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VOJTĚCH JARNÍK'S WORK IN LATTICE POINT THEORY

Břetislav Novák

I was lucky enough to be accompanied by the personality of V. Jarník from the beginning and throughout my University studies. V. Jarník in his inimitable way lectured for us the basic course in Mathematical Analysis and later a number of more advanced courses from this field, and I also attended a number of his elective courses in Number Theory. He was the supervisor of my Master Thesis and later my Ph.D. (Candidate of Science) dissertation and introduced me to his beloved topic of lattice points. He strongly influenced my research work and was my ideal in teaching. I am deeply grateful to him for all.

The most successful periods of Jarník's activities are the years 1928–31 (devoted almost exclusively to more than twenty papers on lattice points in more-dimensional ellipsoids) and the period 1934–35.

Further information concerning the work of V. Jarník and his successors can be found in the monographs F. Fricker, Einfuehrung in die Gitterpunktentheorie, Birkhauser 1982, A. Walfisz, Gitterpunkte in mehrdimensionalen Kugeln, Warszawa 1957, MM 33. and also in the paper Schwarz, W., Zassenhaus, H.: In memoriam Bohuslav Diviš, December 20, 1942–July 26, 1976. J. Number Theory 9, v–vii (1977).

Let us start with an analysis of Jarník's contributions to the theory of lattice points. Having in mind their number and extent (27 papers of more than five hundred pages) we restrict ourselves only to an incomplete account of the most important and most easily describable results and to the indication of the basic connections.

The oldest and most commonly known is the "circle" problem of Gauss. For $x > 0$ let $A(x)$ denote the number of couples of integers u and v, for which $u^2 + v^2 \leqslant x$ holds. The quantity $A(x)$ for $x > 0$ has the meaning of the number of lattice points belonging to the (closed) circle with center at the origin and with radius $\sqrt{x}$. In the most natural way we will try to approximate the function $A(x)$ by the area of the circle, i.e. by the expression πx.

The “circle” problem consists now in finding the best possible approximation of the difference $P(x) = A(x) - V(x)$ in the following sense:

An elementary argument shows that $P(x) = O(\sqrt{x})$.[1] In 1906 the Polish mathematician Sierpiński proved that $P(x) = O(x^{\frac{1}{3}})$. In 1915 Hardy with Landau arrived at an important result showing that $P(x) = \Omega(x^{\frac{1}{4}})$.[2] This yields that for the “proper” exponent of the estimates of the function $P(x)$, i.e. for the value

$$f = \limsup_{x\to+\infty} \frac{\lg |P(x)|}{\lg x}, \tag{1}$$

the inequality $\frac{1}{4} \leqslant f \leqslant \frac{1}{3}$ holds. (Let us mention that (1) means exactly the same as the validity of the relations $P(x) = O(x^{f+\varepsilon})$,$P(x) = \Omega(x^{f-\varepsilon})$ for every $\varepsilon > 0$.) Therefore the “circle” problem in fact consists in determining the value of (1).

It was the work of van der Corput who showed in 1923 that $f \leqslant \frac{37}{112}$ which meant a turning point of the research. Let us note that in spite of great efforts by many excellent mathematicians throughout the past fifty years, the problem has not been solved yet (e.g. it is known that $f \leqslant \frac{13}{40}$, but all attempts to improve the inequality $f \geqslant \frac{1}{4}$) have failed. We are mentioning these results in order to demonstrate the long effort which was needed for a relatively “slight” decrease of the estimate of the exponent; in this way the importance and difficulty of Jarník’s work in this field will perhaps stand out more clearly.

The attention of mathematicians was then oriented to various generalizations of the “circle” problem. Let us mention only two directions which became the subject of intensive interest of Prof. Jarník.

The first generalization is the following. Let us remain in the plane, but consider instead of the circle $u^2 + v^2 \leqslant x$ a convex compact set, circumscribed by a curve of length at most $\sqrt{x}$, which possessess a continuously varying tangent and a nonzero radius of curvature which does not exceed the value $c\sqrt{x}$ with a positive constant c. For this class of sets van der Corput in 1919 established the estimate $P(x) = O(x^{\frac{1}{3}})$, where $P(x)$ has a meaning analogous to the case of the circle.

At this stage it seemed that also for this generalization it is possible to expect analogous improvements as for the “circle” problem. Therefore Jarník’s result from [9][3] was surprising and unexpected; in this paper (besides further finer results)

[1] The relation $f(x) = O(g(x))$ means that $g(x) > 0$ for x sufficiently large and that the relation $\limsup\limits_{x\to+\infty} \frac{|f(x)|}{g(x)} < +\infty$ holds.

[2] The relation $f(x) = \Omega(g(x))$ means that the relation $f(x) = o(g(x))$ does not hold, i.e. g is positive for x sufficiently large and $\limsup\limits_{x\to+\infty} \frac{|f(x)|}{g(x)} > 0$.

[3] References correspond to the list of papers of V. Jarník given on pp. 133–142.

he constructs a set having the given properties, for which $P(x) = \Omega(x^{\frac{1}{3}})$ holds. The excellence of this result is documented also by the response in literature. E. Landau devoted a full chapter to this construction in his work "Vorlesungen über Zahlentheorie", and the result is quoted also in other monographs, e.g. in the book of Gelfond and Linnik.

The "plane" problem is treated also in further papers of Jarník. In [7] Jarník generalizes the Ω-estimate of Landau and Hardy presented above to a sufficiently general class of regions in the plane (which is described in a way slightly different from the one given above). The paper [12] tackles the problem from a slightly different point of view. In the angle $0 \leqslant u \leqslant \frac{y}{x} \leqslant v \leqslant 1$, $x > 0$ let us consider a sufficiently "reasonable" curve L which circumscribes a certain sector and let $0 < \lambda_1 < \lambda_2 < \dots$ be the sequence of all positive numbers λ for which the image of the curve L in the homothety with center at the origin and the quotient λ contains at least one lattice point. Jarník considers the interesting question concerning the behaviour of the difference $\lambda_{n+1} - \lambda_n$ for large n, special attention being paid to the case when L is part of a quadratic curve. The "plane" case is considered also in the papers [8], [31] and [71].

The other direction of generalization can be characterized as follows: from the "circle" problem we keep only the fact that u^2+v^2 is a positively definite quadratic form. The problem is the following: if $r \geqslant 0$ is a natural number, $Q(u_1, u_2, \dots, u_r)$ a positively definite quadratic form with a symmetric matrix of coefficients, for $x \geqslant 0$ let $A(x)$ be the number of lattice points in the domain $Q(u_1, u_2, \dots, u_r) \leqslant x$. Denote by $V(x)$ the volume of this ellipsoid, i.e. $V(x) = \frac{\pi^{\frac{r}{2}} x^{\frac{r}{2}}}{\sqrt{D}\Gamma(\frac{1}{2}r+1)}$, where D is the determinant of the form Q, and let $P(x) = A(x) - V(x)$. We are looking—analogously to the case of the "circle" problem—for the possibly best O and Ω estimates.

In 1905 Minkowski showed in an elementary way that

$$P(x) = O(x^{\frac{r}{2}-\frac{1}{2}}).$$

In the years 1915–1924 this problem (in a much more general setting) was studied by E. Landau who proved that

$$P(x) = O(x^{\frac{r}{2}-\frac{r}{r+1}}), \quad P(x) = \Omega(x^{\frac{r-1}{4}}). \tag{2}$$

(Bentkus, V.; Goetze, E. in 1997 published in Acta Arithmetica (80, No. 2, 101–125) that for $r \geqslant 9$ the relation $P(x) = O(x^{\frac{r}{2}-1})$ holds. Their proof is based on a new method.) Let us observe that for the case of the circle this yields the above mentioned results of Sierpiński, Landau and Hardy. The gap between both of these estimates is—especially for large r—substantial and we are facing e.g. the problem of finding the value $f_Q = f$ which is defined by (1).

The first progress was done by the results of A. Walfisz and E. Landau in the period 1924–25. Their result was

$$P(x) = O(x^{\frac{r}{2}-1}) \tag{3}$$

for $r \geqslant 5$, provided Q is a rational form (i.e. its coefficients are integer multiples of the same real number). For $r = 4$ and a rational Q Landau showed that the estimate

$$P(x) = O(x \lg^2 x) \tag{4}$$

holds.

Let us mention at this point that the problems for "small" dimensions ($r = 2, 3, 4$) are of different nature and require other methods in comparison with those for "big" dimensions ($r \geqslant 5$), the case $r = 4$ being in a certain sense "transient".[4] This follows, roughly speaking, e.g. from the fact, that every natural number can be expressed as the sum of four squares of integers and that this number cannot be less in general.

The method used by Walfisz and Landau for deriving their O-estimates is based on the following idea (for the sake of simplicity assume that the coefficients of Q are integers). The function

$$f(z) = \sum z^{Q(m_1, m_2, \dots, m_r)}$$

(the summation is done over all integers $m_1, m_2, \dots, m_r$) is clearly holomorphic in the circle $|z| < 1$ and the coefficients in its Taylor expansion

$$\sum_{m=0}^{\infty} a_m z^m$$

clearly represent the number of possible expressions of (the number) m by the form Q. We evidently have

$$a_m = \frac{1}{2\pi i} \int_{\varphi} f(z) z^{-m-1} \, \mathrm{d}z,$$

[4] As was shown by A. Walfisz, for rational forms and $r = 4$ we have $P(x) = O(x \lg^{\frac{2}{3}})$ and $P(x) = \Omega(x(\lg\lg x)^{\tau})$ where τ depends on Q only.

where φ is a positively oriented circle with center at the origin and radius $r < 1$. Using now some transformation properties of the function f, which enable us to find its approximate expression in the vicinity of the point $re^{\frac{2\pi ih}{k}}$ (h, k are integers), $r = 1 - \frac{1}{m}$, we get the relation

$$a_m = \frac{\pi^{\frac{r}{2}} m^{\frac{r}{2}-1}}{\sqrt{D}\Gamma(\frac{1}{2}r)} \sum_{k=1}^{\infty} \sum_{\substack{h=0 \\ (h,k)=1}}^{k} \frac{S_{h,k}}{k^r} e^{-\frac{2\pi imh}{k}} + O(m^{\frac{r}{4}}) \tag{5}$$

(for $m \to +\infty$), where $S_{h,k}$ are certain sums (the so called generalized Gauss sums), for which the inequality $|S_{h,k}| \leqslant ck^{\frac{r}{2}}$ holds.[5] Since $A(x) = \sum_{m \leqslant x} a_m$, we obtain from this easily (3) also for $r \geqslant 5$, as well as the estimate (4). It was not clear whether these estimates are definitive and there was no effective Ω-method available (except the method of Landau, which leads to the estimate (2)). In this situation Jarník directed his attention to the surprising fact, which escaped all, that for rational ellipsoids the estimate

$$P(x) = \Omega(x^{\frac{r}{2}-1})$$

can be proved in a quite elementary manner. In this way the first definitive result in this theory was reached.

Jarník's result provided a new impetus for a series of investigations (Landau, Müntz, Petterson, Walfisz, Jarník) which exploited various methods. Let us shortly mention Jarník's contribution in this direction (for the sake of simplicity, instead of rational ellipsoids we are looking for ellipsoids with integer coefficients, $r \geqslant 5$).

In the papers [20] and [21] it is shown that there exists a constant c such that each of the inequalities

$$P(n) > (M + c)n^{\frac{r}{2}-1}, \quad P(n) < (M - c)n^{\frac{r}{2}-1}, \quad M = \frac{\pi^{\frac{r}{2}}}{2\Gamma(\frac{1}{2})\sqrt{D}}$$

holds for an infinite number of natural n, even for all terms of a certain arithmetic sequence. So if we denote

$$\varrho_Q(n) = \frac{P(n)}{n^{\frac{r}{2}-1}},$$

we immediately obtain that

$$\liminf_{n \to +\infty} \varrho_Q(n) < M - c, \quad \limsup_{n \to +\infty} \varrho_Q(n) > M + c$$

holds.

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[5] Here and also in the sequel the letter c means (in general different) positive constants depending on Q only.

If $Q(u) = u_1^2 + u_2^2 + \ldots + u_r^2$ (the case of the r-dimensional ball) a more detailed investigation of the sequence $\varrho_Q(n) = \varrho_r(n)$ is possible. For example (see [16]), its $\liminf$ and $\limsup$ can be asymptotically expressed (with respect to r), it can be shown that the sequence possesses infinitely many accumulation points ([31]) or (for $r \geqslant 8$ even) to investigate its accumulation points, if n runs over a certain infinite set of natural numbers (see [24]). The importance of these and further results of Jarník is evident from the fact that in the well-known monograph "Gitterpunkte in mehrdimensionalen Kugeln" of Walfisz the whole third chapter is compiled from them.

The starting point for a majority of the just cited results was the relation (5). At this point let us mention the paper [32], even if it is in fact the "discrete" form of the mean value theorem which will be mentioned later. Jarník shows that for every constant E there exists a positive constant C_E such that

$$\sum_{n \leqslant x} (P(n) - En^{\frac{r}{2}-1})^2 = C_E x^{r-1} + g(x)$$

holds where $g(x) = O(x^{r-2})$ for $r > 8$, $O(x^{\frac{r}{2}-1} \lg x)$ for $r = 8$, $O(x^{\frac{3r}{4}} \lg x)$ for $r = 5, 6, 7$ and $g(x) = \Omega(x^{r-2})$ for $E = 0$, $r \geqslant 5$. This again gives the estimate (6) and moreover we can see that even in "average" the values of $P(n)$ cannot be approximated by the expression $En^{\frac{r}{2}-1}$ with a constant E.

It is understandable that when by means of the relations (3) and (6) the problem was solved for rational ellipsoids (even if for the case $r \geqslant 5$ only) all the interest was concentrated on irrational ellipsoids. The only existing effective method was starting from the relation (5). With a great effort Walfisz succeeded in using this method for the proof that forms

$$\alpha u_1^2 + Q_1(u_2, u_3, \ldots, u_r)$$

(α rational, Q_1 a rational form) for $r \geqslant 10$ satisfy the estimate

$$P(x) = o(x^{\frac{r}{2}-1}) \tag{7}$$

and that it cannot be improved. However, at the same time he showed also that for almost all $\alpha > 0$,[6] $r \geqslant 10$ even

$$P(x) = O(x^{\frac{r}{2}-\frac{6}{5}} \lg^{\frac{1}{4}} x) \tag{8}$$

holds.

[6] Almost all in the sense of the Lebesgue measure (similarly also for the forthcoming formulations).

There have been two consequences of Walfisz's work: irrational ellipsoids behave evidently in a completely different way than rational ellipsoids and secondly, further effective use of the method described above cannot give essentially new results. Let us quote the best specialist in the field—A. Walfisz, who on September 26, 1929 at the First Congress of Slavonian mathematicians said: "Although the estimates (7) and (8) brought a certain insight into the theory of lattice points in irrational ellipsoids, it was clear from the very beginning that the arguments leading to it were merely tools for orientation due to the lack of better ones. Therefore it was necesssary to abandon the singular series and to find something completely different. I had the idea that this will take some time. The more I was surprised—and surely not only myself—by the discoveries of Jarník. In a series of treatises, which have been published since the middle of the last year and which by their originality, deep ideas and techniques belong to the most remarkable work in modern research, Jarník was dealing with the problem using powerful and substantial tools, and he obtained a whole series of results of surprising exactness."[7]

The contribution of Prof. Jarník's work to the theory of lattice points in ellipsoids consists mainly in his elaborating new, highly effective O- and Ω-methods, which helped him to succeed in proving a whole series of definitive results—a rare phenomenon in number theory and especially in the theory of lattice points . It is possible even to say that all the definitive results in this field either belong to him or relied heavily on his work.

Let us now shortly treate the fundamental ideas of Jarník's methods of studying the O- and Ω-estimates of the function $P(x)$. Jarník's Ω-method is basically very simple. The function $A(x)$ is piecewise constant; the changes of the function $P(x)$ are therefore "mostly" given by the change of the function $V(x) = x^{\frac{r}{2}}$. In a more detailed form: assume that $A(x)$ is constant on the intervals $[\lambda_n, \lambda_n + h_n]$, where $0 < h_n < \lambda_n$, $\lim\limits_{n\to+\infty} \lambda_n = +\infty$. Hence

$$|P(\lambda_n + h_n) - P(\lambda_n)| = c((\lambda_n + h_n)^{\frac{r}{2}} - \lambda_n^{\frac{r}{2}}) \geqslant \frac{1}{2} c r h_n \lambda_n^{\frac{r}{2}-1}.$$

If we now have the information that for some $\beta \geqslant 0$ we have $h_n \lambda_n^{\beta} \geqslant c$, we obtain

$$|P(\lambda_n + h_n) - P(\lambda_n)| \geqslant c\lambda_n^{\frac{r}{2}-1-\beta}.$$

Therefore we get either

$$|P(\lambda_n)| \geqslant c\lambda_n^{\frac{r}{2}-1-\beta}$$

[7] See Časopis pro pěst. mat. 59 (1929), 200–223.

or

$$|P(\lambda_n + h_n)| \geqslant c\lambda_n^{\frac{r}{2}-1-\beta} \geqslant c(\lambda_n + h_n)^{\frac{r}{2}-1-\beta},$$

i.e.

$$P(x) = \Omega(x^{\frac{r}{2}-1-\beta}).$$

If e.g. Q is a form with integer coefficients, the choice $\lambda_n = n$, $h_n = \frac{1}{2}$, $\beta = 0$ leads to the estimate (6). How can this idea be applied to the case of irrational ellipsoids? Let us restrict ourselves in the sequel (as Jarník does) to forms of "almost diagonal" shape

$$\begin{aligned}(9)\qquad Q(u) = a_1Q_1(u_1, u_2, \dots, u_{r_1}) + a_2Q_2(u_{r_1+1}, \dots, u_{r_1+r_2}) + \dots \\ + a_\sigma Q_\sigma(u_{r_1+r_2+\dots+r_{\sigma-1}+1}, \dots, u_r)\end{aligned}$$

where $\sigma, r_1, r_2, \dots, r_\sigma$ are natural numbers, $r = r_1 + \dots + r_\sigma$, $Q_1, Q_2, \dots, Q_\sigma$ are positively definite quadratic forms with integer coefficients, $a_1, a_2, \dots, a_\sigma$ are positive reals. Now it can be shown that if the inequalities

$$\left|\frac{a_j}{a_1}q - p_j\right| \leqslant q^{-\gamma}, \quad j = 2, \dots, \sigma$$

possess infinitely many solutions in the integer numbers $p_2, p_3, \dots, p_\sigma, q$, it is possible to choose λ_n and h_n with the properties given above for $\beta = \frac{1}{\gamma}$. This immediately yields the estimate

$$P(x) = \Omega(x^{\frac{r}{2}-1-\frac{1}{\gamma}}).$$

This method was modified by Jarník several times. At the same time the results showed that the value (1) probably will be closely related to the simultaneous approximation of the numbers $\frac{a_2}{a_1}, \frac{a_3}{a_1}, \dots, \frac{a_\sigma}{a_1}$.

Jarník's O-method is much more complicated. The method mentioned above has power series as its starting point and cannot be used for irrational ellipsoids. Therefore Jarník introduces the Dirichlet series

$$(10)\qquad \Theta_Q(s) = \sum_{m_1,\dots,m_r} e^{-sQ(m_1,m_2,\dots,m_r)} = \sum_{m=0}^{\infty} a_m e^{-\lambda_m s},$$

by which a holomorphic function in the domain $\mathrm{Re}s > 0$ is defined. The numbers λ_m are the values of the form Q at the lattice points, the number a_m determines the number of lattice points on the surface $Q(u) = \lambda_m$. As is known, we can write

(for the sake of simplicity we neglect the fact that the equality holds only for $x \neq \lambda_m$, $m = 0, 1, 2, \ldots$)

$$A(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs}\Theta(s)}{s}\,\mathrm{d}s \tag{11}$$

for an arbitrary positive a, where we integrate over the line $\operatorname{Re} s = a$. We omit also the fact that this integral is not absolutely convergent and therefore direct estimations cannot be established.[8] From (11) we immediately get the expression

$$P(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs}F(s)}{s}\,\mathrm{d}s, \tag{12}$$

where $F(s) = \Theta(s) - \frac{\pi^{\frac{r}{2}}}{\sqrt{D}s^{\frac{r}{2}}}$. With respect to the term e^{xs} it is seemingly advantageous to set $a = \frac{1}{x}$. With help of transformation properties of the function (10) it is now easy to show that the integral over a "small" part of the integration path in the vicinity of the real axis is relatively small. The transformation properties of the function (10) enable us to find—for rational Q—an estimate of the form

$$|\Theta_Q(s)| \leqslant c\left(\frac{x}{k(1 + x|t - \frac{2\pi h}{k}|)}\right)^{\frac{r}{2}}, \tag{13}$$

which is valid provided the difference $|t - \frac{2\pi h}{k}|$ is sufficiently small, and where $h \neq 0$ and k have no common divisor, $0 < k \leqslant \sqrt{x}$. Forms of the shape (9) can be written as

$$\Theta_Q(s) = \Theta_{Q_1}(a_1 s)\Theta_{Q_2}(a_2 s)\ldots\Theta_{Q_\sigma}(a_\sigma s),$$

hence we easily get the estimate

$$|F(s)| \leqslant c\prod_{j=1}^{\sigma}\left(\frac{x}{k_j(1 + x|t - \frac{2\pi h_j}{a_j k_j}|)}\right)^{\frac{r_j}{2}} \tag{14}$$

provided $|t - \frac{2\pi h_j}{a_j k_j}|$ is sufficiently small, h_j, k_j being integers, $(h_j, k_j) = 1$, $h_j \neq 0$, $0 < k_j \leqslant \sqrt{x}$, $j = 1, 2, \ldots, \sigma$. From the estimate (14) one can see that the expression on its right hand side will be large if the differences $|t - \frac{2\pi h_j}{a_j k_j}|$ are very small. But in this case also the differences $|\frac{h_j}{a_j k_j} - \frac{h_1}{a_1 k_1}|$ will be small and therefore the numbers $\frac{a_2}{a_1}, \frac{a_3}{a_1}, \ldots, \frac{a_\sigma}{a_1}$ can be fairly well approximated simultaneously by rational

[8] This difficulty can be avoided by considering the integral $\int_0^x P(t)\,\mathrm{d}t$.

numbers. This is the first sketch of a similar connection as we had for the case of Ω-estimates.

Now this roughly indicated idea ought to be made more precise. In a very sophisticated manner Jarník divides the integration path into certain parts, on which the magnitude of the differences $|t - \frac{2\pi h_j}{a_j k_j}|$ is precisely described and therefore a fairly good estimate can be found by the relation (14). Now it is necessary to estimate the measure of such sets, to give their fine classification and to finish the whole indicated procedure to the final result (at this point Jarník uses and in most cases derives very fine statements belonging to the theory of diophantine approximations). We hope it can be seen from this short sketch that the difficulties which were overcome by Jarník are of a sort which would probably discourage many a mathematician, even the most gifted ones. It also brings to light the feature which characterizes many of his papers: ingenious combination of different fields of mathematics whose interrelations had not been recognized until then.

Let us proceed now to a short survey of the most important results of Jarník's work on lattice points in ellipsoids. Unfortunately we have to omit the finest of them because their formulation is considerably complicated, but this is in fact the substance of the problem.

In the papers [19] and [27] it is shown that for $r \geqslant 5$ and an irrational form Q the relation

$$P(x) = o(x^{\frac{r}{2}-1}) \tag{15}$$

holds.

The problem whether this estimate can be improved is solved by Jarník negatively in the paper [67] in a very original way using categorial methods, which (as will be clear from the overview of his work in the field of the theory of real functions) was excellently mastered by him: if $\varphi(x)$ is a positive and nonincreasing function, $\lim\limits_{x\to+\infty} \varphi(x) = 0$, $r \geqslant 5$ then for every system $a_1, a_2, \ldots, a_\sigma$ of positive numbers the form (9) satisfies, except for a set of first category, the estimate (15) and

$$P(x) = \Omega(x^{\frac{r}{2}-1}\varphi(x)).$$

In the paper [17] it was shown that for almost all systems $a_1, a_2, \ldots, a_\sigma$ the relation

$$P(x) = O(x^{\frac{r}{2}-\lambda+\varepsilon})$$

holds for every $\varepsilon > 0$, where $\lambda = \sum\limits_{j=1}^{\sigma} \min(1, \frac{1}{4} r_j)$.

Hence if $r_1 = r_2 = \ldots = r_\sigma = 1$ then $f \leqslant \frac{1}{4}r$ for almost all $a_1, a_2, \ldots, a_\sigma$, if $r_1 \geqslant 4, \ldots, r_\sigma \geqslant 4$ then $f \leqslant \frac{1}{2}r - \sigma$ for almost all $a_1, a_2, \ldots, a_\sigma$ and this estimate cannot be improved, because we always have (see [18])

$$P(x) = \Omega(x^{\frac{1}{2}r-\sigma}).$$

Therefore if $r_1 \geqslant 4, \ldots, r_\sigma \geqslant 4$ then

$$f = \frac{1}{2}r - \sigma$$

for almost all $a_1, a_2, \ldots, a_\sigma$ and if $r_1 = r_2 = \ldots = r_\sigma = 1$ then (cf. (2))

$$\frac{r-1}{4} \leqslant f \leqslant \frac{1}{4}r$$

for almost all $a_1, a_2, \ldots, a_\sigma$.

Thus result "rehabilitates" the Ω-estimate (2) of Landau which appeared to be too rough.

So if $r_1 \geqslant 4, \ldots, r_\sigma \geqslant 4$ then we have for the value of (1) the inequality

$$\frac{1}{2}r - \sigma \leqslant f \leqslant \frac{1}{2}r - 1 \tag{16}$$

and this inequality cannot be improved. Of course the problem arises whether for every value of f satisfying (16) there exists a form Q of the shape (9), for which $f_Q = f$. For the case $\sigma = 2$, $r_1, r_2 \geqslant 4$ Jarník shows in [22] that

$$f = \frac{1}{2}r - 1 - \frac{1}{\gamma}$$

holds where $\gamma = \gamma(a_1, a_2)$ denotes the supremum of all $\beta > 0$ for which the inequality

$$|q\frac{a_2}{a_1} - p| \leqslant q^{-\beta}$$

has infinitely many solutions in natural p, q.

For the case $\sigma > 2$ Jarník solved the problem in a "nonconstructive" way using the Hausdorff measure. Namely, in the paper [45] he showed that the Hausdorff dimension[9] of the set of those $\frac{a_2}{a_1}, \ldots, \frac{a_\sigma}{a_1}$ for which the corresponding form (9) satisfies the equality $f = f_Q$, is equal to

$$\Big(1 - \frac{2}{r-2f}\Big)\sigma$$

[9] See pp. 24–25 of Dodson contribution.

(and this holds also for $\sigma = 2$). Let us mention in this context that in extending the validity of the relation (17) also to the case $\sigma > 2$ ($\gamma = \gamma(a_1, a_2, \dots, a_\sigma)$ is defined similarly as above) B. Diviš—a student of Prof. Jarník—succeeded in 1968.[10]

In the paper [70] the case $\sigma = 2$ is studied in detail; e.g., those values of β are considered for which

$$\liminf_{x\to+\infty} \frac{|P(x)|}{x^\beta} < +\infty$$

holds, the changes of sign of the function $P(x)$ are taken into account, etc. This work belongs to the finest in this field. The paper [88] is devoted to an interesting problem which values f assumes if some of a_j's are fixed. Jarník shows that for $\sigma > 3$, a_1, a_2 fixed, $\gamma(a_1, a_2) = 1$ for almost all $a_3, a_4, \dots, a_\sigma$, and for every $\varepsilon > 0$ the estimate

$$P(x) = O(x^{\frac{r}{2}-\lambda+\varepsilon})$$

is valid (under the assumption $r_j \geqslant \frac{2(\gamma+1)}{\gamma}$ this result was generalized by B. Diviš).

The survey presented above—though a short one—shows that in many cases definitive results have not been achieved. Therefore Landau in 1923 (one year before Cramér) started the investigation of the so called averaged value $T(x) = \sqrt{\frac{M(x)}{x}}$, where $M(x) = \int_0^x (P(t))^2 \,\mathrm{d}t$, and demonstrated that for the case of a circle

$$M(x) = cx^{\frac{3}{2}} + O(x^{1+\varepsilon})$$

holds for every $\varepsilon > 0$. Therefore the "average order" of the function $P(x)$ is $\frac{1}{4}$ in this case. Observe that the investigation of the function $M(x)$ is relatively easier than the study of the function $P(x)$ (nonnegativeness, monotony); therefore more precise O and Ω results can be expected. On the other hand, sufficiently good results concerning the function $M(x)$ make it possible to achieve nice results on the function $P(x)$, which would not be reachable otherwise. This was the reason why the investigation of the function $M(x)$ developed into an independent and self-contained part of the lattice point theory.

Jarník paid attention to this problem practically from the beginning. In the paper [8] he generalized Landau's result presented above to the relation

$$\int_0^x P((\sqrt{t}+\alpha)^2)P((\sqrt{t}+\beta)^2)\,\mathrm{d}t = \Phi(\alpha-\beta)x^{\frac{3}{2}} + O(x^{1+\varepsilon})$$

for every $\varepsilon > 0$, where Φ is a certain positive function. From this he derived a series of interesting properties of the function $P(x)$ (analogues of almost-periodicity, etc.). The above mentioned paper [32] belongs in fact also to this group.

[10] See Czechoslov. Math. Journal 20(1970), 130–139, 149–159.

The main contribution of Jarník to the investigation of the function $M(x)$ consists in the fact that he succeeded in modifying his methods of studying the function $P(x)$ into a form suitable also for $M(x)$. The starting point for Jarník is the easily provable equality

$$M(x) = -\frac{1}{4\pi^2} \int_{(a)} \int_{(a)} \frac{e^{x(s+s')F(s)\overline{F(\overline{s}')}}}{ss'(s+s')} \, \mathrm{d}s \, \mathrm{d}s'$$

which can be easily obtained from (12). The integral mentioned above can be now estimated by a similar procedure as was indicated above. It is understandable that for the case of the double integral many specific obstacles and difficulties occur.

We introduce now at least the basic results. Jarník restricts himself—similarly as above—to the case of forms of the type (9). Using an idea of Landau it can be shown that we always have

$$M(x) \geqslant c x^{\frac{r}{2}+\frac{1}{2}},$$

and Jarník's method for rational forms gives

$$M(x) \geqslant c x^{r-1}.$$

For all forms of the type considered the following estimates can be established:

$$M(x) = O(x^{r-1})$$

for $r > 1$,

$$M(x) = O(x^2 \lg^2 x)$$

for $r = 1$ and

$$M(x) = O(x^{\frac{3}{2}} \lg^4 x)$$

for $r = 2$ (for all of them see [33]).

For irrational forms of the investigated type and for $r \geqslant 4$ we have

$$M(x) = o(x^{r-1})$$

and this estimate cannot be improved in general in the same sense as was mentioned above for the function $P(x)$ (see [38], [67]). For almost all forms of the type (9) and for $\sigma = r$ we obtain

$$M(x) = O(x^{\frac{r+1}{2}} \lg^{3r+3} x),$$

i.e. together with the lower estimate presented above we obtain for almost all of these forms

$$\limsup_{x\to+\infty} \frac{\lg T(x)}{\lg x} = \frac{r-1}{4}$$

and this almost completely rehabilitates the Ω-estimate (2) of Landau. Of great importance is the paper [71] in which for $r = 2, 3$ the above mentioned O-estimates are improved to

$$M(x) = O(x^{\frac{3}{2}}) \quad \text{for} \quad r = 2,$$
$$M(x) = O(x^2 \lg x) \quad \text{for} \quad r = 3.$$

This represents a definitive result for $r = 2$. For $r = 3$ the logarithmic factor remains, which cannot be removed as we will see in the sequel.

Perhaps the most interesting result in [69] is the relation

$$M(x) = Kx^2 \lg x + O(x^2 \sqrt{\lg x})$$

which holds for $r = 3$ and a rational Q, $K = c$. Hence for rational forms ve have an unexpected result

$$P(x) = \Omega((x \lg x)^{\frac{1}{2}}).$$

In [69] it is further shown that

$$M(x) = kx^{r-1} + g(x)$$

holds, where $g(x) = O(x^{r-2})$ for $r \geqslant 6$, $g(x) = O(x^3 \lg^2 x)$ for $r = 5$, $g(x) = O(x^{\frac{5}{2}} \lg x)$ for $r = 4$ and $g(x) = \Omega(x^{r-2})$ for $r \geqslant 4$, where K is a constant and Q a rational form.

Finally, we mention just briefly the fine and complicated papers [40] and [68]. In these papers a detailed and deep investigation of the case $\sigma = 2, 3$ is presented. For instance, for $\sigma = 2$ e.g. (using the development of the number $\frac{a_2}{a_1}$ into a continued fraction) a relatively simple function $G(x)$ is found such that $\frac{M(x)}{G(x)}$ for $x > c$ lies between two positive constants.

We conclude the survey of Prof. Jarník's papers on the theory of lattice points by mentioning the paper [89] from 1968 which is by its character somewhat out of the groups decribed above.

We have indicated above that it is of advantage not to investigate the function $P(x)$ itself but rather its integral $P_1(x) = \int_0^x P(t)\,dt$. Landau's method with help of which he proved the results (2) is based on the following idea. We introduce in an obvious way functions $P_0(t) = P(t), P_1(t), \ldots$. It is relatively easy to see that for an integer $\varrho > \frac{1}{2}r$ we get for every form the definitive result

$$P_\varrho(x) = O(x^{\frac{r-1}{4}+\frac{\varrho}{2}}), \quad P_\varrho(x) = \Omega(x^{\frac{r-1}{4}+\frac{\varrho}{2}}).$$

From this (essentially by creating differences) the estimates (2) can be proved.

Jarník in the paper [89] set

$$P_\varrho(x) = \frac{1}{\Gamma(\varrho)} \int_0^x P(t)(x-t)^{\varrho-1}\,\mathrm{d}t$$

for $\varrho \geqslant 0$ and investigated the new problem of the dependence of the O- and Ω-estimates of the function $P_\varrho(x)$ on the parameter ϱ. (Let us mention that for an integer ϱ both processes give the same functions.) From the results of this paper we present only the definitive ones. Jarník reached them by combining his own and Landau's methods. Assume that the form Q is rational. Then

$$P_\varrho(x) = O(x^{\frac{r}{2}-1}), \quad P_\varrho(x) = \Omega(x^{\frac{r}{2}-1})$$

for $0 \leqslant \varrho < \frac{1}{2}r - 2$,

$$P_\varrho(x) = O(x^{\frac{r-1}{4}+\frac{\varrho}{2}}), \quad P_\varrho(x) = \Omega(x^{\frac{r-1}{4}+\frac{\varrho}{2}})$$

for $\varrho > \frac{1}{2}r - \frac{1}{2}$. For $\frac{1}{2}r - 2 \leqslant \varrho \leqslant \frac{1}{2}r - \frac{1}{2}$ or $2\varrho + 1 \leqslant r \leqslant 2\varrho + 4$ there is a certain gap in the results achieved by Jarník, which for $\varrho = 0$ corresponds to the classical unsolved problems for $r = 2, 3, 4$. For the functions $P_\varrho(x)$ we obtain a certain "shift" of the classical problems to another framework and to achieve definitive results would be no less difficult in it. Let us mention the interesting circumstance that the above Ω-estimates hold for all ϱ; for $\varrho = \frac{1}{2}(r-3)$ actually a transition from the former to the latter occurs.

What was presented above surely shows that the work of Prof. Jarník ranks him together with E. Landau and A. Walfisz to the famous researchers which have developed the theory of lattice points in multidimensional ellipsoids to a concise and closed form.