathscr{A}_2$ onto $\mathscr{A}_3$ which assigns to each $\langle r, f \rangle$ the point $r \cdot 1 + f$. Since $\mathscr{A}_2$ is closed in $\mathscr{A}_3$ and $1 \notin \mathscr{A}_2$, the mapping φ is a uniform homeomorphism, in particular $\{\langle r_a, f_a \rangle\}$ is a Cauchy net; hence $\{r_a\}$ is a Cauchy net in R, and thus $\{r_a \cdot 1\}$ is a Cauchy net.

Combining Lemma 41 C.18 with the Stone-Weierstrass Theorem for proximity spaces (25 E.2) we obtain immediately the following result.

41 C.19. Theorem. *Suppose that a proximity space $\mathscr{P}$ is projectively generated by a set $\mathscr{F}$ of bounded functions. If $\mathscr{A}$ is the smallest closed subalgebra of* **F***($\mathscr{P}$, R) *containing $\mathscr{F}$ then either $\mathscr{A} =$* **P***($\mathscr{P}$, R) *or $\mathscr{A}$ is a maximal ideal in* **P***($\mathscr{P}$, R).

Combining Lemma 41 C.18 with 41 C.13 and 41 C.15 we obtain immediately the following results.

41 C.20. Theorem. (a) *The following condition is necessary and sufficient for a uniformizable space* $\mathscr{P}$ *to be induced by a unique uniformizable proximity:*

If a collection $\mathscr{F} \subset \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *projectively generates* $\mathscr{P}$, *then the smallest closed subalgebra of* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *containing* $\mathscr{F}$ *either coincides with* $|\mathbf{C}^*(\mathscr{P}, \mathrm{R})|$ *or is a maximal ideal of* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$.

(b) *The following condition is necessary and sufficient for a uniformizable space* $\mathscr{P}$ *to be compact:*

If a collection $\mathscr{F} \subset \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *distinguishes separated points of* $\mathscr{P}$, *then the smallest closed subalgebra of* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *containing* $\mathscr{F}$ *either coincides with* $|\mathbf{C}^*(\mathscr{P}, \mathrm{R})|$ *or is a maximal ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$.

The usual formulation of the Stone-Weierstrass Theorem for compact spaces involves a description of maximal ideals in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ which will now be given. If $X \subset |\mathscr{P}|$ and L_X is the set of all bounded continuous functions which vanish on X, then clearly L_X is a closed ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$. We shall prove that a space $\mathscr{P}$ is compact if and only if each closed ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ is of this form. First we shall introduce some terminology.

41 C.21. Definition. Let $\mathscr{A}$ be a subalgebra of a normed algebra $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$, where $\mathscr{P}$ is a non-void struct. A *fixed ideal in* $\mathscr{A}$ is an ideal L in $\mathscr{A}$ such that $L = \mathbf{E}\{f \mid f \in \mathscr{A}, f[X] \subset (0)\}$ for some $X \subset |\mathscr{P}|$. An ideal L is said to be *free* if it is not fixed.

If L is a proper fixed ideal then there exists a point x such that each function of L vanishes at x. Indeed, if $L = \mathbf{E}\{f \mid f[X] \subset (0)\}$ and $X = \emptyset$, then clearly L coincides with $|\mathbf{C}^*(\mathscr{P}, \mathrm{R})|$. In particular, if L is a maximal ideal, then each function f of L vanishes at a point independent of f.

41 C.22. Theorem. *Each of the following two conditions is necessary and sufficient for a uniformizable space* $\mathscr{P} = \langle P, u\rangle$ *to be compact:*

(a) *Each closed ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *is fixed.*

(b) *Each maximal ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *is fixed.*

Proof. I. Each maximal ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ is closed and hence (a) implies (b). It remains to show that (a) is necessary and (b) is sufficient.

II. Assume that $\mathscr{P}$ is compact, and let L be a closed ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$. Consider the set X of all $x \in |\mathscr{P}|$ such that $fx = 0$ for each f in L, and the fixed ideal L_1 consisting of all $f \in \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ which vanish on X. We shall show that $L_1 = L$. Let us consider the set $\mathscr{A}$ of all $f \in \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ which are constant on X, and the closure v projectively generated by the family $\{f : P \to \mathrm{R} \mid f \in \mathscr{A}\}$. Clearly $\mathscr{A}$ is a closed subalgebra of $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ and v is coarser than u, and hence v is compact. Let ϱ be the mapping of $\mathbf{F}^*(\mathscr{P}, \mathrm{R})$ onto $\mathbf{F}^*(\langle|\mathscr{P}|, v\rangle, \mathrm{R})$ which carries each f into $f : \langle P, v\rangle \to \mathrm{R}$; thus ϱ is a normed-algebra-isomorphism, the collection $\varrho[\mathscr{A}]$ projectively generates the compact space $\langle P, v \rangle$, and $\varrho[\mathscr{A}]$ is a closed subalgebra of $\mathbf{F}^*(\langle P, v \rangle, \mathrm{R})$ containing all constant functions, and therefore by 41 C.13, $\varrho[\mathscr{A}] = \mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$. To prove $L = L_1$ it is sufficient to show that $\varrho[L] = \varrho[L_1]$, i.e. that $\varrho[L]$ consists of all $f \in \mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$ vanishing on X. Notice that $\varrho[L]$ is a closed subalgebra of $\mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$. Let $\mathscr{B}$ be the smallest closed algebra containing all constant functions and $\varrho[L]$. If we show that $\mathscr{B} = \mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$, then each element f of $\mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$ must have the form $r \,.\, 1 + g$, where $g \in \varrho[L]$, and hence $f \in \varrho[L]$ if and only if f vanishes on X, and this is precisely that what is needed. Thus it remains to prove that $\mathscr{B} = \mathbf{C}^*(\langle P, v \rangle, \mathrm{R})$. Since $\langle P, v \rangle$ is compact, it is sufficient to show that $x \notin v(y)$ implies that $fx \neq fy$ for some f in $\varrho[L]$ (by 41 C.15). Suppose $x \notin v(y)$. Clearly at least one of the points x and y does not belong to X, say y, and hence we can choose a g in L such that $gy \neq 0$. Next, u is finer than v and hence $x \notin u(y)$, which implies that $hx \neq hy$ for some h in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$; clearly we may assume that $hx = 0$. Since L is an ideal, the function $h \,.\, g$ belongs to L and $(h \,.\, g)\, x = 0$, $(h \,.\, g)\, y \neq 0$. Thus $f = \varrho(h \,.\, g)$ belongs to $\varrho[L]$ and $fx \neq fy$.

III. Assuming (b) let us consider a proper filter $\mathscr{X}$ of sets on $\mathscr{P}$ and the set L of all bounded continuous functions f on $\mathscr{P}$ such that f vanishes on some X of $\mathscr{X}$. Clearly L is an ideal. Let L_1 be a maximal ideal containing L. Since L_1 is fixed, there exists a point x such that $fx = 0$ for each f in L_1, in particular, $fx = 0$ for each f in L. Thus, given an X in $\mathscr{X}$, if a bounded continuous function f vanishes on X, then $fx = 0$; consequently $x \in uX$. It follows that $x \in \bigcap u[\mathscr{X}]$. The proof is complete.

It seems to be in place to summarize the preceding characterizations of compactness in the class of all uniformizable spaces. It should be noted that some conditions have not been explicitly stated up to now.

41 C.23. Theorem. *Each of the following conditions is necessary and sufficient for a uniformizable closure space to be compact:*

(a) *The intersection of any centered collection of exact closed sets is non-void (see* 41 C.4*).*

(b) *Each cover of $\mathscr{P}$ consisting of exact open sets has a finite subcover.*

(c) *If a collection $\mathscr{F}$ of bounded continuous functions distinguishes separated points of $\mathscr{P}$, then $\mathscr{F}$ projectively generates $\mathscr{P}$.*

(d) *If a uniformizable closure v is coarser than u and the quasi-discrete modifications of u and v coincide, then u and v coincide.*

(e) *Each closed ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *is fixed.*

(f) *Each maximal ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *is fixed.*

(g) *If a collection $\mathscr{F}$ of bounded continuous functions distinguishes separated points of $\mathscr{P}$, then the smallest algebra containing $\mathscr{F}$ and all the constant functions is dense in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$.

(h) *If a collection $\mathscr{F}$ of bounded continuous functions distinguishes separated points of $\mathscr{P}$, then the smallest closed algebra containing $\mathscr{F}$ either coincides with*

$\mathbf{C}^*(\mathscr{P}, \mathsf{R})$ *or consists of all bounded continuous functions* f *vanishing at a point* x *(independent of* f*).*

Remark. The necessity of (h), the so-called Stone-Weierstrass Theorem, is the generalization of the Weierstrass Theorem due to M. H. Stone.

D. COMPACTIFICATIONS

Some of the results obtained earlier will nov be applied to the theory of compactification.

41 D.1. Definition. A *compactification of a closure space* $\mathscr{P}$ is a uniformizable compact space $\mathscr{Q}$ containing $\mathscr{P}$ as a dense subspace such that each point of $|\mathscr{Q}| - |\mathscr{P}|$ is closed in $\mathscr{Q}$; and hence, if $x \in |\mathscr{Q}| - |\mathscr{P}|$, $y \in |\mathscr{Q}| - (x)$, then the points x and y are separated (because $\mathscr{Q}$ is uniformizable). A *compactification* $\mathscr{Q}_1$ *of* $\mathscr{P}$ is said to be *finer than a compactification* $\mathscr{Q}_2$ *of* $\mathscr{P}$ if there exists a continuous mapping f of $\mathscr{Q}_1$ into $\mathscr{Q}_2$ such that $fx = x$ for each $x \in \mathscr{P}$, i.e. $f : \mathscr{P} \to \mathscr{P}$ is the identity homeomorphism of $\mathscr{P}$.

41 D.2. Remarks. (a) A space $\mathscr{P}$ is uniformizable if and only if it has a compactification. Indeed, if $\mathscr{P}$ has a compactification then $\mathscr{P}$ is uniformizable as a subspace of a uniformizable space. If $\mathscr{P}$ is uniformizable, then the closure structure of $\mathscr{P}$ is induced by a totally bounded uniformity $\mathscr{U}$ and an augmentation-separated completion $\langle Q, \mathscr{V} \rangle$ of $\langle |\mathscr{P}|, \mathscr{U} \rangle$ is a complete totally bounded uniform space, and hence $\langle Q, v \rangle$ is a compact space where v is the closure induced by $\mathscr{V}$; clearly $\langle Q, v \rangle$ is a compactification of $\mathscr{P}$ (since $\langle Q, \mathscr{V} \rangle$ is an augmentation-separated completion of $\langle |\mathscr{P}|, \mathscr{U} \rangle$).

(b) If $\langle Q, v \rangle$ is a compactification of $\langle P, u \rangle$ and X is a compact subspace of $\langle P, u \rangle$, then the closure of X in $\langle Q, v \rangle$ contains no point of $Q - P$. In fact, if $X \subset P$ and $x \in uX - P$ and if $\mathscr{Y}$ is the neighborhood system of x in $\langle Q, v \rangle$, then $\bigcap\{vY \mid Y \in \mathscr{Y}\} = (x)$ (because x and each other point of $\langle Q, v \rangle$ are separated, i.e. if $y \neq x$ then there exists an Y in $\mathscr{Y}$ such that $y \notin vY$), and hence $\bigcap\{X \cap v(X \cap Y) \mid Y \in \mathscr{Y}\} = \emptyset$; since $X \cap [\mathscr{Y}]$ is a proper filter on X, X is not compact.

(c) A uniformizable space $\mathscr{P}$ is compact if and only if $\mathscr{P}$ is the unique compactification of $\mathscr{P}$. In fact, if $\mathscr{P}$ is a compactification of $\mathscr{P}$, then $\mathscr{P}$ is compact, and if $\mathscr{Q}$ is a compactification of $\mathscr{P}$ and $\mathscr{Q} \neq \mathscr{P}$, then $\mathscr{P}$ is not compact by (b).

(d) If X is a closed subspace of a space $\mathscr{P}$ and $\mathscr{Q}$ is a compactification of P, then the closure Y of X in $\mathscr{Q}$ is a compactification of X. Indeed, X is dense in Y, Y is compact as a closed subspace of a compact space, and each point of $Y - X$ is closed because $Y - X \subset |\mathscr{Q}| - |\mathscr{P}|$.

(e) If $\mathscr{Q}$ is a compactification of $\mathscr{P}$, then $\mathscr{Q}$ is a compactification of each subspace $\mathscr{R}$ of $\mathscr{Q}$ such that $|\mathscr{P}| \subset |\mathscr{R}|$.

(f) If $\mathscr{Q}$ is a compactification of $\mathscr{P}$ and $\mathscr{R}$ is a compact subspace of $\mathscr{Q}$ such that $|\mathscr{P}| \subset |\mathscr{R}|$, then $\mathscr{R} = \mathscr{Q}$.

(g) If f is a continuous mapping of a compactification $\mathscr{Q}$ of $\mathscr{P}$ into $\mathscr{Q}$ such that $fx = x$ for each $x \in |\mathscr{P}|$, then f is the identity mapping of $\mathscr{Q}$ onto $\mathscr{Q}$. We shall prove that $fx = x$ for each $x \in \mathscr{Q}$. If $x \in |\mathscr{Q}| - |\mathscr{P}|$ and $y \in |\mathscr{Q}| - (x)$, then we can choose neighborhoods U of x and V of y such that $U \cap V = \emptyset$. Since $x \in \overline{U \cap |\mathscr{P}|}$, $fx \in \in \overline{f[U \cap |\mathscr{P}|]} = \overline{U \cap |\mathscr{P}|}$ and hence $fx \neq y$; this proves $fx = x$.

(h) If $\mathscr{Q}$ is a compactification of $\mathscr{P}$ and $\mathscr{P}$ is separated, then $\mathscr{Q}$ is separated. If $\mathscr{Q}$ is a separated uniformizable compact space, then $\mathscr{Q}$ is a compactification of each dense subspace of $\mathscr{Q}$, e.g. $[\![0, 1]\!]$ is a compactification of $]\!] 0, 1 [\![$ as well as of $\mathrm{Q} \cap \cap [\![0, 1]\!]$.

In the following theorem we shall describe relations between compactifications of $\mathscr{P}$, totally bounded uniform structures of $\mathscr{P}$ and completions of totally bounded uniform structures of $\mathscr{P}$. Recall that, by 41 C.8, if $\mathscr{Q}$ is a compact uniformizable space, then there exists a unique uniformity inducing the closure structure of $\mathscr{Q}$.

41 D.3. (a) *Let $\mathscr{P}$ be a uniformizable space and let K be the class of all compactifications of $\mathscr{P}$. The relation* $\mathbf{E}\{\langle \mathscr{Q}_1, \mathscr{Q}_2 \rangle \mid \mathscr{Q}_1$ *is a compactification of $\mathscr{P}$ finer than the compactification* $\mathscr{Q}_2\}$ *is a quasi-order on K, and $\mathscr{Q}_1$ is equivalent with $\mathscr{Q}_2$ (i.e. $\mathscr{Q}_1$ is finer than $\mathscr{Q}_2$ and $\mathscr{Q}_2$ is finer than $\mathscr{Q}_1$) if and only if there exists a homeomorphism f of $\mathscr{Q}_1$ onto $\mathscr{Q}_2$ such that $fx = x$ for each $x \in |\mathscr{P}|$.* (b) *For each $\mathscr{Q}$ in K let $\mathscr{U}_{\mathscr{Q}}$ be the relativization to $|\mathscr{P}|$ of the unique uniformity $\mathscr{V}_{\mathscr{Q}}$ inducing the closure structure of $\mathscr{Q}$. Then $\mathscr{U}_{\mathscr{Q}}$ is totally bounded and $\mathscr{V}_{\mathscr{Q}}$ is an augmentation-separated completion of $\mathscr{U}_{\mathscr{Q}}$ for each $\mathscr{Q} \in K$. If $\mathscr{V}$ is an augmentation-separated completion of a totally bounded uniformity $\mathscr{U}$ inducing the closure structure of $\mathscr{P}$ then $\mathscr{V} = \mathscr{V}_{\mathscr{Q}}$ and $\mathscr{U} = \mathscr{U}_{\mathscr{Q}}$ for some $\mathscr{Q}$ in K. Stated more formally, the relation $\{\mathscr{Q} \to \mathscr{U}_{\mathscr{Q}} \mid \mathscr{Q} \in K\}$ ranges on the set of all totally bounded uniformities inducing the closure structure of $\mathscr{P}$, $\{\mathscr{Q} \to \mathscr{V}_{\mathscr{Q}} \mid \mathscr{Q} \in K\}$ ranges on the class of all augmentation-separated completions of totally bounded uniformities inducing the closure structure of $\mathscr{P}$.* (c) *$\mathscr{Q}_1$ is finer than $\mathscr{Q}_2$ if and only if $\mathscr{U}_{\mathscr{Q}_1}$ is uniformly finer than $\mathscr{U}_{\mathscr{Q}_2}$.* (d) *There exists a finest compactification of $\mathscr{P}$; it corresponds to the Čech uniformity of $\mathscr{P}$.*

Proof. I. Proof of (a): The transitivity of the relation in question follows from the fact that the composite of two continuous mappings is a continuous mapping. If $\mathscr{Q}_1$ is finer than $\mathscr{Q}_2$ and $\mathscr{Q}_2$ is finer than $\mathscr{Q}_1$ and if f and g are the corresponding mappings, then $g \circ f = h$ is a continuous mapping of $\mathscr{Q}_1$ into itself such that $hx = x$ for x in $|\mathscr{P}|$. By 41 D.2 (g), h is the identity mapping of $\mathscr{Q}_1$ onto itself. The second statement of (a) follows.

II. Proof of (b): If $\mathscr{Q} \in K$, then $\mathscr{V}_{\mathscr{Q}}$ is a totally bounded uniformity and hence $\mathscr{U}_{\mathscr{Q}}$, the relativization of $\mathscr{V}_{\mathscr{Q}}$ to $|\mathscr{P}|$, is totally bounded as well. Next, since $\mathscr{Q}$ is compact, $\mathscr{V}_{\mathscr{Q}}$ is complete and hence $\langle |\mathscr{Q}|, \mathscr{V}_{\mathscr{Q}} \rangle$ is a completion of $\langle |\mathscr{P}|, \mathscr{U}_{\mathscr{Q}} \rangle$. Next, each point of $|\mathscr{Q}| - |\mathscr{P}|$ is closed and hence $\langle |\mathscr{Q}|, \mathscr{V}_{\mathscr{Q}} \rangle$ is an augmentation-separated completion of $\langle|\mathscr{P}|, \mathscr{U}_{\mathscr{Q}}\rangle$. If $\mathscr{U}$ is a totally bounded uniformity inducing the closure structure of $\mathscr{P}$ and if $\langle Q, \mathscr{V}\rangle$ is an augmentation-separated completion of $\langle|\mathscr{P}|, \mathscr{U}\rangle$, then $\langle Q, \mathscr{V}\rangle$ is a complete and totally bounded uniform space and the induced space $\langle Q, v\rangle$ is compact. Clearly $\mathscr{Q} = \langle Q, v\rangle$ is a compactification of $\mathscr{P}$ and $\mathscr{U} = \mathscr{U}_{\mathscr{Q}}$, $\mathscr{V} = \mathscr{V}_{\mathscr{Q}}$.

III. Proof of (c): If $\mathscr{Q}_1$ is finer than $\mathscr{Q}_2$ and f is a continuous mapping of $\mathscr{Q}_1$ into $\mathscr{Q}_2$ such that $fx = x$ for each $x \in |\mathscr{P}|$, then $f: \langle|\mathscr{Q}_1|, \mathscr{V}_{\mathscr{Q}_1}\rangle \to \langle|\mathscr{Q}_2|, \mathscr{V}_{\mathscr{Q}_2}\rangle$ is uniformly continuous because $\mathscr{V}_{\mathscr{Q}_1}$ is the fine uniformity of $\mathscr{Q}_1$. Consequently, the restriction $f: \langle|\mathscr{P}|, \mathscr{U}_{\mathscr{Q}_1}\rangle \to \langle|\mathscr{P}|, \mathscr{U}_{\mathscr{Q}_2}\rangle$ is also uniformly continuous, which shows that $\mathscr{U}_{\mathscr{Q}_1}$ is uniformly finer than $\mathscr{U}_{\mathscr{Q}_2}$ (because $f: |\mathscr{P}| \to |\mathscr{P}|$ is the identity mapping). Conversely, if $\mathscr{U}_{\mathscr{Q}_1}$ is uniformly finer than $\mathscr{U}_{\mathscr{Q}_2}$, then the mapping $\mathrm{J}: \langle|\mathscr{P}|, \mathscr{U}_{\mathscr{Q}_1}\rangle \to \langle|\mathscr{P}|, \mathscr{U}_{\mathscr{Q}_2}\rangle$ is uniformly continuous, and has an extension to a uniformly continuous mapping f of $\langle|\mathscr{Q}_1|, \mathscr{V}_{\mathscr{Q}_1}\rangle$ into $\langle|\mathscr{Q}_2|, \mathscr{V}_{\mathscr{Q}_2}\rangle$ (by theorem 41 B.1). Clearly $f: \mathscr{Q}_1 \to \mathscr{Q}_2$ has the required properties.

IV. Proof of (d): Since the Čech uniformity of $\mathscr{P}$ is the finest totally bounded uniformity inducing the closure structure of $\mathscr{P}$, statement (d) follows from statement (c).

41 D.4. Definition. A *Čech-Stone compactification of a uniformizable space* $\mathscr{P}$ is a compactification $\mathscr{Q}$ of $\mathscr{P}$ such that the Čech uniformity of $\mathscr{P}$ is a relativization of the Čech uniformity of $\mathscr{Q}$ (this is the unique uniformity inducing the closure structure of $\mathscr{Q}$). A Čech-Stone compactification of $\mathscr{P}$ is usually denoted by $\beta\mathscr{P}$.

41 D.5. Main theorem on Čech-Stone compactifications. *Every uniformizable space has a Čech-Stone compactification, and each of the following conditions is necessary and sufficient for a compactification $\mathscr{Q}$ of a uniformizable space $\mathscr{P}$ to be a Čech-Stone compactification:*

(a) *$\mathscr{Q}$ is a finest compactification of $\mathscr{P}$.*

(b) *The Čech uniformity of $\mathscr{Q}$ is a completion of the Čech uniformity of $\mathscr{P}$.*

(c) *Each bounded continuous function on $\mathscr{P}$ has a continuous extension to $\mathscr{Q}$.*

(d) *Each continuous mapping of $\mathscr{P}$ into a uniformizable compact space has a continuous domain-extension to $\mathscr{Q}$.*

Proof. Existence follows from 41 D.3. Again by 41 D.3, conditions (a) and (b) are only restatements of the definition of the Čech-Stone compactification. Clearly (d) implies (c) (recall that every closed bounded interval of reals is compact). Since the Čech uniformity is projectively generated by the collection of all bounded continuous functions, condition (c) implies that the Čech uniformity of $\mathscr{P}$ is a relativization of the Čech uniformity of $\mathscr{Q}$; by definition 41 D.4, $\mathscr{Q}$ is the Čech-Stone compactification of $\mathscr{P}$. It remains to show that if $\mathscr{Q}$ is a Čech-Stone compactification, then (d) holds. Let f be a continuous mapping of $\mathscr{P}$ into a uniformizable compact space $\mathscr{R}$, and let $\mathscr{W}$ be the unique uniformity inducing the closure structure of $\mathscr{R}$, $\mathscr{U}$ be the Čech uniformity of $\mathscr{P}$ and $\mathscr{V}$ be the Čech uniformity of $\mathscr{Q}$. Since $\mathscr{U}$ is the finest continuous totally bounded uniformity for $\mathscr{P}$ and $\mathscr{W}$ is totally bounded, the mapping $g = f : \langle|\mathscr{P}|, \mathscr{U}\rangle \to \langle|\mathscr{R}|, \mathscr{W}\rangle$ is uniformly continuous. Since $\langle|\mathscr{R}|, \mathscr{W}\rangle$ is complete and $\langle|\mathscr{P}|, \mathscr{U}\rangle$ is dense in $\langle|\mathscr{Q}|, \mathscr{V}\rangle$, by 41 B.1 there exists a uniformly continuous domain-extension of g to $\langle|\mathscr{Q}|, \mathscr{V}\rangle$. Clearly $g : \mathscr{Q} \to \mathscr{R}$ is a continuous extension of f.

Remark. The results of Theorem 41 D.5 are one of the most profound results of this book. The Čech-Stone compactifications have many interesting properties, some of which will be given in the exercises.

Every uniformizable space has a finest compactification, the so-called Čech-Stone compactification. On the other hand a space need not have a coarse compactification. This may be seen from the following theorem.

41 D.6. Theorem. *The following conditions on a uniformizable space $\mathscr{P}$ are equivalent:*

(a) *There exists a coarsest compactification of $\mathscr{P}$.*

(b) *There exists a (unique) uniformly coarsest uniformity inducing the closure structure of $\mathscr{P}$, i.e. $\mathscr{P}$ has a coarse uniformity.*

(c) *There exists a (unique) proximally coarsest uniformizable proximity inducing the closure structure of $\mathscr{P}$, i.e. $\mathscr{P}$ has a coarse uniformizable proximity.*

(d) *The proximity $p = \{X \to Y \mid X \subset |\mathscr{P}|, Y \subset |\mathscr{P}|$, if either $\overline{X}$ or $\overline{Y}$ is compact then $\overline{X} \cap \overline{Y} \neq \emptyset\}$ is uniformizable and induces the closure structure of $\mathscr{P}$.*

(e) *$\mathscr{P}$ is feebly locally compact, i.e. each point of $\mathscr{P}$ has a neighborhood which is a compact subspace of $\mathscr{P}$.*

(f) *$|\mathscr{P}|$ is open in a compact uniformizable space.*

(g) *$|\mathscr{P}|$ is open in a Čech-Stone compactification of $\mathscr{P}$.*

(h) *$\mathscr{P}$ is compact or there exists a one-point compactification of $\mathscr{P}$ (i.e., a compactification $\mathscr{Q}$ of $\mathscr{P}$ such that $|\mathscr{Q}| - |\mathscr{P}|$ is a singleton).*

Remark. If the equivalent conditions of (a)–(h) are fulfilled, then the proximity p of (d) is the coarse uniformizable proximity of $\mathscr{P}$, and any compactification $\mathscr{Q}$ of $\mathscr{P}$ such that $|\mathscr{Q}| - |\mathscr{P}|$ has at most one point is a coarse compactification of $\mathscr{P}$.

For the proof we shall need the following two propositions each of which is also useful in other situations.

41 D.7. (a) *If $\mathscr{P}$ is a regular topological feebly locally compact space, then the relation $p = \{X \to Y \mid X \subset |\mathscr{P}|, Y \subset |\mathscr{P}|$, if either $\overline{X}$ or $\overline{Y}$ is compact, then $\overline{X} \cap \overline{Y} \neq \emptyset\}$ is a uniformizable proximity inducing the closure structure of $\mathscr{P}$, and p is the proximally coarsest uniformizable proximity inducing the closure structure of $\mathscr{P}$.*

(b) *If $\mathscr{Q}$ is a coarse compactification of a space $\mathscr{P}$, then $|\mathscr{Q}| - |\mathscr{P}|$ has at most one point.*

41 D.8. *If $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are compactifications of $\mathscr{P}$ and $\mathscr{Q}_2$ is coarser than $\mathscr{Q}_1$, then there exists a unique continuous mapping f of $\mathscr{Q}_1$ into $\mathscr{Q}_2$ leaving fixed the points of $\mathscr{P}$ (i.e. such that $fx = x$ for each x in $\mathscr{P}$). This mapping is surjective and carries the points of $|\mathscr{Q}_1| - |\mathscr{P}|$ into the points of $|\mathscr{Q}_2| - |\mathscr{P}|$, i.e. $f[|\mathscr{Q}_1| - |\mathscr{P}|] = |\mathscr{Q}_2| - |\mathscr{P}|$.*

Remark. Notice that the uniqueness assertion is a generalization of 41 D.2 (g).

Proof of 41 D.6. We shall prove (a) ⇒ (h) ⇒ (g) ⇒ (f) ⇒ (e) ⇒ (d) ⇒ (c) ⇒ ⇒ (b) ⇒ (a).

I. The implication (a) ⇒ (h) follows from 41 D.7 (b).

II. If $\mathscr{Q}$ is a compactification of $\mathscr{P}$ such that $X = |\mathscr{Q}| - |\mathscr{P}|$ has at most one point and f is a continuous mapping of a $\beta\mathscr{P}$ into $\mathscr{Q}$ which leaves fixed the points of $\mathscr{P}$, then (by 41 D.8) the set $|\beta\mathscr{P}| - |\mathscr{P}| = f^{-1}[X]$ is closed because X is closed in $\mathscr{Q}$, and hence $|\mathscr{P}|$ is open in $\beta\mathscr{P}$. Thus (h) ⇒ (g).

III. Clearly (g) ⇒ (f), and (f) ⇒ (e) follows from the fact that an open subspace of a regular feebly locally compact topological space is feebly locally compact.

IV. The implication (e) ⇒ (d) was stated in 41 D.7 (a).

V. The proximity p of (d) is a coarse uniformizable proximity of $\mathscr{P}$. In fact, if q is any uniformizable proximity inducing the closure structure of $\mathscr{P}$, XqY and $\overline{X} \cap \overline{Y} = \emptyset$, then neither $\overline{X}$ nor $\overline{Y}$ is compact (by 41 C.11) and hence XpY, which shows that p is proximally coarser than q. Thus (d) implies (c) and the first statement of the remark to 41 D.6 is proved (and the second is evident).

VI. If p is the coarse uniformizable proximity of $\mathscr{P}$ and $\mathscr{U}$ is the proximally coarse semi-uniformity of $\langle|\mathscr{P}|, p\rangle$, then clearly $\mathscr{U}$ is a coarse uniformity of $\mathscr{P}$. Thus (c) implies (b).

VII. The implication (b) ⇒ (a) follows from 41 D.3.

Proof of 41 D.7 (a). Let us notice that if two sets X and Y are distant in $\langle|\mathscr{P}|, p\rangle$, then $\overline{X} \cap \overline{Y} = \emptyset$ and at least one of the sets $\overline{X}$ and $\overline{Y}$ is compact.

First let us prove the following statement: if U is a neighborhood of a compact set K then there exists a closed compact neighborhood K_1 of K such that $K_1 \subset \operatorname{int} U$. For each x in K let V_x be a closed neighborhood contained in the intersection of $\operatorname{int} U$ with a compact neighborhood of x. Thus $\{\operatorname{int} V_x \mid x \in K\}$ is an open cover of K, and K being compact, some finite subfamily $\{\operatorname{int} V_x \mid x \in F\}$ also covers K. Clearly the set $K_1 = \bigcup\{V_x \mid x \in F\}$ has the required property. Now the proof is almost evident. In fact, if U is a neighborhood of x in $\mathscr{P}$ then (x) is compact and hence there exists a compact closed neighborhood K of x such that $K \subset \operatorname{int} U$. Clearly $\overline{K} \cap \overline{|\mathscr{P}| - U} = K \cap (|\mathscr{P}| - \operatorname{int} U) = \emptyset$ and $\overline{K} = K$ is compact. Thus U is a proximal neighborhood of K and hence U is a proximal neighborhood of x. Thus p induces the closure structure of $\mathscr{P}$. If X non p Y, then one of the sets X and Y, say X, is compact, and of course $U = |\mathscr{P}| - Y$ is a neighborhood of X. By our auxiliary proposition we can choose a compact closed neighborhood K of X such that $K \subset \operatorname{int} U$, and then we can choose a compact closed neighborhood K_1 of K such that $K_1 \subset \operatorname{int} U$. Clearly K is a proximal neighborhood of X and $|\mathscr{P}| - K$ is a proximal neighborhood of $|\mathscr{P}| - K_1$ and hence of $Y \subset |\mathscr{P}| - K_1$. Thus the proximity p fulfils condition (prox 5). By 25 B.2, p is uniformizable.

Proof of 41 D.7 (b). It suffices to prove that if $\mathscr{Q}$ is a compactification of $\mathscr{P}$ and $|\mathscr{Q}| - |\mathscr{P}|$ has at least two distinct points, say x and y, then $\mathscr{Q}$ is not coarse. To prove this we shall construct a compactification $\mathscr{R}$ which is coarser than $\mathscr{Q}$ but which is not finer than $\mathscr{Q}$. Identifying the points x and y of $\mathscr{Q}$ we obtain evidently a compactification $\mathscr{R}$ of $\mathscr{P}$ which is coarser than $\mathscr{Q}$ because the canonical mapping f of $\mathscr{Q}$ onto $\mathscr{R}$ is continuous and leaves fixed the points of $|\mathscr{P}|$. On the other hand, $\mathscr{R}$ is not finer than $\mathscr{Q}$. In fact, assuming the contrary, take a continuous mapping g of $\mathscr{R}$ into $\mathscr{Q}$ which leaves fixed the points of $|\mathscr{P}|$; thus $h = g \circ f$ is a mapping of $\mathscr{Q}$ into $\mathscr{Q}$ which leaves fixed the points of $|\mathscr{P}|$; by 41 D.2 (g) the mapping h is the identity mapping of $\mathscr{Q}$ and hence the equality $hx = hy$ (which follows from the equality $fx = fy$) implies that $x = y$, and this contradicts our assumption.

Proof of 41 D.8. Let f be a continuous mapping of $\mathscr{Q}_1$ onto $\mathscr{Q}_2$ which is an extension of $\mathsf{J} : \mathscr{P} \to \mathscr{P}$. The set $\mathsf{E}f$ is compact in $\mathscr{Q}_2$ and contains $|\mathscr{P}|$. By 41 D.2 (f), $\mathsf{E}f = |\mathscr{Q}_2|$, and hence f is surjective. Let x be any point of $|\mathscr{Q}_1| - |\mathscr{P}|$; we shall show that $fx \notin |\mathscr{P}|$. Consider the neighborhood system $\mathscr{A}$ of x in $\mathscr{Q}_1$ and the collection $\mathscr{B}$ of all $A \cap |\mathscr{P}|$, $A \in \mathscr{A}$. Of course $\mathscr{B}$ is a proper filter on $|\mathscr{P}|$ and the intersection of all $\bar{B}^{\mathscr{P}}$, $B \in \mathscr{B}$, is empty. We have $x \in \bar{B}^{\mathscr{Q}_1}$ for each B in $\mathscr{B}$, and hence $fx \in \bar{B}^{\mathscr{Q}_2}$ for each B in $\mathscr{B}$ (recall that f is continuous). Finally, $\bar{B}^{\mathscr{Q}_2} \subset \bar{B}^{\mathscr{P}} \cup (|\mathscr{Q}_2| - |\mathscr{P}|)$, and consequently $fx \notin |\mathscr{P}|$. Thus f has the required properties. We shall show that f is the unique continuous mapping which leaves fixed the points of $|\mathscr{P}|$. Suppose that there are two such mappings, say f and g, and choose an x such that $fx \neq gx$. Evidently $x \in |\mathscr{Q}_1| - |\mathscr{P}|$, and hence both fx and gx belong to $|\mathscr{Q}_2| - |\mathscr{P}|$. The points fx and gx are separated and therefore we can choose neighborhoods U of fx and V of gx such that $U \cap V = \emptyset$. The sets $f^{-1}[U]$ and $g^{-1}[V]$ are neighborhoods of x in $\mathscr{Q}_1$ and hence $W = f^{-1}[U] \cap \cap g^{-1}[V]$ is a neighborhood of x. Clearly $W \cap |\mathscr{P}| = U \cap V \cap |\mathscr{P}| = \emptyset$, which contradicts the fact that $|\mathscr{P}|$ is dense in $\mathscr{Q}_1$.

Now we may present various characterizations of uniformizable spaces with a unique uniformizable proximity.

41 D.9. Theorem. *The following conditions on a uniformizable space $\mathscr{P}$ are equivalent:*

(a) *There exists a unique uniformizable proximity inducing the closure structure of $\mathscr{P}$, i.e. $\upsilon\mathbf{P}(\mathscr{P})$ is a singleton.*

(b) *$|\beta\mathscr{P}| - |\mathscr{P}|$ has at most one point.*

(c) *Any compactification of $\mathscr{P}$ is a Čech-Stone compactification, i.e., any two compactifications of $\mathscr{P}$ are equivalent.*

(d) *If two closed sets X and Y are functionally separated (i.e. distant with respect to the Čech proximity), then at least one of the sets X and Y is compact.*

(e) *There exists a unique uniformity inducing the closure structure of* $\mathscr{P}$, *i.e.* $\upsilon\mathbf{U}(\mathscr{P})$ *is a singleton.*

Proof. I. The equivalence of (a), (b) and (c) follows immediately from Theorem 41 D.3 which describes the relationship between compactifications and uniformizable proximities. Condition (d) states that the proximity p of 41 D.6 (d) coincides with the Čech proximity, i.e. that p is uniformizable, induces the closure structure of $\mathscr{P}$ and coincides with the Čech proximity; by 41 D.6, p is the coarse uniformizable proximity of $\mathscr{P}$. Thus (d) states that the Čech proximity is the coarse uniformizable proximity of $\mathscr{P}$ and this is a restatement of (a). Thus the conditions (a)–(d) are equivalent.

II. Clearly (e) implies (a), and (a) implies that there exists a unique proximally coarse uniformity (i.e. a totally bounded uniformity) inducing the closure structure of $\mathscr{P}$. It remains to show that if (d) holds, then each uniformity inducing the closure structure of $\mathscr{P}$ is proximally coarse. We shall prove somewhat more, namely that (d) implies that each continuous pseudometric on $\mathscr{P}$ is totally bounded. Assuming that a continuous pseudometric d is not totally bounded we can find a positive real r and an infinite set M such that $d\langle x, y\rangle \geqq r$ for each x and y in M, $x \neq y$. If X and Y are any two non-void disjoint subsets of M, then the distance from X to Y (in $\langle|\mathscr{P}|, d\rangle$) is positive, namely at least r, and hence $\overline{X}$ and $\overline{Y}$ are functionally separated in $\langle|\mathscr{P}|, d\rangle$ and so certainly in $\mathscr{P}$. On the other hand, the closure (in $\mathscr{P}$) of no infinite subset X of M is compact; in fact, the relativization of d to $\overline{X}$ is not totally bounded.

If $\mathscr{P}$ is a normal space, then any two disjoint closed sets are functionally separated and therefore condition (d) of the last theorem can be stated as follows: if X and Y are disjoint closed sets, then either X or Y is compact. Thus we have proved

41 D.10. Corollary. *A normal space* $\mathscr{P}$ *has a unique uniformizable proximity if and only if at least one of any two disjoint closed sets is compact.*

41 D.11. Example. The space $\mathscr{P}$ of all countable cardinals has a unique uniformizable proximity. Indeed, $\mathscr{P}$ is normal, and if X and Y are two disjoint closed sets then either X or Y is bounded (and therefore compact), because in the opposite case we could construct two sequences, $\{x_n\}$ in X and $\{y_n\}$ in Y, such that $x_n \leqq y_n \leqq x_{n+1}$; but this is impossible because such sequences would have a common limit point, namely $\sup\{x_n\} = \sup\{y_n\}$ which must belong to $X \cap Y$. For further examples see the exercises.

In conclusion we shall describe locally compact uniformizable spaces and spaces with a unique uniformity by means of continuous functions with compact support. In accordance with 19 F.11, the closed support (in what follows, simply support) of a function on a space is the closure of the set of all points at which the function does not vanish. Notice that each continuous function with a compact support is bounded.

41 D.12. Theorem. *Let* $\mathscr{P}$ *be a uniformizable closure space, and let* A *be the set of all continuous functions on* $\mathscr{P}$ *with compact supports.* (a) A *is an ideal in* $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *and* $A = \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ *if and only if* $\mathscr{P}$ *is compact.* (b) *The space* $\mathscr{P}$ *is locally compact* *if and only if the collection A projectively generates $\mathscr{P}$.* (c) *The space $\mathscr{P}$ has a unique uniformity if and only if either $A = \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ (this is the case when $\mathscr{P}$ is compact) or A is dense in a maximal ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$.*

Proof. Statement (a) is evident. I. The "only if" in (b) follows from the description of projective generating families by means of neighborhoods (32 A.6). Conversely, assume that A projectively generates $\mathscr{P}$ and let us consider the set X of all points x of $\mathscr{P}$ such that $fx = 0$ for each f in A. If $y \in |\mathscr{P}| - X$ and if f is an element of A such that $fy \neq 0$, then the support of f is a compact neighborhood of y, and hence $\mathscr{P}$ is feebly locally compact (and so certainly locally compact) at each point of $|\mathscr{P}| - X$. We shall prove that $X = \emptyset$. First we shall observe that the complement of any open neighborhood of any point of X is compact. Given an open neighborhood G of an $x \in X$, choose $f_1, \ldots, f_n$ in A and neighborhoods U_i of $f_i x = 0$ such that $V = \bigcap\{f_i^{-1}[U_i] \mid i \leqq n\} \subset G$. Clearly the complement of V is contained in the union K of the supports of the functions f_i. The set K is compact (as the finite union of compact sets) and the complement of G is contained in K. Thus $|\mathscr{P}| - G$ is compact (as a closed set in a compact set). The proof is now straightforward. If $\mathscr{P}$ is compact then the constant function $\{x \to 1\}$ belongs to A, and hence $X = \emptyset$. It remains to observe that $X \neq \emptyset$ implies that $\mathscr{P}$ is compact. If $\mathscr{X}$ is any proper filter on $\mathscr{P}$ and if $x \in X$, then either x is a cluster point of $\mathscr{X}$ or the complement of some neighborhood of x belongs to $\mathscr{X}$; consequently x is a cluster point of $\mathscr{X}$ or $\mathscr{X}$ contains a compact set (namely the complement of an open neighborhood of x, which is compact). In both cases $\mathscr{X}$ has a cluster point, and hence $\mathscr{P}$ is compact.

II. If $A = \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ then $\mathscr{P}$ is compact (by (a)), and hence $\mathscr{P}$ has a unique uniformity. Suppose that the closure B of A is a maximal ideal in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$, and consider a Čech-Stone compactification $\mathscr{Q}$ of $\mathscr{P}$. We shall prove that $|\mathscr{Q}| - |\mathscr{P}|$ is a singleton, and hence $\mathscr{Q}$ is a coarse compactification of $\mathscr{P}$; it will follow from 41 D.9 (b) that $\mathscr{P}$ has a unique uniformity. It is sufficient to show that each continuous function on $\mathscr{Q}$ is constant on $|\mathscr{Q}| - |\mathscr{P}|$. Consider the canonical isomorphism h of $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ onto $\mathbf{C}^*(\mathscr{Q}, \mathrm{R})$ which assigns to each $f \in \mathbf{C}^*(\mathscr{P}, \mathrm{R})$ the unique continuous domain-extension of f to $\mathscr{Q}$. Clearly each hf, $f \in A$, is zero at each point of $|\mathscr{Q}| - |\mathscr{P}|$, in particular, each hf is constant on $|\mathscr{Q}| - |\mathscr{P}|$. Since h is an isomorphism, the same is true for each $f \in B$, and $h[B]$ is a maximal ideal in $\mathbf{C}^*(\mathscr{Q}, \mathrm{R})$. Consequently, each continuous function on $\mathscr{Q}$ is the sum of an hf, $f \in B$, and a constant function. Thus each continuous function on $\mathscr{Q}$ is constant on $|\mathscr{Q}| - |\mathscr{P}|$.

III. Assuming that a non-compact space $\mathscr{P}$ has a unique uniformity we shall prove that the closure B of A in $\mathbf{C}^*(\mathscr{P}, \mathrm{R})$ is a maximal ideal. Consider any compactification $\mathscr{Q}$ of $\mathscr{P}$; thus $\mathscr{Q}$ is a Čech-Stone compactification and simultaneously a coarse compactification. Consider the isomorphism $h : \mathbf{C}^*(\mathscr{P}, \mathrm{R}) \to \mathbf{C}^*(\mathscr{Q}, \mathrm{R})$ defined in II. We shall show that the set $h[B]$ consists of all continuous functions on $\mathscr{Q}$ which vanish at the unique point x of $|\mathscr{Q}| - |\mathscr{P}|$. Clearly $hfx = 0$ for each $f \in A$, and hence for each $f \in B$. Conversely, assuming that $hfx = 0$ we shall prove that $f \in B$. Clearly we may and shall assume that $f \geqq 0$. For each n in N and each y in $\mathscr{P}$ let $f_n y = \max((n+1)^{-1}, fy) - (n+1)^{-1}$. Clearly the sequence $\{f_n\}$ converges to f in $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$. It is easily seen that $f_n \in A$ for each n, and hence $f \in B$; it suffices to observe that the closure of any set $\mathbf{E}\{y \mid |fy| \geqq r\}$, $r > 0$, is compact in $\mathscr{P}$ (because the closure of this set in $\mathscr{Q}$ does not contain x). The proof is complete.

Remark. If $\mathscr{P}$ has a unique uniformity then the set A need not be closed.

E. SPACES OF IDEALS

If $\langle P, u\rangle$ is a uniformizable space and p is a uniformizable proximity which induces u, then there exists a compactification $\langle K, v\rangle$ of $\langle P, u\rangle$ such that p is a relativization of the unique uniformizable proximity inducing v, i.e. XpY if and only if $X \subset P$, $Y \subset P$ and $vX \cap vY \neq \emptyset$. This important result has been proved by means of completions of uniform spaces. In this subsection we shall give a new proof by means of the structure space of maximal ideals of the algebra of all bounded proximally continuous functions.

If p_1 and p_2 are uniformizable proximites inducing the closure structure of a given space $\mathscr{P}$ and if $\mathscr{K}_1$ and $\mathscr{K}_2$ are any corresponding compactifications, then there exists a continuous mapping of $\mathscr{K}_1$ onto $\mathscr{K}_2$ which leaves fixed the points of $\mathscr{P}$ if and only if p_2 is proximally coarser than p_1. The "only if" is evident and "if" was proved by means of the theorem on extension of uniformly continuous mappings into a complete uniform space. Here we shall give an alternate proof.

Let $\mathscr{R} \doteq \langle R, +, \cdot\rangle$ be a commutative semi-ring with a unit. In 18 E we defined the structure space $\mathfrak{M}(\mathscr{R})$ of maximal ideals of $\mathscr{R}$. By 18 E.6 the space $\mathfrak{M}(\mathscr{R})$ is the set of all maximal ideals of $\mathscr{R}$ endowed with a topological closure operation such that the sets

$$ha = \mathbf{E}\{L \mid L \text{ is a maximal ideal containing } a\}$$

form a closed base. The set ha is called the *hull* of a in $\mathfrak{M}(\mathscr{R})$.

41 E.1. Theorem. *The structure space $\mathfrak{M}(\mathscr{R})$ of a commutative semi-ring $\mathscr{R}$ with a unit* 1 *is compact.*

Proof. It is sufficient to show that the intersection of any centered family $\{ha \mid a \in A\}$ where $A \subset |\mathscr{R}|$ is non-void. Let us consider the set B of all elements of the form $b_0 a_0 + \ldots + b_n a_n$ where $n \in \mathsf{N}$, $a_i \in A$ and $b_i \in |\mathscr{R}|$. We shall prove that $1 \notin B$, i.e. $1 \neq \sum_{i \leqq n} b_i a_i$ for each n, b_i and a_i; then B will be a proper ideal of $\mathscr{R}$ containing A, and if L is any maximal ideal containing B (such a maximal ideal exists because $\mathscr{R}$ has a unit), then necessarily L belongs to each ha, $a \in A$, i.e. the intersection in question is non-void. Let $b = b_0 a_0 + \ldots + b_n a_n$ be any element of B. By our assumption the set $\bigcap\{ha_i \mid i \leqq n\}$ is non-void and hence there exists a maximal ideal L' containing all a_i, $i \leqq n$. Consequently $b \in L'$ and hence $b \neq 1$.

41 E.2. Theorem. *Let $\mathscr{P}$ be a closure space and let $\mathscr{A}$ be a closed subspace of* $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ *which contains all constant functions. If $f^{-1}[0]$ is closed in $\mathscr{P}$ for each f in $\mathscr{A}$ then each function in $\mathscr{A}$ is continuous.*

Proof. It is sufficient to show that $g \circ f \in \mathscr{A}$ provided that $f \in \mathscr{A}$ and g is a continuous function on R. In fact, if F is closed in R, then we can choose a continuous function g on R such that $gx = 0$ if and only if $x \in F$; and then $f^{-1}[F] = (g \circ f)^{-1}[0]$ for each f in $\mathscr{A}$, which shows that $f^{-1}[F]$ is closed in $\mathscr{P}$ for each f in $\mathscr{A}$ and each closed set F in R, that is to say, each $f \in \mathscr{A}$ is continuous. To prove our auxiliary proposition let us consider the proximity p projectively generated by $\mathscr{A}$. By the Stone-Weierstrass Theorem for proximity spaces, the set $\mathscr{A}$ consists of all f such that $f : \langle|\mathscr{P}|, p\rangle \to \mathsf{R}$ is proximally continuous. If g is a continuous functions on R and $f \in \mathscr{A}$ then $f : \langle|\mathscr{P}|, p\rangle \to \mathsf{R}$ is proximally continuous and $\mathbf{E}f$ is a bounded subset of R, and hence $f : \langle|\mathscr{P}|, p\rangle \to K$ is defined and proximally continuous for some compact subspace K of R. The function $g : K \to \mathsf{R}$ is proximally continuous because K is compact. Thus $g \circ f : \langle|\mathscr{P}|, p\rangle \to \mathsf{R}$ is proximally continuous. Consequently $g \circ f \in \mathscr{A}$. The proof is complete.

We now proceed to the subject proper of this subsection.

41 E.3. *Let $\mathscr{P} = \langle P, p\rangle$ be a uniformizable proximity space, and let us consider the structure space $\mathfrak{M}(\mathbf{P}^*(\mathscr{P}, \mathsf{R}))$ of maximal ideals of the algebra* $\mathbf{P}^*(\mathscr{P}, \mathsf{R})$. For brevity we shall write $\mathfrak{M}_{\mathscr{P}}$ instead of $\mathfrak{M}(\mathbf{P}^*(\mathscr{P}, \mathsf{R}))$. If $x \in |\mathscr{P}|$ then the fixed ideal L_x consisting of all $f \in \mathbf{P}^*(\mathscr{P}, \mathsf{R})$ such that $fx = 0$ is a maximal ideal; indeed, each f can be written as $r \,.\, 1 + g$ with g in L_x and this decomposition is unique (see the notes preceding 41 C.18). Let ι, more precisely $\iota_{\mathscr{P}}$, be the mapping of $\mathscr{P}$ into $\mathfrak{M}_{\mathscr{P}}$ which assigns to each x the fixed maximal ideal L_x.

If L is a maximal ideal then any $f \in \mathbf{P}^*(\mathscr{P}, \mathsf{R})$ can be written as $r \,.\, 1 + g$ with g in L and this decomposition is unique. Let $\{r_{fL} \mid f \in \mathbf{P}^*(\mathscr{P}, \mathsf{R}), L \in \mathfrak{M}_{\mathscr{P}}\}$ be the family of reals such that $f = r_{fL} \,.\, 1 + g_L$ in L for each f and L. For each f let f^* be the function on $\mathfrak{M}_{\mathscr{P}}$ which assigns to each L the number r_{fL}. If L is a fixed ideal ιx, then $f^*L = fx$; in fact $f = r_{fL} \,.\, 1 + g_L$ with g_L in L, i.e. $g_L x = 0$, and hence $fx = r_{fL}$. Thus $f^* \circ \iota = f$ for each f. We shall prove

(a) *The mapping $\{f \to f^*\} : \mathbf{P}^*(\mathscr{P}, \mathsf{R}) \to \mathbf{F}^*(\mathfrak{M}_{\mathscr{P}}, \mathsf{R})$ is a normed-algebra-embedding.*

Since $f^* \circ \iota = f$ we have $\|f^*\| \geqq \|f\|$ (of course the norms are taken in the corresponding algebras). On the other hand, it is easily seen that $|r_{fL}| \leqq \|f\|$ (for each x we have $|r_{fL}| \leqq |fx| + |g_L x| \leqq \|f\| + |g_L x|$, and $\inf \{|g_L x|\} = 0$ because g_L belongs to a maximal ideal), and hence $\|f^*\| \leqq \|f\|$. Thus the mapping is norm-preserving. Clearly, $r_{f+gL} = r_{fL} + r_{gL}$, $r_{fgL} = r_{fL} \,.\, r_{gL}$ and $r_{afL} = a \,.\, r_{fL}$ for each f, g, L and $a \in \mathsf{R}$, and hence the mapping is an algebra-homomorphism. Finally, the mapping is injective, because if f^* is the zero-function then $\|f^*\| = 0$, and hence $\|f\| = 0$, which implies that f is the zero-function.

(b) *The set* $\mathscr{A}$ *of all* f^* *is a closed subalgebra of* $\mathbf{F}^*(\mathfrak{M}_{\mathscr{P}}, \mathrm{R})$ (because $\mathbf{P}^*(\mathscr{P}, \mathrm{R})$ is complete and thus $\mathscr{A}$ is complete). Next, $f^{*-1}[0]$ consists of all maximal ideals containing f. In fact, $f^*L = 0$ if and only if $f = 0 \,.\, 1 + g$ with g in L. Thus $f^{*-1}[0]$ is the hull of f. Consequently $f^{*-1}[0]$ is closed in $\mathfrak{M}_{\mathscr{P}}$. By 41 E.2 each f^* is continuous. Finally, the functions f^* distinguish the points of $\mathfrak{M}_{\mathscr{P}}$; in fact, if L_1 and L_2 are distinct maximal ideals then we can choose an f in $L_2 - L_1$, and then $f^*L_2 = 0, f^*L_1 \neq 0$. We have proved that the set $\mathscr{A}$ of all f^* is a closed algebra of continuous functions on a compact separated topological, hence uniformizable, space $\mathfrak{M}_{\mathscr{P}}$ and each constant function belongs to $\mathscr{A}$; by the Stone-Weierstrass Theorem $\mathscr{A}$ coincides with the set of all continuous functions on $\mathfrak{M}_{\mathscr{P}}$, i.e. $\mathscr{A} = \mathbf{C}^*(\mathfrak{M}_{\mathscr{P}}, \mathrm{R})$. Combining this with (a) we obtain

(c) *The mapping* $\{f \to f^*\} : \mathbf{P}^*(\mathscr{P}, \mathrm{R}) \to \mathbf{C}^*(\mathfrak{M}_{\mathscr{P}}, \mathrm{R})$ *is a normed-algebra-isomorphism, and* $\mathfrak{M}_{\mathscr{P}}$ *is separated and uniformizable.*

The space $\mathfrak{M}_{\mathscr{P}}$ is compact and uniformizable, and hence there exists a unique uniformizable proximity q inducing the closure structure of $\mathfrak{M}_{\mathscr{P}}$. Consider $\mathfrak{M}_{\mathscr{P}}$ as a proximity space $\langle|\mathfrak{M}_{\mathscr{P}}|, q\rangle$. A function on $\mathfrak{M}_{\mathscr{P}}$ is proximally continuous if and only if it is continuous. The functions f^* projectively generate the proximity space $\mathfrak{M}_{\mathscr{P}}$, and the functions $f = f^* \circ \iota$ projectively generate the proximity space $\mathscr{P}$; projective generation is associative (39 A.5), and therefore

(d) *The mapping* ι *of* $\mathscr{P}$ *into the proximity space* $\mathfrak{M}_{\mathscr{P}}$ *is a projective generating mapping; in particular, if* $\mathscr{P}$ *is separated then* ι *is a proximal embedding.*

Finally let us observe that

(e) $\mathbf{E}\iota$ *(i.e. the set of all fixed maximal ideals) is dense in* $\mathfrak{M}_{\mathscr{P}}$.

In the opposite case there would exist a non-zero continuous functions on $\mathfrak{M}_{\mathscr{P}}$ which is zero on $\mathbf{E}\iota$; however, such a function does not exist because $\{f \to f^*\}$: $\mathbf{P}^*(\mathscr{P}, \mathrm{R}) \to \mathbf{C}^*(\mathfrak{M}_{\mathscr{P}}, \mathrm{R})$ is surjective (by (c)).

The results of 41 E.3 will be summarized in the following theorem.

41 E.4. Theorem. *Let* $\mathscr{P}$ *be a uniformizable proximity space and let* ι *be the mapping of* $\mathscr{P}$ *into* $\mathfrak{M}(\mathbf{P}^*(\mathscr{P}, \mathrm{R}))$ *which assigns to each* x *the fixed maximal ideal of functions vanishing at* x. *Let* $\tilde{\iota}$ *be the mapping of* $\mathbf{P}^*(\mathfrak{M}(\mathbf{P}^*(\mathscr{P}, \mathrm{R})), \mathrm{R})$ *into* $\mathbf{P}^*(\mathscr{P}, \mathrm{R})$ *which assigns to each* g *the function* $g \circ \iota$. *Then*

(a) $\tilde{\iota}$ *is a normed algebra-isomorphism.*

(b) ι *is a proximal projective generating mapping; in particular, if* $\mathscr{P}$ *is separated then* ι *is a proximal embedding.*

(c) $\mathfrak{M}(\mathbf{P}^*(\mathscr{P}, \mathrm{R}))$ *is a uniformizable compact space and* $\mathbf{E}\iota$ *is dense.*

Proof: $\tilde{\iota}$ is the mapping with graph $\{f^* \to f\}$.

It may be noted that 41 E.4 gives a new proof of existence of compactifications. More precisely

41 E.5. Corollary. *If p is a uniformizable proximity inducing the closure structure of a space $\mathscr{P}$ then there exists a compactification $\mathscr{K}$ of $\mathscr{P}$ such that XpY if and only if $|X| \subset |\mathscr{P}|$, $Y \subset |\mathscr{P}|$ and $\overline{X}^{\mathscr{K}} \cap \overline{Y}^{\mathscr{K}} \neq \emptyset$, i.e. p is a relativization of the unique uniformizable proximity which induces the closure structure of $\mathscr{K}$.*

Proof. Let K consist of all points of $|\mathscr{P}|$ and all free maximal ideals of $\mathscr{R} = \mathbf{P}^*(\langle|\mathscr{P}|, p\rangle, \mathsf{R})$, and consider the mapping φ of K into $\mathfrak{M}(\mathscr{R})$ such that $\varphi x = \iota x$ if $x \in |\mathscr{P}|$ and $\varphi x = x$ otherwise. The space $\mathscr{K}$ projectively generated by φ has the required properties.