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THE WORK OF PROFESSOR JARNÍK IN REAL ANALYSIS

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Ladies and gentlemen,

first, let me thank Professor Novák for entrusting me with the pleasant task of discussing Professor Jarník's contribution to Real Analysis at this meeting. I would also like to thank Professor Zajíček for helping me with preparation for this talk, in particular for explanation of his recent results continuing some of Professor Jarník's work. Indeed, much of the work of Prague school of Real Analysis, including my own, has its roots, directly or indirectly, in Jarník's research, and all members of this school have been deeply influenced by the approach to analysis stemming from Jarník's teaching and from his textbooks.

Let me also recall that, as a beginning student, I had an opportunity to attend Professor Jarník's lecture on simultaneous diofantine approximations. For a few months, we could follow an excellent lecture, an interesting subject, and a detailed and very technical analysis of problems, which I found highly interesting both in itself and in the results it led to. I thoroughly enjoyed the precise style, in which every little piece had to fit exactly in its place, and which in analysis leads to preferring $\varepsilon/2$ to ε, a trick much hated by some, but much admired but others (including myself). This lecture series turned out to be the last given by Professor Jarník; his health did not allow him to finish it, even though, when it became clear that he would not be able to continue it at the university, he tried to do it at his home.

Let me come back to Professor Jarník's papers in Real Analysis, or as some prefer to say, in Theory of Real Functions. Of course, this subject is not very precisely defined, as can be clearly seen by a casual perusal of a few recent articles in the current leading journal in this field, *Real Analysis Exchange*. However, even the purest real analysts would agree that more than twenty of Professor Jarník's papers belong to their field; others might count another ten as belonging there as well.

It seems that Professor Jarník's interest in the theory of functions of real variable started with his study of the so-called Bolzano's function (2). This function was constructed by Bernard Bolzano about 1830 to show that there are functions disagreeing with the naive concept of continuity: He argues that his function is continuous in an interval $[a, b]$ (interestingly enough, although Bolzano understood the need for uniform convergence to preserve continuity, he argues incorrectly using pointwise convergence only; but that is another story) and yet has not a finite derivative at any point of a certain set dense in $[a, b]$. Jarník in (2) proves not only that Bolzano's function is actually an example of a continuous nowhere differentiable (finitely or infinitely) function, but he also studies in great detail its Dini derivatives. Since the Dini derivatives (and their generalizations) will be important later, let us recall that the upper right Dini derivative (or derived number) of a function f at a point x is defined by

$$D^+ f(x) = \limsup_{y \to x+} \frac{f(y) - f(x)}{y - x};$$

the other three Dini derivatives (denoted by D^-, D_+, and D_-) are defined in an obvious way.

It was probably his study of Dini derivatives of one particular function that led Professor Jarník in (4) to a very interesting result:

If $f\colon [a, b] \to \mathbb{R}$ has unbounded variation in each interval then at each point of a set dense in $[a, b]$ at least one of the four Dini derivatives equals $+\infty$.

As Professor Jarník remarks, this result is applicable to every nowhere differentiable function. The reason why I find this statement remarkable is that it seems to be very close to the way in which K. M. Garg started in 1970 the new wave of interest in typical functions! (I will return to this later.)

For the next ten years Professor Jarník worked on different questions (some of them also in Real Analysis; see later), but he returned to nowhere differentiable functions after he learned about the famous result proved independently by Banach and Mazurkiewicz in 1931:

Typical continuous function $f\colon [a, b] \to \mathbb{R}$ is nowhere differentiable.

(Following the "modern" terminology, I use the phrase "typical continuous function $f\colon [a, b] \to \mathbb{R}$ has property P" to replace the statement "the set of all continuous functions $f\colon [a, b] \to \mathbb{R}$ not having the property P is a first category subset of the space of all continuous functions on $[a, b]$".)

Professor Jarník immediately recognized the importance of this theorem. (For Czech readers he gave an account of its proof as a part of (42).) Indeed, Professor Jarník's investigation of typical continuous functions belongs to his most referred-to works from real analysis. (I do not dare to say 'the most referred-to', not so much because I have not attempted to count the references, but mainly because of the recent popularity of Jarník points, to which we will come in a short while.) Let us look first at the results about Dini derivatives of typical functions known after the publication of (39) in 1933:

For a typical continuous function f,

(a) $[D_-f(t), D^-f(t)] \cup [D_+f(t), D^+f(t)] = [-\infty, +\infty]$ for every t (Jarník (39)),

(b) $\max(|D_+f(t)|, |D^+f(t)|) = \max(|D_-f(t)|, |D^-f(t)|) = \infty$ for every t (Banach (1931), Mazurkiewicz (1931)),

(c) there are uncountable sets M_-, M^-, M_+, and M^+ such that

$$\begin{aligned} D_-f(t) &= +\infty \text{ for every } t \in M_-, \\ D^-f(t) &= -\infty \text{ for every } t \in M^-, \\ D_+f(t) &= +\infty \text{ for every } t \in M_+, \text{ and} \\ D^+f(t) &= -\infty \text{ for every } t \in M^+ \end{aligned}$$ (Saks (1952), and

(d) $D_-f(t) = D_+f(t) = -\infty$ and $D^-f(t) = D^+f(t) = +\infty$ for almost every t (Jarník (39)).

(One should recall that A. S. Besicovitch (1925) constructed a continuous function without unilateral derivatives (finite or infinite) at any point and that the Banach-Mazurkiewicz result seemed to suggest that typical ("most") functions have the worst possible behaviour. For this reason the "Saks' rarity theorem" (c) has always been considered as very surprising.)

In a different direction the result of (a) was generalized by Marcinkiewicz (1935), whose work has been continued more recently by Schultz (1972). But it was only in 1970 when the next significant contribution to the study of Dini derivatives of typical continuous functions appeared. In that year K. M. Garg improved (d) by showing that for a typical continuous function the set of non-knot points is also of the first category. (Further results in this direction were proved by G. Petruska (1985) and by L. Zajíček (1989); for related results see also Wos (1986), Kozyrev (1992) and Zajíček (1987).) An interesting feature of Garg's work is that his proofs of the statements about typical functions are based upon general theorems about the behaviour of nowhere monotone functions (compare with Jarník's paper (4)!) which are then combined with Jarník's results. As an example I may give a simple argument (belonging essentially to Garg) showing that (c) follows from (d). First, let us recall a (rather easy) monotonicity theorem: If $f\colon (a,b) \to \mathbb{R}$ is continuous, $D^+f \geqslant 0$ almost everywhere and $D^+f > -\infty$ for all but countably many points, then f is non-decreasing. From this, since (d) implies that a typical f fulfils $D^+f \geqslant 0$ a.e. but is not non-decreasing, we infer that $D^+f(t) = -\infty$ for t

belonging to an uncountable set. (It should be noted that the original proof of the "Saks' rarity theorem" is considered to be considerably more complicated than Jarník's proof of (d).) Garg in 1970 also gives a partial answer to the natural problem whether "Jarník's relations" (i.e. (a) and (b)) are all relations among Dini derivatives of typical functions holding at every point.

I would also like to point out that (39) (supplemented by (55)) also contains results about the "typical" behaviour of "generalized Dini derivatives" defined by

$$D^+_\gamma f(t) = \limsup_{h \to 0_+} \frac{f(t+h) - f(t)}{\gamma(h)}.$$

The statements corresponding to (a), (b) and (d) again hold, but there is a small surprise connected with (c): If $\lim_{h \searrow 0} \gamma(h)/h = \infty$, then for a typical continuous function $D^+_\gamma f(t) \geqslant 0 \geqslant D_{\gamma +} f(t)$ for every t. This implies, for example, that, even though there are points at which, say

$$\lim_{h \searrow 0} \left(f(t+h) - f(t) \right) / h = +\infty,$$

there are no points at which

$$\lim_{h \searrow 0} \left(f(t+h) - f(t) \right) / \sqrt{h} = +\infty.$$

The next huge group of Jarník's results about typical functions is concerned with approximate derivatives, approximate Dini derivatives, etc. To explain them, let us recall that the upper right density of a measurable set $E \subset \mathbb{R}$ at a point x is defined by

$$d^+(E, x) = \limsup_{h \searrow 0} \frac{\operatorname{meas}(E \cap (x, x+h))}{h}.$$

(Other kinds of densities are defined and denoted similarly.) The most important "approximate notion" is that of approximate derivative: A function f is said to have an approximate derivative at a point x if there is a measurable set E with $d_-(E, x) = d_+(E, x) = 1$ such that

$$f_E'(x) = \lim_{y \to x, y \in E} \frac{f(y) - f(x)}{y - x}$$

exists. (This limit is, of course, called the approximate derivative of f at x.) Motivated by this, let us call a number c (possibly $+\infty$ or $-\infty$) a right approximate derived number of a function f with upper density α if there is a set E such that $d^+(E, x) = \alpha$ and $f_E'(x) = c$.

The usual knowledge about Jarník's "approximate results" for typical functions seems to be restricted to the statement that typical continuous functions do not have approximative derivative at any point. But Jarník (44, 46, 48) proves much more! Indeed: Typical continuous function has the following properties:

(I) For every t,

  (1) $+\infty$ or $-\infty$ is an approximate right derived number of f with upper density one,

  (2) $+\infty$ or $-\infty$ is an approximate left derived number of f with upper density one,

  (3) both $+\infty$ or $-\infty$ are approximate derived numbers of f with upper symmetric density $\frac{1}{2}$ each, and

  (4) at least two of the numbers $-\infty$, 0, $+\infty$ are approximate right derived numbers of f at t, with upper density $\frac{1}{4}$, or at least two of these numbers are approximate left derived numbers of f at t, with upper density $\frac{1}{4}$.

(II) For almost every t each number is a bilateral derived number with upper density one.

As an example of corollaries let us just note that a typical continuous function cannot have both unilateral approximate derivatives at any point (though it has one unilateral ordinary derivative at some points!).

This research has been very recently continued by J. Malý and L. Zajíček, who found both valid (Zajíček (1993)) and invalid (Malý and L. Zajíček (1991)) analogues of (a) for approximate derivatives. Moreover, Zajíček (1996) proved that for certain further weakening of the notion of the approximate derivative (called weak preponderant derivative) a result as surprising as Saks' rarity theorem holds: Typical function has this (bilateral) derivative at some points.

I cannot forget at least to mention also Jarník's papers about typical functions in other function spaces. In (47) he studies typical continuous functions from $[a, b]$ to $\mathbb{R}^k$ (a similar study has been done independently by W. Hurewicz (1933)) and in (52) he proves an unexpected statement about Dini derivatives of typical functions of Baire class α. The questions similar to those asked in (52) have been considered again only very recently (see, e.g., Ceder and Pearson (1983) and Mustafa (1983)). In his last paper about typical functions Jarník answers a problem of Mikusinski by showing that for typical pairs of continuous functions f and g the convolution $f * g$ is nowhere differentiable.

However, the interest of Professor Jarník in real analysis was far from being confined to the behaviour of typical functions. Even in the papers mentioned above one can find many "non-typical" statements and/or proofs. For example, we should not forget that (46) contains also a construction of a continuous functions for which

$$\operatorname*{aplim}_{y\to x}\left|\frac{f(y)-f(x)}{y-x}\right| = \infty$$

almost everywhere. Points at which this equation holds are called Jarník points of f; they seem to be hard to come by, since from (II) we know that typical functions cannot have many of them (but they do have some, see Malý and Zajíček (1991)). Something so rare, and so difficult to imagine, may seem to be just one of those strange mathematical curiosities that do not have any "real existence"; but no, it turned out much later that it is exactly this type of behaviour that is typical for trajectories of one-dimensional Brownian motion: Every point of such a trajectory is its Jarník point! This phenomenon has deserved much attention, see, for example, Geman and Horowitz (1980), Bertoin (1988), or Anderson, Horowitz and Pitt (1991). (For a deterministic improvement of Jarník's example see Malý and Zajíček (1991).)

At this moment, I would like to come to three papers concerned not with non-differentiability but rather with differentiability. The first of them, (5), published in 1923, proves considerably more than that if a function f is differentiable (possibly to $+\infty$ or $-\infty$) everywhere on an interval, then f' is a function of the first class (i.e., it is a pointwise limit of a sequence of continuous functions). (One should note that the result is obvious only if f' is finite.) If you ask a real analyst when this statement was proved, he would probably say that sometime in the years 1941–1950 when the fundamental Zahorski's papers appeared. (As far as I know, the first other reference is Gleyzal in 1941.) Since the extension to approximate derivatives was proved by myself in 1971, to symmetric derivatives by L. Larson in 1984, and since this kind of questions became important only after Zahorski's 1950 paper, I can only admire Jarník's intuition.

The fate of the second paper from this group (6) which appeared in the same year is similar. Here Professor Jarník asks the question whether there is an extension of a function defined on a closed set to the whole line such that at the points of the original set the Dini derivatives of the function and of the extension coincide. There are several ways how one should consider this coincidence at the end points of the contiguous intervals. Jarník considers several problems of this sort and gives a complete solution. (A small question left open was settled in one of my very first papers.) However, even for the simplest case of Jarník's results, namely for the statement that a differentiable function on a perfect set has an extension to a differentiable function on the line, the usual (and only) reference is to Petruska and Laczkovich (1972)! On the other hand, Jarník's result has been partly extended by V. Aversa, M. Laczkovich, and myself to higher dimensions in 1986.

The third and last paper from this group is (41). Compared to the previous two, this was a lucky paper since it influenced the most important development in modern real analysis, namely the work of Z. Zahorski. Let us consider an everywhere differentiable function f. From (6) and from a result of S. Banach (1921) we see that the set $\{x\colon\ f'(x) = +\infty\}$ is of type G_δ and has measure zero. Conversely, assume that E is a G_δ set of measure zero. Is there an everywhere differentiable function f such that $E = \{x\colon\ f'(x) = +\infty\}$? This was the question asked by Professor Jarník (41) in 1933. He did not succeed in giving a complete answer, but he constructed a function f for which $E = \{x\colon\ f'(x) = +\infty\}$ and such that all the Dini derivatives at all points outside E are finite. Jarník's proof is very close to the later Zahorski's (1941) and Choquet's (1946) complete affirmative solution; the main difference seems to be that Jarník proved only a weaker version of the so-called Luzin-Menchoff lemma. (Interestingly enough, the first proof of this lemma appeared already in 1924 in a paper by V. S. Bogomolova, who attributes its statement to Luzin and Menchoff. Its use in a way similar to that used later by Zahorski and Choquet appears in Ward (1933).) It was also the further development of Zahorski's solution of Jarník's problem which led in Zahorski's (1950) paper to results which are influencing the field of real analysis till now.

The next two papers I want to speak about also belong among the well known achievements of Professor Jarník. Let us consider a differentiable function $\varphi\colon\ \mathbb{R} \to \mathbb{R}$, and let us denote $f = \varphi'$ and

$$F(x,y) = \frac{\varphi(x) - \varphi(y)}{x - y}.$$

Then an easy exercise shows that

$$(*) \qquad f(t) = \lim_{\substack{(x,y)\to(t,t) \\ x\neq y,\,(x-t)(y-t)\leqslant 0}} F(x,y)$$

for each $t \in \mathbb{R}$. We know already from (5) that f is a function of the first class. Hence for this special choice of f the formula $(*)$ gives a function of the first class. The problem Jarník asked and answered in (10) was to which extent the special conditions upon F are necessary. The answer might seem to be surprising: A function $f\colon\ \mathbb{R} \to \mathbb{R}$ is of the first class if and only if there is a function $F\colon\ \mathbb{R}^2 \to \mathbb{R}$ such that $(*)$ holds.

This is obviously a result about the boundary behaviour of functions of two variables. However, it can be used to deduce information about functions of one variable (namely that derivatives are of the first class). This type of argument is nowadays usually called the Jarník-Blumberg method.

I will come to the Blumberg's and further Jarník's contributions in a short while. Before that, let me just mention that Jarník proves in (10) much more than only the above result: He gives a similar statement for real-valued functions on perfect subsets of $\mathbb{R}^k$ and considers in $(*)$ also limits with other restrictions on (x, y). There is also a well known generalization of Jarník's result given in a series of papers by Snyder, who extended Jarník's theorem to more general spaces, showed that, if f is of the first class, F can be chosen to be continuous on $\mathbb{R}^2 \setminus \{(x,x)\colon\ x \in \mathbb{R}\}$, and, probably most importantly, extended Jarník's result to the approximate case.

The next significant contribution to the Jarník-Blumberg method came from H. Blumberg and from his student M. Schmeisser. Using the boundary behaviour of functions of two variables, they proved that for an arbitrary function $f\colon\ \mathbb{R} \to \mathbb{R}$ all $t \in \mathbb{R}$ but countably many have the property that $D^+f(t) \geqslant D_-f(t)$. Jarník in (57) proved a stronger theorem about boundary behaviour of functions of two variables: If $F\colon\ \mathbb{R}^2 \to \mathbb{R}$ and S is a line in $\mathbb{R}^2$ then all but countably many $t \in S$ have the property that for any two half-lines starting from t to the same component of $\mathbb{R}^2 \setminus S$ the cluster sets of F at t along these half-lines have non-empty intersection.

Jarník also gives the corresponding improvement of the results of Blumberg and Schmeisser for functions of one variable and he notes that the statement does not hold if two half-lines are replaced by three half-lines. Improvements of this example showing that in this case all $t \in S$ might be "exceptional" were given by Bagemihl, Piranian, and Young (1959) and by Erdös and Piranian (1960). In this connection one should notice that such counterexamples to the "three half-lines" version of the problem are obviously impossible if the real-valued function F is continuous on $\mathbb{R}^2 \setminus S$ but that the situation is far from being clear, say, for bounded $\mathbb{R}^2$-valued continuous maps on $\mathbb{R}^2 \setminus S$. In fact, the question if in this case the exceptional set may have positive measure is one of the forms of the famous still open "three segment problem".

Among the continuations of Jarník's work (57) let me mention a generalization from straight lines to curves due to Bagemihl (1955) and Layek's (1988) discussion of direct essential analogues of Jarník's results.

There remains quite a number of Jarník's papers in real analysis which I cannot consider here in much detail. However, I simply have to say at least a few words on the existence of a dense semigroup of continuous functions generated by finitely many functions.

In 1934 Schreier and Ulam proved that there are five continuous functions $g_1, \dots, g_5\colon\ [0,1] \to [0,1]$ such that the set of all functions obtained by their compositions is dense in $C([0,1],[0,1])$. In the same year W. Sierpiński improved the result by showing that: For every sequence $f_1, f_2, \ldots \in C([0,1],[0,1])$ there are four functions $g_1, \ldots, g_4$ (from the same set) such that each of the functions f_i can be obtained from them by compositions.

In (50) Jarník and Knichal reduced four to two (which is surely the smallest possible) and in (51) they obtained very interesting results for monotonic functions from $[0,1]$ to $[0,1]$, where the optimal number is three! Their method was further used by W. Sierpiński (1935) to solve a more general question. For a recent analogue for non-compact intervals see S. Subbiah (1983).

Clearly, I did not even mention several papers of Professor Jarník in real analysis that might be considered as more interesting and/or important. Except for Jarník's contribution to the questions of Hausdorff dimension explained in the excellent talk by Professor Dodson one should mention Jarník's contribution to the general theory of Hausdorff measures (28), his papers about Riemann integral (15), (25), and his excellent detailed study of rearrangements of non-absolutely convergent series. I find very interesting also the papers (61) (which belongs to general topology) and (85), also because they are results of the interaction of two of the greatest men in our mathematics, Professor Vojtěch Jarník and Professor Eduard Čech.

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