7 Local and boundary properties of phases

In this paragraph we discuss some further properties of phases of a differential equation (q). Mainly we shall be concerned with first phases, and consequently we shall refer to these simply as phases; when a second phase is being considered, this will be explicitly stated and in such a case we shall assume that the function q does not vanish in the interval j.

7.1 Unique determination of a phase from the Cauchy initial conditions

Let us consider a differential equation (q). We have the following result.

Theorem. Let $t_0 \in j$; X_0, $X_0' \neq 0$, X_0'' be arbitrary numbers. Then there exists precisely one phase α of the differential equation (q) *which satisfies at the point t_0 the Cauchy initial conditions:*

$$\alpha(t_0) = X_0, \qquad \alpha'(t_0) = X_0', \qquad \alpha''(t_0) = X_0''. \tag{7.1}$$

This phase α is included in the phase system of the basis (u, v) of the differential equation (q)*:*

$$\left.\begin{aligned} u(t) &= \left(X_0' \cos X_0 - \frac{1}{2}\frac{X_0''}{X_0'} \sin X_0\right) u_0(t) + \sin X_0 \cdot v_0(t), \\ v(t) &= -\left(X_0' \sin X_0 + \frac{1}{2}\frac{X_0''}{X_0'} \cos X_0\right) u_0(t) + \cos X_0 \cdot v_0(t), \end{aligned}\right\} \tag{7.2}$$

in which u_0, v_0 are those integrals of (q) *determined by the initial values*

$$u_0(t_0) = 0, \qquad u_0'(t_0) = 1; \qquad v_0(t_0) = 1, \qquad v_0'(t_0) = 0.$$

Proof. It is sufficiently general to carry out the proof for the case $X_0 = 0$ and then transform the basis found by means of the orthogonal substitution of (5.41), taking the value $\lambda = X_0$.

We therefore assume that there is a phase α of the differential equation (q) with the initial values 0, X_0' ($\neq 0$), X_0'' and let

$$\begin{aligned} u(t) &= c_{11}u_0(t) + c_{12}v_0(t), \\ v(t) &= c_{21}u_0(t) + c_{22}v_0(t) \end{aligned}$$

be the first and second elements of a corresponding basis; u_0, v_0 have the significance given in the statement of the theorem, while naturally c_{11}, c_{12}, c_{21}, c_{22} represent appropriate constants. By an easy calculation, the following formulae are seen to hold at the point t_0:

$$r^2 = c_{12}^2 + c_{22}^2; \qquad rr' = c_{11}c_{12} + c_{21}c_{22}; \qquad -w = c_{11}c_{22} - c_{12}c_{21}$$
$$(r^2 = u^2 + v^2;\ w = uv' - u'v).$$

Now, on our assumptions, the functions α, α', α'' take the values 0, X_0', X_0'' at the point t_0. It follows, on making use of (5.14), that

$$c_{12} = 0, \qquad X_0' = \frac{c_{11}}{c_{22}}, \qquad X_0'' = -2\frac{c_{11}}{c_{22}}\frac{c_{21}}{c_{22}}.$$

Obviously $c_{22} \neq 0$. We can take $c_{22} = 1$ because if we were to multiply the integrals u, v by $1/c_{22}$ we would merely obtain a basis proportional to (u, v) and consequently derive the same phase system (§ 5.17). Then we have

$$u(t) = X_0' u_0(t), \qquad v(t) = -\frac{1}{2}\frac{X_0''}{X_0'}u_0(t) + v_0(t). \tag{7.3}$$

There is therefore at most one phase α with the initial values 0, X_0', X_0'' and this must be included in the phase system determined by the formula (3). Now it is easy to calculate that that phase α which is included in this phase system and vanishes at the point t_0 does indeed satisfy the given initial conditions. This completes the proof.

It should be noticed that the above theorem implies the following formula

$$\tan\alpha(t) = \frac{\sin X_0 + \left(X_0'\cos X_0 - \frac{1}{2}\frac{X_0''}{X_0'}\sin X_0\right)\tan\alpha_0(t)}{\cos X_0 - \left(X_0'\sin X_0 + \frac{1}{2}\frac{X_0''}{X_0'}\cos X_0\right)\tan\alpha_0(t)} \tag{7.4}$$

In this $\alpha_0(t)$ represents an arbitrary phase of the basis (u_0, v_0). This formula is valid for all values $t \in j$, with the exception of zeros of $\cot\alpha_0(t)$, $\cot\alpha(t)$, at which points it has no meaning.

7.2 Boundary values of phases

Let α be a phase of the differential equation (q). Since we know that α is an increasing or decreasing function in the interval j $(= (a, b))$, there exist finite or infinite limits

$$c = \lim_{t\to a+}\alpha(t), \qquad d = \lim_{t\to b-}\alpha(t). \tag{7.5}$$

We call these numbers c and d (which may possibly be infinite) the *left* and *right boundary values* of the phase α. Obviously the boundary value c (d) is finite if the phase α is bounded in a right (left) neighbourhood of the left (right) end point a (b) of j; moreover (§ 5.4) the left (right) boundary value c (d) of α is finite if the differential equation (q) is of finite type or right (left) oscillatory; it is infinite if (q) is left (right) oscillatory or oscillatory. From § 3.4 we conclude also that if the differential equation (q) possesses 1-conjugate numbers, then the boundary value c (d) of α is finite if and only if the left (right) 1-fundamental number $r_1(s_1)$ of (q) is proper. Obviously $c < d$ or $c > d$ according as the phase α is increasing or decreasing; $\operatorname{sgn}(d - c) = \operatorname{sgn} \alpha'$.

For any constant λ, obviously the phase $\alpha + \lambda$ of (q) has the boundary values $c + \lambda$, $d + \lambda$. In particular, the left (right) boundary values of the phases of the phase system of every basis of (q) differ from each other by an integral multiple of π.

The number $|c - d|$ we call the *oscillation* of the phase α in the interval j. Our notation is: $O(\alpha|j)$ or more briefly $O(\alpha)$. The oscillation $O(\alpha)$ is finite and positive, or is infinite, according as α is bounded or unbounded in j. All phases of the complete phase system $[\alpha]$ clearly have the same oscillation $O(\alpha)$ (§ 5.17).

Two phases α, $\bar{\alpha}$ of the differential equation (q) are linked, as we know, by the formula (5.39). Consequently if the boundary values c, d of the phase α differ by an integral multiple of π, then the same is true for the boundary values $\bar{c}$, $\bar{d}$ of $\bar{\alpha}$. If the oscillation $O(\alpha)$ of α is an integral multiple of π, then the same is true for the oscillation of every phase of the differential equation (q). Such a value of $O(\alpha)$ can naturally only occur when the differential equation (q) is of finite type (m), $m \geqslant 1$. In the case $m \geqslant 2$ it will be shown (§ 7.16) that the differential equation (q) is general or special according as

$$(m - 1)\pi < O(\alpha) < m\pi \qquad \text{or} \qquad O(\alpha) = m\pi$$

where $O(\alpha)$ is the oscillation of every phase α.

This result prompts the following definition: we call a differential equation (q) of type (1) *general* or *special* according as, for the oscillation $O(\alpha)$ of each of its phases α, we have $0 < O(\alpha) < \pi$ or $O(\alpha) = \pi$.

Then we have the following theorem.

Theorem. The differential equation (q) *of finite type* (m), $m \geqslant 1$, *is general or special according as, for the oscillation of each of its phases, we have:* $(m - 1)\pi < O(\alpha) < m\pi$ *or* $O(\alpha) = m\pi$.

7.3 Normalized boundary values of phases

We shall continue in this paragraph to make use of the above notation.

Let $\bar{a}$, $\bar{b}$ denote the numbers a, b or b, a according as $\operatorname{sgn} \alpha' > 0$ or < 0. Correspondingly, let $\bar{c}$, $\bar{d}$ denote the numbers c, d or d, c. Explicitly:

$$\left.\begin{aligned} \bar{a} &= \frac{1}{2}(1 + \varepsilon)\, a + \frac{1}{2}(1 - \varepsilon) b; & \bar{b} &= \frac{1}{2}(1 - \varepsilon) a + \frac{1}{2}(1 + \varepsilon) b; \\ \bar{c} &= \frac{1}{2}(1 + \varepsilon) c + \frac{1}{2}(1 - \varepsilon) d; & \bar{d} &= \frac{1}{2}(1 - \varepsilon) c + \frac{1}{2}(1 + \varepsilon)\, d \end{aligned}\right\} \tag{7.6}$$

$$(\varepsilon = \operatorname{sgn} \alpha').$$

We call $\bar{c}$, $\bar{d}$ *normalized boundary values* of the phase α. Clearly

$$\lim_{t \to \bar{a}} \alpha(t) = \bar{c}; \qquad \lim_{t \to \bar{b}} \alpha(t) = \bar{d}, \tag{7.7}$$

in which naturally we are considering the corresponding right or left limit, and moreover

$$\bar{c} < \bar{d}. \tag{7.8}$$

It is convenient to call the numbers $\bar{a}$, $\bar{b}$ the *normalized ends* of the interval j with respect to the phase α.

7.4 Singular phases

In this and the following §§ 7.5–7.16 we are concerned, as noted above, with *first* phases and with conjugate numbers, fundamental numbers, fundamental integrals, fundamental sequences and singular bases, always of the *first* kind.

By a *singular phase* of the differential equation (q) we mean a phase of a singular basis of (q), that is to say a basis (u, v) whose first term u is a left or right fundamental integral of (q) (§ 3.9).

In the transformation theory which we are going to consider, the phase concept is of fundamental importance. Consequently, it is convenient to utilize those phases which are most closely associated with the differential equation (q) under consideration. A particular claim to this position is possessed by the singular phases, since the corresponding (singular) bases are largely determined by the type and kind of a given differential equation (q).

Let (q) be a differential equation with conjugate numbers and proper fundamental numbers, and let the corresponding left (right) fundamental number $r_1(s_1)$ be proper. The differential equation (q) consequently admits of left (right) fundamental integrals, the left (right) fundamental sequence $r_1 = a_1 < a_2 < a_3 < \ldots$ $(s_1 = b_{-1} > b_{-2} > b_{-3} > \ldots)$ and naturally also left (right) principal bases.

The fundamental theorem is the following.

Theorem. The left (*right*) *boundary values of the phases included in the phase system* (α) *of a basis* (u, v) *of the differential equation* (q) *are integral multiples of* π *if and only if* (u, v) *is a left* (*right*) *principal basis.*

Proof. (a) Let us assume that the left (right) boundary values of phases of the basis (u, v) are integral multiples of π. Then precisely one phase α of (u, v) has zero as its left (right) boundary value. For brevity, we call this phase the left (right) *null phase* of (u, v).

We now assert that α takes, at the point $r_1(s_1)$, the value $\varepsilon\pi$ $(-\varepsilon\pi)$; that is, $\alpha(r_1) = \varepsilon\pi$ $(\alpha(s_1) = -\varepsilon\pi)$, where $\varepsilon = \operatorname{sgn} \alpha'$. If this is so, then $r_1(s_1)$ is a zero of u (§ 5.3) and consequently (u, v) represents a left (right) principal basis of (q).

For simplicity, we shall assume for definiteness that α is the left null phase of (u, v).

We know, (§ 5.13) that the function

$$y(t) = k_1 \frac{\sin(\alpha(t) + k_2)}{\sqrt{|\alpha'(t)|}}$$

constructed with arbitrary constants k_1, k_2 represents the general integral of (q). Since the differential equation (q) admits of conjugate numbers, $O(\alpha) > \pi$; moreover, $c = 0$ by hypothesis; it follows that α takes the value $\varepsilon\pi$ at some point $x \in j$. We thus have to show that every number $t_0 \in j$ such that $t_0 > x$ possesses left conjugate numbers, while no number $t_0 \in (a, x)$ has left conjugate numbers.

Let $t_0 \in j$ be arbitrary and $n \geqslant 0$ the integer determined by the inequalities

$$n\varepsilon\pi \lesseqgtr \alpha(t_0) \underset{=}{\lessgtr} (n+1)\varepsilon\pi.$$

The symbol $\gtrless$ denotes $<$ or $>$ according as $\varepsilon = 1$ or $\varepsilon = -1$.

We associate with the number t_0 the integral y $(= y_0)$ constructed with the constants

$$k_1 = 1, \qquad k_2 = (n+1)\varepsilon\pi - \alpha(t_0);$$

we then have $0 \underset{=}{\lessgtr} k_2 \lesseqgtr \varepsilon\pi$, and the integral y_0 vanishes at the point t_0.

Now let $t_0 > x$. If $k_2 = 0$, then by the relationship $\alpha(x) = \varepsilon\pi$, $y_0(x) = 0$, so x is a left conjugate number of t_0. In the case $0 \lesseqgtr k_2 \lesseqgtr \varepsilon\pi$ we have $\alpha(x) + k_2 \gtrless \varepsilon\pi$ and also, since $c = 0$, $\alpha(t) + k_2 \lesseqgtr \varepsilon\pi$ for appropriate numbers $t \in (a, x)$. The function $\alpha + k_2 - \varepsilon\pi$ has therefore a zero in the interval (a, x) which is obviously a zero of y_0 and consequently a left conjugate number of t_0.

Now let $t_0 < x$. Then we have $\alpha(t_0) + k_2 = \varepsilon\pi$. It follows that for $t \in (a, t_0)$, we have $0 \lesseqgtr \alpha(t) + k_2 \lesseqgtr \varepsilon\pi$ and $y_0(t) \neq 0$. There is therefore no left conjugate number of t_0.

(b) Now let (u, v) be a left (right) principal basis of (q). Consider a phase α of (u, v) and put $\varepsilon = \operatorname{sgn} \alpha'$, and let c (d) be the left (right) boundary value of α. Since u is a left (right) fundamental integral of (q), we have (§ 5.3) $\alpha(r_1) = n\varepsilon\pi$ $(\alpha(s_1) = -n\varepsilon\pi)$; n integral, $n \geqslant 0$.

But $\bar{\alpha}(t) = \alpha(t) - c$ $(\bar{\alpha}(t) = \alpha(t) - d)$ is the left (right) null phase of a basis $(\bar{u}, \bar{v})$ of (q). So, from (a),

$$\varepsilon\pi = \bar{\alpha}(r_1) = \alpha(r_1) - c = n\varepsilon\pi - c \qquad (-\varepsilon\pi = \bar{\alpha}(s_1) = \alpha(s_1) - d = -n\varepsilon\pi - d),$$

hence

$$c = (n-1)\varepsilon\pi, \qquad (d = -(n-1)\varepsilon\pi),$$

and the proof is complete.

7.5 Properties of singular phases

We now examine more closely the properties of singular phases. Let (u, v) be a left (right) principal basis of the differential equation (q) and (α) its first phase system. u is therefore a left (right) fundamental integral, and consequently one which vanishes at the points $a_1 < a_2 < \ldots$ $(b_{-1} > b_{-2} > \ldots)$, while v is an integral of the differential equation (q) independent of u. In every interval $(a_\mu, a_{\mu+1})$ $((b_{-\mu-1}, b_{-\mu}))$ this integral has precisely one zero x_μ $(x_{-\mu})$; $\mu = 0, 1, \ldots$; $a_0 = a(b_0 = b)$:

$$(a =)\, a_0 < x_0 < a_1 < x_1 < a_2 < \cdots \quad ((b =)\, b_0 > x_0 > b_{-1} > x_{-1} > b_{-2} > \cdots). \tag{7.9}$$

The phases $\alpha \in (\alpha)$ increase or decrease according as the Wronskian w of (u, v) is negative or positive; $\operatorname{sgn} \alpha' = \operatorname{sgn}(-w)$. (5.14). We set $\varepsilon = \operatorname{sgn}(-w)$.

7.6 Boundary values of null phases

From the theorem of § 7.4, the phase system (α) contains precisely one left (right) null phase α_0; this takes the value $\varepsilon\pi$ ($-\varepsilon\pi$) at the point a_1 ($= r_1$) (b_{-1} ($= s_1$)) and consequently at the points (9) it takes the values

$$0, \frac{1}{2}\varepsilon\pi, \varepsilon\pi, \frac{3}{2}\varepsilon\pi, 2\varepsilon\pi, \ldots \quad \left(0, -\frac{1}{2}\varepsilon\pi, -\varepsilon\pi, -\frac{3}{2}\varepsilon\pi, -2\varepsilon\pi, \ldots\right); \tag{7.10}$$

$$\alpha_0(a_\mu) = \mu\varepsilon\pi, \quad \alpha_0(x_\mu) = \left(\mu + \frac{1}{2}\right)\varepsilon\pi \quad \left(\alpha_0(b_{-\mu}) = -\mu\varepsilon\pi, \alpha_0(x_{-\mu}) = -\left(\mu + \frac{1}{2}\right)\varepsilon\pi\right);$$

$\mu = 0, 1, \ldots$; naturally, $\alpha_0(a_0)$ ($\alpha_0(b_0)$) denotes the left (right) boundary value.

We now determine the right (left) boundary value d_0 (c_0) of the null phase α_0. This boundary value depends upon the type and kind of the differential equation (q).

I. First, let the differential equation (q) be of finite type (m), $m \geqslant 2$.

In this case, both fundamental numbers r_1, s_1 are proper, both fundamental sequences

$$(r_1 =)\, a_1 < a_2 < \cdots < a_{m-1}, \qquad (s_1 =)\, b_{-1} > b_{-2} > \cdots > b_{-m+1}$$

contain precisely $m - 1$ terms, and relationships hold such as those of (3.2).

From (10) we obtain

$$(m-1)\varepsilon\pi \lessgtr d_0 \leqq\!\!\!\!\geqq m\varepsilon\pi \qquad (-m\varepsilon\pi \leqq\!\!\!\!\geqq c_0 \lessgtr -(m-1)\varepsilon\pi). \tag{7.11}$$

Now the equality sign holds if and only if the differential equation (q) is special, for if $d_0 = m\varepsilon\pi$ ($c_0 = -m\varepsilon\pi$) then the theorem of § 7.4 shows that u is a right (left) fundamental integral of (q), and consequently vanishes at both the points r_1, s_1 (s_1, r_1). The fundamental numbers r_1, s_1 are therefore conjugate, which implies that the differential equation (q) is special.

Conversely, if the differential equation (q) is special, then the fundamental numbers r_1, s_1 are conjugate, and consequently the left (right) fundamental integral u is also a right (left) fundamental integral. From the theorem of § 7.4 we deduce that $d_0(c_0)$ is an integral multiple of π, and from (11) we obtain

$$d_0 = m\varepsilon\pi \qquad (c_0 = -m\varepsilon\pi).$$

(a) Let (q) be general. In this case the conditions (11) hold, but equality is not possible. The right (left) boundary value d_0 (c_0) of α_0 depends upon the second element v of the left (right) principal basis (u, v). Now we show that if v is a right (left) fundamental integral, so that (u, v) is a principal basis of the differential equation (q), then we have

$$d_0 = \left(m - \frac{1}{2}\right)\varepsilon\pi \qquad \left(c_0 = -\left(m - \frac{1}{2}\right)\varepsilon\pi\right). \tag{7.12}$$

For in this case we have $x_\mu = b_{-m+\mu+1}$ ($x_{-\mu} = a_{m-\mu-1}$), $\mu = 0, \ldots, m-2$ and from (10) the phase α_0 takes the value $(m - \frac{3}{2})\varepsilon\pi$ ($-(m - \frac{3}{2})\varepsilon\pi$) at the point $b_{-1}(a_1)$.

The function $\bar{\alpha}_0 = \alpha_0 - d_0$ ($\bar{\alpha}_0 = \alpha_0 - c_0$) is obviously a right (left) null phase of (q) and so (from (10)) takes the value $-\varepsilon\pi$ ($\varepsilon\pi$) at the point $b_{-1}(a_1)$. Hence

$$-\varepsilon\pi = \left(m - \frac{3}{2}\right)\varepsilon\pi - d_0 \qquad \left(\varepsilon\pi = -\left(m - \frac{3}{2}\right)\varepsilon\pi - c_0\right),$$

and the relation (12) follows immediately.

(b) Let (q) be special. In this case we have, as was shown above,

$$d_0 = m\varepsilon\pi \qquad (c_0 = -m\varepsilon\pi), \tag{7.13}$$

so that d_0 is independent of the choice of the second element v of the corresponding left principal (and also right principal) basis (u, v).

II. Secondly, let the differential equation (q) be of infinite type. If (q) is right (left) oscillatory, then it admits only of left (right) infinite fundamental sequences $a_1 < a_2 < \ldots$ $(b_{-1} > b_{-2} > \ldots)$; then (9) is an infinite sequence.

In this case, obviously,

$$d_0 = \varepsilon\infty \qquad (c_0 = -\varepsilon\infty). \tag{7.14}$$

7.7 Boundary values of other phases

We now turn back to the phase system (α) of the left (right) principal basis (u, v) of (q), considered in § 7.5.

The system (α) is obviously formed from the phases

$$\alpha_\nu(t) = \alpha_0(t) - \nu\varepsilon\pi \qquad (\alpha_\nu(t) = \alpha_0(t) + \nu\varepsilon\pi)$$
$$(\nu = 0, \pm 1, \pm 2, \ldots).$$

At the points a_μ, x_μ $(b_{-\mu}, x_{-\mu})$ the phase α_ν takes the following values

$$\left.\begin{aligned} &\alpha_\nu(a_\mu) = (\mu - \nu)\varepsilon\pi, \qquad &&\alpha_\nu(x_\mu) = \left(\mu - \nu + \frac{1}{2}\right)\varepsilon\pi \\ &\Big(\alpha_\nu(b_{-\mu}) = -(\mu - \nu)\varepsilon\pi, \qquad &&\alpha_\nu(x_{-\mu}) = -\left(\mu - \nu + \frac{1}{2}\right)\varepsilon\pi\Big) \end{aligned}\right\} \quad 7.15)$$
$$(\mu = 0, 1, \ldots).$$

The left (right) boundary value c_ν (d_ν) of the phase α_ν is

$$c_\nu = -\nu\varepsilon\pi \qquad (d_\nu = \nu\varepsilon\pi). \tag{7.16}$$

The right (left) boundary value d_ν (c_ν) of the phase α_ν according to the type and kind of the differential equation (q) is as follows,

I. Finite type (m), $m \geqslant 2$.

(a) General (q):

$$(m - \nu - 1)\varepsilon\pi \lesseqgtr d_\nu \lesseqgtr (m - \nu)\varepsilon\pi \qquad (-(m - \nu)\varepsilon\pi \lesseqgtr c_\nu \lesseqgtr -(m - \nu - 1)\varepsilon\pi). \tag{7.17}$$

If, in particular, v is a right (left) fundamental integral and consequently (u, v) a principal basis of the differential equation (q) then we have, independently of our choice of the fundamental integral v,

$$d_\nu = \left(m - \nu - \frac{1}{2}\right)\varepsilon\pi \qquad \left(c_\nu = -\left(m - \nu - \frac{1}{2}\right)\varepsilon\pi\right). \tag{7.18}$$

(b) Special (q):

$$d_\nu = (m - \nu)\varepsilon\pi \qquad (c_\nu = -(m - \nu)\varepsilon\pi). \tag{7.19}$$

II. Infinite type.

(a) Right oscillatory (q):

$$d_\nu = \varepsilon\infty. \tag{7.20}$$

(b) Left oscillatory (q):

$$c_\nu = -\varepsilon\infty. \tag{7.21}$$

7.8 Normal phases

The null phases of a differential equation (q) are characterized by the fact that they are always non-zero in the interval j. We now consider, instead, phases which have one (and naturally only one) zero in j. A phase of the differential equation (q) which vanishes at a point of j will in what follows be called a *normal phase*.

We consider a differential equation (q). Let (u, v) be a basis of (q) and (α) the phase system of this basis (u, v).

We observe first, that the phase system (α) contains normal phases if and only if the integral u has zeros in the interval j. In this case every zero of the integral u is a zero of a normal phase of (α); conversely, the zero of every normal phase of (α) coincides with one of the zeros of the integral u. It follows that:

If the differential equation (q) is of finite type (m) $(m \geqslant 1)$ then the phase system (α) contains $m - 1$ or m normal phases, while if (q) is of infinite type then it contains a countable infinity of normal phases; the zeros of these phases coincide with those of the integral u. If, in particular, the differential equation (q) is oscillatory, then the system (α) has only normal phases.

7.9 Structure of the set of singular normal phases of a differential equation (q)

By a *singular normal phase* of a differential equation (q) we naturally mean (§ 7.4) a normal phase of a singular basis of (q). Singular normal phases thus occur in differential equations (q) of finite type (m), $m \geqslant 2$, and in left or right oscillatory differential equations (q) but only in these cases. Let (q) be a differential equation with conjugate numbers and the corresponding left (right) proper fundamental number $r_1(s_1)$.

We start from the situation considered in § 7.5, letting (u, v) be a left (right) principal basis of the differential equation (q) and (α) its first phase system.

The zeros of the integrals u coincide with the singular numbers $a_1 < a_2 < \ldots$ $b_{-1} > b_{-2} > \ldots$ of the differential equation (q); hence to every number $a_r(b_{-r})$ $(r = 1, 2, \ldots)$ there corresponds precisely one normal phase in the phase system (α) with the zero a_r (b_{-r}). From (15) we see that this is the phase given by

$$\alpha_r(t) = \alpha_0(t) - r\varepsilon\pi \quad (\alpha_r(t) = \alpha_0(t) + r\varepsilon\pi). \tag{7.22}$$

Conversely, every normal phase included in the phase system (α) is one of these phases $\alpha_1, \alpha_2, \ldots$

By a *phase bundle with the apex* a_r (b_{-r}) or more briefly, an *a_r-bundle* (*b_{-r}-bundle*) of the differential equation (q) we mean the subset comprising all normal phases of all left (right) principal bases of (q) which vanish at the point $a_r(b_{-r})$, $r = 1, 2, \ldots$. Obviously the set of all singular normal phases of the differential equation (q) is the union of the phase bundles with the apices $a_1, a_2, \ldots$ $(b_{-1}, b_{-2}, \ldots)$.

7.10 Structure of a phase bundle

We now examine the structure of the phase bundles; for brevity we shall confine ourselves to left principal bases. These occur, as we know, only in differential equations (q) of finite type (m), $m \geqslant 2$, and in right oscillatory differential equations (q). (The study of right principal bases is entirely analogous).

Let (u, v) be a left principal basis of (q) and a_r a term of the left fundamental sequence of (q). We know (§ 3.9) that the left principal bases of (q) form the three-parameter system $(\rho u, \sigma v + \bar{\sigma} u)$, $\rho\sigma \neq 0$. Now for every choice of values given to the parameters ρ, σ, $\bar{\sigma}$ the bases $(\rho u, \sigma v + \bar{\sigma} u)$ and $\left(\frac{\rho}{\sigma} u, v + \frac{\bar{\sigma}}{\sigma} u\right)$ are proportional, and therefore have the same phase system and consequently the same normal phase vanishing at the point a_r. So the phase bundle of the differential equation (q) with the apex a_r obviously comprises precisely those normal phases of the two parameter basis system $(\rho u, v + \sigma u)$, $\rho \neq 0$, which vanish at the point a_r. Naturally, for every basis of this two-parameter system there exists precisely one normal phase of (q) with the zero a_r.

The basis system $(\rho u, v + \sigma u)$ now separates into one-parameter systems each of which is determined by a fixed value σ_0 of σ. Such a one-parameter system thus comprises the left principal bases $(\rho u, v + \sigma_0 u)$, $\rho \neq 0$. Those normal phases of the left principal bases of this one-parameter system which are included in the a_r-bundle, form a one-parameter sub-system $P(a_r|\sigma_0)$ of the a_r-bundle. We call $P(a_r|\sigma_0)$ the *phase bunch with the apex a_r* or more briefly the *a_r-bunch* of the differential equation (q).

The phase bundle of the differential equation (q) with apex a_r consequently consists of a one-parameter system of a_r-bunches $P(a_r|\sigma)$, each of which is formed from those normal phases of the left principal bases $(\rho u, v + \sigma u)$, $0 \neq \rho$, arbitrary, σ fixed, which vanish at the point a_r. Thus the study of the structure of the phase bundles is reduced to that of the phase bunches.

7.11 Structure of a phase bunch

Now we wish to study the structure of the phase bunches. We consider a phase bunch $P(a_r|\sigma)$, with the apex a_r, of the differential equation (q). Without loss of generality we can take $\sigma = 0$ and also assume that w $(= uv' - u'v) < 0$ For brevity we shall write $P(a_r)$ instead of $P(a_r|0)$ The phase bunch $P(a_r)$ consists of the normal phases of the left principal bases $(\rho u, v)$, $\rho \neq 0$ which vanish at the point a_r.

For every value ρ $(\neq 0)$, ρu is a left fundamental integral, consequently vanishing at the points $a_1 < a_2 < \ldots$, while v represents an integral of the differential equation (q) independent of u. This integral v has precisely one zero x_μ, $\mu = 0, 1, \ldots$ in each interval $(a_\mu, a_{\mu+1})$, and we have ordering relationships similar to (9). For brevity we shall denote the intervals (a_μ, x_μ), $(x_\mu, a_{\mu+1})$ by j_μ and j'_μ respectively, i.e. $j_\mu = (a_\mu, x_\mu)$, $j'_\mu = (x_\mu, a_{\mu+1})$; if the differential equation (q) is of finite type (m) $(m \geqslant 2)$, then naturally we only have to consider the intervals $j_0, j'_0, j_1, j'_1, \ldots, j_{m-1}$, where x_{m-1} stands for the upper end point b of j.

For every number ρ $(\neq 0)$ we write $\alpha_{r,\rho}$ or more briefly α_ρ for the normal phase of the basis $(\rho u, v)$ contained in the phase bunch $P(a_r)$, and $c_{r,\rho}$ or $d_{r,\rho}$, more briefly c_ρ or d_ρ, for its left or right boundary value. Obviously we have $\tan \alpha_\rho = \rho u/v$ and consequently

$$\tan \alpha_\rho = \rho \tan \alpha_1. \tag{7.23}$$

The Wronskian of the basis $(\rho u, v)$ is obviously ρw. We deduce, using our assumption $w < 0$, that the phase α_ρ increases for positive values of ρ and decreases for negative values of ρ, i.e. $\operatorname{sgn}(\alpha'_\rho) = \operatorname{sgn} \rho$ $(= \varepsilon)$.

At the points

$$x_0, a_1, x_1, \ldots, x_{r-1}, a_r, x_r, \ldots$$

the phase α_ρ takes the following values, which are independent of $|\rho|$:

$$-\left(r - \frac{1}{2}\right)\varepsilon\pi, \quad -(r-1)\varepsilon\pi, \qquad -\left(r - \frac{3}{2}\right)\varepsilon\pi, \ldots, -\frac{1}{2}\varepsilon\pi, 0, \frac{1}{2}\varepsilon\pi, \ldots$$

and consequently

$$\alpha_\rho(a_{\mu+1}) = -(r - \mu - 1)\varepsilon\pi, \qquad \alpha_\rho(x_\mu) = -\left(r - \mu - \frac{1}{2}\right)\varepsilon\pi \tag{7.24}$$

$$(\mu = 0, 1, \ldots).$$

From (16) we have

$$c_\rho = -r\varepsilon\pi. \tag{7.25}$$

For the right boundary value d_ρ of α_ρ we obtain from (17), (18), (19), (20) the following inequalities, according to the type and kind of the differential equation (q):

I. Finite type (m), $(m \geqslant 2)$:

(a) General differential equation (q):

$$(m - r - 1)\varepsilon\pi \lesseqgtr d_\rho \lesseqgtr (m - r)\varepsilon\pi. \tag{7.26}$$

If in particular v is a right fundamental integral and consequently $(\rho u, v)$ a principal basis of the differential equation (q) then $x_\mu = b_{-m+\mu+1}$ $(\mu = 0, \ldots, m-2)$ and moreover

$$d_\rho = \left(m - r - \frac{1}{2}\right) \varepsilon\pi. \tag{7.27}$$

(b) Special differential equation (q):

$$d_\rho = (m - r)\varepsilon\pi. \tag{7.28}$$

II. Infinite type. Right oscillatory differential equation (q):

$$d_\rho = \varepsilon\infty. \tag{7.29}$$

The phase bunch $P(a_r)$ separates into two sub-bunches, of which one, $P_1(a_r)$, (say) comprises the increasing and the other, $P_{-1}(a_r)$, the decreasing phases. The individual phases in the respective sub-bunches $P_1(a_r)$ and $P_{-1}(a_r)$ take the same values at the points $a_{\mu+1}, x_\mu (\mu = 0, 1, \ldots)$, namely $-(r - \mu - 1)\pi$, $-(r - \mu - \frac{1}{2})\pi$ and $(r - \mu - 1)\pi$, $(r - \mu - \frac{1}{2})\pi$ respectively and have the same left boundary value $-r\pi$ or $r\pi$. Their right boundary values are generally dependent on the individual phases, but not in the following cases:

When the differential equation (q) is of:

I. Finite type (m), $m \geqslant 2$,

(a) General, and v is a right fundamental integral of (q)

(b) Special

II. Infinite type, right oscillatory.

In these cases the individual phases of $P_1(a_r)$ and $P_{-1}(a_r)$ also have the same right boundary value; in case I(a) they are $(m - r - \frac{1}{2})\pi$, $-(m - r - \frac{1}{2})\pi$, respectively; in case I(b) they are $(m - r)\pi$, $-(m - r)\pi$, and in case II they are $-\infty$, ∞.

Clearly this situation can be described as follows:

All the curves $[t, \alpha_\rho(t)]$, $t \in j$, determined by the phases α_ρ of the sub-bunch $P_\varepsilon(a_r)$ $(\varepsilon = \pm 1)$ go through the points

$$\left(x_\mu, -\left(r - \mu - \frac{1}{2}\right) \varepsilon\pi\right), \qquad (a_{\mu+1}, -(r - \mu - 1)\varepsilon\pi) \quad (\mu = 0, 1, \ldots) \tag{7.30}$$

and tend on the left to the point $(a, -r\varepsilon\pi)$. In the cases I(a), I(b), II they also tend on the right to a common point, in fact towards $(b, (m - r - \frac{1}{2})\varepsilon\pi)$, $(b, (m - r)\varepsilon\pi)$, $(b, \varepsilon\infty)$ respectively. Moreover all these curves lie in the region B_ε formed by the union of the open rectangular regions

$$\left.\begin{aligned} j_\mu \times \left(-(r - \mu)\varepsilon\pi, -\left(r - \mu - \frac{1}{2}\right) \varepsilon\pi\right) \\ j'_\mu \times \left(-\left(r - \mu - \frac{1}{2}\right) \varepsilon\pi, -(r - \mu - 1)\varepsilon\pi\right) \quad \mu) = 0, 1, \ldots). \end{aligned}\right\} \tag{7.31}$$

To complete the picture, we shall establish the fact that through every point of the region B_ε there passes precisely one curve $[t, \alpha_\rho(t)]$; in other words the curves considered, $[t, \alpha_\rho(t)]$ fill the region B_ε completely and simply.

For, let $P_0(t_0, X_0)$ be a point which we may take, for instance, to lie in the rectangular region $j_\mu \times (-(r-\mu)\varepsilon\pi), \quad -(r-\mu-\frac{1}{2})\varepsilon\pi)$ so that $t_0 \in (a_\mu, x_\mu)$, $X_0 \in (-(r-\mu)\varepsilon\pi, -(r-\mu-\frac{1}{2})\varepsilon\pi.)$

If the curve $[t, \alpha_\rho(t)]$ determined by a phase $\alpha_\rho \in P_\varepsilon(a_r)$ passes through the point P_0, then we have $\alpha_\rho(t_0) = X_0$ and moreover $\rho = \tan X_0/\tan \alpha_1(t_0)$ $(= \rho_0)$. We have therefore at most to consider the phase $\alpha_{\rho_0}(t)$. Moreover, from (23) we have $\tan \alpha_{\rho_0}(t_0) = \tan X_0$; consequently, since both the numbers $\alpha_{\rho_0}(t_0)$, X_0 lie in the interval $(-(r-\mu)\varepsilon\pi, -(r-\mu-\frac{1}{2})\varepsilon\pi)$, we have $\alpha_{\rho_0}(t_0) = X_0$. This proves our assertion.

We also see that for any two phases $\alpha_\rho, \alpha_{\bar{\rho}} \in P_\varepsilon(a_r)$, there holds the following relation in the interval j

$$|\alpha_\rho - \alpha_{\bar{\rho}}| < \frac{\pi}{2}. \tag{7.32}$$

The situation in the case Ia is shown in figure 2.

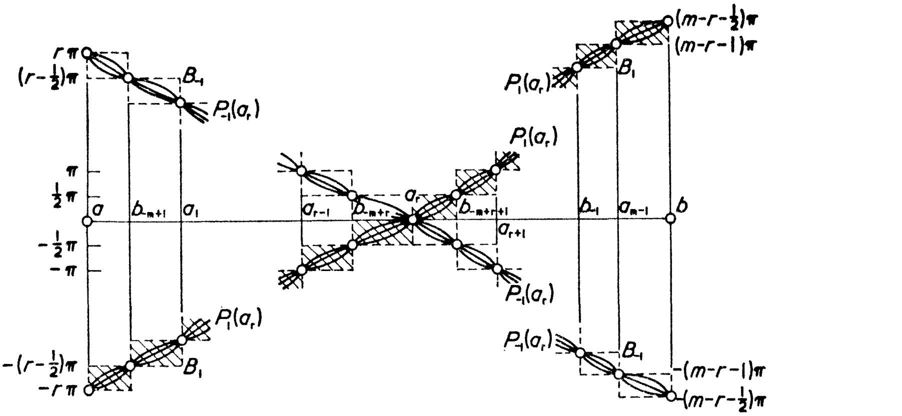

Figure 2

7.12 The mapping of ρ into the phase α_ρ

We now wish to study more closely the mapping $L: \rho \to \alpha_\rho$. Let I_1, I_{-1} denote respectively the positive and negative real half-lines.

1. *The mapping* L *maps the interval* I_ε *on the sub-bunch* $P_\varepsilon(a_r)$ $(\varepsilon = \pm 1)$.
2. *The mapping* L *is simple.*

For, from $\rho, \bar{\rho} \in I_\varepsilon$, $\rho \neq \bar{\rho}$, it follows that $\alpha_\rho \neq \alpha_{\bar{\rho}}$. Conversely, if $\alpha_\rho \neq \alpha_{\bar{\rho}}$ for two phases $\alpha_\rho, \alpha_{\bar{\rho}} \in P_\varepsilon(a_r)$ and at the same time $\rho = \bar{\rho}$, then the formula (23) gives $\alpha_\rho = \alpha_{\bar{\rho}} + k\pi$, where k is an integer $\neq 0$. This is impossible, since both phases $\alpha_\rho, \alpha_{\bar{\rho}}$ vanish at the point a_r.

3. *For $\rho, \bar{\rho} \in I_\varepsilon$, $\rho < \bar{\rho}$ in every interval j_μ or j'_μ ($\mu = 0, 1, \ldots$) there holds the relation $\alpha_\rho < \alpha_{\bar{\rho}}$ or $\alpha_\rho > \alpha_{\bar{\rho}}$ respectively.*

For, obviously $\tan \alpha_1 > 0$ or $\tan \alpha_1 < 0$ in every interval j_μ or j'_μ respectively; then, taking account of (23), our assertion follows.

The sub-bunch $P_\varepsilon(a_r)$ admits of the following ordering relation, denoted by $\prec$: for $\alpha, \bar{\alpha} \in P_\varepsilon(a_r)$ we have $\alpha \prec \bar{\alpha}$ if and only if the inequality $\alpha < \bar{\alpha}$ or $\alpha > \bar{\alpha}$ holds in every interval j_μ or j'_μ, respectively.

The mapping L is order-preserving with respect to this ordering.

4. We define a metric in the set $P_\varepsilon(a_r)$, by means of the formula

$$d(\alpha_\rho, \alpha_{\bar{\rho}}) = \sup_{t \in j} |\alpha_\rho(t) - \alpha\bar{\rho}(t)|.$$

In the interval I_ε we adopt the usual Euclidean metric.

We now show that:

The mapping L is homeomorphic.

Proof. (a) Let $\rho, \bar{\rho} \in I_\varepsilon$ be arbitrary numbers. At every point $t \in j$ other than $a_{\mu+1}, x_\mu$ ($\mu = 0, 1, \ldots$) we have the relation

$$\tan(\alpha_\rho - \alpha_{\bar{\rho}}) = \frac{\tan \alpha_\rho - \tan \alpha_{\bar{\rho}}}{1 + \tan \alpha_\rho \tan \alpha_{\bar{\rho}}} = \frac{\rho - \bar{\rho}}{\dfrac{v}{u} + \rho\bar{\rho}\dfrac{u}{v}}$$

whence, taking account of (32),

$$|\alpha_\rho - \alpha_{\bar{\rho}}| \leqslant \tan|\alpha_\rho - \alpha_{\bar{\rho}}| = \frac{|\rho - \bar{\rho}|}{\left|\dfrac{v}{u}\right| + \rho\bar{\rho}\left|\dfrac{u}{v}\right|} = \frac{|\rho - \bar{\rho}|}{\left(\sqrt{\left|\dfrac{v}{u}\right|} - \sqrt{\rho\bar{\rho}}\sqrt{\left|\dfrac{u}{v}\right|}\right)^2 + 2\sqrt{\rho\bar{\rho}}}.$$

From these relations it follows that

$$d(\alpha_\rho, \alpha_{\bar{\rho}}) \leqslant \frac{1}{2}\frac{|\rho - \bar{\rho}|}{\sqrt{\rho\bar{\rho}}}. \tag{7.33}$$

so that the mapping L is continuous at every point $\bar{\rho} \in I_\varepsilon$.

(b) Let $\alpha_\rho, \alpha_{\bar{\rho}} \in P_\varepsilon(a_r)$ be arbitrary phases. Obviously, there exists a number $t_0 \in j$ such that $\tan \alpha_1(t_0) = \delta$ ($= \pm 1$). At this point t_0 we have, from (23),

$$\delta(\rho - \bar{\rho}) = \tan \alpha_\rho - \tan \alpha_{\bar{\rho}} = (1 + \rho\bar{\rho}) \tan(\alpha_\rho - \alpha_{\bar{\rho}})$$

and further

$$\frac{|\rho - \bar{\rho}|}{1 + \rho\bar{\rho}} = \tan|\alpha_\rho - \alpha_{\bar{\rho}}| \leqslant \tan d(\alpha_\rho, \alpha_{\bar{\rho}}).$$

From these relationships it follows that

$$\frac{|\rho - \bar{\rho}|}{1 + \rho\bar{\rho}} \leqslant \tan d(\alpha_\rho, \alpha_{\bar{\rho}}); \tag{7.34}$$

hence the mapping L^{-1} is continuous at every point $\alpha_{\bar{p}} \in P_\varepsilon(a_r)$. This completes the proof.

7.13 Relations between zeros and boundary values of normal phases

In the course of our study of singular normal phases we encountered certain relations between the zeros and boundary values of these phases. For instance, Figure 2 shows that in the case we have considered every increasing or decreasing normal phase with the zero a_r possesses the boundary values $-r\pi$, $(m - r - \frac{1}{2})\pi$ or $r\pi$, $-(m - r - \frac{1}{2})\pi$. We now study in greater generality the relations between zeros and boundary values of normal phases.

Consider a differential equation (q) of finite type (m), $m \geq 1$, or a left or right oscillatory differential equation. The left and right 1-fundamental sequences of (q), if they exist, we denote by

$$(a <)\, r_1 = a_1 < a_2 < \cdots \qquad \text{and} \qquad (b >)\, s_1 = b_{-1} > b_{-2} > \cdots.$$

Let α be a normal phase of (q), $t_0 \in j$ its zero, and c, d its left and right boundary values. For simplicity, we put $\varepsilon = \operatorname{sgn} \alpha'$ and use the symbol $\lessgtr$ to mean $<$ when $\varepsilon = +1$, $>$ when $\varepsilon = -1$; similarly $\gtrless$ means $>$ when $\varepsilon = +1$, and $<$ when $\varepsilon = -1$.

(a) First we assume that the differential equation (q) is of finite type or is right oscillatory. In these cases the boundary value c is finite.

If the differential equation (q) is of type (1), and so without conjugate numbers of the first kind, then obviously we have the relations

$$-\pi\varepsilon \lessgtr c \lessgtr 0.$$

We now assume that the differential equation (q) admits of 1-conjugate numbers.

Let $r \geq 0$ be the integer defined by the stipulation $t_0 \in (a_r, a_{r+1}]$; $a_0 = a$. We consider the left null phase α_0 included in the system $[\alpha]$:

$$\alpha_0(t) = \alpha(t) - c.$$

We clearly have the relations

$$\alpha_0(a_r) \lessgtr \alpha_0(t_0) \underline{\lessgtr} \alpha_0(a_{r+1}),$$

in which equality holds if and only if $t_0 = a_{r+1}$. It follows, when we take account of the monotonicity of α_0 and formula (10), that the two following relations hold simultaneously

$$a_r < t_0 \leq a_{r+1}; \qquad -(r + 1)\pi\varepsilon \underline{\lessgtr} c \lessgtr -r\pi\varepsilon.$$

(b) Secondly, we assume that the differential equation (q) is of finite type or left oscillatory. In these cases the boundary value d is finite.

As above, we find that if the differential equation (q) is of type (1) then we have

$$0 \lessgtr d \lessgtr \pi\varepsilon.$$

If the differential equation (q) admits of 1-conjugate numbers, then there hold simultaneously the formulae

$$b_{-s-1} \leqslant t_0 \lesseqgtr b_{-s}; \qquad s\pi\varepsilon \lesseqgtr d \underset{=}{\lesseqgtr} (s+1)\pi\varepsilon \qquad (s \geqslant 0;\ b_0 = b),$$

in which both equality signs must be taken at the same time.

These results may be summed up:

Theorem. Between the zero $t_0 \in j$ and the boundary values c, d of a normal phase α of the differential equation (q), *there hold the relations set out below, corresponding to the following table of type and kind of the differential equation* (q):

I. *finite type* (m), $m \geqslant 1$; (a) *general* (b) *special;*
II. *infinite type;* (a) *right oscillatory* (b) *left oscillatory* (c) *oscillatory;*

I. *The boundary values c, d are finite.*

(a) *There hold the relations* $(m-1)\pi\varepsilon \lesseqgtr d - c \lesseqgtr m\pi\varepsilon$, *and when*

$$m = 1:\ t_0 \in j,\ -\pi\varepsilon \lesseqgtr c \lesseqgtr 0 \lesseqgtr d \lesseqgtr \pi\varepsilon;$$
$$m \geqslant 2:\quad a_r < t_0 \leqslant b_{-m+r+1}; \quad -(r+1)\pi\varepsilon \lesseqgtr c \lesseqgtr -r\pi\varepsilon;$$
$$(m-r-1)\pi\varepsilon \underset{=}{\lesseqgtr} d \lesseqgtr (m-r)\pi\varepsilon$$

or

$$b_{-m+r+1} < t_0 \leqq a_{r+1}; \quad -(r+1)\pi\varepsilon \underset{=}{\lesseqgtr} c \lesseqgtr -r\pi\varepsilon;$$
$$(m-r-2)\pi\varepsilon \lesseqgtr d \lesseqgtr (m-r-1)\pi\varepsilon.$$

(b) *We have* $d - c = m\pi\varepsilon$ *and*

$$a_r < t_0 \leqslant a_{r+1}; \quad -(r+1)\pi\varepsilon \underset{=}{\lesseqgtr} c \lesseqgtr -r\pi\varepsilon; \quad d = c + m\pi\varepsilon.$$

II. *At least one of the boundary values c, d is infinite.*

(a) $a_r < t_0 \leqslant a_{r+1}; \quad -(r+1)\pi\varepsilon \underset{=}{\lesseqgtr} c \lesseqgtr -r\pi\varepsilon; \quad d = \varepsilon\infty.$

(b) $b_{-r-1} \leqslant t_0 < b_{-r}; \quad r\pi\varepsilon \lesseqgtr d \underset{=}{\lesseqgtr} (r+1)\pi\varepsilon; \quad c = -\varepsilon\infty.$

(c) $t_0 \in j; \quad c = -\varepsilon\infty, \quad d = \varepsilon\infty.$

7.14 Boundary characteristics of normal phases

Let α be a normal phase of the differential equation (q), $t_0 \in j$ its zero and c, d its left and right boundary values. The ordered triad $(t_0; c, d)$ we shall call the *boundary characteristic* of the normal phase α; we shall use for it the notation $\chi(\alpha)$, or more briefly χ. The elements c, d are the *essential terms* of $\chi(\alpha)$. It is convenient also to speak of the values c, d when they denote the symbols $+\infty$ or $-\infty$.

From the above results, we see that there exist certain relationships between the terms of $\chi(\alpha)$, according to the type of the differential equation (q). In particular, for all types of the differential equation (q), we have $\alpha(t_0) = 0$, sgn $\alpha' = -$ sgn $c =$ sgn d.

We now wish to study how far the normal phase α is determined by these relations.

By a *characteristic triple* of the differential equation (q) we mean an ordered triad $(\bar{t}_0; \bar{c}, \bar{d})$ whose terms satisfy the appropriate conditions I(a)–II(c) of § 7.13, according to the type and kind of the differential equation (q). Naturally, $t_0 \in j$ and $\bar{c}, \bar{d}$ may denote the symbols $\pm\infty$; also $\varepsilon = \operatorname{sgn}(\bar{d} - \bar{c})$.

Obviously, the boundary characteristic $\chi(\alpha)$ represents a characteristic triple of (q). The question which we now examine is: given any characteristic triple χ of the differential equation (q), do there correspond normal phases with the boundary characteristic χ? If so, how may these normal phases be determined?

7.15 Determination of normal phases with given characteristic triple

Let $\chi = (t_0; c, d)$ be a characteristic triple of the differential equation (q). We assume that there does indeed exist a normal phase α of (q) with the boundary characteristic χ. Let (u_0, v_0) be the basis of the differential equation (q) determined by the initial values

$$u_0(t_0) = 0, \quad u_0'(t_0) = -\operatorname{sgn} c; \quad v_0(t_0) = 1, \quad v_0'(t_0) = 0$$

and $\bar{\alpha}_0$ be the phase of (u_0, v_0) which vanishes at the point t_0: $\bar{\alpha}_0(t_0) = 0$. Obviously, $\bar{\alpha}_0$ is a normal phase of (q) and we have $\operatorname{sgn} \bar{\alpha}_0' = -\operatorname{sgn} c$. We denote by $\chi(\bar{\alpha}_0) = (t_0; \bar{c}_0, \bar{d}_0)$ the boundary characteristic of $\bar{\alpha}_0$. We have $-\operatorname{sgn} \bar{c}_0 = \operatorname{sgn} \bar{\alpha}_0' = -\operatorname{sgn} c$ and consequently $\operatorname{sgn} \bar{c}_0 = \operatorname{sgn} c$, $\operatorname{sgn} \bar{d}_0 = \operatorname{sgn} d$.

The functions $\alpha, \bar{\alpha}_0$ are, however, phases of the same differential equation (q); it follows that if one of the numbers $c, \bar{c}_0$ or $d, \bar{d}_0$ is finite, so is the other. Further, the following formula holds in the interval j, except at the singular points of $\tan \alpha(t)$, $\tan \bar{\alpha}_0(t)$;

$$\tan \alpha(t) = \frac{c_{11} \tan \bar{\alpha}_0(t) + c_{12}}{c_{21} \tan \bar{\alpha}_0(t) + c_{22}},$$

with appropriate constants $c_{11}, c_{12}, c_{21}, c_{22}$. (See § 5.17, corollary 4).

Now $c_{12} = 0$, since both phases $\alpha, \bar{\alpha}_0$ vanish at the point t_0. Obviously $c_{22} \neq 0$, and we can assume that $c_{22} = 1$, because the numerator and denominator may be divided by c_{22}. So we have

$$\tan \alpha(t) = \frac{c_{11} \tan \bar{\alpha}_0(t)}{c_{21} \tan \bar{\alpha}_0(t) + 1}.$$

This formula can be put into the following form, valid for all $t \in j$

$$\sin \bar{\alpha}_0(t) \cdot [c_{11} \cos \alpha(t) - c_{21} \sin \alpha(t)] = \cos \bar{\alpha}_0(t) \sin \alpha(t).$$

If the numbers $c, \bar{c}_0$ or $d, \bar{d}_0$ are finite then this relation gives, respectively,

$$\left.\begin{aligned} \sin \bar{c}_0 \cdot [c_{11} \cos c - c_{21} \sin c] &= \cos \bar{c}_0 \cdot \sin c, \\ \sin \bar{d}_0 \cdot [c_{11} \cos d - c_{21} \sin d] &= \cos \bar{d}_0 \cdot \sin d. \end{aligned}\right\} \tag{7.35}$$

We observe that the constants c_{11}, c_{21}, satisfy one or both of the linear equations (35) according to the type and kind of the differential equation (q) and are more or less determined by the boundary values c, $\bar{c}_0$ or d, $\bar{d}_0$. We now wish to study this situation in the various cases.

We use the notation of § 7.13. In particular, we put $\varepsilon = \operatorname{sgn} \bar{\alpha}_0' = \operatorname{sgn} \alpha'$ and denote the 1-fundamental sequences, if they exist, by $(a <)\, r_1 = a_1 < a_2 < \cdots$, $(b >)\, s_1 = b_{-1} > b_{-2} > \cdots$.

I. Finite type (m), $m \geqslant 1$

$$c, d \text{ nite}, (m-1)\pi\varepsilon \lessgtr d - c \lesseqqgtr m\pi\varepsilon.$$

(a) General case:

$$(m-1)\pi\varepsilon \lessgtr d - c \lessgtr m\pi\varepsilon. \tag{7.36}$$

1. $m = 1$: $\sin c \sin d \neq 0$, $t_0 \in j$, $\sin \bar{c}_0 \sin \bar{d}_0 \neq 0$.

2. $m \geqslant 2$:

α) $\sin c \sin d \neq 0$, $b_{-m+r+1} \neq t_0 \neq a_{r+1}$, $\sin \bar{c}_0 \sin \bar{d}_0 \neq 0$,

β) $\sin c \neq 0$, $\sin d = 0$; $t_0 = b_{-m+r+1} < a_{r+1}$; $\sin \bar{c}_0 \neq 0$, $\sin \bar{d}_0 = 0$,

γ) $\sin c = 0$, $\sin d \neq 0$; $b_{-m+r+1} < t_0 = a_{r+1}$; $\sin \bar{c}_0 = 0$, $\sin \bar{d}_0 \neq 0$.

In cases 1 and 2α) the equations (35) can be written as follows

$$\left.\begin{aligned} c_{11} \cot c - c_{21} &= \cot \bar{c}_0, \\ c_{11} \cot d - c_{21} &= \cot \bar{d}_0. \end{aligned}\right\} \tag{7.37}$$

From (36) we have $\cot c - \cot d \neq 0$. It is clear that the constants c_{11}, c_{21} are uniquely determined.

In the respective cases 2β) (2γ)) the first (second) equation (35) can be replaced by the first (second) equation (37) while the second (first) equation (35) is satisfied identically. Clearly, of the two constants c_{11}, c_{21}, one is undetermined.

(b) Special case:

$$d - c = m\pi\varepsilon = \bar{d}_0 - \bar{c}_0.$$

1. $m = 1$: $\sin c \sin d \neq 0$, $t_0 \in j$, $\sin \bar{c}_0 \sin \bar{d}_0 \neq 0$.

2. $m \geqslant 2$:

α) $\sin c \sin d \neq 0$; $a_r < t_0 < a_{r+1}$, $\sin \bar{c}_0 \sin \bar{d}_0 \neq 0$.

β) $\sin c = \sin d = 0$; $t_0 = a_{r+1}$, $\sin \bar{c}_0 = \sin \bar{d}_0 = 0$.

In the cases 1 and 2α) the equations (35) are linearly dependent and can be replaced by one of the equations (37). Clearly, of the constants c_{11}, c_{21}, one is undetermined. In the case 2β) the equations (35) are satisfied identically: the constants c_{11}, c_{21} are arbitrary.

II. Infinite type

At least one of the boundary values c, d is infinite.

(a) Right oscillatory differential equation:

$$-(r+1)\pi\varepsilon \lesseqqgtr c \lessgtr -r\pi\varepsilon; \quad d = \varepsilon\infty.$$

1. $\sin c \neq 0$, $a_r < t_0 < a_{r+1}$; $\sin \bar{c}_0 \neq 0$.

2. $\sin c = 0$, $t_0 = a_{r+1}$; $\sin \bar{c}_0 = 0$.

Clearly, in case 1 one of the constants c_{11}, c_{21} is undetermined, in case 2 both are arbitrary.

(b) Left oscillatory differential equation:

$$s\pi\varepsilon \lessgtr d \lesseqgtr (s+1)\pi\varepsilon; \quad c = -\varepsilon\infty.$$

1. $\sin d \neq 0$, $b_{-s-1} < t_0 < b_{-s}$; $\sin \bar{d}_0 \neq 0$.
2. $\sin d = 0$, $t_0 = b_{-s-1}$; $\sin \bar{d}_0 = 0$.

Clearly, in case 1 one of the constants c_{11}, c_{21} is undetermined, in case 2 both are arbitrary.

(c) Oscillatory differential equation:

$$t_0 \in j; \quad c = -\varepsilon\infty, \quad d = \varepsilon\infty.$$

In this case both the constants c_{11}, c_{21} are arbitrary.

These results may be collected together as follows:

Theorem. For every characteristic triple $\chi = (t_0; c, d)$ *of the differential equation* (q) *there exist corresponding normal phases of* (q) *with the boundary characteristic* χ. *According to the type and kind of the differential equation* (q) *and according as to whether* t_0 *is singular or not, there is precisely one normal phase or there is precisely one system of normal phases* (*containing one or two parameters*) *of the differential equation* (q) *with the boundary characteristic* χ. *More precisely:*

There is precisely one normal phase of the differential equation (q) *with boundary characteristic* χ, *if* (q) *is a general differential equation either of type* (1), *or of type* (m), $m \geqslant 2$, t_0 *not being singular.*

There are precisely ∞^1 *normal phases of the differential equation* (q) *with the boundary characteristic* χ, *if* (q) *is a differential equation of type* (m), $m \geqslant 2$, *and* t_0 *is singular, or if the differential equation* (q) *is special of type* (1) *or of type* (m), $m \geqslant 2$, t_0 *not being singular; this is also true if the differential* (q) *is left or right oscillatory and* t_0 *is not singular.*

There are precisely ∞^2 *normal phases of the differential equation* (q) *with boundary characteristic* χ, *if* (q) *is special of type* (m), $m \geqslant 2$, *and* t_0 *is singular; this is also true if the differential equation* (q) *is left or right oscillatory and the number* t_0 *is singular; finally this is also true if the differential equation* (q) *is oscillatory.*

7.16 Determination of the type and kind of the differential equation (q) by means of the boundary values of a phase of (q)

We now show that, given the boundary values of a phase of the differential equation (q) the type and kind of (q) is uniquely determined.

Let (q) be a differential equation, α a phase of (q) and c, d the left and right boundary values of α.

Theorem. Let the numbers c, d be finite; then the differential equation (q) *is of finite type* (m), $m \geqslant 1$, *and is general with*† $m = [|d - c|/\pi] + 1$ *or special with* $m = |d - c|/\pi$ *according as* $|d - c|$ *cannot or can be divided by* π. *If c is finite and d infinite, then the differential equation* (q) *is right oscillatory; if c is infinite and d finite then* (q) *is left oscillatory. If c, d are both infinite then* (q) *is oscillatory.*

Proof. We first assume the numbers c, d are finite. Then, from the theorem of § 7.13, the differential equation (q) is of finite type (m), $m \geqslant 1$.

We choose λ so that $c^* = c + \lambda$, $d^* = d + \lambda$ have different signs. Then c^*, d^* are the boundary values of the normal phase $\alpha^* = \alpha + \lambda$ of (q); let t_0 be the zero of α^*.

If the differential equation (q) is general, then from the theorem of § 7.13 we have

$$(m - 1)\pi\varepsilon \lessgtr d^* - c^* \lessgtr m\pi\varepsilon \quad (\varepsilon = \operatorname{sgn}(d^* - c^*));$$

from which it follows that $|d - c|$ is not divisible by π and m has the value $[|d - c|/\pi] + 1$.

If the differential equation is special, then from the same theorem we have

$$d^* - c^* = m\pi\varepsilon;$$

in this case $|d - c|$ is divisible by π and m has the value $|d - c|/\pi$.

Secondly we assume that at least one of the numbers c, d is infinite. In this case our assertion follows immediately from the theorem of § 7.13.

In particular we have: a differential equation (q) of finite type (m), $m \geqslant 2$, is general or special according as the oscillation $O(\alpha)$ of each of its phases α satisfies the relations

$$(m - 1)\pi < O(\alpha) < m\pi \qquad \text{or} \qquad O(\alpha) = m\pi.$$

From the definition of § 7.2, this statement is also true for $m = 1$.

7.17 Properties of second phases

In the study of local and boundary properties of second phases of the differential equation (q), we make use of similar ideas and methods to those employed with respect to the first phases. We shall therefore abbreviate the discussion and bring out only a few of the relevant concepts and results. We assume from now on that the carrier q of the differential equation (q) is always non-zero in the interval j, and satisfies further properties according to circumstances. In particular, we shall recall that in the case $q \in C_2$ the associated differential equation ($\hat{q}_1$) of (q) exists and the first phases of this differential equation ($\hat{q}_1$) represent the second phases of (q) (§ 5.11).

(a) Theorem on the unique determination of a second phase from the Cauchy initial conditions.

Let $t_0 \in j$; Z_0, $Z_0' \neq 0$, Z_0'' *be arbitrary numbers. We assume the existence of* $q'(t_0)$.

There exists precisely one second phase β *of the differential equation* (q) *which satisfies at the point* t_0 *the Cauchy initial conditions*:

$$\beta(t_0) = Z_0, \quad \beta'(t_0) = Z_0', \quad \beta''(t_0) = Z_0''. \tag{7.38}$$

† $[x]$ denotes here, of course, the greatest integer not exceeding x. (Trans.)

This phase β is included in the second phase system of the basis (u, v) of the differential equation (q):

$$\left.\begin{aligned} u(t) &= \sin Z_0 \cdot u_0(t) + \frac{1}{q(t_0)}\left[Z_0' \cos Z_0 + \frac{1}{2}\left(\frac{q'(t_0)}{q(t_0)} - \frac{Z_0''}{Z_0'}\right)\sin Z_0\right] v_0(t), \\ v(t) &= \cos Z_0 \cdot u_0(t) + \frac{1}{q(t_0)}\left[-Z_0' \sin Z_0 + \frac{1}{2}\left(\frac{q'(t_0)}{q(t_0)} - \frac{Z_0''}{Z_0'}\right)\cos Z_0\right] v_0(t), \end{aligned}\right\} \tag{7.39}$$

in which u_0, v_0 are integrals of (q) *determined by the initial values*

$$u_0(t_0) = 0, \quad u_0'(t_0) = 1; \quad v_0(t_0) = 1, \quad v_0'(t_0) = 0,$$

(b) Let β be a second phase of the differential equation (q).

Assuming, as above, that $q(t) \neq 0$, $t \in j$, β is an increasing or decreasing function in the interval j $(= (a, b))$. The finite or infinite limiting values

$$c' = \lim_{t \to a+} \beta(t) \qquad \text{and} \qquad d' = \lim_{t \to b-} \beta(t)$$

are called respectively the *left* and *right boundary values of* β.

The circumstances under which these boundary values are finite or infinite are analogous to those for the first phases. In particular, we have the following fact: let the differential equation (q) possess 2-conjugate numbers; then the boundary value c' (d') of β is finite if and only if the left (right) 2-fundamental number $r_2(s_2)$ of (q) is proper.

The number $|c' - d'|$ is known as the *oscillation* of the phase β in the interval j. The notation used is $O(\beta|j)$, more briefly $O(\beta)$.

(c) The left (right) boundary values of the second phases of a left (right) principal basis of the second kind of the differential equation (q) are integral multiples of the number π.

The right (left) boundary values of the second phases of a 2-principal basis, whose first term is a left (right) 2-fundamental integral, are odd multiples of $\frac{1}{2}\pi$.

(d) We define second normal phases of the differential equation (q) and their boundary characteristics in a similar manner to those of first phases. With regard to the structure of the set of singular second normal phases, and the determination of second normal phases by their boundary characteristics, there are analogous results to those for first phases (§§ 7.9–7.15).

7.18 Relations between the boundary values of a first and second phase of the same basis

Let (u, v) be a basis of the differential equation (q) and α, β be first and second phases of (u, v). We choose the phases so that in the interval j the relations

$$0 < \beta(t) - \alpha(t) < \pi \tag{7.40}$$

hold; that is to say, we are dealing with two neighbouring phases of the mixed phase system of (u, v) (§ 5.14).

We know that the functions $y(t)$, $y'(t)$ constructed with arbitrary constants k_1, k_2

$$\left.\begin{aligned} y(t) &= k_1 \frac{\sin[\alpha(t)+k_2]}{\sqrt{|\alpha'(t)|}}, \\ y'(t) &= \pm k_1 \sqrt{|q(t)|}\,\frac{\sin[\beta(t)+k_2]}{\sqrt{|\beta'(t)|}} \end{aligned}\right\} \tag{7.41}$$

represent the general integral of (q) and its derivative.

We now assume that the function q is negative throughout the interval j. Then the phases α, β either both increase or both decrease, i.e. $\operatorname{sgn}\alpha' \operatorname{sgn}\beta' = 1$.

Let c, c' and d, d' be the left and right boundary values of α, β. We know that the two numbers c, c' and also d, d' are simultaneously finite or infinite.

We consider the first case and assume from here onwards that the boundary values c, c' or d, d' are finite. It then follows from (40) that

$$0 \leqq c' - c \leqq \pi, \quad \text{or} \quad 0 \leqq d' - d \leqq \pi.$$

We now show that:

The relation $c' = c$ ($d' = d$) holds if and only if the left (right) 3-fundamental number r_3 (s_3) of (q) *is improper. In this case, we have therefore the situation described in* § 3.8. $a = r_3$, $r_4 = r_2 < r_1$ ($b = s_3$, $s_4 = s_2 > s_1$).

The relation $c' = c + \pi$ ($d' = d + \pi$) holds if and only if the left (right) 4-fundamental number r_4 (s_4) of (q) *is improper. In this case, we have $a = r_4$, $r_3 = r_1 < r_2$ ($b = s_4$, $s_3 = s_1 > s_2$).*

Proof. We restrict ourselves to proving the statements regarding the left boundary values c, c'.

First we note that each of the relations $c' = c$, $c' = c + \pi$ is invariant with respect to a transformation $\bar{\alpha}(t) = \alpha(t) + \lambda$, $\bar{\beta}(t) = \beta(t) + \lambda$ of the phases α, β, where λ is arbitrary. One can, in particular, take $\lambda = -c$, so without loss of generality we may assume $c = 0$. We shall also assume, for definiteness, that $\operatorname{sgn}\alpha' = \operatorname{sgn}\beta' = 1$.

(a) Let $c' = c = 0$. Let us select a number $x \in j$; we thus have to show that there exists a number which is left 3-conjugate with x.

From the fact that $c = 0$ and $\operatorname{sgn}\alpha' = 1$ we have $\alpha(x) > 0$. In the formulae (41) let us choose the constants k_1, k_2 as $k_1 = 1$, $k_2 = -\alpha(x)$. Then we have an integral y of (q) which vanishes at the point x, and its derivative y'.

From (40), we have the inequality

$$\beta(x) - \alpha(x) > 0. \tag{7.42}$$

Since $c' = 0$ and $\operatorname{sgn}\beta' = 1$, the inequality $\beta(t) < \frac{1}{2}\alpha(x)$ holds for every $t \in j$ in a right neighbourhood of a, and consequently

$$\beta(t) - \alpha(x) < 0. \tag{7.43}$$

From (42) and (43) we conclude that the left side of (43), regarded as a function of t, takes the value 0 at some point $t_0 \in (a, x)$. The number t_0 is obviously a zero of y' and consequently is a left 3-conjugate number of x.

(b) Let $a = r_3$. Then every number $x \in j$ possesses a left 3-conjugate number. We have to show that $c' = 0$.

We assume that $c' > 0$ and choose a number $x \in j$ satisfying the inequality $\alpha(x) < \frac{1}{2}c'$; this is possible since $c = 0$ and $\operatorname{sgn} \alpha' = 1$. Next, in the formulae (41), we choose the constants k_1, k_2 as $k_1 = 1$, $k_2 = -\alpha(x)$. Then we have an integral y of (q) which vanishes at the point x, and its derivative y'.

Now from the definition of c' and the fact that $\operatorname{sgn} \beta' = 1$ it follows that at every point $t \in (a, x)$

$$0 < \frac{1}{2}c' = c' - \frac{1}{2}c' < \beta(t) - \alpha(x) < \beta(x) - \alpha(x) < \pi,$$

and therefore $0 < \beta(t) - \alpha(x) < \pi$.

Obviously, the derivative y' of y has no zero to the left of x; consequently x has no left 3-conjugate number, which is a contradiction.

(c) Let $c' = \pi$. Let us choose a number $x \in j$; we then have to show that x has a left 4-conjugate number. Since $c' = \pi$ and $\operatorname{sgn} \beta' = 1$ we have $\beta(x) > \pi$.

In the formulae (41) we choose the constants k_1, k_2 as $k_1 = 1$, $k_2 = -\beta(x)$. Then we have an integral y of (q) whose derivative y' vanishes at the point x.

From (40) we have the inequality

$$\alpha(x) - \beta(x) > -\pi. \tag{7.44}$$

Since $c = 0$ and $\operatorname{sgn} \alpha' = 1$ the inequality $\alpha(t) < \frac{1}{2}(\beta(x) - \pi)$ holds for all t in a left neighbourhood of a, and consequently we also have

$$\alpha(t) - \beta(x) < -\pi. \tag{7.45}$$

From (44) and (45) we conclude that: the left side of (45), regarded as a function of t, takes the value $-\pi$ at a point $t_0 \in (a, x)$. This number t_0 is obviously a zero of y and so represents a left 4-conjugate number of x.

(d) Let $a = r_4$. Then to every number $x \in j$ there corresponds a left 4-conjugate number of x. We have to show that $c' = \pi$.

We assume that $c' < \pi$ and choose a number $x \in j$ satisfying the inequality $c' < \beta(x) < \pi$; since $c' < \pi$ and $\operatorname{sgn} \beta' = 1$, such a choice is possible.

Next, in the formulae (41) we choose the constants k_1, k_2 as $k_1 = 1$, $k_2 = -\beta(x)$. Then we have an integral y of (q) whose derivative y' vanishes at the point x.

Now, since $c = 0$ and $\operatorname{sgn} \alpha' = 1$, at every point $t \in (a, x)$ we have

$$-\pi < \alpha(t) - \pi < \alpha(t) - \beta(x) < \alpha(x) - \beta(x) < 0,$$

hence $-\pi < \alpha(t) - \beta(x) < 0$.

The integral y has therefore no zero to the left of x. Consequently there is no left 4-conjugate number of x, which is a contradiction.