2. Let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$. The set of all elements $p \in \mathfrak{G}$ such that $p\mathfrak{A} = \mathfrak{A}p$ is a subgroup $\mathfrak{N}$ of $\mathfrak{G}$, the so-called *normalizer* of $\mathfrak{A}$. The latter is the greatest supergroup of $\mathfrak{A}$ in which $\mathfrak{A}$ is invariant; that is to say, $\mathfrak{A}$ is invariant in $\mathfrak{N}$ and each subgroup of $\mathfrak{G}$ in which $\mathfrak{A}$ is invariant is a subgroup of $\mathfrak{N}$.
3. The center of $\mathfrak{G}$ is an invariant subgroup of $\mathfrak{G}$.
4. If there exists, in a finite group of order N ($\geqq 2$), a subgroup of order $\frac{1}{2}N$, then the latter is invariant in the former. For example, in the diedric permutation group of order $2n$ ($n \geqq 3$) there is an invariant subgroup of order n consisting of all the elements of the group corresponding to the rotations of the vertices of a regular n-gon about its center (19.7.2).
5. Associating, with every element $p \in \mathfrak{G}$, any element $x^{-1}px \in \mathfrak{G}$ with $x \in \mathfrak{G}$ arbitrary, we obtain a symmetric congruence on $\mathfrak{G}$. The decomposition $\bar{G}$ corresponding to the latter is called the *fundamental decomposition of* $\mathfrak{G}$. The field of each invariant subgroup of $\mathfrak{G}$ is the sum of certain elements of $\bar{G}$. $\bar{G}$ is complementary to every generating decomposition of $\mathfrak{G}$.
6. Let $p \in \mathfrak{G}$ be an arbitrary point and $\mathring{\mathfrak{G}}$ the (p)-group associated with $\mathfrak{G}$ (19.7.11). Consider a subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$ and the subgroup $\mathring{\mathfrak{A}}$ of $\mathring{\mathfrak{G}}$, lying on the field $p\mathfrak{A} = \mathfrak{A}p$ (20.3.3; 21.8.7). Show that: a) $\mathring{\mathfrak{A}}$ is invariant in $\mathring{\mathfrak{G}}$; b) all generating decompositions of $\mathring{\mathfrak{G}}$ coincide with the generating decompositions of $\mathfrak{G}$.

25. Factor groups

25.1. Definition

Let us now consider a factoroid $\bar{\mathfrak{G}}$ on $\mathfrak{G}$. According to the definition of a factoroid, the field of $\bar{\mathfrak{G}}$ is a generating decomposition of $\mathfrak{G}$ and is therefore generated by a suitable subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$ (24.3.2). The product $p\mathfrak{A} \,.\, q\mathfrak{A}$ of an element $p\mathfrak{A} \in \bar{\mathfrak{G}}$ and an element $q\mathfrak{A} \in \bar{\mathfrak{G}}$ is, by the definition of multiplication in a factoroid, the element of $\bar{\mathfrak{G}}$ that contains the set $p\mathfrak{A} \,.\, q\mathfrak{A}$. Since the latter coincides, as we know, with $pq\mathfrak{A} \in \bar{\mathfrak{G}}$, the multiplication in $\bar{\mathfrak{G}}$ is given by the following formula:

$$p\mathfrak{A} \circ q\mathfrak{A} = pq\mathfrak{A}. \tag{1}$$

Now we shall show that $\bar{\mathfrak{G}}$ *is a group whose unit is the field of the invariant subgroup* $\mathfrak{A}$ *and the element inverse of an arbitrary element* $a\mathfrak{A}$ *is* $a^{-1}\mathfrak{A}$.

In fact, first, by 15.6.3, $\bar{\mathfrak{G}}$ is associative. Next, by 18.7.5, the field A of the invariant subgroup $\mathfrak{A}$ is the unit of $\bar{\mathfrak{G}}$. Finally we have:

$$p\mathfrak{A} \circ p^{-1}\mathfrak{A} = pp^{-1}\mathfrak{A} = \underline{1}\mathfrak{A} = A$$

and so $p^{-1}\mathfrak{A} \in \bar{\mathfrak{G}}$ is the inverse element of $p\mathfrak{A} \in \bar{\mathfrak{G}}$.

Every factoroid $\bar{\mathfrak{G}}$ on $\mathfrak{G}$ is therefore a group and is uniquely determined by a subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$; the field of $\bar{\mathfrak{G}}$ is the decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$. $\bar{\mathfrak{G}}$ is called a *factor group* or a *group of cosets* and is said to be *generated by the invariant subgroup* $\mathfrak{A}$; notation: $\mathfrak{G}/\mathfrak{A}$.

25.2. Factoroids on a group

From the result in 24.3.2 we have the following information about all the possible factoroids on a group $\mathfrak{G}$:

All factoroids on $\mathfrak{G}$ *are precisely the factor groups on* $\mathfrak{G}$ *generated by the individual invariant subgroups of* $\mathfrak{G}$.

Note that *the greatest (least) factoroid on* $\mathfrak{G}$ *is the greatest (least) factor group* $\mathfrak{G}/\mathfrak{G}$ $(\mathfrak{G}/\{\underline{1}\})$; *it is generated by the greatest (least) invariant subgroup of* $\mathfrak{G}$, *namely, the subgroup* $\mathfrak{G}$ $(\{\underline{1}\})$.

25.3. Properties of factor groups

The properties of factor groups follow from the properties of the generating decompositions of groups (24.4).

Let $\mathfrak{G}/\mathfrak{A}$, $\mathfrak{G}/\mathfrak{B}$ be arbitrary factor groups on $\mathfrak{G}$.

$\mathfrak{G}/\mathfrak{A}$ *and* $\mathfrak{G}/\mathfrak{B}$ *are the covering and the refinement of the factor groups* $\mathfrak{G}/\mathfrak{B}$ *and* $\mathfrak{G}/\mathfrak{A}$, *respectively, if and only if* $\mathfrak{A} \supset \mathfrak{B}$.

The greatest common refinement $(\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{B})$ *of the factor groups* $\mathfrak{G}/\mathfrak{A}$, $\mathfrak{G}/\mathfrak{B}$ *is the factor group* $\mathfrak{G}/(\mathfrak{A} \cap \mathfrak{B})$.

The least common covering $[\mathfrak{G}/\mathfrak{A}, \mathfrak{G}/\mathfrak{B}]$ *of the factor groups* $\mathfrak{G}/\mathfrak{A}$, $\mathfrak{G}/\mathfrak{B}$ *is the factor group* $\mathfrak{G}/\mathfrak{A}\mathfrak{B}$.

$\mathfrak{G}/\mathfrak{A}$ *and* $\mathfrak{G}/\mathfrak{B}$ *are complementary.*

On every group the system of factor groups is closed with regard to the operations ( ), [ ]. *Together with the multiplications associating with each ordered pair of factor groups either their least common covering or their greatest common refinement, this system is a modular lattice with extreme elements. The latter are the greatest and the least corresponding factor groups.*

Note, in particular, that *the groups belong to the class of groupoids on which every two factoroids are complementary.*

25.4. Factor groups in groups

1. *Intersections and closures.* Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C}$ be subgroups of $\mathfrak{G}$ and $\mathfrak{B}$ invariant in $\mathfrak{A}$. Consider the factoroids $\mathfrak{A}/\mathfrak{B} \sqcap \mathfrak{C}$ and $\mathfrak{C} \sqsubset \mathfrak{A}/\mathfrak{B}$. From 24.5.1 we know that the subgroups $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{B}$ are interchangeable and that the subgroups $\mathfrak{B} \cap \mathfrak{C}$ and $\mathfrak{B}$ are invariant in $\mathfrak{A} \cap \mathfrak{C}$ and $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$, respectively. Moreover, the fields of the factoroids in question are given by the generating decompositions $(\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C})$ and $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{B}$ (21.2.1).

Consequently:

$$\mathfrak{A}/\mathfrak{B} \sqcap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{C})/(\mathfrak{B} \cap \mathfrak{C}), \quad \mathfrak{C} \sqsubset \mathfrak{A}/\mathfrak{B} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/\mathfrak{B} \tag{1}$$

from which we conclude that:

The factoroids $\mathfrak{A}/\mathfrak{B} \sqcap \mathfrak{C}$ and $\mathfrak{C} \sqsubset \mathfrak{A}/\mathfrak{B}$ are factor groups given by the formulae (1).

In particular (for $\mathfrak{A} = \mathfrak{G}$), we have the following theorem:

Assuming $\mathfrak{B}$, $\mathfrak{C}$ to be arbitrary subgroups of $\mathfrak{G}$, $\mathfrak{B}$ invariant in $\mathfrak{G}$, the factoroids $\mathfrak{G}/\mathfrak{B} \sqcap \mathfrak{C}$ and $\mathfrak{C} \sqsubset \mathfrak{G}/\mathfrak{B}$ are factor groups and there holds:

$$\mathfrak{G}/\mathfrak{B} \sqcap \mathfrak{C} = \mathfrak{C}/(\mathfrak{B} \cap \mathfrak{C}), \quad \mathfrak{C} \sqsubset \mathfrak{G}/\mathfrak{B} = \mathfrak{C}\mathfrak{B}/\mathfrak{B}.$$

2. *Special five-group theorem.* Let us return to the situation described in 24.5.2. Consider the factoroids $\mathring{\mathfrak{A}}$, $\mathring{\mathfrak{C}}$ (15.3.3) which are, as we know, the coverings of the following factor groups, enforced by the factor group $\overline{\mathfrak{B}} = (\mathfrak{A} \cap \mathfrak{C})/\mathfrak{U}$:

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{B} \sqcap (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D} = (\mathfrak{A} \cap \mathfrak{C})/(\mathfrak{C} \cap \mathfrak{B}).$$

$$(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/\mathfrak{D} \sqcap (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} = (\mathfrak{C} \cap \mathfrak{A})/(\mathfrak{A} \cap \mathfrak{D}).$$

The fields of $\mathring{\mathfrak{A}}$, $\mathring{\mathfrak{C}}$ are the (generating) decompositions

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B} \text{ and } (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}$$

(24.5.2). Consequently:

$$\mathring{\mathfrak{A}} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{U}\mathfrak{B}, \quad \mathring{\mathfrak{C}} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/\mathfrak{U}\mathfrak{D},$$

hence $\mathring{\mathfrak{A}}$, $\mathring{\mathfrak{C}}$ are factor groups given by these formulae.

From 15.3.3 we know that $\mathring{\mathfrak{A}}$, $\mathring{\mathfrak{C}}$ are coupled and therefore isomorphic (16.1.2).

Thus we have arrived at *the* so-called *special five-group theorem*:

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ be subgroups of $\mathfrak{G}$ with $\mathfrak{B}$ and $\mathfrak{D}$ invariant in $\mathfrak{A}$ and $\mathfrak{C}$, respectively. Then $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are invariant in $\mathfrak{A} \cap \mathfrak{C}$. Now let $\mathfrak{U}$ be an invariant subgroup of $\mathfrak{A} \cap \mathfrak{C}$ such that

$$\mathfrak{A} \cap \mathfrak{C} \supset \mathfrak{U} \supset (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}).$$

Then $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{U}$ are interchangeable with both $\mathfrak{B}$ and $\mathfrak{D}$ and the subgroup $\mathfrak{U}\mathfrak{B}$ or $\mathfrak{U}\mathfrak{D}$ is invariant in $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ or $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}$, respectively. Moreover, the subgroups

$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{U}\mathfrak{B}$ *and* $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/\mathfrak{U}\mathfrak{D}$ *are coupled, hence isomorphic, so we have:*

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/\mathfrak{U}\mathfrak{B} \simeq (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/\mathfrak{U}\mathfrak{D}.$$

In particular (for $\mathfrak{U} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$), there applies *the four-group theorem* (H. ZASSENHAUS):

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ *be subgroups of* $\mathfrak{G}$, *with* $\mathfrak{B}$ *invariant in* $\mathfrak{A}$ *and* $\mathfrak{D}$ *in* $\mathfrak{C}$. *Then the subgroups* $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ *are invariant in* $\mathfrak{A} \cap \mathfrak{C}$. *Moreover,* $\mathfrak{A} \cap \mathfrak{C}$ *and* $\mathfrak{A} \cap \mathfrak{D}$ *are interchangeable with* $\mathfrak{B}$ *and* $\mathfrak{C} \cap \mathfrak{A}$, $\mathfrak{C} \cap \mathfrak{B}$ *with* $\mathfrak{D}$. *The subgroup* $(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}$ *is invariant in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ *and* $(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}$ *in* $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}$. *The factor groups* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}$ *and* $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}$ *are coupled and therefore isomorphic, so we have:*

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \cong (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}.$$

25.5. Further properties of factor groups

1. *Enforced coverings of factor groups.* Let $\mathfrak{B}$ denote an invariant subgroup of $\mathfrak{G}$ and $\mathfrak{B}_1$ an invariant subgroup of the factor group $\mathfrak{G}/\mathfrak{B}$. Thus the elements of $\mathfrak{B}_1$ are cosets with regard to $\mathfrak{B}$ and one of them is the field B of the invariant subgroup $\mathfrak{B}$. This is true because B is, as we know from 25.1, the unit of the factor group $\mathfrak{G}/\mathfrak{B}$ and is therefore an element of each subgroup of $\mathfrak{G}/\mathfrak{B}$. The sum of all elements of $\mathfrak{B}_1$ is, consequently, a certain supergroup A of B, containing the unit $\underline{1}$ of $\mathfrak{G}$, hence: $\underline{1} \in B \subset A$. The subgroup $\mathfrak{B}_1$ generates, on $\mathfrak{G}/\mathfrak{B}$, a factor group $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1$ and, in accordance with 15.4.1, the latter enforces a certain covering $\overline{\mathfrak{A}}$ of $\mathfrak{G}/\mathfrak{B}$. Note that $\overline{\mathfrak{A}}$ is a factoroid on $\mathfrak{G}$, each of its elements being the sum of all elements of $\mathfrak{G}/\mathfrak{B}$ that are contained in the same element of the factor group $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1$. In particular, the set A is an element of $\overline{\mathfrak{A}}$ and as it contains the unit $\underline{1}$ of $\mathfrak{G}$ it is, by 24.3.2, the field of an invariant subgroup $\mathfrak{A}$ of $\mathfrak{G}$; furthermore, $\overline{\mathfrak{A}}$ is the factor group $\mathfrak{G}/\mathfrak{A}$. Since $\mathfrak{B}$ is invariant in $\mathfrak{G}$, it is also invariant in $\mathfrak{A}$ and it is easy to see that $\mathfrak{B}_1 = \mathfrak{A}/\mathfrak{B}$.

The result:

The covering of the factor group $\mathfrak{G}/\mathfrak{B}$, *enforced by the factor group* $(\mathfrak{G}/\mathfrak{B})/\mathfrak{B}_1$, *is the factor group* $\mathfrak{G}/\mathfrak{A}$; *the field of* $\mathfrak{A}$ *is the sum of all the elements of* $\mathfrak{G}/\mathfrak{B}$ *that are comprised in the invariant subgroup* $\mathfrak{B}_1$ *of* $\mathfrak{G}/\mathfrak{B}$. $\mathfrak{B}_1$ *is the factor group* $\mathfrak{A}/\mathfrak{B}$.

2. *Series of factor groups.* Consider a series of factoroids $(\overline{\mathfrak{A}})$ on $\mathfrak{G}$, namely

$$\big((\overline{\mathfrak{A}}) = \big)\, \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha \; (\alpha \geqq 1).$$

By 25.2, each member $\overline{\mathfrak{A}}_\gamma$ of this series is a factor group $\mathfrak{G}/\mathfrak{A}_\gamma$ of $\mathfrak{G}$, generated by a subgroup $\mathfrak{A}_\gamma$ invariant in $\mathfrak{G}$ ($\gamma = 1, \ldots, \alpha$). The series $(\overline{\mathfrak{A}})$ therefore consists of the factor groups on $\mathfrak{G}$:

$$\big((\overline{\mathfrak{A}}) =\big)\ \mathfrak{G}/\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/\mathfrak{A}_\alpha .$$

Note that the subgroups $\mathfrak{A}_\gamma$ generate a series $(\mathfrak{A})$ (25.3):

$$\big((\mathfrak{A}) =\big)\ \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha .$$

$(\overline{\mathfrak{A}})$ is said to be a *series of factor groups on* $\mathfrak{G}$; notation $(\mathfrak{G}/\mathfrak{A})$.

The theory of series of factor groups on $\mathfrak{G}$ is a special case of the theory of the series of factoroids developed in Chapter 17. The novum of this case consists in the fact that in the theory of the series of factoroids certain situations have to be postulated, whereas in the theory of the series of factor groups they occur automatically. In comparison with the theory of the series of factoroids, this new theory has therefore become simpler and clearer.

Since any two series of factor groups on $\mathfrak{G}$ are complementary (25.3), there holds (17.6; 25.3) the following theorem:

Let

$$\big((\mathfrak{G}/\mathfrak{A}) =\big)\ \mathfrak{G}/\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/\mathfrak{A}_\alpha ,$$
$$\big((\mathfrak{G}/\mathfrak{B}) =\big)\ \mathfrak{G}/\mathfrak{B}_1 \geqq \cdots \geqq \mathfrak{G}/\mathfrak{B}_\beta$$

be series of factor groups on $\mathfrak{G}$, *of lengths* $\alpha, \beta \geqq 1$, *respectively. The series* $(\mathfrak{G}/\mathfrak{A})$ *and* $(\mathfrak{G}/\mathfrak{B})$ *have co-basally joint refinements* $(\mathfrak{G}/\mathfrak{A}_*)$, $(\mathfrak{G}/\mathfrak{B}_*)$ *with coinciding initial and final members.* $(\mathfrak{G}/\mathfrak{A}_*)$ *and* $(\mathfrak{G}/\mathfrak{B}_*)$ *are given by the construction described in* 17.6. *Their members* $\mathring{\mathfrak{A}}_{\gamma,\nu} = \mathfrak{G}/\mathfrak{A}_{\gamma,\nu}$ *and* $\dot{\mathfrak{B}}_{\delta,\mu} = \mathfrak{G}/\mathfrak{B}_{\delta,\mu}$, *respectively, are factor groups generated by the invariant subgroups*

$$\mathfrak{A}_{\gamma,\nu} = \mathfrak{A}_\gamma(\mathfrak{A}_{\gamma-1} \cap \mathfrak{B}_\nu) \quad (= \mathfrak{A}_{\gamma-1} \cap \mathfrak{A}_\gamma\mathfrak{B}_\nu)$$

and

$$\mathfrak{B}_{\delta,\mu} = \mathfrak{B}_\delta(\mathfrak{B}_{\delta-1} \cap \mathfrak{A}_\mu) \ (= \mathfrak{B}_{\delta-1} \cap \mathfrak{B}_\delta\mathfrak{A}_\mu),$$

where $\gamma, \mu = 1, 2, \ldots, \alpha + 1$; $\delta, \nu = 1, 2, \ldots, \beta + 1$ *and, furthermore,* $\mathfrak{A}_0 = \mathfrak{B}_0 = \mathfrak{G}$, $\mathfrak{A}_{\alpha+1} = \mathfrak{B}_{\beta+1} = \mathfrak{A}_\alpha \cap \mathfrak{B}_\beta$.

25.6. Exercises

1. The order of a factor group on a finite group of order N is a divisor of N.
2. Consider the complete group $\mathfrak{G}$ of Euclidean motions on a straight line (in a plane); the subgroup of $\mathfrak{G}$, consisting of all Euclidean motions $f[a]$ ($f[\alpha; a, b]$) is invariant in $\mathfrak{G}$ (19.7.1). The corresponding factor group has exactly two elements; one consists of all Euclidean motions $f[a]$ ($f[\alpha; a, b]$), the other of $g[a]$ ($g[\alpha; a, b]$).
3. Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ be subgroups of $\mathfrak{G}$ with $\mathfrak{B}$ and $\mathfrak{D}$ invariant in $\mathfrak{A}$ and $\mathfrak{C}$, respectively. Then the factor groups $\mathfrak{A}/\mathfrak{B}$, $\mathfrak{C}/\mathfrak{D}$ are adjoint with regard to the subgroups $\mathfrak{B}$, $\mathfrak{D}$ (15.3.4; 23.3).
4. Every two chains of factor groups in $\mathfrak{G}$, from $\mathfrak{G}$ to $\{\underline{1}\}$, have isomorphic refinements (Jordan-Hölder-Schreier's theorem) (see 16.4.4).