19 Linear mapping of the integral spaces of the differential equations (q), (Q) on each other

19.1 Fundamental concepts

We consider two oscillatory differential equations (q), (Q) in the intervals $j = (a, b)$ and $J = (A, B)$. We denote their integral spaces by r, R while (u, v), (U, V) will be arbitrary bases of r, R and w, W the Wronskians of these bases.

Every integral $y \in r$ of (q) has determinate constant coordinates c_1, c_2 with respect to the basis (u, v), such that $y = c_1u + c_2v$, and the same is true of every integral $Y \in R$ of (Q): $Y = C_1U + C_2V$. Conversely, given any ordered pair of constants c_1, c_2 or C_1, C_2 there corresponds precisely one integral $y \in r$ of (q) or precisely one integral $Y \in R$ of (Q), with the coordinates c_1, c_2 or C_1, C_2.

We now define a linear mapping $\boldsymbol{p}$ of the integral space r onto the integral space R in the following manner; given an integral $y \in r$ of (q).

$$y = \lambda u + \mu v,$$

we associate with this the integral $Y \in R$ of (Q) formed with the same constants λ, μ:

$$Y = \lambda U + \mu V.$$

The image Y of y under this mapping $\boldsymbol{p}$ will be denoted by $\boldsymbol{p}y$, consequently $Y = \boldsymbol{p}y$; it is convenient also to write $y \to Y\,(\boldsymbol{p})$, or more briefly $y \to Y$. The bases (u, v), (U, V) we call the *first* and *second* bases of the linear mapping $\boldsymbol{p}$. Obviously $u \to U(\boldsymbol{p})$, $v \to V(\boldsymbol{p})$. We say that the linear mapping p is determined by the bases (u, v), (U, V) (in this order), and represent this by writing $\boldsymbol{p} = [u \to U, v \to V]$. The ratio w/W is called the *characteristic* of the linear mapping $\boldsymbol{p}$, and is denoted by the symbol $\chi\boldsymbol{p}$.

The linear mapping $\boldsymbol{p}$ is also determined by an arbitrary first basis $(\bar{u}, \bar{v})$ and second basis $(\boldsymbol{p}\bar{u}, \boldsymbol{p}\bar{v})$, hence $\boldsymbol{p} = [\bar{u} \to \boldsymbol{p}\bar{u}, \bar{v} \to \boldsymbol{p}\bar{v}]$. The Wronskians of the bases $(\bar{u}, \bar{v})(\boldsymbol{p}\bar{u}, \boldsymbol{p}\bar{v})$ differ from w, W by the same (non-zero) multiplicative constant, so the characteristic of the linear mapping $\boldsymbol{p}$ is independent of the choice of bases of $\boldsymbol{p}$.

In the linear mapping $\boldsymbol{p}$, two linearly independent integrals of the differential equation (q) are mapped onto linearly independent integrals of (Q). The images of two dependent integrals of the differential equation (q) are also dependent, and differ from each other by the same multiplicative constant as their originals.

When speaking of two linear mappings of r onto R we say that they have the same or the opposite character according as their characteristics have the same sign or not.

Let $c \neq 0$ be an arbitrary number. The mapping determined by the bases (u, v), (cU, cV) we denote by $c\boldsymbol{p}$, that is to say $c\boldsymbol{p} = [u \to cU, v \to cV]$. This linear mapping maps every integral $y \in r$ of (q) onto the integral $c.\boldsymbol{p}y \in R$, and thus onto an integral of

(Q) which is linearly dependent upon py. We say that the linear mapping cp is linearly dependent or more briefly dependent upon p; it is convenient to call this a *variation* of p. The characteristic of cp is $(1/c^2)(w/W)$, so $\chi cp = (1/c^2)\chi p$. Consequently the linear mappings p and cp have the same character. The zeros of the images of each integral $y \in r$ of (q) in the linear mappings p and cp are obviously the same.

19.2 Composition of mappings

Alongside the linear mapping p, we now consider a linear mapping P of the integral space R of (Q) onto the integral space $\bar{R}$ of another differential equation ($\bar{\text{Q}}$) in an interval $\bar{J} = (\bar{A}, \bar{B})$.

As the first basis of P we choose (U, V); let the second be $(\bar{U}, \bar{V})$; $P = [U \to \bar{U}, V \to \bar{V}]$. We denote by $\bar{W}$ the Wronskian of $(\bar{U}, \bar{V})$, so we have $\chi P = W/\bar{W}$.

Now we show that:

The composed mapping $\bar{P} = Pp$ of the integral space r onto the integral space $\bar{R}$ is the linear mapping $\bar{P} = [u \to \bar{U}, v \to \bar{V}]$. Its characteristic is the product of χP, χp, thus

$$\chi Pp = \chi P \cdot \chi p. \tag{19.1}$$

For, let $y \in r$ be an arbitrary integral of (q) and let $Y = py$, $\bar{Y} = PY$. Then with appropriate choice of the constants λ, μ we have the following relations

$$y = \lambda u + \mu v, \qquad Y = \lambda U + \mu V, \qquad \bar{Y} = \lambda \bar{U} + \mu \bar{V},$$

and the first part of our statement follows from this. The characteristic $\chi\bar{P}$ is obviously $w/\bar{W} = (w/W)(W/\bar{W})$ which proves the second part.

The inverse mapping to p, i.e. p^{-1}, of the integral space R onto the integral space r is the linear mapping $p^{-1} = [U \to u, V \to v]$. Its characteristic is the reciprocal of the characteristic of p, that is

$$\chi p^{-1} = (\chi p)^{-1}. \tag{19.2}$$

For the inverse of the mapping p, which is the mapping p^{-1} of the integral space R to the integral space r, is so defined that $p^{-1}p$ is the identity mapping e of the integral space r onto itself: $p^{-1}p = e$. From this the first part of our statement follows. Now obviously $e = [u \to u, v \to v]$, $\chi e = 1$, and when we use (1), with $P = p^{-1}$, the second part also follows.

19.3 Mapping of an integral space into itself

The above results naturally remain valid when two (or more) of the differential equations (q), (Q), ($\bar{\text{Q}}$) coincide, so that the corresponding integral spaces r, R, $\bar{R}$ coincide also.

Consider in particular the case $\bar{Q} = Q = q$, $\bar{R} = R = r$. We are then concerned with a linear mapping of the integral space r of the differential equation (q) onto itself.

Such a linear mapping p is determined by a first and second basis (u, v), (U, V) of the differential equation (q) in the above sense: $p = [u \to U, v \to V]$. Given two linear mappings $p = [u \to U, v \to V]$, $P = [U \to \bar{U}, V \to \bar{V}]$ of the integral space r onto itself, their composed mapping Pp is the linear mapping $Pp = [u \to \bar{U}, v \to \bar{V}]$ of r onto itself, and a formula such as (1) holds. The linear mapping $p^{-1} = [U \to u, V \to v]$ is the linear mapping inverse to p of the integral space r onto itself, and there holds a formula such as (2). Moreover, $p^{-1}p = e$, $\chi e = 1$.

19.4 Determination of the linear mappings from the first phases

Let $p = [u \to U, v \to V]$ be a linear mapping of the integral space r of (q) on the integral space R of (Q).

We choose arbitrary first phases α, A of the bases (u, v), (U, V) of the linear mapping p. Then from (5.26) we have

$$\left.\begin{aligned} u &= \varepsilon\sqrt{|w|}\,\frac{\sin\alpha}{\sqrt{|\alpha'|}}, & v &= \varepsilon\sqrt{|w|}\,\frac{\cos\alpha}{\sqrt{|\alpha'|}},\\ U &= \mathrm{E}\sqrt{|W|}\,\frac{\sin \mathrm{A}}{\sqrt{|\dot{\mathrm{A}}|}}, & V &= \mathrm{E}\sqrt{|W|}\,\frac{\cos \mathrm{A}}{\sqrt{|\dot{\mathrm{A}}|}}; \end{aligned}\right\} \tag{19.3}$$

in which ε, E take the values $+1$ or -1 according as the phases α, A are proper or not with respect to the bases (u, v), (U, V).

Let us take arbitrary coordinates $\lambda = \gamma\cos k_2$, $\mu = \gamma\sin k_2$ ($\gamma > 0$, $0 \leqslant k_2 < 2\pi$) and using these coordinates with respect to the bases (u, v), (U, V) let us form the integrals $y = \lambda u + \mu v$ ($\in r$), $Y = \lambda U + \mu V$ ($\in R$) of (q), (Q). Then y, Y may be expressed as follows:

$$y = k_1\frac{\sin(\alpha + k_2)}{\sqrt{|\alpha'|}}, \qquad Y = \frac{\varepsilon E}{\sqrt{|\chi p|}}\cdot k_1\frac{\sin(\mathrm{A} + k_2)}{\sqrt{|\dot{\mathrm{A}}|}} \qquad (k_1 = \varepsilon\sqrt{|w|}\gamma). \tag{19.4}$$

We see that for every choice of the phases α, A of the bases (u, v), (U, V) of p, the linear mapping p is given by the formula $y \to Y$; here y, Y always represent two integrals of the differential equations (q), (Q) defined by the formulae (4) with arbitrary constants k_1, k_2, such that $k_1 \neq 0$, $0 \leqslant k_2 < 2\pi$. We call an ordered pair (α, A) of phases of the bases (u, v), (U, V) of the linear mapping p, a *phase basis* of p. For every choice of the phase basis (α, A) of the linear mapping p, any two integrals $y \in r$, $Y = py \in R$ are expressible by the formulae (4) with the same constants k_1, k_2.

From (3) we obtain the relationship

$$\operatorname{sgn}\chi p = \operatorname{sgn}\alpha'\cdot\operatorname{sgn}\dot{\mathrm{A}}, \tag{19.5}$$

that is, the characteristic χp is positive if both phases α, A increase or decrease; it is negative if one of these increases and the other decreases.

We have seen (§ 19.1) that the linear mapping $\boldsymbol{p}$ can also be determined from an arbitrary first basis $(\bar{u}, \bar{v})$ of (q) and the second basis $(\boldsymbol{p}\bar{u}, \boldsymbol{p}\bar{v})$. As first term of a phase basis (α, **A**) of the linear mapping $\boldsymbol{p}$ we can therefore choose an arbitrary phase α of (q); then the second term **A** is determined uniquely up to an integral multiple of π.

19.5 Phase bases of composed mappings

Once again we consider, in addition to the linear mapping $\boldsymbol{p}$, a linear mapping $\boldsymbol{P}$ of the integral space R of (Q) on to the integral space $\bar{R}$ of a further differential equation $(\bar{\mathrm{Q}})$ in an interval $\bar{J} = (\bar{A}, B)$.

Let (α, **A**) be a phase basis of $\boldsymbol{p}$ and (**A**, $\bar{\mathbf{A}}$) a phase basis of $\boldsymbol{P}$. We let ε, **E**, $\bar{\mathbf{E}}$ have their usual significance in respect of the phases α, **A**, $\bar{\mathbf{A}}$.

It is easy to verify that:

The composed linear mapping $\boldsymbol{Pp}$ of the integral space r on the integral space $\bar{R}$ admits of the phase basis (α, $\bar{\mathbf{A}}$) and for any two integrals $y \in r$, $\bar{Y} = \boldsymbol{Ppy} \in \bar{R}$ we have the formulae

$$y = k_1 \frac{\sin(\alpha + k_2)}{\sqrt{|\alpha'|}}, \qquad \bar{Y} = \frac{\varepsilon \bar{E}}{\sqrt{|\chi \boldsymbol{P}| \cdot |\chi \boldsymbol{p}|}} k_1 \frac{\sin(\bar{\mathbf{A}} + k_2)}{\sqrt{|\dot{\bar{\mathbf{A}}}|}}. \tag{19.6}$$

The mapping $\boldsymbol{p}^{-1}$ (inverse to the mapping $\boldsymbol{p}$) of the integral space R onto the integral space r admits of the phase basis (**A**, α) and for any two integrals $Y \in R, y = \boldsymbol{p}^{-1} Y \in r$, we have

$$Y = k_1 \frac{\sin(\mathbf{A} + k_2)}{\sqrt{|\dot{\mathbf{A}}|}}, \qquad y = \varepsilon E \sqrt{|\chi \boldsymbol{p}|}\, k_1 \frac{\sin(\alpha + k_2)}{\sqrt{|\alpha'|}}. \tag{19.7}$$

Every phase basis of the identity mapping $\boldsymbol{e}$ of the integral space r onto itself is obviously of the form (α, $\alpha + n\pi$), where n is an integer and α is an arbitrary phase of the differential equation (q).

19.6 Determination of a mapping from arbitrary phases

In the above work we have associated with every linear mapping $\boldsymbol{p}$ of the integral space r onto the integral space R an ordered pair of phases α, **A** of the equations (q), (Q), namely the phase basis of $\boldsymbol{p}$, and this association has been done in such a way that it allows us to represent every integral $y \in r$ and its image $Y = \boldsymbol{p}y \in R$ by means of the formulae (4).

Conversely, it follows from the formulae (3), (4) that:

Arbitrary phases α, **A** of the differential equations (q), (Q) form a phase basis (α, **A**) of infinitely many linearly dependent mappings $c\boldsymbol{p}$ ($c \neq 0$), each mapping the integral space r onto the integral space R. We obtain the bases (u, v), (U, V) of a linear mapping $\boldsymbol{p}$ of this system by means of the formulae (3) with arbitrary choice of the constants ε, $\mathrm{E} = \pm 1$; w, $W (\neq 0)$, and for every integral $y \in r$ and its image $Y = \boldsymbol{p}y \in R$ there hold formulae such as (4).

19.7 Normalized linear mappings

We shall continue to use the above notation.

Let $\boldsymbol{p} = [u \to U, v \to V]$, $\boldsymbol{P} = [U \to \bar{U}, V \to \bar{V}]$ be arbitrary linear mappings of the integral space r of (q) onto the integral space R of (Q) and of the integral space R onto the integral space $\bar{R}$ of $(\bar{\mathrm{Q}})$. Moreover, let (α, A), $(\mathrm{A}, \bar{\mathrm{A}})$ be any phase bases of these linear mappings $\boldsymbol{p}$, $\boldsymbol{P}$.

Let $z \in j$, $Z \in J$ be arbitrary numbers.

We call the linear mapping $\boldsymbol{p}$ *normalized with respect to the numbers* z, Z (in this order) if it maps every integral $y \in r$ of (q) which vanishes at the point z onto an integral $Y \in R$ of (Q) which vanishes at the point Z; that is to say, when $y(z) = 0 \Rightarrow \boldsymbol{p}y(Z) = 0$.

Obviously, the linear mapping $\boldsymbol{p}$ is normalized with respect to the numbers z, Z if it maps a single integral $y \in r$ of (q) which vanishes at the point z onto an integral $Y \in R$ of (Q) which vanishes at the point Z. If therefore $y \in r$ is an arbitrary integral of (q) and $Y \in R$ its image under the linear mapping $\boldsymbol{p}$, then the linear mapping $\boldsymbol{p}$ is normalized with respect to every zero of y and every zero of Y. If, moreover, the linear mapping $\boldsymbol{p}$ is normalized with respect to the numbers z, Z then every mapping $c\boldsymbol{p}$ $(c \neq 0)$ which is dependent upon $\boldsymbol{p}$ has the same property.

We now have the important theorem:

Theorem. The linear mapping $\boldsymbol{p}$ *is normalized with respect to the numbers* z, Z *if and only if the values* $\alpha(z)$, $\mathrm{A}(Z)$ *of the terms of* (α, A) *at the points* z, Z *differ by an integral multiple of* π, *that is* $\alpha(z) - \mathrm{A}(Z) = n\pi$, *n integral.*

Proof. Let $y \in r$ be an integral of (q) which vanishes at the point z, and $Y = \boldsymbol{p}y \in R$ its image under the linear mapping $\boldsymbol{p}$. There hold therefore formulae such as (4) formed with appropriate constants k_1, k_2 and the number $\alpha(z) + k_2$ is an integral multiple of π.

If the linear mapping $\boldsymbol{p}$ is normalized with respect to the numbers z, Z, then $Y(Z) = 0$. Then the number $\mathrm{A}(Z) + k_2$ is an integral multiple of π, and the same is obviously true also of $\alpha(z) - \mathrm{A}(Z)$.

If, conversely, $\alpha(z) - \mathrm{A}(Z)$ is an integral multiple of π, then this is also true for $\mathrm{A}(Z) + k_2$, hence $Y(Z) = 0$. This completes the proof.

We can also show that: If the linear mapping $\boldsymbol{p}$ is normalized with respect to the numbers z, Z then it is also normalized with respect to every two numbers $\bar{z} \in j$, $\bar{Z} \in J$, such that $\bar{z}$ and $\bar{Z}$ are 1-conjugate with z and Z respectively. Conversely, let $\boldsymbol{p}$ be normalized with respect to z, Z and also with respect to two other numbers $\bar{z} \in j$, $\bar{Z} \in J$, where either $\bar{z}$ is conjugate with z or $\bar{Z}$ is conjugate with Z; then both these conjugacies hold, i.e. $\bar{z}$ is conjugate with z and $\bar{Z}$ with Z.

Proof. Let the linear mapping $\boldsymbol{p}$ be normalized with respect to the numbers z, Z, so that $\alpha(z) - \mathrm{A}(Z) = n\pi$, n being an integer.

(a) Let $\bar{z} \in j$, $\bar{Z} \in J$ be arbitrary numbers, with $\bar{z}$ conjugate to z and $\bar{Z}$ to Z. We have therefore $\bar{z} = \phi_\nu(z)$, $\bar{Z} = \Phi_N(Z)$, in which ϕ_ν, Φ_N are appropriate central dispersions of the differential equations (q) and (Q). Now the Abel functional equation (13.21) gives

$$\alpha(\bar{z}) = \alpha(z) + \nu\pi \cdot \operatorname{sgn} \alpha', \quad \mathbf{A}(\bar{Z}) = \mathbf{A}(Z) + N\pi \cdot \operatorname{sgn} \dot{\mathbf{A}},$$

and we see that the number $\alpha(\bar{z}) - \mathbf{A}(\bar{Z})$ is an integral multiple of π.

(b) Let $\boldsymbol{p}$ be normalized with respect to the numbers $\bar{z}$, $\bar{Z}$ and, for definiteness, assume that $\bar{z}$ is conjugate with z. Then $\alpha(\bar{z}) - \mathbf{A}(\bar{Z})$ is an integral multiple of π, and from the Abel functional equation it follows that the same is true for the number $\alpha(\bar{z}) - \alpha(z)$. We have therefore $\mathbf{A}(\bar{Z}) - \mathbf{A}(Z) = m\pi$, m an integer, and when we take account of the Abel functional equation, this gives $\bar{Z} = \Phi_M(Z)$ with $M = m \operatorname{sgn} \dot{\mathbf{A}}$. This completes the proof.

It is easy to verify the following statements:—

(i) Let the linear mappings $\boldsymbol{p}$, $\boldsymbol{P}$ be normalized with respect to z, Z and Z, $\bar{Z}$ respectively, then the composed linear mapping $\boldsymbol{Pp}$ is normalized with respect to z, $\bar{Z}$.

(ii) If the linear mapping $\boldsymbol{p}$ is normalized with respect to z, Z then the inverse linear mapping $\boldsymbol{p}^{-1}$ is normalized with respect to Z, z.

(iii) If the identity mapping $\boldsymbol{e}$ of the linear space r on itself is normalized with respect to the numbers z, Z then these are conjugate.

19.8 Canonical phase bases

We now show that:—

If the linear mapping $\boldsymbol{p}$ of the integral space r of (q) onto the integral space R of (Q) is normalized with respect to the numbers $z \in j$, $Z \in J$, then it possesses a phase basis $(\alpha, \mathbf{A})$ whose terms vanish at the point z and Z respectively, i.e. $\alpha(z) = 0$, $\mathbf{A}(Z) = 0$.

Proof. Assume that $\boldsymbol{p}$ is *normalized* with respect to the numbers z, Z.

We know that it is possible to choose an arbitrary phase of (q) as first term of a phase basis of $\boldsymbol{p}$, whereupon the second term is uniquely determined up to an integral multiple of π. Let us therefore choose as first term of a phase basis $(\alpha, \mathbf{A}_0)$ of $\boldsymbol{p}$ any phase α of (q) which vanishes at the point z: $\alpha(z) = 0$. Then, since $\boldsymbol{p}$ is *normalized* with respect to z, Z, we have $\mathbf{A}_0(Z) = n\pi$, n being an integer. If we now replace the phase $\mathbf{A}_0$ of (Q) by $\mathbf{A} \equiv \mathbf{A}_0 - n\pi$, then the phases α, $\mathbf{A}$ form a phase basis of $\boldsymbol{p}$ with the desired property.

We call a phase basis $(\alpha, \mathbf{A})$ of a linear mapping $\boldsymbol{p}$ which is normalized with respect to the numbers z, Z a *canonical phase basis with respect to the numbers* z, Z, if its terms α, $\mathbf{A}$ vanish at the points z, Z, i.e. $\alpha(z) = 0$, $\mathbf{A}(Z) = 0$.

The following facts are easily seen:

Let $(\alpha, \mathbf{A})$, $(\mathbf{A}, \bar{\mathbf{A}})$ be canonical phase bases of the linear mappings $\boldsymbol{p}$, $\boldsymbol{P}$ with respect to the numbers z, Z and Z, $\bar{Z}$ respectively. Then $(\alpha, \bar{\mathbf{A}})$ is a canonical phase basis of the linear mapping $\boldsymbol{Pp}$ with respect to the numbers z, $\bar{Z}$.

If $(\alpha, \mathbf{A})$ is a canonical phase basis of the linear mapping $\boldsymbol{p}$ with respect to the numbers z, Z then $(\mathbf{A}, \alpha)$ is a canonical phase basis of the inverse linear mapping $\boldsymbol{p}^{-1}$ with respect to the numbers Z, z.

Every canonical phase basis of the identity mapping $\boldsymbol{e}$ of the integral space r on itself with respect to the numbers z, $\phi_\nu(z)$ has the form $(\alpha, \alpha\phi_{-\nu})$, in which α is a phase of the differential equation (q) which vanishes at the point z: $\alpha(z) = 0$.