12 Introduction to the theory of central dispersions

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We now wish to study certain functions of an independent variable which we shall call central dispersions of the first, second, third and fourth kinds. The central dispersion of the κ-th kind ($\kappa = 1, 2, 3, 4$) occur only in differential equations with κ-conjugate numbers. In order to simplify our study we shall for the rest of this Chapter A always assume that the differential equation (q) under consideration has conjugate numbers of all four kinds; we shall also assume that the carrier q is always negative in its interval of definition: $q \leqslant 0$. This assumption is not necessary when considering conjugate numbers of the first kind.

12.1 Some preliminaries

We consider a differential equation (q), $t \in j = (a, b)$. According to our assumption the differential equation (q) admits of conjugate numbers of all four kinds, and we have $q < 0$ for all $t \in j$. According to § 3.11, for each kind κ (= 1, 2, 3, 4), the numbers $t \in j$ which possess a ν-th left or right κ-conjugate number form an open interval $i_{\kappa,\nu}$ or $j_{\kappa,\nu}$; $\nu = 1, 2, \ldots$. These intervals $i_{\kappa,\nu}$, $j_{\kappa,\nu}$ were fully described in that paragraph. We know that each interval $i_{\kappa,\nu}$, $j_{\kappa,\nu}$ is a sub-interval of j, and we recall the following property: if the differential equation (q) is left or right oscillatory, then all the intervals $i_{\kappa,\nu}$ or $j_{\kappa,\nu}$ respectively coincide with j; if the differential equation (q) is oscillatory then all the intervals $i_{\kappa,\nu}, j_{\kappa,\nu}$ coincide with j ($\kappa = 1, 2, 3, 4$; $\nu = 1, 2, \ldots$).

12.2 Definition of the central dispersions

Let κ be one of the numbers 1, 2, 3, 4 and, let n_κ (= n) be a positive integer; we assume that in the interval j there are numbers for which the n-th right or left κ-conjugate number exists; such numbers consequently make up the interval $j_{\kappa,n}$ or $i_{\kappa,n}$. If, for instance, the differential equation (q) is of finite type (m), $m \geqslant 2$, then we have $n_1 \leqslant m$.

1. Let $\kappa = 1$. In the interval $j_{1,n}(i_{1,n})$ we define the function ϕ_n (ϕ_{-n}) as follows: For $t \in j_{1,n}$ ($t \in i_{1,n}$) let $\phi_n(t)$ ($\phi_{-n}(t)$) be the n-th right (left) number conjugate of the first kind with t. $\phi_n(t)$ ($\phi_{-n}(t)$) is therefore the n-th zero, lying to the right (left) of t, of any integral of the differential equation (q) which vanishes at the point t.

We call the function ϕ_n (ϕ_{-n}) the n-th ($-n$-th) *central dispersion of the first kind* or the 1-*central dispersion with the index* n ($-n$). In the particular case $n = 1$ we speak of the *fundamental dispersion of the first kind.* The fundamental dispersion of the first kind, ϕ_1, is therefore defined in the interval $j_{1,1}$ and its value $\phi_1(t)$ represents the first zero after t of every integral of the differential equation (q) which vanishes at the point t.

2. Let $\kappa = 2$. In the interval $j_{2,n}$ ($i_{2,n}$) we define the function ψ_n (ψ_{-n}) as follows: For $t \in j_{2,n}$ ($t \in i_{2,n}$) let $\psi_n(t)$ ($\psi_{-n}(t)$) be the n-th right (left) number conjugate of the second kind with t. $\psi_n(t)$ ($\psi_{-n}(t)$) is therefore the n-th zero lying to the right (left) of t of the derivative of any integral of the differential equation (q) whose derivative vanishes at the point t.

We call the function ψ_n (ψ_{-n}) the n-th ($-n$-th) *central dispersion of the second kind or the* 2*-central dispersion with the index* n ($-n$). For $n = 1$ we speak of *the fundamental dispersion of the second kind.* The fundamental dispersion of the second kind, ψ_1, is therefore defined in the interval $j_{2,1}$, and its value $\psi_1(t)$ represents the first zero following t of the derivative of every integral of the differential equation (q) whose derivative vanishes at the point t.

We see that *if the differential equation* (q) *has the associated differential equation* ($\hat{q}_1$) *then the* 2*-central dispersions of* (q) *coincide with the* 1*-central dispersions of* ($\hat{q}_1$).

3. Let $\kappa = 3$. In the interval $j_{3,n}$ ($i_{3,n}$) we define the functions χ_n (χ_{-n}) as follows: For $t \in j_{3,n}$ ($t \in i_{3,n}$) let $\chi_n(t)$ ($\chi_{-n}(t)$) be the n-th right (left) conjugate number of the third kind with t. $\chi_n(t)$ ($\chi_{-n}(t)$) is therefore the n-th zero to the right (left) of t of the derivative of any integral of the differential equation (q) which vanishes at the point t.

We call the function χ_n (χ_{-n}) the n-th ($-n$-th) *central dispersion of the third kind* or the 3*-central dispersion with the index* n ($-n$). For $n = 1$ we speak of the *fundamental dispersion of the third kind.* The fundamental dispersion of the third kind, χ_1 is therefore defined in the interval $j_{3,1}$ and its value $\chi_1(t)$ represents the first zero occurring after t of the derivative of every integral of the differential equation (q) which vanishes at the point t.

4. Finally let $\kappa = 4$. In the interval $j_{4,n}$ ($i_{4,n}$) we define the function ω_n (ω_{-n}) as follows: For $t \in j_{4,n}$ ($t \in i_{4,n}$) let $\omega_n(t)$ ($\omega_{-n}(t)$) be the n-th right (left) conjugate number of the fourth kind with t. $\omega_n(t)$ ($\omega_{-n}(t)$) is therefore the n-th zero lying to the right (left) of t of every integral of the differential equation (q) whose derivative vanishes at the point t.

We call the function ω_n (ω_{-n}) the n-th ($-n$-th) *central dispersion of the fourth kind* or the 4*-central dispersion with the index* n ($-n$). For $n = 1$ we speak of the *fundamental dispersion of the fourth kind.* The fundamental dispersion of the fourth kind, ω_1, is therefore defined in the interval $j_{4,1}$ and its value $\omega_1(t)$ represents the first zero after t of each integral of the differential equation (q) whose derivative vanishes at the point t.

The terminology used for central dispersions is intended as a reminder of the distribution or dispersion of the zeros of integrals of the differential equation (q) and their derivatives. The adjective "central" refers to certain properties of central dispersions of the first and second kinds which are related to the group-theoretical concept of the "centre" (§ 21.6, 4, § 21.7).

12.3 Central dispersions of oscillatory differential equations (q)

The central dispersions which we have just defined exist in various different intervals, according to the kind and index, these intervals generally being proper sub-intervals of j. If the differential equation (q) is oscillatory then the interval of definition of every

central dispersion coincides with j. Because of this simplification we shall concern ourselves in what follows with oscillatory differential equations only. We shall therefore assume that the differential equation (q) is oscillatory in its interval of definition $j = (a, b)$, also that $q < 0$ for all $t \in j$.

In this case the integrals of the differential equation (q) have infinitely many zeros which cluster towards a and b. Moreover in the interval j there exist four countable systems of central dispersions, namely the central dispersions of the first, second, third and fourth kinds, ϕ_ν, ψ_ν, χ_ν, ω_ν: $\nu = \pm 1, \pm 2, \ldots$. It is convenient also to introduce the zero-th central dispersions of the first and second kinds by setting $\phi_0(t) = t$, $\psi_0(t) = t$ for all $t \in j$.

By the above definitions, the values of the central dispersions at an arbitrary point $t \in j$ represent the zeros of integrals of the differential equation (q) or of their derivatives; specifically, of integrals which either vanish or have their derivatives vanishing at the point t. See Fig. 3.

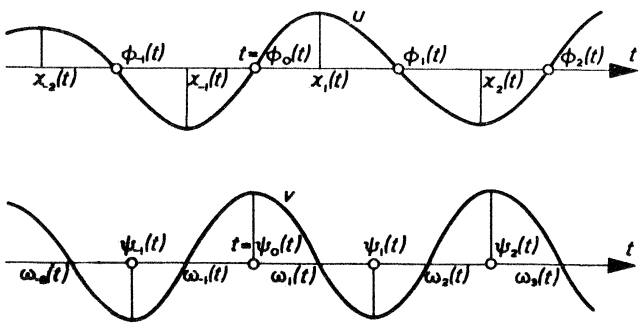

Figure 3

12.4 Relations between central dispersions

Obviously, for every point $t \in j$ we have the following relations

$$\left.\begin{aligned} &\cdots < \chi_{-2}(t) < \phi_{-1}(t) < \chi_{-1}(t) < t < \chi_1(t) < \phi_1(t) < \chi_2(t) < \cdots, \\ &\cdots < \omega_{-2}(t) < \psi_{-1}(t) < \omega_{-1}(t) < t < \omega_1(t) < \psi_1(t) < \omega_2(t) < \cdots. \end{aligned}\right\} \quad (12.1)$$

From now on we shall often employ the following notation: For two functions f, g defined in the interval j, we shall denote the composite function $f[g(t)]$, by fg. Also, f^{-1} will denote the inverse function to f, when this exists. If ν is an integer, then f^ν denotes the ν-th or $-\nu$-th iterated function f or f^{-1}, according as $\nu > 0$ or $\nu < 0$; that is $\underbrace{ff \ldots f}_{\nu}$ or $\underbrace{f^{-1}f^{-1} \ldots f^{-1}}_{-\nu}$. Finally we set $f^0 = t$.

Moreover let Φ, Ψ, X, Ω denote respectively the set of all central dispersions of the first, second, third, fourth kinds and Γ the union of all these sets: $\Gamma = \Phi \cup \Psi \cup X \cup \Omega$.

Between central dispersions of the same kind and those of different kinds there exist various relationships resulting from the composition of these functions. We set out these relationships schematically in the following "multiplication table":—

	Φ	Ψ	X	Ω	
Φ	Φ	—	—	Ω	
Ψ	—	Ψ	X	—	
X	X	—	—	Ψ	
Ω	—	Ω	Φ	—	

The significance of this table is as follows: Composition of two central dispersions $a \in A$, $b \in B$ (A, B each denoting one of the sets Φ, Ψ, X, Ω) either gives a function which is not a central dispersion or gives a central dispersion ab from the set C which stands at the intersection of the A row and B column; i.e. $ab \in C$.

Now we give these relationships more precisely.

Let μ, ν and $\rho \neq 0$, $\sigma \neq 0$ be arbitrary integers.

1. $\phi_\mu \phi_\nu = \phi_{\mu+\nu}$. From this relation, it follows that

$$\left.\begin{aligned} \phi_0\phi_\nu = \phi_\nu\phi_0 = \phi_\nu, \qquad & \phi_\nu\phi_{-\nu} = \phi_0\,(= t), \\ \phi_1\phi_\nu = \phi_\nu\phi_1 = \phi_{\nu+1}, \qquad & \phi_\nu = \phi_1^\nu. \end{aligned}\right\} \tag{12.2}$$

2. $\psi_\mu\psi_\nu = \psi_{\mu+\nu}$. Hence

$$\left.\begin{aligned} \psi_0\psi_\nu = \psi_\nu\psi_0 = \psi_\nu, \qquad & \psi_\nu\psi_{-\nu} = \psi_0\,(= t). \\ \psi_1\psi_\nu = \psi_\nu\psi_1 = \psi_{\nu+1}, \qquad & \psi_\nu = \psi_1^\nu. \end{aligned}\right\} \tag{12.3}$$

3. $\varphi_\nu\omega_\rho = \begin{cases} \omega_{\nu+\rho} & \text{for } \rho > 0,\ \nu \geqslant -\rho+1 \text{ and for } \rho < 0,\ \nu \leqq -\rho-1; \\ \omega_{\nu+\rho-1} & \text{for } \rho > 0,\ \nu \leqslant -\rho; \\ \omega_{\nu+\rho+1} & \text{for } \rho < 0,\ \nu \geqslant -\rho. \end{cases}$

4. $\psi_\nu\chi_\rho = \begin{cases} \chi_{\nu+\rho} & \text{for } \rho > 0,\ \nu \geqslant -\rho+1 \text{ and for } \rho < 0,\ \nu \leqq -\rho-1; \\ \chi_{\nu+\rho-1} & \text{for } \rho > 0,\ \nu \leqslant -\rho; \\ \chi_{\nu+\rho+1} & \text{for } \rho < 0,\ \nu \geqslant -\rho. \end{cases}$

5. $\chi_\rho\varphi_\nu = \begin{cases} \chi_{\rho+\nu} & \text{for } \rho > 0,\ \nu \geqslant -\rho+1 \text{ and for } \rho < 0,\ \nu \leqq -\rho-1; \\ \chi_{\rho+\nu-1} & \text{for } \rho > 0,\ \nu \leqslant -\rho; \\ \chi_{\rho+\nu+1} & \text{for } \rho < 0,\ \nu \geqslant -\rho. \end{cases}$

6. $\omega_\rho\psi_\nu = \begin{cases} \omega_{\rho+\nu} & \text{for } \rho > 0,\ \nu \geqslant -\rho+1 \text{ and for } \rho < 0,\ \nu \leqq -\rho-1; \\ \omega_{\rho+\nu-1} & \text{for } \rho > 0,\ \nu \leqslant -\rho; \\ \omega_{\rho+\nu+1} & \text{for } \rho < 0,\ \nu \geqslant -\rho. \end{cases}$

$$7.\ \chi_\rho\omega_\sigma = \begin{cases} \psi_{\rho+\sigma} & \text{for } \rho > 0,\ \sigma < 0 \text{ and for } \rho < 0,\ \sigma > 0; \\ \psi_{\rho+\sigma-1} & \text{for } \rho > 0,\ \sigma > 0; \\ \psi_{\rho+\sigma+1} & \text{for } \rho < 0,\ \sigma < 0. \end{cases}$$

In particular, for $\sigma = -\rho$

$$\chi_\rho\omega_{-\rho} = \psi_0\,(= t). \tag{12.4}$$

$$8.\ \omega_\rho\chi_\sigma = \begin{cases} \phi_{\rho+\sigma} & \text{for } \rho > 0,\ \sigma < 0 \text{ and for } \rho < 0,\ \sigma > 0; \\ \phi_{\rho+\sigma-1} & \text{for } \rho > 0,\ \sigma > 0; \\ \phi_{\rho+\sigma+1} & \text{for } \rho < 0,\ \sigma < 0. \end{cases}$$

In particular, for $\sigma = -\rho$

$$\omega_\rho\chi_{-\rho} = \phi_0\,(= t). \tag{12.5}$$

From the above relations we have the following corollaries:

$$\phi_\nu\phi_{-\nu} = t, \quad \psi_\nu\psi_{-\nu} = t, \quad \chi_\rho\omega_{-\rho} = t, \quad \omega_\rho\chi_{-\rho} = t, \tag{12.6}$$

and moreover

$$\left.\begin{array}{llll} \phi_n = \phi_1\phi_{n-1}, & \psi_n = \psi_1\psi_{n-1}, & \chi_n = \chi_1\phi_{n-1}, & \omega_n = \omega_1\psi_{n-1}, \\ \phi_{-n} = \phi_{-1}\phi_{-n+1}, & \psi_{-n} = \psi_{-1}\psi_{-n+1}, & \chi_{-n} = \chi_1\phi_{-n}, & \omega_{-n} = \omega_1\psi_{-n} \end{array}\right\} \tag{12.7}$$

$$\left.\begin{array}{lll} \phi_n = \phi_1^n, & \psi_n = \psi_1^n, \quad \chi_n = \chi_1\phi_1^{n-1} = \psi_1^{n-1}\chi_1, & \\ & \omega_n = \omega_1\psi_1^{n-1} = \phi_1^{n-1}\omega_1, & \\ \phi_{-n} = \phi_{-1}^n, & \psi_{-n} = \psi_{-1}^n, \quad \chi_{-n} = \chi_{-1}\phi_{-1}^{n-1} = \psi_{-1}^{n-1}\chi_{-1}, & \\ & \omega_{-n} = \omega_{-1}\psi_{-1}^{n-1} = \phi_{-1}^{n-1}\omega_{-1} & \\ & (n = 1, 2, \ldots). & \end{array}\right\} \tag{12.8}$$

The formulae 6 show that to every central dispersion there corresponds another central dispersion which is its inverse, and more precisely the following central dispersions are pairs of inverses: $\phi_\nu, \phi_{-\nu}$; $\psi_\nu, \psi_{-\nu}$; $\chi_\rho, \omega_{-\rho}$; $\omega_\rho, \chi_{-\rho}$. From (8) it follows that every central dispersion can be obtained by composition of the fundamental dispersions and their inverses.

12.5 Algebraic structure of the set of central dispersions

In the set $\Gamma = \Phi \cup \Psi \cup X \cup \Omega$ we introduce a binary operation (multiplication) by defining the product of two elements as their composition. It is clear from the above table that certain ordered pairs of central dispersions $a \in A$, $b \in B$ (A, B each representing one of the sets Φ, Ψ, X, Ω) have a product $ab = c \in \Gamma$, while other ordered pairs of central dispersions do not possess any product in the set Γ. The set Γ under this operation forms an algebraic structure, a so-called *semi-groupoid.*

From the formulae (6), (8) we see that the set Φ, under the multiplication considered, forms an infinite cyclic group generated by the element ϕ_1. The unit element 1 of the group Φ is the element $\phi_0\,(= t)$. For every integer ν, the central dispersions ϕ_ν and $\phi_{-\nu}$ are inverse elements of the group Φ.

The structure of the set Ψ is similar; it forms an infinite cyclic group generated by the element ψ_1. The unit element $\underline{1}$ of the group Ψ is ϕ_0 $(= t)$ and so coincides with that of the group Φ. For every integer ν, ψ_ν and $\psi_{-\nu}$ are inverse elements of the group Ψ. The groups Φ, Ψ consequently have the unit element 1 in common.

Further, formula (6) shows that each of the two sets X, Ω consists of elements which are the inverses of elements of the other. Any two elements $\chi_\rho \in X$ and $\omega_{-\rho} \in \Omega$ are inverse to each other; that is, their product gives the unit element: $\chi_\rho \omega_{-\rho} = \omega_{-\rho}\chi_\rho = \underline{1}$. Obviously the sets X, Ω have no element in common with the groups Φ, Ψ.

To sum up: the semi-groupoid Γ is formed from two infinite cyclic groups Φ, Ψ, which have the unit element $\underline{1} = t$ in common, and also from two countable sets X, Ω, disjoint from the former two, whose elements are inverses of each other in pairs. Moreover, multiplication in the semi-groupoid is given by the formulae of § 12.4.