Linear Differential Transformations of the Second Order

26 Existence and generality of complete transformations

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Complete transformations

This chapter is devoted to questions relating to the existence and generality of complete transformations of two differential equations (q), (Q) and to a study of the structure of the set of such transformations. The relevant theory takes its origin from the results obtained in § 9.5 relating to similar phases of two differential equations, so we shall use the notation which we employed there. In particular, we shall denote the left and right 1-fundamental sequences of the differential equations (q), (Q), when they exist, by

$$(a <)\, a_1 < a_2 < \cdots; \qquad (b >)\, b_{-1} > b_{-2} > \cdots$$
$$(A <)\, A_1 < A_2 < \cdots; \qquad (B >)\, B_{-1} > B_{-2} > \cdots$$

26 Existence and generality of complete transformations

26.1 Formulation of the problem

The starting point of the following study is provided by the existence and uniqueness theorem for solutions of the differential equation (Qq) (§ 24.1) and the remark made on pages 211–2. From this theorem we know that there is precisely one broadest solution $Z(t)$ of the differential equation (Qq) in a certain interval k ($\subset j$) with the initial conditions specified. The range K of this broadest solution $Z(t)$ forms a subinterval of J; $K \subset J$. It is important for our further study to note that, in general, neither does the interval k coincide with j nor does K coincide with J. This means that integrals Y of the differential equation (Q) are in general not transformed by the function $Z(t)$ over their whole domains into integrals y of the differential equation (q) but only portions of one into portions of the other.

We call a *solution* $X(t)$ of the differential equation (Qq) *complete* if its domain of definition k coincides with j and its range K coincides with J. Similarly we speak of *complete transformations* of integrals Y of the differential equation (Q) into integrals y of the differential equation (q) when these are formed from complete solutions $X(t)$ of the equation (Qq) according to formula (23.7). Complete solutions of the differential equation (Qq) are obviously characterized by the property that the corresponding curves pass from one corner of the rectangular region $j \times J$ to the diagonally opposite corner.

By means of complete solutions of the differential equation (Qq), i.e. by complete transformations, integrals of the equation (Q) are transformed into integrals of the equation (q) in their whole domain. Obviously, if X is a complete solution of (Qq) then the inverse solution x of (qQ) is also complete.

We can now formulate the question with which we shall be concerned in this section as follows:

To determine necessary and sufficient conditions for the existence of complete solutions of the differential equation (Qq), *and to determine the number of these solutions.*

26.2 Preliminary

Let us consider two differential equations (q), (Q); their intervals of definition will again be denoted by $j = (a, b)$, $J = (A, B)$.

We know (§ 24.1) that every solution X of the differential equation (Qq) is uniquely determined by two suitable first phases α, $\mathbf{A}$ of the differential equations (q), (Q) by means of the formula

$$\alpha(t) = \mathbf{A}X(t) \qquad (t \in k \subset j). \tag{26.1}$$

We called such phases the generators of X. One of the generators, say α, can be chosen arbitrarily from among the phases of (q); the other, $\mathbf{A}$, is then determined uniquely by this choice of α and the function X.

This holds, in particular, for every complete solution of the differential equation (Qq), in so far as such exist.

26.3 The existence problem for complete solutions of the differential equation (Qq)

The following theorem is fundamental for the study of existence questions relative to complete solutions of the differential equation (Qq):

Theorem 1. Two phases α, $\mathbf{A}$ *of the differential equations* (q), (Q) *generate a complete solution* X *of the differential equation* (Qq) *if and only if they are similar.*

Proof. (a) Let X be a complete solution of the differential equation (Qq) and α, $\mathbf{A}$ be the generating phases. Then we have $X(j) = J$, and for $t \in j$ the relation (1) holds. Clearly, therefore the ranges of the phases α, $\mathbf{A}$ coincide in their intervals of definition j, J so the phases α, $\mathbf{A}$ are similar (§ 9.5).

(b) Let α, $\mathbf{A}$ be similar phases of the equations (q), (Q). Then the ranges of α, $\mathbf{A}$, in their intervals of definition j, J, form the same interval L. It follows that in the interval j the range of the function X constructed from the formula $X(t) = \mathbf{A}^{-1}\alpha(t)$ coincides with J and further that for $t \in j$ the relation (1) holds. The phases α, $\mathbf{A}$ are therefore generators of the complete solution X of the differential equation (Qq). This completes the proof.

From (1) it follows that for $t \in j$, $X'(t) = \alpha'(t)/\dot{\mathbf{A}}X(t)$, and so:

According as the generating phases α, $\mathbf{A}$ of a complete solution X of the differential equation (Qq) are directly or indirectly similar, X represents an increasing or decreasing function in the interval j.

From theorem 1 we deduce that the differential equation (Qq) has complete solutions X if and only if the differential equations (q), (Q) admit of similar phases. This leads, using § 9.6, to the following result:

Theorem 2. The differential equation (Qq) *has complete solutions* X *if and only if the differential equations* (q), (Q) *are of the same character.*

26.4 The multiplicity of the complete solutions of the differential equation (Qq)

We now assume that the equations (q), (Q) are of the same character. By theorem 2, the equation (Qq) therefore admits of complete solutions.

If we apply the theorem of § 9.6, we obtain the following result: Let $t_0 \in j$ be arbitrary, and let $X_0 \in J$ be a number associated directly or indirectly with t_0 with respect to the differential equations (q), (Q). There exist, respectively, increasing or decreasing complete solutions X of the differential equation (Qq), which take the value X_0 at the point t_0. According to the character of the differential equations (q), (Q) and according to whether the numbers t_0, X_0 are singular or not, there is either precisely one complete solution X or a 1- or 2-parameter system of complete solutions of the differential equation (Qq).

More precisely (making use of § 9.6) we have:

There exists precisely one complete solution X, if the differential equations (q), (Q) are general differential equations either of type (1) or of type (m), $m \geqslant 2$, with the numbers t_0, X_0 not singular.

There exist ∞^1 complete solutions X, if the differential equations (q), (Q) are general of type (m) $m \geqslant 2$, and the numbers t_0, X_0 are singular; also if the differential equations (q), (Q) are special of type (1) or of type (m), $m \geqslant 2$, and the numbers t_0, X_0 are not singular; finally if the differential equations (q), (Q) are oscillatory on one side and the numbers t_0, X_0 are not singular.

There exist precisely ∞^2 complete solutions X, if the differential equations (q), (Q) are special of type (m), $m \geqslant 2$, and the numbers t_0, X_0 are singular; also if the differential equations (q), (Q) are oscillatory on one side and the numbers t_0, X_0 are singular; finally if the differential equations (q), (Q) are oscillatory.

This result naturally makes possible also statements relating to the existence and generality of complete solutions of the differential equation (qq), and hence on the possibility and number of complete transformations of integrals of the equation (q) into themselves ($Q = $ q). We observe that the equation (qq) always admits of complete solutions. If (q) is of finite type (m), $m \geqslant 1$, or is oscillatory, then there exist always both increasing and decreasing complete solutions of the differential equation (qq); if however it is oscillatory on one side, then there exist only increasing complete solutions of (qq). We leave it to the reader to prove these results in detail.