23 Transformation properties of solutions of the differential equation (Qq)

23.1 Relations between solutions of the differential equations (Qq), (qQ), (qq), (QQ)

We are only interested in regular solutions of these differential equations, that is to say in solutions $X \in C_3$ with non-vanishing derivative X'. If therefore X is such a solution of one of these differential equations in a partial interval k of j or J, and $K = X(k)$ is its range, then in this interval K there exists the inverse function $x \in C_3$ of X. This has a derivative $\dot{x}$ which is always non-zero in this interval. The range of x is naturally the interval k; $x(K) = k$. We use the term homologous (with respect to the relevant differential equation) to describe any two numbers $t \in k$, $T \in K$ which are linked by the relationships $T = X(t)$, $t = x(T)$.

1. Let $X(t)$, $t \in i$ $(\subset j)$ be a solution of the differential equation (Qq). Then the function inverse to X, $x(T)$, $T \in I\,(= X(i) \subset J)$ is a solution of the differential equation (qQ).

Proof. Let $t \in i$, $T \in I$ be two homologous numbers. Since X is a solution of (Qq), at the point t we have the relation

$$-\frac{\{X, t\}}{X'} + Q(X)X' = \frac{q(t)}{X'}. \tag{23.1}$$

From this, taking account of formulae (1.10), (1.6) we have

$$\frac{\{x, T\}}{\dot{x}} + Q(T)\frac{1}{\dot{x}} = q(x)\dot{x}$$

and further

$$-\{x, T\} + q(x)\dot{x}^2 = Q(T).$$

This completes the proof.

2. Let X, x be inverse solutions of the differential equations (Qq), (qQ) with intervals of definition i $(\subset j)$, I $(\subset J)$. Then at any two homologous points $t \in i$, $T \in I$ there hold the symmetric relations

$$Q(X)X' - \frac{1}{2}\frac{\{X, t\}}{X'} = q(x)\dot{x} - \frac{1}{2}\frac{\{x, T\}}{\dot{x}}, \tag{23.2}$$

$$Q(X)X' + \frac{1}{4}\left(\frac{1}{X'}\right)'' = q(x)\dot{x} + \frac{1}{4}\left(\frac{1}{\dot{x}}\right)^{\cdot\cdot}. \tag{23.3}$$

To see this, we start from the formula (1); from this and (1.6) it follows that

$$Q(X)X' - \frac{1}{2}\frac{\{X, t\}}{X'} = q(x)\dot{x} + \frac{1}{2}\frac{\{X, t\}}{X'} \tag{23.4}$$

and thence, using (1.10), we get the relation (2). Formula (3) is obtained from (4) and (1.16).

3. We continue to employ the symbols X, x with the above meaning. Let f, F be two functions constructed in the intervals i, I with arbitrary constants a_0, a_1; A_0, A_1 as follows

$$\left.\begin{aligned} f(t) &= a_0 + a_1 t + \frac{1}{4}\cdot\frac{1}{X'(t)} + \int_{X(t_0)}^{X(t)} [t - x(\eta)]Q(\eta)\,d\eta, \\ F(T) &= A_0 + A_1 T + \frac{1}{4}\cdot\frac{1}{\dot{x}(T)} + \int_{x(T_0)}^{x(T)} [T - X(H)]q(H)\,dH; \end{aligned}\right\} \tag{23.5}$$

where $t_0 \in i$, $T_0 \in I$ denote arbitrarily chosen homologous numbers.

Then, at any two homologous points $t \in i$, $T \in I$ we have the relationship

$$f''(t) = \ddot{F}(T), \tag{23.6}$$

the proof of which follows from (3) above.

In order to formulate the following theorems more simply, we shall denote the functions Q, q by Q_1, Q_2 and the differential equations (QQ), (Qq), (qQ), (qq) respectively by (Q_{11}), (Q_{12}), (Q_{21}), (Q_{22}).

4. Let X, Y be arbitrary solutions of the differential equations $(Q_{\alpha\beta})$, $(Q_{\beta\gamma})$ $(\alpha, \beta, \gamma = 1, 2)$. Let i, k be the intervals of existence of the functions X, Y and let I, K be the ranges of the latter. Moreover let $i \cap K \neq \varnothing$, so that the composite function $Z = XY$ is defined in a certain interval $\bar{k}$ $(\subset k)$.

We can show that the function Z is a solution of the equation $(Q_{\alpha\gamma})$ in the interval k. For, by our assumptions, in the interval $\bar{k}$ we have:

$$-\{Y, t\} + Q_\beta(Y)Y'^2 = Q_\gamma(t),$$
$$-\{X, Y\} + Q_\alpha(Z)X'^2(Y) = Q_\beta(Y),$$

and at the same time from (1.17) we have

$$\{Z, t\} = \{X, Y\}Y'^2(t) + \{Y, t\}.$$

From these relationships it follows that

$$-\{Z, t\} + Q_\alpha(Z)Z'^2 = Q_\gamma(t),$$

and the proof is complete.

23.2 Reciprocal transformations of integrals of the differential equations (q), (Q)

We now return to the situation considered in § 22.2 and concern ourselves with the question of how far the transformations of the equation (Q) into the equation (q) are determined by the solutions of the equation (Qq).

Let X be a solution of (Qq) with the interval of definition i ($\subset j$). We know that the function x inverse to X, with the definition interval ($X(i) =$) I ($\subset J$), satisfies the differential equation (qQ) (§ 23.1).

We choose an arbitrary number $t_0 \in i$, and denote the values of the functions X, X', X'' at the point t_0 by X_0, X_0' ($\neq 0$), X_0''; analogously, x_0, $\dot{x}_0$ ($\neq 0$), $\ddot{x}_0$ denote the values of x, $\dot{x}$, $\ddot{x}$ at the point $T_0 \in I$ homologous to t_0. The numbers X_0, X_0', X_0'' are inter-related, since $X_0 = T_0$, $x_0 = t_0$, and the formulae (1.6) hold.

1. If Y is an integral of the differential equation (Q), then the function $\bar{y}$, defined in the interval i by means of the formula

$$\bar{y}(t) = \frac{Y[X(t)]}{\sqrt{|X'(t)|}} \tag{23.7}$$

satisfies the differential equation (q), and this solution $\bar{y}$ is that portion lying in the interval i of the integral y of (q) which is determined by the Cauchy initial conditions

$$\left.\begin{aligned} y(t_0) &= \frac{Y(X_0)}{\sqrt{|X_0'|}}, \\ y'(t_0) &= \frac{\dot{Y}(X_0)}{\sqrt{|X_0'|}} \cdot X_0' - \frac{1}{2}\frac{Y(X_0)}{\sqrt{|X_0'|}} \cdot \frac{X_0''}{X_0'}. \end{aligned}\right\} \tag{23.8}$$

Proof. Clearly, the function $\bar{y}$ is everywhere twice differentiable in the interval i, and it is easy to verify that the following formulae hold:

$$\left.\begin{aligned} \bar{y}'(t) &= \frac{\dot{Y}(X)}{\sqrt{|X'|}} \cdot X' + Y(X)\left(\frac{1}{\sqrt{|X'|}}\right)', \\ \bar{y}''(t) &= \frac{\ddot{Y}(X)}{\sqrt{|X'|}} \cdot X'^2 - \frac{Y(X)}{\sqrt{|X'|}} \cdot \{X, t\}. \end{aligned}\right\} \tag{23.9}$$

Since the functions Y, X satisfy respectively the differential equations (Q) and (Qq), at every point $t \in i$ we have

$$\begin{aligned} \ddot{Y}(X) &= Q(X)\,Y(X), \\ -\{X, t\} &= -Q(X)X'^2 + q(t). \end{aligned}$$

We have therefore

$$\bar{y}''(t) = \frac{Y(X)}{\sqrt{|X'|}}\,Q(X)X'^2 + \frac{Y(X)}{\sqrt{|X'|}}\,[-Q(X)X'^2 + q(t)]$$

and consequently

$$\bar{y}''(t) = q(t)\bar{y}.$$

so the function $\bar{y}$ is a solution of the equation (q). The values $\bar{y}(t_0)$, $\bar{y}'(t_0)$ are given from (7) and (9) by means of (8).

2. Let Y, y be the integrals of the differential equations (Q), (q) considered in Theorem 1 above. Then the portion $\bar{Y}$ of Y defined in the interval I is given by the inverse formula to (7), namely

$$\bar{Y}(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}} \tag{23.10}$$

and the Cauchy initial values $Y(T_0)$, $Y'(T_0)$ are

$$\left.\begin{aligned} Y(T_0) &= \frac{y(x_0)}{\sqrt{|\dot{x}_0|}}, \\ \dot{Y}(T_0) &= \frac{y'(x_0)}{\sqrt{|\dot{x}_0|}} \cdot \dot{x}_0 - \frac{1}{2}\frac{y(x_0)}{\sqrt{|\dot{x}_0|}} \cdot \frac{\ddot{x}_0}{\dot{x}_0}. \end{aligned}\right\} \tag{23.11}$$

Proof. Since y is an integral of (q) and x a solution of (qQ), the theorem above shows that the function

$$\tilde{Y}(T) = \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}$$

is a solution of the differential equation (Q) in the interval I.

Now, at two homologous points $T \in I$, $t \in i$ there hold the relations

$$\tilde{Y}(T) = \frac{\bar{y}(t)}{\sqrt{|\dot{x}(T)|}} = \frac{Y[X(t)]}{\sqrt{|\dot{x}(T) \cdot X'(t)|}} = Y(T) = \bar{Y}(T).$$

Consequently, $\tilde{Y}$ is the portion $\bar{Y}$ of the integral Y defined in I. From formula (8) the Cauchy initial conditions for Y are given by the formulae (11).

The above study thus yields the following theorem.

Theorem. The ordered pair of functions, $w(t) = k/\sqrt{|X'(t)|}$, $X(t)$ *constructed with an arbitrary constant* k ($\neq 0$), *represents a transformation* $[w, X]$ *of the differential equation* (Q) *into the differential equation* (q). *At the same time, the ordered pair of functions* $W(T) = k^{-1}/\sqrt{|\dot{x}(T)|}$, $x(T)$ *represents a transformation of the differential equation* (q) *into* (Q).

Every integral Y of the differential equation (Q) *is transformed by means of the transformation* $[w, X]$ *into its image*

$$\bar{y}(t) = k\,\frac{Y[X(t)]}{\sqrt{|X'(t)|}}, \tag{23.12}$$

which forms a portion of the image integral y of Y determined by the initial values

$$\left.\begin{aligned} y(t_0) &= k\,\frac{Y(X_0)}{\sqrt{|X_0'|}}, \\ y'(t_0) &= k\left[\frac{\dot{Y}(X_0)}{\sqrt{|X_0'|}} \cdot X_0' - \frac{1}{2}\frac{Y(X_0)}{\sqrt{|X_0'|}} \cdot \frac{X_0''}{X_0'}\right]. \end{aligned}\right\} \tag{23.13}$$

At the same time, the integral y of the differential equation (q) *is transformed by means of the transformation* $[W, x]$ *into its image*

$$\bar{Y}(T) = \frac{1}{k} \frac{y[x(T)]}{\sqrt{|\dot{x}(T)|}}, \tag{23.14}$$

which forms a portion of the image integral Y of y determined by the initial values

$$\left.\begin{aligned} Y(T_0) &= \frac{1}{k} \frac{y(x_0)}{\sqrt{|\dot{x}_0|}}, \\ \dot{Y}(T_0) &= \frac{1}{k} \left[\frac{y'(x_0)}{\sqrt{|\dot{x}_0|}} \cdot \dot{x}_0 - \frac{1}{2} \frac{y(x_0)}{\sqrt{|\dot{x}_0|}} \cdot \frac{\ddot{x}_0}{\dot{x}_0} \right]. \end{aligned}\right\} \tag{23.15}$$

3. Let Y, y be the integrals of the differential equations (Q), (q), considered in the above theorem. Then at every two homologous points $T \in I$, $t \in i$ there hold the relations

$$\left.\begin{aligned} \sqrt[4]{|\dot{x}(T)|} \cdot k\, Y(T) &= \sqrt[4]{|X'(t)|} \cdot y(t), \\ \frac{k}{\sqrt[4]{|\dot{x}(T)|}} \left[\dot{Y}(T) + \frac{1}{4}\, Y(T) \cdot \frac{\ddot{x}(T)}{\dot{x}(T)} \right] &= \frac{\varepsilon}{\sqrt[4]{|X'(t)|}} \left[y'(t) + \frac{1}{4} y(t) \frac{X''(t)}{X'(t)} \right] \end{aligned}\right\} \tag{23.16}$$

with $\varepsilon = \operatorname{sgn} X_0' = \operatorname{sgn} \dot{x}_0$.

These relations can be obtained from the formulae (12), (14) and their derivatives, by application of (1.6).

4. The image integrals u, v of two independent integrals U, V of the differential equation (Q) formed by the transformation $[w, X]$ are independent, and an analogous statement holds for the transformation $[W, x]$. This follows immediately from the formulae (13) and (15).

23.3 Transformations of the derivatives of integrals of the differential equations (q), (Q)

The above results can be used to determine transformations of the integrals (or of their first derivatives) of one of the differential equations (q), (Q) into portions of integrals (or their derivatives) of the other equation.

We assume that the carriers q, Q of the equations (q), (Q) $\in C_2$ and are always non-zero in their intervals of definition j, J. Then the differential equations (q), (Q) admit of associated differential equations ($\hat{\mathrm{q}}_1$), ($\hat{\mathrm{Q}}_1$), as in § 1.9. Their carriers $\hat{q}_1$, $\hat{Q}_1$ are determined by means of (1.18) and (1.20) while the relation between the derivatives y', $\dot{Y}$ of the integrals y, Y of the differential equations (q), (Q) and the integrals y_1, Y_1 of ($\hat{\mathrm{q}}_1$), ($\hat{\mathrm{Q}}_1$) is that of (1.21).

When we apply the above results to the differential equations ($\hat{\mathrm{q}}_1$), (Q) and ($\hat{\mathrm{q}}_1$), ($\hat{\mathrm{Q}}_1$) we obtain information about transformations of integrals y, Y of the differential equations (q), (Q) and their derivatives y', $\dot{Y}$. The transformations corresponding to the relations (12), (14) are

$$\left.\begin{aligned} \bar{y}'(t) &= k\sqrt{|q(t)|}\,\frac{Y[X_1(t)]}{\sqrt{|X_1'(t)|}}, & \bar{Y}(T) &= \frac{1}{k}\,\frac{1}{\sqrt{|q[x_1(T)]|}}\,\frac{y'[x_1(T)]}{\sqrt{|\dot{x}_1(T)|}},\\ \bar{y}'(t) &= k\sqrt{\left|\frac{q(t)}{Q[X_2(t)]}\right|}\,\frac{\dot{Y}[X_2(t)]}{\sqrt{|X_2'(t)|}}, & \dot{\bar{Y}}(T) &= \frac{1}{k}\sqrt{\left|\frac{Q(T)}{q[x_2(T)]}\right|}\,\frac{y'[x_2(T)]}{\sqrt{|\dot{x}_2(T)|}}; \end{aligned}\right\} \tag{23.17}$$

X_1, x_1 here represent mutually inverse solutions of the differential equations ($Q\hat{q}_1$), ($\hat{q}_1Q$) and X_2, x_2 are mutually inverse solutions of ($\hat{Q}_1\hat{q}_1$), ($\hat{q}_1\hat{Q}_1$).

23.4 Relations between solutions of the differential equation (Qq) and first phases of the differential equations (q), (Q)

The phases of the differential equations (q), (Q) considered in this paragraph are always first phases so we shall speak in what follows simply of phases instead of first phases.

We continue to use the symbols X, x, etc. as in § 23.2.

1. Let A be a phase of the differential equation (Q). Then the function $\bar{\alpha}$ defined in the interval i by means of the formula

$$\bar{\alpha}(t) = \mathrm{A}[X(t)] \tag{23.18}$$

is a portion of a phase α of the differential equation (q) and this phase α is determined by the Cauchy initial conditions

$$\alpha(t_0) = \mathrm{A}(X_0);\quad \alpha'(t_0) = \dot{\mathrm{A}}(X_0)X_0';\quad \alpha''(t_0) = \ddot{\mathrm{A}}(X_0)X_0'^2 + \dot{\mathrm{A}}(X_0)X_0''. \tag{23.19}$$

Obviously, the phases α, A are linked (§ 9.2).

Proof. The phase A is contained in the phase system of a basis (U, V) of the differential equation (Q), (§ 5.6). Consequently, we have the relation $\tan \mathrm{A} = U/V$ holding in the interval J, except at the zeros of V.

We consider the transformation $w(t) = 1/\sqrt{|X'(t)|}$, $X(t)$ of the differential equation (Q) into the differential equation (q). Let u, v be the image integrals of U, V under this transformation $[w, X]$. From § 23.2, 4 (u, v) is a basis of (q); let α_0 be a phase of this basis. Then we have, for $t \in i$ (apart from the singular points),

$$\tan \alpha_0(t) = \frac{u(t)}{v(t)} = \frac{U[X(t)]}{V[X(t)]} = \tan \mathrm{A}[X(t)],$$

and hence $\alpha_0(t) + m\pi = \mathrm{A}[X(t)]$, m being an appropriate integer. Now, the function $\alpha = \alpha_0 + m\pi$ represents a phase of the basis (u, v) and $\bar{\alpha}$ is the portion of α defined in i. By differentiating (18) we obtain the initial values $\alpha'(t_0)$, $\alpha''(t_0)$ as stated in (19). This completes the proof.

Naturally, the solution x inverse to X of the differential equation (qQ) transforms the phase α into a portion $\bar{\mathbf{A}}$ of $\mathbf{A}$:

$$\bar{\mathbf{A}}(T) = \alpha[x(T)] \quad (T \in I = X(i)). \tag{23.20}$$

2. Let α, $\mathbf{A}$ be arbitrary linked phases of the differential equations (q), (Q) and let

$$L = a(j) \cap \mathbf{A}(J); \quad k = \alpha^{-1}(L), \quad K = \mathbf{A}^{-1}(L). \tag{23.21}$$

Then corresponding to every number $t \in k$ or $T \in K$ there is precisely one number $Z(t) \in K$ or $z(T) \in k$ satisfying respectively the equation

$$\alpha(t) = \mathbf{A}[Z(t)] \qquad \text{or} \qquad \alpha(z(T)] = \mathbf{A}(T). \tag{23.22}$$

The functions $Z(t) = \mathbf{A}^{-1}\alpha(t)$, $z(T) = \alpha^{-1}\mathbf{A}(T)$, which are defined by (22) in the intervals k, K and are obviously inverse functions, belong to the class C_3 and represent regular solutions of the differential equations (Qq), (qQ) respectively. The curves defined by the functions Z, z go from boundary to boundary of the rectangular region $(a, b) \times (A, B)$.

Proof. (a) Let $t \in k$ be arbitrary. Then $\alpha(t) \in L = \mathbf{A}(K)$, and since $\mathbf{A}$ increases or decreases, there is precisely one number $Z(t) \in K$ satisfying the first equation (22). A similar result holds for the second equation (22).

(b) From $Z(t) = \mathbf{A}^{-1}\alpha(t)$, $z(T) = \alpha^{-1}\mathbf{A}(T)$ it follows that the functions Z, z belong to the class C_3 and their derivatives Z', $\dot{z}$ are always non-zero. If we take the Schwarzian derivative of (22) it is clear that the functions Z, z satisfy the differential equations (Qq), (qQ).

(c) The validity of the last statement follows from the result of § 9.2 relating to the intervals k, K. This completes the proof.

We call Z, z the solutions of the differential equations (Qq), (qQ) *generated by* the phases α, $\mathbf{A}$.

The solution X of the differential equation (Qq) considered in 1 above is obviously that portion with domain of definition i of the solution Z of the differential equation (Qq) generated by the phases α, $\mathbf{A}$.

23.5 Reciprocal transformations of first and second phases of the differential equations (q), (Q)

From § 23.4, 1, a solution X of the differential equation (Qq) transforms each first phase $\mathbf{A}$ of the equation (Q) into a portion $\bar{\alpha}$ of a first phase α of the equation (q), according to the formula (18). An analogous statement holds for a solution x of the equation (qQ) and each first phase α of (q): the function x similarly transforms the phase α into a portion $\bar{\mathbf{A}}$ of a first phase of (Q). From § 23.4, 2 any two linked first phases α, $\mathbf{A}$ of the equations (q), (Q) generate inverse solutions of the equations (Qq), (qQ).

Now we assume that the functions q, $Q \in C_2$ and are always non-zero in their intervals of definition j, J, so that the differential equations (q), (Q) admit of associated differential equations $(\hat{q}_1)$, $(\hat{Q}_1)$. Then every first phase α_1 of $(\hat{q}_1)$ represents a second phase β of (q), and similarly every first phase $\mathbf{A}_1$ of $(\hat{\mathrm{Q}}_1)$ represents a second phase $\mathbf{B}$ of (Q): $\alpha_1 = \beta$, $\mathbf{A}_1 = \mathbf{B}$. If we apply the results of § 23.4, 1 and 2, to the differential equations $(\hat{\mathrm{q}}_1)$, (Q) and $(\hat{\mathrm{q}}_1)$, $(\hat{\mathrm{Q}}_1)$, we obtain results relating to transformations of first and second phases α, $\mathbf{A}$ or β, $\mathbf{B}$ of the differential equations (q), (Q) into each other. The transformation formulae corresponding to the relations (18), (20) are

$$\bar{\beta}(t) = \mathbf{A}[X_1(t)]; \qquad \bar{\mathbf{A}}(T) = \beta[x_1(T)];$$
$$\bar{\beta}(t) = \mathbf{B}[X_2(t)]; \qquad \bar{\mathbf{B}}(T) = \beta[x_2(T)];$$

in which X_1, x_1, represent mutually inverse solutions of the equations $(\mathrm{Q}\hat{\mathrm{q}}_1)$, $(\hat{\mathrm{q}}_1\mathrm{Q})$ and X_2, x_2 are similarly mutually inverse solutions of $(\hat{\mathrm{Q}}_1\hat{\mathrm{q}}_1)$, $(\hat{\mathrm{q}}_1\hat{\mathrm{Q}}_1)$.