LÖWIG'S WORKS ON FUNCTIONAL EQUATIONS

The defense of Heinrich Löwig's doctoral thesis *Über periodische Differenzengleichungen* took place at the German University in Prague, December 1927. The thesis itself is no longer extant, but its contents were summarised in a short four-page outline [L1]. More importantly, the results were published in a series of two extensive papers [L2, L3], which appeared in Acta Mathematica in 1931. According to Löwig, these papers represent a revised and extended version of his original thesis. Although their titles refer to "difference equations", we prefer to use the term "functional equations", which seems to be more appropriate in the context of current terminology.

The topic was inspired by earlier investigations of Émile Picard published in Acta Mathematica under the title *Sur une classe des transcendantes nouvelles.* The object of Picard's study was a system of functional equations

$$f_k(z+h) = Q_k(f_1(z), \dots, f_n(z)), \quad k \in \{1, \dots, n\},$$

where $Q_1, \dots, Q_n$ are given and $f_1, \dots, f_n$ are unknown functions, z is a complex variable, and h is a nonzero complex number. Under certain assumptions, Picard was able to prove the existence of ω-periodic solutions of the given system of equations for every nonzero $\omega \in \mathbb{C}$ which is not a real multiple of h. This result might be interpreted as a generalisation of elliptic functions, i.e. functions with two linearly independent complex periods h and ω.

Löwig succeeded in generalising Picard's results to systems of the form

$$f_k(z+h) = Q_k(z, f_1(z), \dots, f_n(z)), \quad k \in \{1, \dots, n\}, \tag{1}$$

where the right-hand sides now depend not only on $f_1, \dots, f_n$, but also on the variable z. The paper [L2] is devoted to linear functional equations, which represent an important special case of the system (1); the results obtained there were subsequently used in [L3] to analyse the general nonlinear system (1). The following sections briefly summarise the contents of both Löwig's papers.

1. Linear equations

A major part of the article [L2] is devoted to the study of a linear system of functional equations

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z) + B_k(z), \quad k \in \{1, \dots, n\}, \tag{2}$$

where q_{kl}, k, $l \in \{1, \dots, n\}$ and B_k, $k \in \{1, \dots, n\}$ are given functions, f_k, $k \in \{1, \dots, n\}$ are unknown functions, z is a complex variable, and h is a given nonzero complex number. This system represents a special case of the general nonlinear system (1). Löwig assumed that all the functions q_{kl} and B_k are ω-periodic, where ω is a nonzero complex number which is not a real multiple of h. The main results of the work [L2] are concerned with the existence and uniqueness of ω-periodic solutions of the system (2), and also the existence of an ω-periodic fundamental set of solutions for the corresponding homogeneous system

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z), \quad k \in \{1, \dots, n\}. \tag{3}$$

Before we proceed further, it might be helpful to review some facts from complex function theory. Given a nonzero number $\omega \in \mathbb{C}$ and a pair of numbers a, $b \in \mathbb{R}$, $a < b$, denote

$$T_\omega(a,b) = \left\{ z \in \mathbb{C};\ a < \operatorname{Im}\left(\frac{z}{\omega}\right) < b \right\}.$$

Thus $T_1(a,b)$ is a strip in the plane between the lines $y = a$ and $y = b$. We have $z \in T_1(a,b)$ if and only if $\omega z \in T_\omega(a,b)$, i.e.

$$T_\omega(a,b) = \{\omega z;\ z \in T_1(a,b)\}.$$

The geometric meaning of $T_\omega(a,b)$ is now clear: if we interpret ω as a vector in the complex plane, then $T_\omega(a,b)$ is a strip between two lines parallel to ω. It is occasionally useful to allow the cases $a = -\infty$ or $b = \infty$; the corresponding set $T_\omega(a,b)$ is then either a half-plane, or the whole complex plane. We note that $z \in T_\omega(a,b)$ implies $z + \omega \in T_\omega(a,b)$; this is the reason why the sets $T_\omega(a,b)$ occur frequently as the domains of ω-periodic functions.

We now describe an analogy of the classical Fourier expansion for ω-periodic holomorphic functions. First, consider a holomorphic function f which is defined on $T_1(a,b)$ and has the period 1, i.e. $f(z+1) = f(z)$ for every $z \in T_1(a,b)$. Using the transformation $z \mapsto e^{2\pi i z}$, the values of f can be mapped into the annulus A bounded by two circles that are centred at the origin and have radii $e^{-2\pi b}$ and $e^{-2\pi a}$. In this way, we obtain a function g defined for every $w \in A$ with $w = e^{2\pi i z}$ by the formula $g(w) = f(z)$. Note that if $w = e^{2\pi i z_1} = e^{2\pi i z_2}$, then $z_1 - z_2$ is an integer; since f is 1-periodic, we conclude that $f(z_1) = f(z_2)$, which confirms that the definition of g makes sense. Since g is holomorphic in the annulus A, it has a Laurent series expansion

$$g(w) = \sum_{n=-\infty}^{\infty} c_n w^n, \quad w \in A,$$

where c_n are certain complex numbers. Consequently,

$$f(z) = g(e^{2\pi i z}) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z}, \quad z \in T_1(a,b).$$

More generally, given an ω-periodic holomorphic function f defined on $T_\omega(a,b)$, we use the substitution $f^*(z/\omega) = f(z)$ to obtain a 1-periodic function f^* defined on $T_1(a,b)$. According to the previous result,

$$f(z) = f^*(z/\omega) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/\omega}, \quad z \in T_\omega(a,b),$$

which is the promised analogy of Fourier series expansion. As we will see later, Löwig often worked with ω-periodic holomorphic functions written in this form. We refer the reader to [2] and [7] for a more detailed treatment of Fourier series in the complex domain.

We now return to the discussion of Löwig's work [L2] on linear functional equations. The paper is divided into four sections, whose contents are analysed below. We recall that h, ω is a pair of nonzero complex numbers such that $h/\omega \notin \mathbb{R}$. Following Löwig's notation, let

$$\lambda = e^{2\pi i h/\omega}$$

and observe that $h/\omega \notin \mathbb{R}$ implies $|\lambda| \neq 1$.

The first section contains a number of auxiliary propositions. Using the Weierstrass sigma and zeta functions (see e.g. [2]), Löwig was able to construct an ω-periodic function f that is meromorphic in the entire complex plane and satisfies the functional equation

$$f(z+h) = tf(z), \tag{4}$$

where $t \in \mathbb{C}$ is an arbitrary fixed nonzero number. He then proceeded to show that if a holomorphic function f satisfies (4) and $t \neq \lambda^n$ for every $n \in \mathbb{Z}$, then f is identically zero. Finally, using the Weierstrass sigma function again, he constructed a sequence of ω-periodic meromorphic functions $\{f_n\}_{n=0}^{\infty}$ such that

$$f_0(z+h) = tf_0(z), \quad f_k(z+h) = tf_k(z) + kf_{k-1}(z), \quad k \geq 1. \tag{5}$$

The second section is already concerned with the nonhomogeneous linear system of functional equations

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z) + B_k(z), \quad k \in \{1, \dots, n\}, \tag{6}$$

where q_{kl} and B_k are given ω-periodic functions. It is assumed that q_{kl} satisfy the following conditions:

- There exists a number $R > 0$ such that the functions q_{kl} are holomorphic in the half-plane given by the inequality $|e^{2\pi i z/\omega}| \leq R$, and can be expressed in the form

$$q_{kl}(z) = \sum_{\alpha=0}^{\infty} q_{kl\alpha} e^{2\pi i \alpha z/\omega}, \quad k, l \in \{1, \dots, n\}.$$

We remark that $|e^w| = e^{\operatorname{Re} w}$ for every $w \in \mathbb{C}$, i.e.

$$|e^{2\pi iz/\omega}| = e^{\operatorname{Re}(2\pi iz/\omega)} = e^{-2\pi \operatorname{Im}(z/\omega)}.$$

Consequently, the inequality $|e^{2\pi iz/\omega}| \le R$ represents the half-plane $\operatorname{Im}(z/\omega) \ge -\frac{\ln R}{2\pi}$.

- $\det\{q_{kl0}\}_{k,l=1}^{n} \neq 0$.
- There exists a positive number $r \le R$ such that

$$\det\{q_{kl}(z)\}_{k,l=1}^{n} \neq 0$$

for all z in the half-plane $|e^{2\pi iz/\omega}| \le r$.

Initially, the functions B_k are assumed to satisfy the following condition, which will be weakened later:

- There exists a number $\rho > 0$ such that the functions B_k are holomorphic in the half-plane $|e^{2\pi iz/\omega}| \le \rho$ and have the series expansions

$$B_k(z) = \sum_{\alpha=\mu}^{\infty} B_{k\alpha} e^{2\pi i\alpha z/\omega}, \quad k \in \{1, \dots, n\},$$

where μ is an integer. If $|\lambda| > 1$, then $\rho \le R$, and if $|\lambda| < 1$, then $\rho \le r$.

Löwig employed the method of undetermined coefficients to find an ω-periodic solution of the system (6). He assumed that the unknown functions f_k can be expressed in the form

$$f_k(z) = \sum_{\alpha=\mu}^{\infty} a_{k\alpha} e^{2\pi i\alpha z/\omega}, \quad k \in \{1, \dots, n\}. \tag{7}$$

By substituting the series expansions of f_k, B_k and q_{kl} into (6) and equating the corresponding coefficients on both sides, we obtain

$$a_{k\alpha}\lambda^k = \sum_{\beta=\mu}^{\alpha} \sum_{l=1}^{n} q_{kl\alpha-\beta} a_{l\beta} + B_{k\alpha} \tag{8}$$

for every $k \in \{1, \dots, n\}$ and $\alpha \ge \mu$. The last equation is in turn equivalent to

$$\sum_{l=1}^{n} (q_{kl0} - \delta_{kl}\lambda^{\alpha}) a_{l\alpha} + \sum_{l=1}^{n} \sum_{\beta=\mu}^{\alpha-1} q_{kl\alpha-\beta} a_{l\beta} + B_{k\alpha} = 0,$$

which uniquely determines the coefficients $a_{k\alpha}$ in the case when

$$\det\{q_{kl0} - \delta_{kl}\lambda^{\alpha}\}_{k,l=1}^{n} \neq 0$$

for every $\alpha \geq \mu$ (using the last equation, it is possible to express $a_{1\alpha}, \dots, a_{n\alpha}$ in terms of $a_{l\mu}, \dots, a_{l\alpha-1}$, $l \in \{1, \dots, n\}$). However, the system (8) might have a solution even if some of the above mentioned determinants vanish.

After finding the coefficients $a_{k\alpha}$, it remains to verify that the infinite series in (7) are indeed convergent. Löwig proved that if $a_{k\alpha}$, $k \in \{1, \dots, n\}$, $\alpha \geq \mu$, is an arbitrary set of coefficients satisfying (8), then the infinite series in (7) are convergent in the half-plane $|e^{2\pi i z/\omega}| \leq \rho|\lambda|$ for $|\lambda| > 1$, and in the half-plane $|e^{2\pi i z/\omega}| \leq \rho$ for $|\lambda| < 1$. It follows that the functions $f_1, \dots, f_n$ defined by (7) represent a solution of the nonhomogeneous system (6) in the corresponding half-plane.

Löwig introduced the notation

$$K(t) = \det\{q_{kl0} - \delta_{kl}t\}_{k,l=1}^{n}$$

and referred to $K(t) = 0$ as the characteristic equation; as we already know, the problem of existence and uniqueness of solutions of the system (6) is closely related to the question whether the characteristic equation possesses roots of the form λ^α, $\alpha \in \mathbb{Z}$.

As a next step, Löwig proceeded to demonstrate a generalisation of the theorem from the first section: Assume that the characteristic equation has no roots of the form λ^α, $\alpha \in \mathbb{Z}$, and $f_1, \dots, f_n$ are ω-periodic functions satisfying

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z), \quad k \in \{1, \dots, n\}. \tag{9}$$

If either $|\lambda| < 1$ and $f_1, \dots, f_n$ are holomorphic in the half-plane $|e^{2\pi i z/\omega}| \leq R|\lambda|$, or $|\lambda| > 1$ and $f_1, \dots, f_n$ are holomorphic in the half-plane $|e^{2\pi i z/\omega}| \leq r$, then $f_1, \dots, f_n$ are identically zero. The proof is by induction on n and makes use of the above mentioned theorem from the first section.

Löwig then returned to the nonhomogeneous system (6) and investigated the existence and uniqueness of an ω-periodic solution under the following weaker hypothesis on the functions B_k:

- There exist positive numbers ρ_1, ρ_2 such that $\rho_1 < \rho_2$, the functions B_k are holomorphic in the strip $\rho_1 \leq |e^{2\pi i z/\omega}| \leq \rho_2$, and have the series expansions

$$B_k(z) = \sum_{\alpha=-\infty}^{\infty} B_{k\alpha} e^{2\pi i \alpha z/\omega}, \quad k \in \{1, \dots, n\}. \tag{10}$$

  Assume also that either $|\lambda| > 1$, $\rho_1 \leq r$ and $\rho_2 \leq R$, or $|\lambda| < 1$, $\rho_1 \leq r$ and $\rho_2 \leq r$.

The essential difference is that in the series expansions of B_k, we are now summing over all integers α. As a first step, Löwig showed that if the characteristic equation has no roots of the form λ^α, $\alpha \in \mathbb{Z}$, then the nonhomogenous system (6) has at most one holomorphic solution. Indeed, the difference of any two solutions is a holomorphic solution of the homogeneous system (9), and using the above mentioned theorem, it is possible to show that the difference is identically zero, i.e. the two solutions coincide.

To prove the existence of an ω-periodic solution of the nonhomogeneous system (6), Löwig started by treating the special case obtained by choosing a pair of numbers $m \in \{1, \dots, n\}$, $\gamma \in \mathbb{Z}$, and letting

$$B_k(z) = \begin{cases} 0, & k \neq m \\ e^{2\pi i \gamma z/\omega}, & k = m. \end{cases}$$

Then he used the method of undetermined coefficients to find an n-tuple of functions of the form

$$f_{km\gamma}(z) = \sum_{\alpha=\gamma}^{\infty} b_{km\alpha\gamma} e^{2\pi i \alpha z/\omega}, \quad k \in \{1, \dots, n\},$$

which constitute a solution of the corresponding nonhomogeneous system. Now, given a general nonhomogenous system with functions B_k having the form (10), the natural candidate for an ω-periodic solution is the n-tuple of functions

$$f_k(z) = \sum_{\gamma=-\infty}^{\infty} \sum_{m=1}^{n} B_{m\gamma} f_{km\gamma}(z) = \sum_{\alpha=-\infty}^{\infty} \sum_{m=1}^{n} \sum_{\gamma=-\infty}^{\alpha} b_{km\alpha\gamma} B_{m\gamma} e^{2\pi i \alpha z/\omega}, \quad (11)$$

$k \in \{1, \dots, n\}$. The difficult part of the proof is to verify the convergence of the infinite series on the right-hand side. We remark that Löwig was able to establish the existence of an ω-periodic solution even in the case when the characteristic equation has roots of the form λ^α, $\alpha \in \mathbb{Z}$.

Using the explicit expression (11), Löwig derived the following estimate: There exists a constant $\chi > 0$ such that if $B_1, \dots, B_n$ are ω-periodic and holomorphic in the strip $P = \{z \in \mathbb{C};\ \rho_1 \leq |e^{2\pi i z/\omega}| \leq \rho_2\}$ and $f_1, \dots, f_n$ represent the ω-periodic solution of the corresponding nonhomogeneous system (6), then

$$|f_k(z)| \leq \chi \cdot \max_{l=1,\dots,n} \left(\sup_{z \in P} |B_l(z)| \right), \quad k \in \{1, \dots, n\}.$$

The inequality is valid in the strip $\rho_1 \leq |e^{2\pi i z/\omega}| \leq \rho_2 |\lambda|$ if $|\lambda| > 1$, and in the strip $\rho_2 |\lambda| \leq |e^{2\pi i z/\omega}| \leq \rho_2$ if $|\lambda| < 1$. The statement generalises Picard's estimate given in the paper [5], which corresponds to the special case when q_{kl} are constant functions.

Löwig concluded the second section by a short discussion of the nonhomogeneous system (6) in the case when B_k are no longer ω-periodic functions; his attention was mostly directed toward the functional equation $f(z+h) - f(z) = z$.

The third section is again concerned with the homogeneous system

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z), \quad k \in \{1, \dots, n\}. \tag{12}$$

The main result is the proof of the existence of ω-periodic meromorphic functions which constitute a fundamental set of solutions of the system (12). These functions are defined in the half-plane $|e^{2\pi iz/\omega}| \le R|\lambda|$ if $|\lambda| > 1$, and in the half-plane $|e^{2\pi iz/\omega}| \le r$ if $|\lambda| < 1$.

The terminology is completely similar to the one used in the theory of ordinary differential equations: we say that $f_1^i, \dots, f_n^i$, where $i \in \{1, \dots, n\}$, represent a fundamental set of solutions, if for every $i \in \{1, \dots, n\}$, the functions $f_1^i, \dots, f_n^i$ satisfy (12), and if every solution $f_1, \dots, f_n$ of (12) can be uniquely expressed as

$$f_k(z) = \sum_{i=1}^{n} a_i(z) f_k^i(z), \quad k \in \{1, \dots, n\},$$

where $a_1, \dots, a_n$ are suitable ω-periodic functions. Consequently, every solution of the nonhomogeneous system (6) can be uniquely expressed in the form

$$f_k(z) = f_k^0(z) + \sum_{i=1}^{n} a_i(z) f_k^i(z), \quad k \in \{1, \dots, n\},$$

where $f_1^0, \dots, f_n^0$ is an arbitrary fixed (particular) solution of the nonhomogeneous system (6), and $a_1, \dots, a_n$ are suitable ω-periodic functions. Löwig proved the existence of the ω-periodic meromorphic fundamental set of solutions by induction on n.

If the ratio of any two roots of the characteristic equation is different from λ^α, where α is a nonzero integer, Löwig provided a more detailed description of the fundamental set of solutions, which is based on the knowledge of multiplicities of the roots $t_1, \dots, t_j$ of the characteristic equation, and of the elementary divisors of the matrix $\{q_{kl0} - \delta_{kl} t_i\}_{k,l=1}^n$, $i \in \{1, \dots, j\}$. He also made use of the proposition proved in the first section concerning the existence of functions satisfying the system (5). We omit the technical details, which are too complicated to be reproduced here (only the statement of the corresponding theorem in [L2] extends over two pages).

The final fourth section discusses functional equations of the n-th order

$$\sum_{l=0}^{n} p_l(z) f(z+lh) = A(z), \tag{13}$$

where $p_n(z) = 1$ for every $z \in \mathbb{C}$. It is easily checked that this equation is equivalent to a system of first-order functional equations

$$f_1(z+h) = f_2(z),\ f_2(z+h) = f_3(z), \dots,\ f_{n-1}(z+h) = f_n(z),$$

$$f_n(z+h) = -\sum_{l=1}^{n} p_{l-1}(z) f_l(z) + A(z).$$

Using the earlier results on systems of first-order equations, it is fairly easy to prove existence as well as uniqueness of an ω-periodic solution of the equation (13) and the associated homogeneous equation

$$\sum_{l=0}^{n} p_l(z) f(z+lh) = 0. \tag{14}$$

Löwig assumed that the following conditions are satisfied:

- There exists a number $R > 0$ such that the functions $p_0, \dots, p_n$ have the series expansion

$$p_k(z) = \sum_{\alpha=0}^{\infty} p_{k\alpha} e^{2\pi i\alpha z/\omega}, \quad k \in \{0, \dots, n\},$$

  which is valid in the half-plane $|e^{2\pi iz/\omega}| \le R$. (Obviously, $p_{n0} = 1$ and $p_{n\alpha} = 0$ for $\alpha \ge 1$.)
- $p_{00} \ne 0$.
- There exists a positive number $r \le R$ such that $p_0(z) \ne 0$ for every z in the half-plane $|e^{2\pi iz/\omega}| \le r$.
- There exist positive numbers ρ_1, ρ_2 such that $\rho_1 < \rho_2$ and the function A has the series expansion

$$A(z) = \sum_{\alpha=-\infty}^{\infty} A_\alpha e^{2\pi i\alpha z/\omega}$$

  in the strip $\rho_1 \le |e^{2\pi iz/\omega}| \le \rho_2$.

These conditions guarantee that the system of n first-order equations obtained from the single n-th order equation satisfies the hypotheses listed in the previous sections. An application of the corresponding theorems then gives the existence of an ω-periodic solution of the nonhomogeneous equation (13), existence of an ω-periodic meromorphic fundamental set of solutions of the homogeneous system (14), etc.

The characteristic equation is now

$$\sum_{l=0}^{n} p_{l0} t^l = 0.$$

Assuming there are no roots of the form λ^α, $\alpha \in \mathbb{Z}$, the equation (13) has a unique holomorphic solution.

2. Nonlinear equations

The subsequent paper [L3] is devoted to systems of nonlinear functional equations of the form

$$f_k(z+h) = Q_k(z, f_1(z), \dots, f_n(z)), \quad k \in \{1, \dots, n\}. \tag{15}$$

Again, the question raised by Löwig is the existence of ω-periodic solutions. He assumed that the functions $Q_1, \dots, Q_n$ are ω-periodic in the first argument, and $Q_k(z, 0, \dots, 0) = 0$ for every $k \in \{1, \dots, n\}$. Moreover, for every z in the strip $\rho_1 \le |e^{2\pi i z/\omega}| \le \rho_2$, it was assumed that Q_k as a function of the n variables $f_1, \dots, f_n$ possesses the Taylor series expansion

$$Q_k(z, f_1, \dots, f_n) = \sum_{l=1}^{n} q_{kl}(z) f_l + \sum_{\alpha_1 + \cdots + \alpha_n \ge 2} G_{\alpha_1 \dots \alpha_n}(z) f_1^{\alpha_1} \cdots f_n^{\alpha_n}$$

in a certain neighbourhood of the point $(0, \dots, 0)$.

Without going into technical details, we describe the main idea of Löwig's proof of the existence of ω-periodic solutions. We introduce the notation

$$B_k(z, f_1, \dots, f_n) = \sum_{\alpha_1 + \cdots + \alpha_n \ge 2} G_{\alpha_1 \dots \alpha_n}(z) f_1^{\alpha_1} \cdots f_n^{\alpha_n}.$$

Consequently,

$$Q_k(z, f_1, \dots, f_n) = \sum_{l=1}^{n} q_{kl}(z) f_l + B_k(z, f_1, \dots, f_n), \quad k \in \{1, \dots, n\}.$$

We already know that under certain conditions, the homogeneous linear system

$$f_k(z+h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z), \quad k \in \{1, \dots, n\}$$

has a fundamental set of solutions consisting of meromorphic ω-periodic functions $f_1^i, \dots, f_n^i$, $i \in \{1, \dots, n\}$. For every n-tuple of real numbers $c_1, \dots, c_n$, let

$$f_k^{(0)}(z, c_1, \dots, c_n) = \sum_{s=1}^{n} c_s f_k^s(z), \quad k \in \{1, \dots, n\}.$$

Given a positive integer ν and an n-tuple of sufficiently small numbers $c_1, \dots, c_n$, Löwig defined the functions $f_k^{(\nu)}$, $k \in \{1, \dots, n\}$, to be the ω-periodic solutions of the nonhomogeneous system

$$f_k^{(\nu)}(z+h, c_1, \dots, c_n) = \sum_{l=1}^{n} q_{kl}(z) f_l^{(\nu)}(z, c_1, \dots, c_n) +$$

$$+ B_k(z, f_1^{(\nu-1)}(z, c_1, \dots, c_n), \dots, f_n^{(\nu-1)}(z, c_1, \dots, c_n)).$$

Finally, he proved the existence of the limits

$$f_k(z, c_1, \dots, c_n) = \lim_{\nu\to\infty} f_k^{(\nu)}(z, c_1, \dots, c_n), \quad k \in \{1, \dots, n\}.$$

The functions $z \mapsto f_k(z, c_1, \dots, c_n)$, where $c_1, \dots, c_n$ are sufficiently small fixed numbers, are ω-periodic and satisfy the system (15).

Thus we see that Löwig in fact proved the existence of infinitely many solutions of the system (15). The meaning of the parameters $c_1, \dots, c_n$ becomes clear when we look at the partial derivatives of f_k; it is possible to show that

$$\frac{\partial f_k}{\partial c_s}(z, 0, \dots, 0) = f_k^s(z), \quad k, s \in \{1, \dots, n\}.$$

In the final part of [L3], the results are again reformulated for the nonlinear functional equation of the n-th order of the form

$$f(z + nh) = P(z, f(z), f(z + h), \dots, f(z + (n-1)h)),$$

which is clearly equivalent to the system of first-order equations

$$f_1(z + h) = f_2(z),\ f_2(z + h) = f_3(z), \dots,\ f_{n-1}(z + h) = f_n(z),$$

$$f_n(z + h) = P(z, f_1(z), \dots, f_n(z)).$$

3. Conclusion

Even our brief overview indicates that Löwig's works on functional equations are quite involved from the technical point of view. His theorems usually have numerous assumptions and, as a consequence, the statements often exceed one printed page (for example, the statement of Theorem 13 in [L2] fills three pages). One source of problems is the necessity of distinguishing between the cases $|\lambda| > 1$ and $|\lambda| < 1$; even the proofs are often different for each case.

Using vector and matrix notation, it would be possible to rewrite some of the statements in a more compact form. For example, the nonhomogenous system

$$f_k(z + h) = \sum_{l=1}^{n} q_{kl}(z) f_l(z) + B_k(z), \quad k \in \{1, \dots, n\}$$

can be written more succinctly as $f(z + h) = Q(z)f(z) + B(z)$, where f, B are vector functions and Q is a matrix function. However, this notation was still not common among mathematicians at the time.

Consisting of 101 printed pages, the paper [L2] is slightly unusual, but not quite exceptional; in the 1930s, the journal Acta Mathematica published other papers of similar length. Even today, the journal continues to publish very long papers written by leading mathematicians.

The proof of the existence of ω-periodic solutions of the nonlinear system is based on the well-known method of successive approximations due to Picard. Although it is without doubt that Löwig possessed an excellent knowledge of both real and complex analysis, the proofs in both papers are altogether elementary with no revolutionary ideas or methods. Of course, we have to keep in mind that [L2, L3] represent only a slightly modified and extended version of Löwig's doctoral thesis. From this point of view, it can be said that the thesis was of very good quality.

It seems that Löwig's works on functional equations did not have a definitive influence on further research within this field. However, his achievements were not completely forgotten, as evidenced by the citations of his work e.g. in the monograph [3] or in the papers [1, 4, 8].

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Functional analysis arose in the early twentieth century and gradually, conquering one stronghold after another, became a nearly universal mathematical doctrine, not merely a new area of mathematics, but a new mathematical world view. Its appearance was the inevitable consequence of the evolution of all of nineteenth-century mathematics, in particular classical analysis and mathematical physics. Its original basis was formed by Cantor's theory of sets and linear algebra. Its existence answered the question of how to state general principles of a broadly interpreted analysis in a way suitable for the most diverse situations.

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