Linear Differential Transformations of the Second Order

28 An abstract algebraic model for the transformation theory of Jacobian oscillatory differential equations

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# 28 An abstract algebraic model for the transformation theory of Jacobian oscillatory differential equations

## 28.1 Structure of the group of second order regular matrices over the real number field

Let $\mathfrak{M}$ be the group of regular second-order matrices (i.e. $2 \times 2$ square matrices) over the field $R$ of real numbers. We first observe that the group $\mathfrak{M}$ contains the subgroup $\mathfrak{M}_0$ consisting of all elements of $\mathfrak{M}$ with positive determinant. This is invariant in $\mathfrak{M}$ and has index 2; the factor group $\mathfrak{M}/\mathfrak{M}_0$ consists therefore of two classes, $\mathfrak{M}_0$ and $M_1$. Moreover, the group $\mathfrak{M}$ contains the subgroup $\mathfrak{U}$ consisting of all unimodular elements.

The centre $\mathfrak{C}$ of $\mathfrak{M}$ consists of all matrices $\mu E = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$; here $\mu \in R$ ($\mu \neq 0$) and $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit element of $\mathfrak{M}$; $\mathfrak{C}$ is invariant in $\mathfrak{M}$. Moreover, $\mathfrak{C}$ contains the subgroup $\mathfrak{C}_0$ consisting of all matrices $\lambda E$ with $\lambda > 0$; this subgroup $\mathfrak{C}_0$ is invariant in $\mathfrak{M}$ and has index 2 in $\mathfrak{C}$; the factor group $\mathfrak{C}/\mathfrak{C}_0$ consists of two classes, $\mathfrak{C}_0$, $C_1$.

Any class $\bar{M} \in \mathfrak{M}/\mathfrak{C}$ comprises all matrices of the form $\mu M$ where $\mu \in R$ ($\mu \neq 0$) and $M$ is an arbitrary element of $\bar{M}$. The determinants of all matrices of $\bar{M}$ have the same sign, namely sgn det $M$, and $\bar{M} \subset \mathfrak{M}_0$ or $\bar{M} \subset M_1$ according as $M \in \mathfrak{M}_0$ or $M \in M_1$ respectively.

The class $\bar{M}$ separates into two disjoint sets $\bar{M}_0, \bar{M}_1 \in \mathfrak{M}/\mathfrak{C}_0$; i.e. $\bar{M} = \bar{M}_0 \cup \bar{M}_1$, $\bar{M}_0 \cap \bar{M}_1 = \emptyset$. If $\bar{M}_0$ contains the matrix $M$ then $\bar{M}_1$ contains the matrix $-M$. The set $\bar{M}_\alpha$ ($\alpha = 0, 1$) consists of all matrices of the form $\lambda (-1)^\alpha M$, with $\lambda > 0$. Both sets $\bar{M}_0$, $\bar{M}_1$ are simultaneously contained either in $\mathfrak{M}_0$ or in $M_1$. In each of these sets $M_\alpha$ there is precisely one unimodular matrix $U_\alpha$ ($\in \mathfrak{U}$), namely

$$U_\alpha = \frac{(-1)^\alpha}{\sqrt{(\text{abs det } M)}} M,$$

and both matrices $U_0$, $U_1$ obviously have the same determinant, sgn det $M$.

## 28.2 An abstract phase group

Let $\mathfrak{G}$ be an abstract group with the properties 1, 2(a), (b), (c) set out below:

(1) $\mathfrak{G}$ *contains a subgroup* $\mathfrak{G}_0$, *invariant in it, of index* 2.

The factor group $\mathfrak{G}/\mathfrak{G}_0$ thus consists of two classes, $\mathfrak{G}_0$ and $G_1$. For $a \in \mathfrak{G}$ we write sgn $a' = 1$ or sgn $a' = -1$ according as $a \in \mathfrak{G}_0$ or $a \in G_1$. Thus we have, for $a, b \in \mathfrak{G}$,

$$\text{sgn } (ab)' = \text{sgn } a' \cdot \text{sgn } b' \tag{28.1}$$

(2) $\mathfrak{G}$ *contains a subgroup* $\mathfrak{E}$ *with the following properties* (a), (b), (c):

(a) *the centre* $\mathfrak{Z}$ *of* $\mathfrak{E} \cap \mathfrak{G}_0$ *is an infinite cyclic group with generator* $c$, (*i.e.* $\mathfrak{Z} = \{c_\nu\}$, $c_\nu = c^\nu$, $\nu = 0, \pm 1, \pm 2, \ldots$, $c_1 = c$) *and for every two elements* $e \in \mathfrak{E}$, $c_\nu \in \mathfrak{Z}$ *there holds the relation*

$$ec_\nu = c_{\nu \operatorname{sgn} e'} e. \tag{28.2}$$

We note that every element $c_\nu \in \mathfrak{Z}$ commutes with every element $e \in \mathfrak{E} \cap \mathfrak{G}_0$, i.e. $ec_\nu = c_\nu e$, and "anticommutes" with every element $e \in \mathfrak{E} \cap G_1$, that is to say $ec_\nu = c_{-\nu} e$.

It follows from (2) that $\mathfrak{Z}$ is invariant in $\mathfrak{E}$. The group $\mathfrak{Z}$ contains the subgroup $\mathfrak{Z}_0$, consisting of all the elements $c_{2\nu}$, i.e. $\mathfrak{Z}_0 \subset \mathfrak{Z}$; $\mathfrak{Z}_0 = \{c_{2\nu}\}$. $\mathfrak{Z}_0$ is invariant in $\mathfrak{E}$ and has index 2 in $\mathfrak{Z}$; the factor group $\mathfrak{Z}/\mathfrak{Z}_0$ consists of the two classes $\mathfrak{Z}_0$, $Z_1$.

Every class $\bar{e} \in \mathfrak{E}/\mathfrak{Z}$ consists of all those elements $c_\nu e$, where $\nu = 0, \pm 1, \pm 2, \ldots$ and $e$ is an arbitrary element of $\bar{e}$: $\bar{e} = \{c_\nu e\}$. Since $\operatorname{sgn} c_\nu' = 1$, it follows that $\operatorname{sgn}(c_\nu e)' = \operatorname{sgn} e'$, and consequently $\bar{e} \subset \mathfrak{G}_0$ or $\bar{e} \subset G_1$ according as $e \in \mathfrak{G}_0$ or $e \in G_1$. The class $\bar{e}$ separates into two disjoint sets $\bar{e}_0$, $\bar{e}_1 \in \mathfrak{E}/\mathfrak{Z}_0$, (i.e. $\bar{e} = \bar{e}_0 \cup \bar{e}_1$, $\bar{e}_0 \cap \bar{e}_1 = \varnothing$). If $\bar{e}_0$ contains the element $e$ ($\in \mathfrak{E}$) then $\bar{e}_1$ contains the element $c_1 e$ ($\in \mathfrak{E}$). The set $\bar{e}_\alpha$ ($\alpha = 0, 1$) coincides with the totality of elements $c_{2\nu+\alpha} e$: $\bar{e}_\alpha = \{c_{2\nu+\alpha} e\}$, and both sets $\bar{e}_0$, $\bar{e}_1$ are simultaneously contained either in $\mathfrak{G}_0$ or in $G_1$.

In what follows we shall make use of the following notation:

For $e \in \mathfrak{E}$ we denote by $\bar{e}$ that class of $\mathfrak{E}/\mathfrak{Z}_0$ which contains the element $e$; that is, $e \in \bar{e} = \mathfrak{Z}_0 e = \{c_{2\nu}\} e = \{c_{2\nu} e\} = \mathfrak{E}/\mathfrak{Z}_0$. Naturally, we also have $\mathfrak{Z}_0 e = e \mathfrak{Z}_0 = e\{c_{2\nu}\} = \{ec_{2\nu}\} \in \mathfrak{E}/\mathfrak{Z}_0$. Similarly, for $M \in \mathfrak{M}$ we denote by $\bar{M}$ that class of $\mathfrak{M}/\mathfrak{C}_0$ which contains the matrix $M$: $M \in \bar{M} = \mathfrak{C}_0 M = \{\lambda E\} M = \{\lambda M\} \in \mathfrak{M}/\mathfrak{C}_0$ ($\lambda > 0$), and at the same time $\mathfrak{C}_0 M = M \mathfrak{C}_0 = M\{\lambda E\} = \{M\lambda\} \in \mathfrak{M}/\mathfrak{C}_0$.

(b) *There exists an isomorphic mapping* $\mathscr{T}$ of $\mathfrak{E}/\mathfrak{Z}_0$ on $\mathfrak{M}/\mathfrak{C}_0$, *with the following properties* (i), (ii):

(i) *For* $e \in \bar{e} \in \mathfrak{E}/\mathfrak{Z}_0$, $M \in \bar{M} = \mathscr{T}\bar{e} \in \mathfrak{M}/\mathfrak{C}_0$, *we have* $\operatorname{sgn} e' = \operatorname{sgn} \det M$.

(ii) $\mathscr{T}\mathfrak{Z}_0 = \mathfrak{C}_0$, $\mathscr{T} Z_1 = C_1$.

The isomorphism $\mathscr{T}$ induces a homomorphic mapping $\mathscr{H}$ of $\mathfrak{E}$ on $\mathfrak{U}$, defined as follows: for $e \in \bar{e} \in \mathfrak{E}/\mathfrak{Z}_0$, we have $\mathscr{H}e = \mathfrak{U} \cap \mathscr{T}\bar{e}$; thus $\mathscr{H}e$ is the unimodular matrix contained in the class $\mathscr{T}\bar{e} \in \mathfrak{M}/\mathfrak{C}_0$. Obviously, $\mathscr{H}$ is a mapping *onto* $\mathfrak{U}$; we see that it is homomorphic by the following argument: from $e_1, e_2 \in \mathfrak{E}$ it follows that

$$\begin{aligned}\mathscr{H}e_1 \mathscr{H}e_2 = (\mathfrak{U} \cap \mathscr{T}\bar{e}_1)(\mathfrak{U} \cap \mathscr{T}\bar{e}_2) = \mathfrak{U} \cap \mathscr{T}\bar{e}_1 \cdot \mathscr{T}\bar{e}_2 = \\ = \mathfrak{U} \cap \mathscr{T}(\bar{e}_1\bar{e}_2) = \mathscr{H}(e_1 e_2),\end{aligned}$$

and on taking account of (ii) we obtain

$$\mathscr{H}c_{2\nu+\alpha} = (-1)^\alpha E \qquad (\nu = 0, \pm 1, \pm 2, \ldots, \alpha = 0, 1). \tag{28.3}$$

We also have, for $e \in \mathfrak{E}$,

$$\mathscr{T}\bar{e} = \{\lambda \mathscr{H} e\}, \qquad (\lambda > 0). \tag{28.4}$$

Finally, we observe that the union of the $\mathscr{T}$-maps of every two classes $\bar{e}_0,\ \bar{e}_1 \in \mathfrak{E}/\mathfrak{Z}_0$, whose union forms a class $\bar{e} \in \mathfrak{E}/\mathfrak{Z}$, (i.e. $\bar{e}_0 \cup \bar{e}_1 = \bar{e} \in \mathfrak{E}/\mathfrak{Z}$), represents a class $\bar{M} \in \mathfrak{M}/\mathfrak{C}$:

$$\mathscr{T}\bar{e}_0 \cup \mathscr{T}\bar{e}_1 = \bar{M} \in \mathfrak{M}/\mathfrak{C}$$

(c) *The normalizer* $\mathfrak{N}_{\mathfrak{E}}$ *of* $\mathfrak{E}$ *in* $\mathfrak{G}$ *coincides with* $\mathfrak{E}$: $\mathfrak{N}_{\mathfrak{E}} = \mathfrak{E}$.

We give the name *abstract phase group* to a group $\mathfrak{G}$ with the above properties 1, 2(a), (b), (c). A subgroup $\mathfrak{E}$ of this with the properties 2(a), (b), (c) we designate a fundamental subgroup of $\mathfrak{G}$. The elements of $\mathfrak{G}$ are called *abstract phases*, but for brevity we generally omit the attribute "abstract" in this connection.

## 28.3 Linear Vector Spaces

We now introduce the notation $\bar{A} = \mathfrak{G}/_r\mathfrak{E}$.

Let $\bar{a} \in \bar{A}$. If $a \in \bar{a}$ is given arbitrarily, then every element of $\bar{a}$ has the form $ea$, for a unique $e \in \mathfrak{E}$; conversely $ea$, with $e$ an arbitrary element of $\mathfrak{E}$, represents an element of $\bar{a}$.

Our object now is to associate simply with every element $\bar{a} \in \bar{A}$ a linear vector space $\mathrm{L}_{\bar{a}}$ of dimension 2 over the field $R$; this gives us a system $\bar{\mathrm{L}}$ of two-dimensional linear vector spaces over $R$ with, naturally, card $\bar{\mathrm{L}}$ = card $\bar{A}$. Every basis of $\mathrm{L}_{\bar{a}}$ is an ordered pair of elements, $U$, $V \in \mathrm{L}_{\bar{a}}$; this will frequently be written in the matrix form $\begin{pmatrix} U \\ V \end{pmatrix}$. If $\mathrm{B}$ is a given basis of $\mathrm{L}_{\bar{a}}$ then every basis of $\mathrm{L}_{\bar{a}}$ has the form $M\mathrm{B}$, for a uniquely determined matrix $M \in \mathfrak{M}$; conversely $M\mathrm{B}$, for any matrix $M \in \mathfrak{M}$, represents a basis of $\mathrm{L}_{\bar{a}}$.

Now we assume that between phases, on the one hand, and the bases of the linear vector spaces $\mathrm{L}_{\bar{a}} \in \bar{\mathrm{L}}$ on the other hand, there are the following relations:

*For every class* $\bar{a} \in \bar{A}$ *and* $\mathrm{L}_{\bar{a}} \in \bar{\mathrm{L}}$ *we have the following properties:*
*With every phase* $a \in \bar{a}$ *there is associated a system* $\bar{\mathrm{B}}_a$ *of bases of* $\mathrm{L}_{\bar{a}}$ *such that*
(a) *the individual bases of* $\bar{\mathrm{B}}_a$ *are constant positive multiples of any one of them,*
(b) *for* $e \in \mathfrak{E}$ *we have* $\bar{\mathrm{B}}_{ea} = \mathscr{H}e \cdot \bar{\mathrm{B}}_a$.

From (a) it follows that, given any $\mathrm{B} \in \bar{\mathrm{B}}_a$, we have $\bar{\mathrm{B}}_a = \{\lambda \mathrm{B}\}$, $\lambda > 0$, while (b) gives, on taking account of (3),

$$\bar{\mathrm{B}}_{c_\nu ea} = (-1)^\nu \bar{\mathrm{B}}_{ea} \qquad (\nu = 0, \pm 1, \pm 2, \ldots).$$

Moreover, for $\mathrm{B} \in \bar{\mathrm{B}}_a$ and $M \in \mathfrak{M}$,

$$M\mathrm{B} \in \bar{\mathrm{B}}_{ea}, \tag{28.5}$$

in which $e = \mathscr{T}^{-1}\bar{M}$.

It follows from the above assumptions that to every $a \in \bar{a}$ there corresponds a basis system $\bar{\mathrm{B}}_a$ of $\mathrm{L}_{\bar{a}}$ with the properties (a), (b). Conversely, every basis system $\bar{\mathrm{B}}$ of $\mathrm{L}_{\bar{a}}$, whose elements differ from one another by a constant positive multiple, coincides with a basis system $\bar{\mathrm{B}}_a$. For, given arbitrary elements $b \in \bar{a}$, $\mathrm{B} \in \bar{\mathrm{B}}$, $\mathrm{B}_b \in \bar{\mathrm{B}}_b$ there correspond elements $M \in \mathfrak{M}$, $e = \mathscr{T}^{-1}\bar{M}$ such that $\mathrm{B} = M\mathrm{B}_b \in \bar{\mathrm{B}}_{eb}$ (from (5)) and consequently $\bar{\mathrm{B}} = \bar{\mathrm{B}}_a$ $(a = eb)$.

If $\mathrm{B} \in \bar{\mathrm{B}}_a$, then we call $a$ a *phase* of B; we also say that B admits of or possesses the phase $a$, and express this by the notation $\mathrm{B} = \mathrm{B}_a$. If $b$ is also a phase of B, then there exists an $e \in \mathfrak{E}$ with $b = ea$ and we have $\bar{\mathrm{B}}_a = \bar{\mathrm{B}}_{ea} = \mathscr{H}e\bar{\mathrm{B}}_a$. Consequently $\mathscr{H}e = E$ and also $e = c_{2\nu}$ for some appropriate $\nu$ $(= 0, \pm 1, \pm 2, \ldots)$; as a consequence, $b = c_{2\nu}a \in \mathfrak{Z}_0 a$; conversely, every element of $\mathfrak{Z}_0 a$ represents a phase of B. Clearly, every basis of a system $\bar{\mathrm{B}}_a$ admits of the same phases and these are precisely the elements of the class $\mathfrak{Z}_0 a$. This class, which is obviously an element of $\mathfrak{G}/_r\mathfrak{Z}_0$, is called the *phase system* of the basis B.

## 28.4 Quasinorms

We now assume that *with every basis* B *of* $\mathrm{L}_{\bar{a}}$ *there is associated a non-zero real number* $\|\mathrm{B}\|$, *known as the quasinorm of* B, *satisfying the conditions:*

(a) $\operatorname{sgn} \|\mathrm{B}\| = -\operatorname{sgn} a'$,
(b) $\|M\mathrm{B}\| = \det M \cdot \|\mathrm{B}\|$,

*where a is a phase of* B *and M an arbitrary matrix of* $\mathfrak{M}$.

We now show that

1. *A basis* B *is uniquely determined if we are given one of its phases and its quasinorm.* For, let one phase $a$ and the quasinorm $\|\mathrm{B}\|$ of a basis B be specified. Let us choose a basis $\mathrm{B}_a \in \bar{\mathrm{B}}_a$. Then we have $\mathrm{B} = \lambda \mathrm{B}_a$ for some $\lambda > 0$, and condition (b) above then shows that $\|\mathrm{B}\| = \lambda^2 \|\mathrm{B}_a\|$, whence $\lambda = \sqrt{(\operatorname{abs}\|\mathrm{B}\|/\operatorname{abs}\|\mathrm{B}_a\|)}$. Hence B is uniquely determined as
$$\mathrm{B} = \sqrt{\left(\frac{\operatorname{abs}\|\mathrm{B}\|}{\operatorname{abs}\|\mathrm{B}_a\|}\right)}\, \mathrm{B}_a. \tag{28.6}$$

2. *Corresponding to every* $k \in R$ $(k \neq 0)$ *with* $\operatorname{sgn} k = -\operatorname{sgn} a'$, *every basis system* $\bar{\mathrm{B}}_a$ *contains precisely one basis* B *with quasinorm k, namely*
$$\mathrm{B} = \sqrt{\left(\frac{\operatorname{abs} k}{\operatorname{abs}\|\mathrm{B}_a\|}\right)}\, \mathrm{B}_a,$$
*in which* $\mathrm{B}_a$ *is an arbitrarily chosen basis in* $\bar{\mathrm{B}}_a$.
For, let $\mathrm{B} \in \bar{\mathrm{B}}_a$ and $\|\mathrm{B}\| = k$; then $\mathrm{B} = \lambda \mathrm{B}_a$ $(\lambda > 0)$, so $k = \lambda^2 \|\mathrm{B}_a\|$, whence necessarily (since $\operatorname{sgn} k = \operatorname{sgn}\|\mathrm{B}_a\|$), $\lambda = \sqrt{(\operatorname{abs} k/\operatorname{abs}\|\mathrm{B}_a\|)}$. This establishes the assertion.
In particular, every basis system $\bar{\mathrm{B}}_a$ contains precisely one basis with the quasinorm $-\operatorname{sgn} a'$. This we call the *unit basis* in $\bar{\mathrm{B}}_a$ and use for it the notation $\mathscr{B}_a$; that is,
$$\mathscr{B}_a = (1/\sqrt{\operatorname{abs}\|\mathrm{B}_a\|})\, \mathrm{B}_a.$$

## 28.5 Kummer transformations of bases

Let us take a basis B of $L_{\bar{a}}$ and an element $x \in \mathfrak{G}$; with these we are going to associate another basis, of some suitable vector space $L_{\bar{b}}$ which we shall denote by $B \circ x$.

*We choose a phase* $a$ *of* B *and specify, by definition,*

$$B \circ x \in \bar{B}_{ax}, \quad \|B \circ x\| = \|B\| \operatorname{sgn} x'.$$

The basis $B \circ x$ thus admits of the phase $ax$ and has quasinorm $\|B\| \operatorname{sgn} x'$; these specifications serve to determine it uniquely, by § 28.4, 1. Since $\bar{B}_{(c_{2\nu}a)x} = \bar{B}_{c_{2\nu}(ax)} = \mathscr{H}c_{2\nu}\bar{B}_{ax} = \bar{B}_{ax}$, the basis $B \circ x$ is independent of the choice of $a$.

Let $\bar{b}$ be the class of $\mathfrak{G}/_r\mathfrak{C}$ containing the element $ax$, i.e. $ax \in \bar{b} \in \mathfrak{G}/_r\mathfrak{C}$, so $B \circ x$ is a basis of $L_{\bar{b}}$. Let us choose $b \in \bar{b}$ and $B_b \in \bar{B}_b$. Clearly, there is a unique element $e \in \mathfrak{C}$ with the property that

$$ax = eb$$

and we have

$$\bar{B}_{ax} = \bar{B}_{eb} = \mathscr{H}e\bar{B}_b = \{\lambda\mathscr{H}eB_b\}, \qquad (\lambda > 0).$$

Since $\operatorname{sgn} \|B\| \operatorname{sgn} x' = -\operatorname{sgn} a' \operatorname{sgn} x' = -\operatorname{sgn} (ax)'$, the system $\bar{B}_{ax}$ contains precisely one basis with quasinorm $\|B\| \operatorname{sgn} x'$, which coincides with $B \circ x$ (see § 28.4, 1). Using (6), we obtain

$$B \circ x = \sqrt{\left(\frac{\operatorname{abs} \|B\|}{\operatorname{abs} \|B_b\|}\right)} \mathscr{H}eB_b. \tag{28.7}$$

*The basis* $B \circ x$ *may thus be represented explicitly by the formula* (7).

The operation $\circ$ thus starts from any basis B of $L_{\bar{a}}$ and any element $x \in \mathfrak{G}$ and associates with them the basis $B \circ x$ of $L_{\bar{b}}$; we call this operation the *Kummer transformation of* B *with* $x$, and $B \circ x$ is itself called the *Kummer transform of* B *with* $x$. Obviously,

$$B \circ x = \sqrt{(\operatorname{abs} \|B\|)}\mathscr{H}e\mathscr{B}_b, \quad \mathscr{B} \circ x = \mathscr{H}e\mathscr{B}_b$$

are special cases of (7).

The only properties of the Kummer transformation which we here need to emphasize are the following:

Let B be a basis of $L_{\bar{a}}$, $x, y$ arbitrary elements of $\mathfrak{G}$ and $M \in \mathfrak{M}$. *Then*

1. $B \circ (xy) = (B \circ x) \circ y$.
   For, let $a$ be a phase of B; then $B \circ (xy)$ admits of the phase $a(xy)$ and has the quasinorm $\|B\| \operatorname{sgn} (xy)'$. But $(B \circ x) \circ y$ admits of the phase $(ax)y$ and has the quasinorm $(\|B\| \operatorname{sgn} x') \operatorname{sgn} y'$, so our statement follows by § 28.4, 1.
2. $(MB) \circ x = M(B \circ x)$.
   For, $MB$ has the phase $ea$, with $e = \mathscr{T}^{-1}M$ (by (5)), hence $(MB) \circ x$ has the phase $(ea)x$ and the quasinorm of $(MB) \circ x$ is $\|MB\| \operatorname{sgn} x' = \det M\|B\| \operatorname{sgn} x'$. But $M(B \circ x)$ has the phase $e(ax)$ and quasinorm $\|M(B \circ x)\| = \det M\|B \circ x\| = \det M\|B\| \operatorname{sgn} x'$, and the statement follows by § 28.4, 1.

## 28.6 Kummer transformations of elements

Now we extend the concept of Kummer transformation to the transformation of elements of $L_{\bar{a}}$.

Let $Y \in L_{\bar{a}}$ and $x \in \mathfrak{G}$ be arbitrary elements of the sets indicated. We select a basis B of $L_{\bar{a}}$; we can then represent $Y$ uniquely with respect to this basis B by means of appropriate coordinates $\beta_1, \beta_2$—that is to say, in the form

$$Y = (\beta_1, \beta_2)\mathrm{B}.$$

*We now write, by definition,*

$$Y \circ x = (\beta_1, \beta_2)(\mathrm{B} \circ x).$$

Thus the operation $\circ$ serves to transform $Y$ into an element $Y \circ x$ of the vector space $L_{\bar{b}}$ containing the basis $\mathrm{B} \circ x$. We call this operation $\circ$ the *Kummer transformation of $Y$ with $x$*, and $Y \circ x$ itself the *Kummer transform of $Y$ with $x$.*

Next we show that $Y \circ x$ *is independent of the choice of* B. For, let C be another basis of $L_{\bar{a}}$; then we have

$$Y = (\gamma_1, \gamma_2)\mathrm{C}$$

for uniquely determined $\gamma_1, \gamma_2 \in R$. Moreover, there is an $M \in \mathfrak{M}$ such that $\mathrm{B} = M\mathrm{C}$ and we have

$$(\gamma_1, \gamma_2)\mathrm{C} = Y = (\beta_1, \beta_2)\mathrm{B} = (\beta_1, \beta_2)M\mathrm{C},$$

hence

$$(\gamma_1, \gamma_2) = (\beta_1, \beta_2)M.$$

Then, on taking account of § 28.5, 2,

$$\begin{aligned} Y \circ x &= (\gamma_1, \gamma_2)(C \circ x) = [(\beta_1, \beta_2)M](\mathrm{C} \circ x) = \\ &= (\beta_1, \beta_2)[M(\mathrm{C} \circ x)] = (\beta_1, \beta_2)[(M\mathrm{C}) \circ x] = \\ &= (\beta_1, \beta_2)(\mathrm{B} \circ x), \end{aligned}$$

which shows that $Y \circ x$ is independent of the choice of B.

## 28.7 Abstract dispersions

This paragraph introduces the concept of *abstract dispersions*; sub-paragraphs (A) and (B) are preliminary to the definition in sub-paragraph (C) of such dispersions and a study of some of their properties. For convenience, the main results are numbered as Lemmas 1 to 6.

Throughout, by the term "classes" we mean elements of $\mathfrak{G}/_r\mathfrak{E}$. The subgroup of $\mathfrak{G}$ conjugate with $\mathfrak{E}$ with respect to $a \in \mathfrak{G}$ will be denoted by $\mathfrak{G}_a$, thus $\mathfrak{G}_a = a^{-1}\mathfrak{E}a$.

(A) *Kummer complexes:*

Let $a, b, A, B$ be elements of $\mathfrak{G}$.

*Lemma 1. The relationship*

$$A^{-1}\mathfrak{E}a = B^{-1}\mathfrak{E}b \qquad (28.8.)$$

*holds if and only if $B = EA$, $b = ea$, where $E, e \in \mathfrak{E}$.*

*Proof.* (i) from $B = EA$, $b = ea$ and $E, e \in \mathfrak{E}$ it follows that

$$B^{-1}\mathfrak{E}b = A^{-1}(E^{-1}\mathfrak{E}e)a = A^{-1}\mathfrak{E}a.$$

(ii) From (8) we have

$$(BA^{-1})\mathfrak{E} = \mathfrak{E}(ba^{-1})$$

and also

$$BA^{-1} = e_0(ba^{-1}), \quad \text{where} \quad e_0 \in \mathfrak{E}. \tag{28.9}$$

This gives

$$(ba^{-1})^{-1}\mathfrak{E}(ba^{-1}) = \mathfrak{E}.$$

Clearly, $ba^{-1}$ is contained in the normalizer $\mathfrak{N}_{\mathfrak{E}}$. On taking account of § 28.2, (2) (c) it follows that $ba^{-1} = e \in \mathfrak{E}$ and, from (9), $BA^{-1} = E \in \mathfrak{E}$ ($E = e_0 e$). We thus have $B = EA$, $b = ea$ and the proof is complete.

It follows from Lemma 1 that to every ordered pair of classes $\bar{A}$, $\bar{a}$ there corresponds a well-defined subset of $\mathfrak{G}$, namely $K(\bar{A}, \bar{a}) = A^{-1}\mathfrak{E}a$, this subset being independent of the choice of the elements $A$, $a$ in the classes $\bar{A}$, $\bar{a}$; we call this subset $K(\bar{A}, \bar{a})$ the *Kummer complex of* $\bar{A}$, $\bar{a}$.

*Lemma 2. The Kummer complex* $K(\bar{A}, \bar{a})$ *is characterized by the property of being the unique common element of the two partitions* $\mathfrak{G}/_l\mathfrak{U}_a$, $\mathfrak{G}/_r\mathfrak{U}_A$.

*Proof.* Obviously we have

$$A^{-1}\mathfrak{E}a = (A^{-1}a)(a^{-1}\mathfrak{E}a) = (A^{-1}a)\mathfrak{U}_a,$$
$$A^{-1}\mathfrak{E}a = (A^{-1}\mathfrak{E}A)(A^{-1}a) = \mathfrak{U}_A(A^{-1}a),$$

hence

$$K(\bar{A}, \bar{a}) \in \mathfrak{G}/_l\mathfrak{U}_a \cap \mathfrak{G}/_r\mathfrak{U}_A$$

Thus $K(\bar{A}, \bar{a})$ occurs as an element in both partitions $\mathfrak{G}/_l\mathfrak{U}_a$, $\mathfrak{G}/_r\mathfrak{U}_A$. But by a known result in group theory ([2*]), the hypothesis of § 28.2, (2)(c) implies that these partitions contain precisely one common element, which thus coincides with $K(\bar{A}, \bar{a})$. We thus have the relation

$$K(\bar{A}, \bar{a}) = \mathfrak{G}/_l\mathfrak{U}_a \cap \mathfrak{G}/_r\mathfrak{U}_A$$

and the proof is complete.

In particular, taking $\bar{A} = \bar{a}$, we see that the Kummer complex of $\bar{a}$, $\bar{a}$ coincides with the subgroup $\mathfrak{U}_a$, i.e. $K(\bar{a}, \bar{a}) = \mathfrak{U}_a$ ($a \in \bar{a}$). This permits us to write, conveniently, $\mathfrak{U}_{\bar{a}}$ in place of $\mathfrak{U}_a$.

(B) *The centre* $\mathfrak{Z}_{\bar{a}}$ *of* $\mathfrak{U}_{\bar{a}} \cap \mathfrak{G}_0$

Now let us consider the centre $\mathfrak{Z}_{\bar{a}}$ of the subgroup $\mathfrak{U}_{\bar{a}} \cap \mathfrak{G}_0$ ($\subset \mathfrak{G}$).

*Lemma 3. For* $a \in \bar{a}$ *we have*

$$\mathfrak{Z}_{\bar{a}} = a^{-1}\mathfrak{Z}a.$$

*Proof.* (i) Let $f \in \mathfrak{Z}_{\bar{a}}$; then $f = a^{-1}e_0 a$ for some appropriate element $e_0 \in \mathfrak{E}$, sgn $e_0' = 1$. Moreover, $f$ commutes with every element $x \in \mathfrak{U}_{\bar{a}} \cap \mathfrak{G}_0$, i.e. every $x$ of the form $x = a^{-1}ea$, where $e \in \mathfrak{E}$, sgn $e' = 1$. Hence

$$xf = (a^{-1}ea)(a^{-1}e_0 a) = a^{-1}(ee_0)a,$$
$$fx = (a^{-1}e_0 a)(a^{-1}ea) = a^{-1}(e_0 e)a;$$

consequently $ee_0 = e_0 e$. Hence $e_0 \in \mathfrak{Z}$ and, finally, $f = a^{-1}e_0 a \in a^{-1}\mathfrak{Z}a$.

(ii) Let $f \in a^{-1}\mathfrak{Z}a$, i.e. $f = a^{-1}e_0 a$ where $e_0 \in \mathfrak{Z}$ is some appropriate element. Then for every $e \in \mathfrak{E}$, sgn $e' = 1$ and we have $ee_0 = e_0 e$, whence $(a^{-1}ea)(a^{-1}e_0 a) = (a^{-1}e_0 a)(a^{-1}ea)$. Thus $f$ commutes with every element $x \in \mathfrak{U}_{\bar{a}} \cap \mathfrak{G}_0$, i.e. $xf = fx$, and the proof is complete.

From Lemma 3, the centre $\mathfrak{Z}_{\bar{a}}$ of $\mathfrak{U}_{\bar{a}} \cap \mathfrak{G}_0$ is an infinite cyclic group with generators $a^{-1}ca$, $a^{-1}c^{-1}a$, i.e. $\mathfrak{Z}_{\bar{a}} = \{a^{-1}c_\nu a\}$, $c_\nu = c^\nu$, $\nu = 0, \pm 1, \pm 2, \ldots$, $c_1 = c$. The individual elements $f_\nu = a^{-1}c_\nu\, a \in \mathfrak{Z}_{\bar{a}}$ depend on the particular element $a \in \bar{a}$ as follows: for $b = ea$, $e \in \mathfrak{E}$ we have $b^{-1}c_\nu b = a^{-1}e^{-1}(c_\nu e)a = a^{-1}e^{-1}(ec_{\nu\,\mathrm{sgn}\,e'})a = a^{-1}c_{\nu\,\mathrm{sgn}\,e'}\,a = f_{\nu\,\mathrm{sgn}\,e'}$.

(C) *Abstract general and special dispersions: abstract central dispersions.*

(1) We shall apply the term *abstract general dispersions of $\bar{A}$, $\bar{a}$* to the elements of $K(\bar{A}, \bar{a})$, but omit the attribute "abstract" when convenient. A general dispersion $x$ of $\bar{A}$, $\bar{a}$ thus has the form $x = A^{-1}ea$ where $A$, $a$ are arbitrary elements of $\bar{A}$, $\bar{a}$ and $e$ is an appropriate element of $\mathfrak{E}$.

Let $\bar{A}$, $\bar{a}$ be arbitrary classes and $\mathrm{L}_{\bar{A}}$, $\mathrm{L}_{\bar{a}}$ the corresponding linear vector spaces.

*Lemma 4. The Kummer transform* $\mathrm{B} \circ x$ *of a basis* $\mathrm{B}$ *of* $\mathrm{L}_{\bar{A}}$ *with an element* $x \in \mathfrak{G}$ *is a basis of* $\mathrm{L}_{\bar{a}}$ *if and only if* $x$ *is a general dispersion of* $\bar{A}$, $\bar{a}$.

*Proof.* Let $\mathrm{B} = \mathrm{B}_A$, $A \in \bar{A}$ and $a \in \bar{a}$.

(i) Assume that $x \in \mathfrak{G}$ has the above property; then $Ax = ea$, $e \in \mathfrak{E}$ and consequently $x = A^{-1}ea \in K(\bar{A}, \bar{a})$.

(ii) Let $x \in K(\bar{A}, \bar{a})$, that is $x = A^{-1}ea$, $e \in \mathfrak{E}$. Then $Ax = A(A^{-1}ea) = ea \in \bar{a}$, whence $\mathrm{B} \circ x = \mathrm{B}_{ea}$ is a basis of $\mathrm{L}_{\bar{a}}$. This proves the Lemma.

A corollary of Lemma 4 is that *the Kummer transform of an element* $Y \in \mathrm{L}_{\bar{A}}$ *with a general dispersion* $x$ *of* $\bar{A}$, $\bar{a}$ *is an element of* $\mathrm{L}_{\bar{a}}$, *i.e.* $Y \circ x \in \mathrm{L}_{\bar{a}}$.

(2) We shall apply the term (*abstract*) *special dispersions of $\bar{a}$* (or merely *dispersions of $\bar{a}$*) to the elements of $K(\bar{a}, \bar{a})$. A special dispersion $x$ of $\bar{a}$ has thus the form $x = a^{-1}ea$ where $a$ is an arbitrary element of $\bar{a}$ and $e$ an appropriate element of $\mathfrak{E}$.

*Lemma 5. The Kummer transform* $\mathrm{B} \circ x$ *of a basis* $\mathrm{B}$ *of* $\mathrm{L}_{\bar{a}}$ *with an element* $x \in \mathfrak{G}$ *is itself a basis of* $\mathrm{L}_{\bar{a}}$ *if and only if* $x$ *is a special dispersion of* $\bar{a}$.

Clearly, this is a special case of Lemma 4. In particular, *the Kummer transform of an element* $Y \in \mathrm{L}_{\bar{a}}$ *with a special dispersion* $x$ *of* $\bar{a}$ *is itself an element of* $\mathrm{L}_{\bar{a}}$, *i.e.* $Y \circ x \in \mathrm{L}_{\bar{a}}$.

(3) We shall apply the term (*abstract*) *central dispersions of $\bar{a}$*, more briefly, *central dispersions*, to the elements of $\mathfrak{Z}_{\bar{a}}$.

Given a fixed element $a$ in the class $\bar{a}$, and the generating element $c$ of $\mathfrak{Z}$, we can associate with every central dispersion $f \in \mathfrak{Z}_{\bar{a}}$ a unique integer $\nu$, known as the *index* of $f$, by means of the formula $f = a^{-1}c^\nu a$; we then write $f = f_\nu$. For a different choice of $a$ or $c$, the index $\nu$ of $f$ either remains unchanged or else changes sign to become $-\nu$. Hence, clearly, the parity of the index of every central dispersion is independent of the choice of the elements $a$, $c$.

*Lemma 6. The Kummer transform* $\mathrm{B} \circ x$ *of a basis* $\mathrm{B}$ *of* $\mathrm{L}_{\bar{a}}$ *with an element* $x \in \mathfrak{G}$ *coincides with* $\mathrm{B}$ *or* $-\mathrm{B}$ *if and only if* $x$ *is a central dispersion of* $a$ *with even or odd index, respectively, i.e.*

$$\mathrm{B} \circ f_\nu = (-1)^\nu \mathrm{B}.$$

*Proof.* (i) Let $f = f_\nu \in \mathfrak{Z}_{\bar{a}}$. Then $af_\nu = a(a^{-1}c_\nu a) = c_\nu a$, and hence, using (28.3), $\mathrm{B} \circ f_\nu = \mathscr{H}c_\nu \mathrm{B} = (-1)^\nu \mathrm{B}$.

(ii) Let $x \in \mathfrak{G}$ and $\mathrm{B} \circ x = (-1)^\alpha \mathrm{B}$, $\alpha = 0, 1$. Then $ax = ea$ and $\mathscr{H}e = (-1)^\alpha E$ $(\mathrm{B} = \mathrm{B}_a)$. Hence $e = c_{2\mu+\alpha}$, $\mu$ integral, and also $x = a^{-1}c_{2\mu+\alpha}a = f_{2\mu+\alpha}$.

It follows from Lemma 6 that *the Kummer transform of an element* $Y \circ \mathrm{L}_{\bar{a}}$ *with a central dispersion* $f_\nu$ *of* $\bar{a}$ *coincides with either* $Y$ *or* $-Y$ *in the sense that*

$$Y \circ f_\nu = (-1)^\nu Y, \qquad (\nu = 0, \pm 1, \pm 2, \ldots) \tag{28.10}$$

## 28.8 Realization of the abstract model in terms of analytical transformation theory

Our object now is to obtain a realization of the above abstract model in terms of the transformation theory of oscillatory differential equations (q). To do this, we must interpret the model elements, considered in 28.1–28.6 above, in this realization.

First, however, we introduce the symbol $\mathscr{B}(t)$ to stand for the particular basis of the differential equation $(-1)$—that is, the equation $y'' = -y$, $t \in j = (-\infty, \infty)$—formed by the elements $\sin t$, $\cos t$. Then for any phase $e(t)$ of $(-1)$ we have, by formula (21.10), the relationship

$$\frac{1}{\sqrt{|e'(t)|}}\mathscr{B}e(t) = \mathscr{H}e\mathscr{B}(t),$$

where $\mathscr{H}e$ is a uniquely determined unimodular matrix; $\det \mathscr{H}e = \operatorname{sgn} e'(t)$.

In our realization of the abstract model, the various elements must be interpreted as follows:

$\mathfrak{G}$: the phase group described in § 10.1.
$\mathfrak{G}_0$: the subgroup comprising the increasing elements of the phase group.
$\mathfrak{E}$: the fundamental subgroup defined in § 10.3.
$\mathfrak{Z}$: the infinite cyclic group with generator $t + \pi$, that is, $\{t + \nu\pi\}$, $(\nu = 0, \pm 1, \pm 2, \ldots)$.
$\mathfrak{Z}_0$: the infinite cyclic group $\{t + 2\nu\pi)$.
$\mathscr{T}$: the operator such that for every phase $e(t)$ of $(-1)$ and $\bar{e} = \{e(t) + 2\nu\pi\}$, we have $\mathscr{T}\bar{e} = \{\lambda\mathscr{H}e\}$, $\lambda > 0$.
$\bar{a}$: the set of all first phases of the differential equation $(\mathrm{q}_a)$ with $q_a(t) = -\{a, t\} - a'^2(t)$, $a(t) \in \bar{a}$.
$\mathrm{L}_{\bar{a}}$: the integral space of $(\mathrm{q}_a)$.
$\bar{\mathrm{B}}_a$: $\left\{\frac{\lambda}{\sqrt{|a'(t)|}}\mathscr{B}a(t)\right\}, \qquad \lambda > 0.$
$\|\mathrm{B}\|$: the Wronskian of $\mathrm{B}$.

We leave it to the reader to verify that the properties assumed for the various elements in the abstract model do in fact hold in the realization. With regard to the properties of § 28.2, 2(c), see [2*].

Now we show that *the operation* $\circ$ *applied to a basis* $B(t) = (U(t), V(t))$ *of an oscillatory differential equation* (q) *in the interval* $j = (-\infty, \infty)$, *and an arbitrary phase function* $x(t)$, *may be realized by means of the Kummer transformation*

$$(U, V) \circ x = \left(\frac{Ux}{\sqrt{|x'|}}, \frac{Vx}{\sqrt{|x'|}}\right). \tag{28.11}$$

For brevity, in the formula (11), and in what follows, we omit the variable $t$.

Let $a$ be a proper first phase of B, $b$ be a phase of $(q_{ax})$ and $e$ be that phase of $(-1)$ determined uniquely by the relation $ax = eb$; finally, let $B_b$ be a basis of $(q_{ax})$ with proper first phase $b$.

Then we have, in the first place, the formula

$$B = \frac{\sqrt{\operatorname{abs} \|B\|}}{\sqrt{|a'|}} \mathscr{B}a, \tag{28.12}$$

and on taking account of (12) and (7), the further relationships

$$\begin{aligned}
\frac{1}{\sqrt{\operatorname{abs} \|B\|}} B \circ x &= \mathscr{H}e \frac{1}{\sqrt{\operatorname{abs} \|B_b\|}} B_b = \\
&= \frac{1}{\sqrt{|b'|}} \mathscr{H}e\mathscr{B}b = \frac{1}{\sqrt{|b'|}} \frac{1}{\sqrt{|e'b|}} \mathscr{B}\, eb = \\
&= \frac{1}{\sqrt{|(eb)'|}} \mathscr{B}eb = \frac{1}{\sqrt{|(ax)'|}} \mathscr{B}ax = \\
&= \frac{1}{\sqrt{|x'|}} \frac{1}{\sqrt{|a'x|}} \mathscr{B}ax \\
&= \frac{1}{\sqrt{\operatorname{abs} \|B\|}} \frac{1}{\sqrt{|x'|}} Bx.
\end{aligned}$$

This establishes the formula (11). In particular, *the operation* $\circ$ *applied to an integral* $Y$ *of* (q) *and a phase function* $x$ *is realized by the Kummer transformation*

$$Y \circ x = \frac{Yx}{\sqrt{|x'|}}. \tag{28.13}$$

Finally, we make the remark that when the concept of abstract dispersions is realized on the above lines, known results from the analytic theory of dispersions reappear; for instance, formula (13.10).