16 Differential equations with coincident central dispersions of the $\varkappa$-th and $(\varkappa + 1)$-th kinds $(\varkappa = 1,3)$

In this paragraph we shall be concerned with differential equations (q) whose central dispersions of the first and second kinds ϕ_ν and ψ_ν or of the third and fourth kinds χ_ρ and ω_ρ coincide ($\nu = 0, \pm 1, \pm 2, \ldots$; $\rho = \pm 1, \pm 2, \ldots$). Obvious examples of such differential equations are those equations (q) whose carrier q is a negative constant in the interval $(-\infty, \infty)$. A carrier q with the property $\phi_\nu = \psi_\nu$ or $\chi_\rho = \omega_\rho$ we shall call, for brevity, an *F-carrier* or an *R-carrier* respectively.

Consider an oscillatory differential equation (q) in the interval $j = (a, b)$ and assume that $q < 0$ for all $t \in j$. We denote by ϕ, ψ, χ, ω the fundamental dispersions of the corresponding kinds; these are thus defined in the entire interval j.

A convenient starting point for the theory of F- and R-carriers is provided by the properties of normalized polar functions (§ 6). Let $\theta(t) = \beta(t) - \alpha(t)$ be a polar function of the carrier q, and $h(\alpha), -k(\beta), p(\zeta)$ be the corresponding 1-, 2-, 3-normalized polar functions. The functions $h, -k, p$ are therefore defined in the interval $J = (-\infty, \infty)$, and the following relations hold at every point $t \in j$

$$\left.\begin{aligned} \beta(t) &= \alpha(t) + h\alpha(t), \quad \alpha(t) = \beta(t) + k\beta(t), \\ \beta(t) - \alpha(t) &= p\zeta(t), \quad \zeta(t) = \alpha(t) + \beta(t), \\ n\pi < h\alpha(t) &= -k\beta(t) = p\zeta(t) < (n+1)\pi; \quad n \text{ integral} \end{aligned}\right\} \tag{16.1}$$

I. Theory of F-Carriers

16.1 Characteristic properties

First we note that from the formulae (12.2), (12.3) it follows that q is an F-carrier if and only if its fundamental dispersions of the 1st and 2nd kinds coincide; $\phi = \psi$ for all $t \in j$.

In the development of the theory which follows we shall confine ourselves generally to the properties of the 1-normalized polar function h. We can reach the same objective by making use of suitable properties of the 2- or 3-normalized polar functions $-k, p$, but we shall content ourselves in this respect with a few comments as opportunity offers.

Theorem. The carrier q is an F-carrier, if and only if the 1*-normalized polar function h has period* π.

Proof. (a) Let q be an F-carrier. Then in the interval j we have $\phi(t) = \psi(t)$. Then, taking account of (1),

$$h[\alpha(t) + \varepsilon\pi] = h\alpha\phi(t) = \beta\phi(t) - \alpha\phi(t) = \beta\psi(t) - \alpha\phi(t)$$
$$= (\beta(t) + \varepsilon\pi) - (\alpha(t) + \varepsilon\pi) = h\alpha(t)$$
$$(\varepsilon = \operatorname{sgn} \alpha' = \operatorname{sgn} \beta'),$$

and consequently $h(\alpha + \pi) = h(\alpha)$ for $\alpha \in (-\infty, \infty)$.

(b) Let the polar function h have period π, so $h(\alpha + \pi) = h(\alpha)$ for $\alpha \in (-\infty, \infty)$. Then at every point $t \in j$,

$$\beta\phi(t) = \alpha\phi(t) + h\alpha\phi(t) = \alpha(t) + \varepsilon\pi + h[\alpha(t) + \varepsilon\pi] = \alpha(t) + \varepsilon\pi + h\alpha(t) = \beta(t) + \varepsilon\pi,$$

and it follows that $\psi(t) = \phi(t)$.

We have thus determined all F-carriers:

The F-carriers are precisely those which are derived by the formula (6.29) *from normalized polar functions h with period π in the interval* $(-\infty, \infty)$ $(h^{\backslash} > -1)$.

Similarly, the F-carriers can be characterized by periodicity with period π or 2π of the 2- or 3-normalized polar functions $-k$ or p.

We have also the following result (due to M. Laitoch [41].

Theorem. The carrier q is an F-carrier if and only if its fundamental dispersion of the first kind, ϕ, is linear:

$$\phi(t) = ct + k \qquad (c > 0,\ k = \text{const}). \tag{16.2}$$

This follows immediately from the above results and (13.32).

16.2 Domain of definition of F-carriers

We now wish to determine the intervals of definition of the F-carriers.

Let q be an F-carrier. The 1-normalized polar function h is therefore periodic with period π, and formula (2) holds. From (13.31) we obtain

$$c = \exp 2\int_0^{\pi} \cot h(\rho)\, d\rho. \tag{16.3}$$

Now the formula (2) gives, for the ν-th central dispersion $\phi_\nu(t)$, $\nu = 0, \pm 1, \pm 2, \ldots$.

$$\phi_\nu(t) = c^\nu t + k\,\frac{c^\nu - 1}{c - 1} \qquad \text{or} \qquad \varphi_\nu(t) + \nu k, \tag{16.4}$$

according as $c \neq 1$ or $c = 1$.

From the facts that $\phi_\nu(t) \to b$ as $\nu \to \infty$, and $\phi_\nu(t) \to a$ as $\nu \to -\infty$, we have (from (4)): in the case $c > 1$

$$b = \infty, \quad a = -k/(c - 1), \qquad \text{hence} \qquad j = (a, \infty), \quad a \text{ finite;}$$

in the case $c < 1$

$$b = k/(1 - c), \quad a = -\infty, \qquad \text{hence} \qquad j = (-\infty, b), \quad b \text{ finite};$$

in the case $c = 1$

$$k > 0, \quad a = -\infty, \quad b = \infty.$$

We have thus determined the intervals of definition of all F-carriers:

The interval of definition j of the F-carrier q is unbounded on one or both sides according as

$$\int_0^{\pi} \cot h(\rho)\, d\rho \neq 0 \qquad \text{or} \qquad = 0.$$

16.3 Elementary carriers

We remind the reader that this term is applied to carriers whose first phases are elementary (§ 8.4). Now we show that:

The carrier q is elementary if and only if the 1*-normalized polar function h has period π and satisfies the following conditions*

$$\left.\begin{aligned} &\int_0^{\pi} \cot h(\rho)\, d\rho = 0, \qquad \int_0^{\varepsilon\pi} \left(\exp 2 \int_0^{\sigma} \cot h(\rho)\, d\rho\right) d\sigma = \pi\alpha_0'. \\ &(\alpha_0' = \alpha'[\alpha^{-1}(0)]; \qquad \varepsilon = \operatorname{sgn} \alpha_0'). \end{aligned}\right\} \tag{16.5}$$

For, if the carrier q is elementary, then its fundamental dispersion of the 1st kind $\phi(t)$ has the form (2) with $c = 1$, $k = \pi$. The 1-normalized polar function h has therefore period π, and from (13.31), (13.30) the relations (5) follow. The second part of the theorem is proved similarly.

We have thus determined all elementary carriers:

The elementary carriers q are precisely those derived by the formula (6.29) *from* 1-*normalized polar functions h defined in the interval* $(-\infty, \infty)$, *having period π, and satisfying the conditions* (5) $(h^{\setminus} > -1)$.

Similarly, the elementary carriers may be expressed in terms of 2- or 3-normalized polar functions $-k$ or p, being given explicitly by the formulae (6.36) or (6.41).

16.4 Kinematic properties of F-carriers

We now make use of the kinematic significance of integrals of the differential equation (q), described in § 1.5, as applied to an F-carrier q.

Let q be an F-carrier. Consider two points P, P' lying on the oriented straight line G, whose motion is given by integrals, u, v of the differential equation (q).

Since the differential equation (q) is oscillatory, the motion of each of these points consists of an oscillation about the fixed point (the origin) O of the straight line G.

We assume that at any instant t_0 at which the point P passes through O, the point P' does not coincide with O and its velocity is zero. At the instant t_0, therefore, the point P' is at a relative maximum distance from O. The times at which the point P passes through the origin O are obviously $\phi_\nu(t_0)$, and those at which the point P' is at a maximum distance from O are $\psi_\nu(t_0)$; $\nu = \ldots, -1, 0, 1, \ldots$. Since q is an F-carrier, we have $\phi_\nu(t_0) = \psi_\nu(t_0)$.

We see therefore that:

The oscillations of the points P, P' about the origin O are such that the point P passes through the origin O when the point P' is at a relative maximum distance from O.

II. Theory of R-Carriers

16.5 Characteristic properties of R-carriers

From the formulae in § 12.4 we have, for all $t \in j$,

$$\left.\begin{aligned} \chi\omega &= \psi, & \omega\chi &= \phi, \\ \omega_n &= \phi^{n-1}\omega, & \omega_{-n} &= \phi_{-1}^{n-1}\chi^{-1}, \\ \chi_n &= \psi^{n-1}\chi, & \chi_{-n} &= \psi_{-1}^{n-1}\omega^{-1} \\ (n = 1, 2, \ldots; &\ \ \phi_{-1} = \phi^{-1}, & \psi_{-1} &= \psi^{-1}). \end{aligned}\right\} \tag{16.6}$$

Hence, from $\chi = \omega$ it follows that $\phi = \psi$ and $\chi_\rho = \omega_\rho$ for $\rho = \pm 1, \pm 2, \ldots$.

This gives the result:

q is an R-carrier if and only if its fundamental dispersions of the third and fourth kinds coincide: $\chi = \omega$ for $t \in j$. An R-carrier is always an F-carrier.

Theorem. The carrier q is an R-carrier if and only if the 1-normalized polar function h satisfies the following relation in the interval $J = (-\infty, \infty)$:

$$h\alpha + h[\alpha + h\alpha - n\pi] = (2n + 1)\pi. \tag{16.7}$$

Proof. If (7) is satisfied, then on applying it at the point $\alpha + h\alpha - n\pi$, there follows the π-periodicity of h:

$$h(\alpha + \pi) = h\alpha. \tag{16.8}$$

We shall now give the proof first for the case $n = 0$. We then have $0 < \beta - \alpha < \pi$, so the corresponding Abel functional equations (13.18), (13.20) hold.

(a) Let q be an R-carrier, so that $\chi = \omega$. Then, in the interval j, we have

$$\beta\chi(t) = \alpha\chi(t) + h\alpha\chi(t)$$

and further, from (13.18), (13.20),

$$\alpha(t) + \pi = \beta(t) + h\left[\beta(t) + \frac{1}{2}(\varepsilon - 1)\pi\right].$$

Since however q is an F-carrier, the function h has period π, and on taking account of (1) the last relationship gives the formula (7) for the case $n = 0$.

(b) Now let the relation (7) be satisfied when $n = 0$; then, from (8), the function h is π-periodic. From (1) and (13.20), we have

$$\beta\omega(t) = \alpha\omega(t) + h\alpha\omega(t) = \beta(t) + \frac{1}{2}(\varepsilon - 1)\pi + h\left[\beta(t) + \frac{1}{2}(\varepsilon - 1)\pi\right]$$
$$= \alpha(t) + \frac{1}{2}(\varepsilon + 1)\pi - \pi + h\alpha(t) + h\left[\alpha(t) + h\alpha(t) + \frac{1}{2}(\varepsilon - 1)\pi\right].$$

Since the function h satisfies (7) and has period π, the last expression, in view of (13.18), equal to $\beta\chi(t)$. We have, therefore $\chi = \omega$ for $t \in j$.

The extension of the proof to the general case, in which n is any integer, is simple. We set

$$h\alpha = h_0\alpha + n\pi. \tag{16.9}$$

Then h_0 is a 1-normalized polar function of the carrier q with the property $0 < h_0 < \pi$.

If q is an R-carrier, then from (a) the function h_0 satisfies the condition

$$h_0\alpha + h_0[\alpha + h_0\alpha] = \pi; \tag{16.10}$$

and from this and (9) the relation (7) follows.

If, conversely, the condition (7) is satisfied, then (10) holds; from that we deduce (using (b)) that q is an F-carrier. This completes the proof.

We have thus determined all the R-carriers;

The R-carriers are precisely those derived by the formula (6.29) *from the* 1-*normalized polar functions h defined in the interval* $(-\infty, \infty)$ *and satisfying* (7) $(h^{\setminus} > -1)$.

Similarly, the R-carriers can be determined by means of 2- or 3-normalized polar functions satisfying the conditions

$$k\beta + k[\beta + k\beta + n\pi] = -(2n + 1)\pi \tag{16.11}$$

and

$$p\zeta + p(\zeta + \pi) = (2n + 1)\pi, \tag{16.12}$$

being given by the formulae (6.36) and (6.41).

16.6 Further properties of R-carriers

The following study takes us further into the properties of R-carriers.

Let q be an R-carrier in the interval j $(= (a, b))$.

We consider an integral curve $\mathfrak{R}$ of the differential equation (q) with the parametric co-ordinates $u(t)$, $v(t)$ in which, for precision, we take the Wronskian $w = uv' - u'v < 0$. We denote the origin of the coordinate system by O.

Let $P, \tilde{P} \in \mathfrak{R}$ be points determined by the parameters t, $\chi(t)$ where $t \in j$ is arbitrary. *Our interest will centre upon the area* Δ *of the triangle* $PO\tilde{P}$.

Obviously

$$2\Delta = r(t) \cdot r\chi(t) \cdot \sin\theta(t); \tag{16.13}$$

where $r(t)$, $r\chi(t)$ are the lengths of the vectors $\overrightarrow{OP}$, $\overrightarrow{O\tilde{P}}$ and $\theta(t)$ is the angle formed by the latter.

Let α be a proper first phase of the basis (u, v). Since $-w > 0$ we have $\alpha' > 0$; we also have $\chi(t) > t$, consequently $\alpha\chi(t) > \alpha(t)$ and since $0 \leqslant \theta(t) < 2\pi$,

$$\theta(t) = \alpha\chi(t) - \alpha(t) + 2n\pi, \qquad 0 \geqslant n \text{ integral.} \tag{16.14}$$

Moreover, let β be the proper second phase of (u, v) neighbouring to α, so that $0 < \beta - \alpha < \pi$.

We write the relation (14) as follows

$$\theta(t) = [\alpha\chi(t) - \beta(t)] + [\beta(t) - \alpha(t)] + 2n\pi$$

and apply the formulae (13.20). Since $\varepsilon = 1$ and $0 < \beta - \alpha < \pi$, we have

$$\theta(t) = \beta(t) - \alpha(t), \tag{16.15}$$

so θ is that polar function of the basis (u, v) generated by α and lying between 0 and π.

To help on the development of this study, it is convenient to quote here the following formulae:

$$\theta\chi = -\theta + \pi, \qquad \alpha' = \beta'\chi \cdot \chi', \qquad \beta' = \alpha'\chi \cdot \chi' \qquad [(13.18), (13.20)] \tag{16.16}$$

$$r\chi \cdot r'\chi = -rr' \quad [(16) \text{ and } (6.8)] \tag{16.17}$$

$$\alpha' = \frac{w \cdot q\chi}{s^2\chi}\chi', \qquad \beta' = \frac{-w}{r^2\chi}\chi' \quad [(16) \text{ and } (5.14), (5.23)] \tag{16.18}$$

Logarithmic differentiation of (13) shows that

$$\frac{\Delta'}{\Delta} = \frac{r'}{r} + \frac{r'\chi}{r\chi}\chi' + \cot\theta \cdot \theta',$$

and the formulae (6.8), (5.14) and then (17), (18) give

$$\cot\theta \cdot \theta' = -\frac{1}{w}rr'(\beta' - \alpha') = -\frac{1}{w}rr' \cdot \beta' - \frac{r'}{r} = -\frac{r'\chi}{r\chi}\chi' - \frac{r'}{r}.$$

Consequently $\Delta' = 0$, and we have the result:

Theorem. The area Δ of the triangle $PO\tilde{P}$ is constant throughout the curve $\mathfrak{R}$.

16.7 Connection between *R*-carriers and Radon curves

From the relationships [(16) and (5.28)]

$$rs \cdot \sin\theta = -w, \qquad r\chi \cdot s\chi \cdot \sin\theta = -w \tag{16.19}$$

there follows, when we take account of (13),

$$r\chi = \frac{2\Delta}{-w}s, \qquad s\chi = \frac{-w}{2\Delta}r. \tag{16.20}$$

Moreover we have from (13.20) and (18)

$$\mathrm{W}\alpha\chi = \mathrm{W}\beta, \qquad \mathrm{W}\beta\chi = \mathrm{W}\alpha \pm \pi, \tag{16.21}$$

in which the sign + or − must be taken according as $0 \leqslant \mathrm{W}\alpha < \pi$ or $\pi \leqslant \mathrm{W}\alpha < 2\pi$.

We now apply to the integral curve $\mathfrak{K}$ the transformation $\boldsymbol{R}$ (§ 6.1) which consists of the inversion $\boldsymbol{K}_{\sqrt{2\Delta}}$, followed by a quarter rotation about O in the positive sense.

The curve $\mathfrak{K}$ is then transformed into a curve $\bar{\mathfrak{K}}$: the point $P \in \mathfrak{K}$ goes over into the point $\bar{P} \in \bar{\mathfrak{K}}$, while the corresponding amplitudes $\bar{r}$, s and angles $\mathrm{W}\alpha$, $\mathrm{W}\beta$; $\bar{\alpha}$, $\bar{\beta}$ are transformed as follows [(6.5)]

$$\bar{r} = \frac{2\Delta}{-w}s, \qquad \bar{\alpha} = \mathrm{W}\beta, \quad \bar{\beta} = \mathrm{W}\alpha \pm \pi, \tag{16.22}$$

in which we take the sign + or − according as $0 \leqslant \mathrm{W}\alpha < \pi$ or $\pi \leqslant \mathrm{W}\alpha < 2\pi$.

Comparing this with (20), (21), gives

$$\bar{r} = r\chi, \qquad \bar{\alpha} = \mathrm{W}\alpha\chi, \qquad \bar{\beta} = \mathrm{W}\beta\chi. \tag{16.23}$$

Clearly, the transformation R takes the curve $\mathfrak{K}$ into itself, so

The integral curves of an R-carrier are Radon curves.

16.8 Connection between *R*- and *F*-carriers

The second formula (18), taken together with (5.23) gives the following formula holding in the interval j

$$\frac{\chi'}{r^2\chi} = -\frac{q}{s^2} \tag{16.24}$$

and moreover, using (20),

$$\chi' = -d^2q \qquad \left(d = \frac{2\Delta}{-w}\right). \tag{16.25}$$

This formula is due to E. Barvínek ([2]). It follows that for t_0, $t \in j$,

$$\chi(t) = \chi(t_0) - d^2\int_{t_0}^{t} q(\sigma)\,d\sigma. \tag{16.26}$$

Similarly the first formula (18) and (5.14) show that

$$\chi' = -\frac{1}{d^2}\frac{1}{q\chi}; \tag{16.27}$$

thus for t_0, $t \in j$,

$$\chi(t) = \chi(t_0) - \frac{1}{d^2}\int_{t_0}^{t}\frac{d\sigma}{q\chi(\sigma)}. \tag{16.28}$$

From (25) and (27) we see that the product of the values of the R-carrier q at any two points t, $\chi(t) \in j$ is constant:

$$q(t)q\chi(t) = \frac{1}{d^4}. \tag{16.29}$$

From the formula (26) we have

$$\chi\chi(t) = \chi(t_0) - d^2 \int_{t_0}^{\chi(t_0)} q(\sigma)\, d\sigma - d^2 \int_{\chi(t_0)}^{\chi(t)} q(\sigma)\, d\sigma.$$

If the last integral is transformed by means of the substitution $\sigma = \chi(\tau)$ and we apply formulae (25) and (29) then we obtain

$$(\phi(t) =)\ \chi\chi(t) = t + k \tag{16.30}$$

with a determinate constant k (> 0). Since $\chi\chi = \phi$, this formula shows that *every R-carrier belongs to the set of F-carriers defined in the interval* $j = (-\infty, \infty)$, (§ 16.2, $c = 1$).

16.9 Kinematic properties of R-carriers

Let q be an R-carrier.

We consider two points P, P' lying on the oriented straight line G, whose motions follow the integrals u, v of the differential equation (q). Let the positions of the points P, P' at an instant t_0 be such that the point P passes through the origin O when P' is at a relative maximum distance from O. Since q is an R-carrier, and consequently also an F-carrier, we have the situation described in § 16.4. Now the instants at which the point P is at its greatest distance from O are $\chi_\rho(t_0)$ and those at which the point P' passes through the origin O are $\omega_\rho(t_0)$: $\rho = \ldots, -1, 1, \ldots$. But since q is an R-carrier, we have $\chi_\rho(t_0) = \omega_\rho(t_0)$. Thus:

The oscillations of the points P, P′ about the origin O are such that each of these passes through the origin at the instant when the other is at a relative maximum distance from the origin.