26. Deformations and the isomorphism theorems for groups

26.1. Deformations of groups

Let $\mathfrak{G}$, $\mathfrak{G}^*$ be groupoids and suppose there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$. If one of these groupoids is a group, what can be said about the other?

1. *Deformation of a group onto a groupoid.* There holds the following theorem:

If $\mathfrak{G}$ is a group, then even $\mathfrak{G}^*$ is a group. Moreover, the $\boldsymbol{d}$-image of the unit of $\mathfrak{G}$ is the unit of $\mathfrak{G}^*$ and to any element $a \in \mathfrak{G}$ there applies $\boldsymbol{d}a^{-1} = (\boldsymbol{d}a)^{-1}$.

To prove this statement, let us first note that, by 13.6.2, the groupoid $\mathfrak{G}^*$ is associative. Let $\underline{1}^*$ stand for the $\boldsymbol{d}$-image of the unit $\underline{1}$ of $\mathfrak{G}$, hence $\underline{1}^* = \boldsymbol{d}\underline{1}$. By 18.7.4, $\underline{1}^*$ is the unit of $\mathfrak{G}^*$. Let, moreover, a^* be an element of $\mathfrak{G}^*$. Since $\boldsymbol{d}$ is a mapping of $\mathfrak{G}$ *onto* $\mathfrak{G}^*$, there exists at least one element $a \in \mathfrak{G}$ such that $a^* = \boldsymbol{d}a$. The equality $aa^{-1} = \underline{1}$ yields $\boldsymbol{d}(aa^{-1}) = \boldsymbol{d}\underline{1}$, i.e., $a^*\boldsymbol{d}a^{-1} = \underline{1}^*$ and, analogously, from $a^{-1}a = \underline{1}$ we have $\boldsymbol{d}(a^{-1}a) = \boldsymbol{d}\underline{1}$, i.e., $\boldsymbol{d}a^{-1}a^* = \underline{1}^*$. Consequently, $\boldsymbol{d}a^{-1}$ is the inverse of a^*, so we have $\boldsymbol{d}a^{-1} = (\boldsymbol{d}a)^{-1}$, which completes the proof. To sum up: Every deformation maps a group again onto a group and preserves the units as well as the inverse elements in both groups.

Consequently, *if any two groupoids $\mathfrak{G}$, $\mathfrak{G}^*$ are isomorphic and one of them is a group, then the other is also a group.* Because, if $\mathfrak{G}$, $\mathfrak{G}^*$ are isomorphic, then there exists an isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}^*$ and, simultaneously, an (inverse) isomorphism of $\mathfrak{G}^*$ onto $\mathfrak{G}$. Thus each of the groupoids $\mathfrak{G}$, $\mathfrak{G}^*$ is an isomorphic image of the other, and so, if one is a group, then the other is also a group. Every isomorphism, naturally, preserves in both groups the units and the inverse elements as well ast he subgroups and, as we can easily verify, the invariant subgroups.

2. *Deformation of a groupoid onto a group.* Let us now omit any further assumptions as regards the groupoid $\mathfrak{G}$ but suppose that $\mathfrak{G}^*$ is a group. By the first isomorphism theorem for groupoids, $\mathfrak{G}^*$ is isomorphic ($\boldsymbol{i}$) with a suitable factoroid $\overline{\mathfrak{G}}$ on $\mathfrak{G}$. The factoroid $\overline{\mathfrak{G}}$ corresponds to the generating decomposition belonging to the deformation $\boldsymbol{d}$ and under the isomorphism $\boldsymbol{i}$ of $\overline{\mathfrak{G}}$ onto $\mathfrak{G}^*$ each element $\bar{a} \in \overline{\mathfrak{G}}$ is mapped onto that element $a^* \in \mathfrak{G}^*$ which is the $\boldsymbol{d}$-image of the individual elements $a \in \bar{a}$. By the above result, $\overline{\mathfrak{G}}$ is a group because $\mathfrak{G}^*$ is a group. The isomorphism $\boldsymbol{i}$ preserves, in both groups, the units as well as the inverse elements; hence, under the isomorphism $\boldsymbol{i}$, the unit $\bar{\underline{1}}$ of $\overline{\mathfrak{G}}$ is mapped onto the unit $\underline{1}^*$ of $\mathfrak{G}^*$ so that $\boldsymbol{i}\bar{\underline{1}} = \underline{1}^*$ and every two inverse elements $\bar{a}$, $\bar{a}^{-1}$ of $\overline{\mathfrak{G}}$ are mapped onto two inverse elements of $\overline{\mathfrak{G}}^*$, hence $\boldsymbol{i}\bar{a} = a^*$, $\boldsymbol{i}\bar{a}^{-1} = a^{*-1}$. As each $\bar{a} \in \overline{\mathfrak{G}}$ consists of all the $\boldsymbol{d}$-inverse images of the element $a^* \in \mathfrak{G}^*$ for which $\boldsymbol{i}\bar{a} = a^*$, the unit $\bar{1}$ of the group $\overline{\mathfrak{G}}$ consists of all the $\boldsymbol{d}$-inverse images of the element $\underline{1}^*$; analogously, two

inverse elements $\bar{a}$, $\bar{a}^{-1}$ of $\bar{\mathfrak{G}}$ consist of all the $\boldsymbol{d}$-inverse images of two inverse elements a^*, a^{*-1} of $\mathfrak{G}^*$. Consequently, there applies the theorem:

If $\mathfrak{G}^*$ is a group, then the factoroid $\bar{\mathfrak{G}}$ on $\mathfrak{G}$, belonging to the deformation $\boldsymbol{d}$, is a group and is isomorphic with $\mathfrak{G}^*$. The unit of $\bar{\mathfrak{G}}$ is the set of all the $\boldsymbol{d}$-inverse images of the unit of $\mathfrak{G}^*$ and any two inverse elements of $\bar{\mathfrak{G}}$ are sets of all the $\boldsymbol{d}$-inverse images of two inverse elements of $\mathfrak{G}^*$.

Let us introduce a simple example to show that if $\mathfrak{G}^*$ is a group, then $\mathfrak{G}$ not only need not be a group but may be an arbitrary groupoid. In fact, let $\mathfrak{G}^*$ denote the group consisting of a single element $\underline{1}^*$, thus $\underline{1}^*\underline{1}^* = \underline{1}^*$, and $\mathfrak{G}$ be an arbitrary groupoid. We are to show that there exists a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$. It is obvious that the mapping associating with each element of $\mathfrak{G}$ the element $\underline{1}^*$ is a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$.

26.2. Cayley's theorem and the realization of abstract groups

1. *Left translations.* Let $\mathfrak{G}$ be a group and a an element of $\mathfrak{G}$. Associating with each element $x \in \mathfrak{G}$ the element $ax \in \mathfrak{G}$, we obtain a mapping of $\mathfrak{G}$ into itself. Since the equation $ax = b$, with b denoting an arbitrary element of $\mathfrak{G}$, has a unique solution $x \in \mathfrak{G}$, it is a simple mapping of $\mathfrak{G}$ onto itself, i.e., a permutation of $\mathfrak{G}$. It is called the *left translation determined by the element a* and denoted by ${}_a\boldsymbol{t}$.

The left translation determined by the element $\underline{1}$ is obviously the identical automorphism on $\mathfrak{G}$. If a, b are different elements of $\mathfrak{G}$, then both left translations ${}_a\boldsymbol{t}, {}_b\boldsymbol{t}$ are different because under ${}_a\boldsymbol{t}$ and ${}_b\boldsymbol{t}$ the element $\underline{1}$ is mapped onto a and b, respectively. Composing ${}_a\boldsymbol{t}$ and ${}_b\boldsymbol{t}$, we obviously obtain the left translation determined by ba, hence ${}_b\boldsymbol{t}\,{}_a\boldsymbol{t} = {}_{ba}\boldsymbol{t}$.

2. *Cayley's theorem.* Let us now consider the groupoid whose field is the set of all left translations determined by the individual elements of $\mathfrak{G}$ and the multiplication defined by the formula ${}_a\boldsymbol{t} \cdot {}_b\boldsymbol{t} = {}_{ab}\boldsymbol{t}$, with ${}_a\boldsymbol{t}$, ${}_b\boldsymbol{t}$ standing for elements of the groupoid. Let us denote it by $\mathfrak{T}_l$. Associating with each element $a \in \mathfrak{G}$ the element ${}_a\boldsymbol{t} \in \mathfrak{T}_l$, we obviously obtain a mapping of $\mathfrak{G}$ onto $\mathfrak{T}_l$; since every two different elements a, $b \in \mathfrak{G}$ are mapped onto two different elements ${}_a\boldsymbol{t}$, ${}_b\boldsymbol{t} \in \mathfrak{T}_l$, the mapping is simple. As the product ab of $a \in \mathfrak{G}$ and $b \in \mathfrak{G}$ is mapped onto ${}_{ab}\boldsymbol{t} \in \mathfrak{T}_l$, i.e., onto the product ${}_a\boldsymbol{t} \cdot {}_b\boldsymbol{t}$ of the image ${}_a\boldsymbol{t}$ of a and the image ${}_b\boldsymbol{t}$ of b, the mapping is a deformation and therefore an isomorphism of $\mathfrak{G}$ onto $\mathfrak{T}_l$. Consequently, $\mathfrak{T}_l$ is a group and, in fact, a permutation group. Thus we have arrived at Cayley's theorem:

Every group is isomorphic with a suitable permutation group.

The importance of this result lies in the fact that, in studying the common properties of isomorphic groups, one may restrict one's attention to the permutation groups.

3. *Realization of abstract groups.* The above considerations suggest the question whether there exists, given an abstract group $\mathfrak{G}$, a permutation group apt to be deformed on it. Every permutation group of that kind is said *to realize the abstract group* $\mathfrak{G}$, and so we ask whether every abstract group can be realized by permutations.

This question can, with regard to the above results, be answered in the affirmative: every abstract group is isomorphic with the corresponding group of the left translations $\mathfrak{T}_l$; consequently, the group $\mathfrak{T}_l$ realizes $\mathfrak{G}$.

For example, let us realize the abstract group of order 4 whose multiplication table is the second in 19.6.1. The corresponding left translations determined by the individual elements are, by the mentioned table, the following permutations

$$\begin{pmatrix} \underline{1} & a & b & c \\ \underline{1} & a & b & c \end{pmatrix}, \quad \begin{pmatrix} \underline{1} & a & b & c \\ a & \underline{1} & c & b \end{pmatrix}, \quad \begin{pmatrix} \underline{1} & a & b & c \\ b & c & \underline{1} & a \end{pmatrix}, \quad \begin{pmatrix} \underline{1} & a & b & c \\ c & b & a & \underline{1} \end{pmatrix};$$

they generate, together with the multiplication $\boldsymbol{p} \,.\, \boldsymbol{q} = \boldsymbol{pq}$, $\boldsymbol{pq}$ being the composite permutation, a permutation group which realizes the group in question.

4. *Right translations.* Given an element $a \in \mathfrak{G}$ and associating with every element $x \in \mathfrak{G}$ the element $xa \in \mathfrak{G}$, we obtain a permutation of $\mathfrak{G}$, the *right translation* $\boldsymbol{t}_a$ *determined by a.*

To the right translations there apply analogous results as to the left. We leave it to the reader to verify this himself.

26.3. The isomorphism theorems for groups

In 16.1 we have discussed isomorphism theorems for groupoids and now we shall specify them for groups. Let $\mathfrak{G}$, $\mathfrak{G}^*$ be arbitrary groups.

1. *First theorem.* Suppose there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$. As we saw in 16.1.1, the factoroid $\bar{\mathfrak{D}}$ corresponding to $\boldsymbol{d}$ is isomorphic with $\mathfrak{G}^*$. By 25.2, $\bar{\mathfrak{D}}$ is the factor group generated by a subgroup of $\mathfrak{G}$ invariant in $\mathfrak{G}$. The field of the latter is the element of $\bar{\mathfrak{D}}$, containing the unit $\underline{1}$ of $\mathfrak{G}$. Since $\underline{1}$ is a $\boldsymbol{d}$-inverse image of the unit $\underline{1}^*$ of $\mathfrak{G}^*$, it is obvious that the element of $\bar{\mathfrak{D}}$, containing $\underline{1}$, consists of all the $\boldsymbol{d}$-inverse images of $\underline{1}^*$. Consequently, the set of all the $\boldsymbol{d}$-inverse images of the unit of $\mathfrak{G}^*$ is the field of an invariant subgroup $\mathfrak{D}$ of $\mathfrak{G}$ and the factor group $\mathfrak{G}/\mathfrak{D}$ is isomorphic with $\mathfrak{G}^*$.

Now let us assume, conversely, that $\mathfrak{G}^*$ is isomorphic with the factor group $\mathfrak{G}/\mathfrak{D}$ on $\mathfrak{G}$ generated by a subgroup $\mathfrak{D}$ invariant in $\mathfrak{G}$. Then there exists an isomorphism $\boldsymbol{i}$ of $\mathfrak{G}/\mathfrak{D}$ onto $\mathfrak{G}^*$. In accordance with 16.1.1, the mapping $\boldsymbol{d}'$ of $\mathfrak{G}$ onto $\mathfrak{G}/\mathfrak{D}$ such that, for $a \in \mathfrak{G}$, $\boldsymbol{d}'a$ is the element $\bar{a} \in \mathfrak{G}/\mathfrak{D}$ containing a, is a deformation of $\mathfrak{G}$

onto $\mathfrak{G}/\mathfrak{D}$. Consequently, $\boldsymbol{d} = \boldsymbol{id}'$ is a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$. By 25.1, the unit of the group $\mathfrak{G}/\mathfrak{D}$ is the field D of $\mathfrak{D}$. Since $\boldsymbol{i}$ maps onto the unit $\underline{1}^*$ of $\mathfrak{G}^*$ precisely the unit of $\mathfrak{G}/\mathfrak{D}$, $\boldsymbol{d}$ maps onto $\underline{1}^*$ exactly those elements of $\mathfrak{G}$ that lie in D. Hence there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$ such that $\mathfrak{D}$ consists of all the $\boldsymbol{d}$-inverse images of the unit of $\mathfrak{G}^*$.

Summing up, we get the first isomorphism theorem for groups:

If there exists a deformation $\boldsymbol{d}$ *of a group* $\mathfrak{G}$ *onto a group* $\mathfrak{G}^*$, *then the set of all* $\boldsymbol{d}$-*inverse images of the unit of* $\mathfrak{G}^*$ *is an invariant subgroup* $\mathfrak{D}$ *of* $\mathfrak{G}$ *and the factor group on* $\mathfrak{G}$, *generated by* $\mathfrak{D}$, *is isomorphic with* $\mathfrak{G}^*$, *i.e.*, $\mathfrak{G}/\mathfrak{D} \simeq \mathfrak{G}^*$. *Conversely, if* $\mathfrak{G}^*$ *is isomorphic with the factor group on* $\mathfrak{G}$, *generated by a subgroup* $\mathfrak{D}$ *invariant in* $\mathfrak{G}$, *then there exists a deformation* $\boldsymbol{d}$ *of* $\mathfrak{G}$ *onto* $\mathfrak{G}^*$ *such that* $\mathfrak{D}$ *consists of all the* $\boldsymbol{d}$-*inverse images of the unit of* $\mathfrak{G}^*$.

2. *Second theorem:*

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ *be subgroups of* $\mathfrak{G}$, *with* $\mathfrak{B}$ *and* $\mathfrak{D}$ *invariant in* $\mathfrak{A}$ *and* $\mathfrak{C}$, *respectively. Moreover, let*

$$\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B},$$

$$\mathfrak{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}, \quad \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}.$$

Then the factor groups $\mathfrak{A}/\mathfrak{B}$, $\mathfrak{C}/\mathfrak{D}$ *are coupled, hence isomorphic and so* $\mathfrak{A}/\mathfrak{B} \simeq \mathfrak{C}/\mathfrak{D}$. *The mapping of either of the factor groups onto the other, realized by the incidence of the elements, is an isomorphism.*

The proof of this theorem directly follows from the results in 23.1 and 16.1.2.

An important special case concerns the closure and the intersection of an arbitrary subgroup and a factor group in $\mathfrak{G}$.

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C}$ be subgroups of $\mathfrak{G}$, with $\mathfrak{B}$ invariant in $\mathfrak{A}$. Then, in accordance with 24.5.1, the subgroups $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{B}$ are interchangeable, $\mathfrak{B} \cap \mathfrak{C}$ is invariant in $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{B}$ in $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$. Let us now apply the above theorem to the groups: $\mathfrak{A}' = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$, $\mathfrak{B}' = \mathfrak{B}$, $\mathfrak{C}' = \mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{D}' = \mathfrak{B} \cap \mathfrak{C}$ which, as it is easy to see, satisfy the corresponding conditions. We obtain $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{B} \simeq (\mathfrak{A} \cap \mathfrak{C})/(\mathfrak{B} \cap \mathfrak{C})$, the isomorphism being realized by the incidence of elements.

Summing up these results, we arrive at the following theorem:

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C}$ *be subgroups of* $\mathfrak{G}$, *with* $\mathfrak{B}$ *invariant in* $\mathfrak{A}$. *Then* $\mathfrak{A} \cap \mathfrak{C}$ *and* $\mathfrak{B}$ *are interchangeable,* $\mathfrak{B} \cap \mathfrak{C}$ *is invariant in* $\mathfrak{A} \cap \mathfrak{C}$ *and* $\mathfrak{B}$ *in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$. *Furthermore, the factor groups* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{B}$ *and* $(\mathfrak{A} \cap \mathfrak{C})/(\mathfrak{B} \cap \mathfrak{C})$ *are coupled, hence isomorphic and thus:*

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{B} \simeq (\mathfrak{A} \cap \mathfrak{C})/(\mathfrak{B} \cap \mathfrak{C});$$

the mapping of either of the factor groups onto the other, realized by the incidence of elements, is an isomorphism.

In particular (for $\mathfrak{A} = \mathfrak{G}$), there holds:

Let $\mathfrak{B}, \mathfrak{C}$ be subgroups of $\mathfrak{G}$, with $\mathfrak{B}$ invariant in $\mathfrak{G}$. Then $\mathfrak{B}$ and $\mathfrak{C}$ are interchangeable and the subgroup $\mathfrak{B} \cap \mathfrak{C}$ is invariant in $\mathfrak{C}$. Moreover, the factor groups $\mathfrak{C}\mathfrak{B}/\mathfrak{B}$ and $\mathfrak{C}/(\mathfrak{B} \cap \mathfrak{C})$ are coupled, hence isomorphic and thus

$$\mathfrak{C}\mathfrak{B}/\mathfrak{B} \simeq \mathfrak{C}/(\mathfrak{B} \cap \mathfrak{C});$$

the mapping of either of the factor groups onto the other, realized by the incidence of elements, is an isomorphism.

3. *Third theorem.* As we know (16.1.3), there exists a third isomorphism theorem for groupoids, concerning coverings of a factoroid.

Let $\mathfrak{B}$ denote an invariant subgroup of $\mathfrak{G}$ and $\mathfrak{B}_1$ an invariant subgroup of the factor group $\mathfrak{G}/\mathfrak{B}$. By the third isomorphism theorem for groupoids, the factor group $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1$ is isomorphic with the covering $\overline{\mathfrak{A}}$ of $\mathfrak{G}/\mathfrak{B}$, enforced by $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1$, i.e., $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1 \simeq \overline{\mathfrak{A}}$; the mapping associating, with every element $\bar{\bar{b}} \in (\mathfrak{G}/\mathfrak{B})\mathfrak{B}_1$, the sum $\bar{a} \in \overline{\mathfrak{A}}$ of all the elements $\bar{b} \in \mathfrak{G}/\mathfrak{B}$ lying in $\bar{\bar{b}}$ is an isomorphism. By 25.5.1, the sum of all the elements of $\mathfrak{G}/\mathfrak{B}$ lying in $\mathfrak{B}_1$ is the field of an invariant subgroup $\mathfrak{A}$ of $\mathfrak{G}$ and $\overline{\mathfrak{A}}$ is the factor group $\mathfrak{G}/\mathfrak{A}$. Moreover, we have $\mathfrak{B}_1 = \mathfrak{A}/\mathfrak{B}$.

Hence follows the third isomorphism theorem for groups:

If $\mathfrak{B}$ and $\mathfrak{B}_1$ are invariant subgroups of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{B}$, respectively, then the sum of the elements of $\mathfrak{G}/\mathfrak{B}$ that lie in $\mathfrak{B}_1$ is the field of a subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$ and there holds:

$$(\mathfrak{G}/\mathfrak{B})/(\mathfrak{A}/\mathfrak{B}) \simeq \mathfrak{G}/\mathfrak{A},$$

the isomorphism associating, with every element $\bar{\bar{b}}$ of the factor group on the left-hand side, the sum of all the elements of the factor group $\mathfrak{G}/\mathfrak{B}$ that lie in $\bar{\bar{b}}$.

26.4. Deformations of factor groups

Let us now start from the results concerning deformations of factoroids (16.2) and consider their particular form in case of factor groups.

Let $\boldsymbol{d}$ be a deformation of a group $\mathfrak{G}$ onto a group $\mathfrak{G}^*$ so that we have $\mathfrak{G}^* = \boldsymbol{d}\mathfrak{G}$.

From 26.3.1 we know that the set of all the $\boldsymbol{d}$-inverse images of the unit of $\mathfrak{G}^*$ is an invariant subgroup $\mathfrak{D}$ of $\mathfrak{G}$ and that the factor group $\mathfrak{G}/\mathfrak{D}$ is isomorphic with $\mathfrak{G}^*$.

The deformation $\boldsymbol{d}$ determines the extended mapping $\boldsymbol{d}$ of the system of all subsets of $\mathfrak{G}$ into the system of all subsets of $\mathfrak{G}^*$; the $\boldsymbol{d}$-image of any subset $A \subset \mathfrak{G}$ is the subset $\boldsymbol{d}A \subset \mathfrak{G}^*$ consisting of the $\boldsymbol{d}$-images of the individual elements $a \in A$ (7.1).

Let $\mathfrak{G}/\mathfrak{A}$ be a factor group on $\mathfrak{G}$, generated by an invariant subgroup $\mathfrak{A}$ of $\mathfrak{G}$.

With regard to 25.3, the factor groups $\mathfrak{G}/\mathfrak{A}$, $\mathfrak{G}/\mathfrak{D}$ are complementary. Consequently, $\mathfrak{G}/\mathfrak{A}$ has, under the extended mapping $\boldsymbol{d}$, the image $\boldsymbol{d}(\mathfrak{G}/\mathfrak{A})$; the latter is a factoroid on $\mathfrak{G}^*$ (16.2.1). The partial extended mapping $\boldsymbol{d}$ of $\mathfrak{G}/\mathfrak{A}$ onto the factoroid $\boldsymbol{d}(\mathfrak{G}/\mathfrak{A})$ is a deformation called the *extended deformation* $\boldsymbol{d}$ (16.2.2).

The $\boldsymbol{d}$-image of the field A of $\mathfrak{A}$ contains the unit of $\mathfrak{G}^*$ (26.1.1). Consequently, $\boldsymbol{d}A \in \boldsymbol{d}(\mathfrak{G}/\mathfrak{A})$ is the field of a subgroup $\boldsymbol{d}\mathfrak{A}$ invariant in $\mathfrak{G}^*$ and the factoroid $\boldsymbol{d}(\mathfrak{G}/\mathfrak{A})$ is the factor group generated by the invariant subgroup $\boldsymbol{d}\mathfrak{A}$ (24.3.2), i.e., $\boldsymbol{d}(\mathfrak{G}/\mathfrak{A}) = \boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$.

The least common covering $[\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{D}]$ of the factor groups $\mathfrak{G}/\mathfrak{A}$, $\mathfrak{G}/\mathfrak{D}$ and the factor group $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ are isomorphic; an isomorphic mapping of the factoroid $[\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{D}]$ onto $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ is obtained by associating, with every element of the factoroid $[\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{D}]$, its image under the extended mapping $\boldsymbol{d}$ (16.2.3). The factoroid $[\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{D}]$ is the factor group $\mathfrak{G}/\mathfrak{A}\mathfrak{D}$ generated by the invariant subgroup $\mathfrak{A}\mathfrak{D}$ (25.3).

The result:

If the group $\mathfrak{G}^*$ is homomorphic ($\boldsymbol{d}$) with the group $\mathfrak{G}$, then the image of every factor group $\mathfrak{G}/\mathfrak{A}$ under the extended mapping $\boldsymbol{d}$ is the factor group $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ and the partial extended mapping $\boldsymbol{d}$ of $\mathfrak{G}/\mathfrak{A}$ onto $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ is a deformation. The factor groups $\mathfrak{G}/\mathfrak{A}\mathfrak{D}$, $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ are isomorphic; an isomorphic mapping of $\mathfrak{G}/\mathfrak{A}\mathfrak{D}$ onto $\boldsymbol{d}\mathfrak{G}/\boldsymbol{d}\mathfrak{A}$ is obtained by associating, with each element of $\mathfrak{G}/\mathfrak{A}\mathfrak{D}$, its image under the extended mapping $\boldsymbol{d}$.

In particular, any factor group which is a covering of $\mathfrak{G}/\mathfrak{D}$ is isomorphic with its image under the extended mapping $\boldsymbol{d}$. An isomorphic mapping is obtained by associating, with each element of the covering, its image under the extended mapping $\boldsymbol{d}$.

26.5. Exercises

1. Realize, by means of permutations, the abstract group of the 4th order whose multiplication table is the first in 19.6.1.

2. Given the multiplication table of a finite group $\mathfrak{G}$, the symbols of the left translations on $\mathfrak{G}$ are obtained by copying, successively, the horizontal heading and writing one line of the table underneath. In a similar way we get, from the vertical heading and the single columns, the symbols of the right translations on $\mathfrak{G}$.

3. A regular octahedron has altogether thirteen axes of symmetry (three of them pass through two opposite vertices, six pass through the centers of two opposite edges and four through the centers of two opposite faces). The rotations of the octahedron about the axes of symmetry which leave the octahedron unaltered form a group of the 24th order, called the *octahedral group* (rotations about the same axis by angles which differ from each other by integer multiples of 360° are considered equal); let us, for the moment, denote the mentioned group by $\mathfrak{O}$. To each rotation which is an element of $\mathfrak{O}$ there corresponds a permutation of the three axes of symmetry passing through two opposite vertices. Associating with each element of $\mathfrak{O}$ the corresponding permutation, we obtain a deformation of $\mathfrak{O}$ onto the symmetric permutation group $\mathfrak{S}_3$. Employing this deformation and taking account of the first and the second isomorphism theorems for groups, prove that $\mathfrak{O}$ contains invariant subgroups of the orders 4 and 12.