24. Invariant (normal) subgroups

24.1. Definition

Let $\mathfrak{A} \supset \mathfrak{B}$ be arbitrary subgroups of $\mathfrak{G}$. If the left and the right coset of every element $a \in \mathfrak{A}$ with regard to $\mathfrak{B}$ coincide, that is to say, if $a\mathfrak{B} = \mathfrak{B}a$, then $\mathfrak{B}$ is said to be *invariant* or *normal in* $\mathfrak{A}$. The left decomposition of $\mathfrak{A}$, generated by $\mathfrak{B}$ is, in that case, the same as the right decomposition; *both decompositions therefore coincide and form a certain decomposition of* $\mathfrak{A}$, called the *decomposition generated by* $\mathfrak{B}$, hence $\mathfrak{A}/_l\mathfrak{B} = \mathfrak{A}/_r\mathfrak{B}$ $(= \mathfrak{A}/\mathfrak{B})$.

In the following study of invariant subgroups lying in the same subgroup $\mathfrak{A}$, we shall restrict our attention to the case $\mathfrak{A} = \mathfrak{G}$. If a subgroup $\mathfrak{B}$ of $\mathfrak{G}$ is invariant in $\mathfrak{G}$, then it is called, for convenience, an *invariant subgroup of* $\mathfrak{G}$.

24.2. Basic properties of invariant subgroups

In $\mathfrak{G}$ there exist at least two (may be coinciding) invariant subgroups: the greatest subgroup, identical with $\mathfrak{G}$ and the least subgroup $\{\underline{1}\}$ consisting of the single element $\underline{1} \in \mathfrak{G}$. They are called the *extreme invariant subgroups of* $\mathfrak{G}$. Groups may also contain subgroups that are not invariant, for example, the subgroup $\mathfrak{A}$ of $\mathfrak{S}_3$ consisting of the two permutations $\underline{1}$, $\boldsymbol{f}$ (notation as in 22.1) is not invariant in $\mathfrak{S}_3$ because, as we have observed, there holds, e.g., $\boldsymbol{a}\mathfrak{A} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\mathfrak{A}\boldsymbol{a} = \{\boldsymbol{a}, \boldsymbol{d}\}$ and so $\boldsymbol{a}\mathfrak{A} \neq \mathfrak{A}\boldsymbol{a}$.

Let $\mathfrak{A} \supset \mathfrak{B}$ be subgroups of $\mathfrak{G}$. If $\mathfrak{B}$ is invariant in $\mathfrak{G}$, then it naturally has the same property in $\mathfrak{A}$. If, conversely, $\mathfrak{B}$ is invariant in $\mathfrak{A}$, it is not necessarily invariant in $\mathfrak{G}$, since the equality $x\mathfrak{B} = \mathfrak{B}x$ may apply to all the elements x of $\mathfrak{A}$ without applying to all the elements of $\mathfrak{G}$. If, for instance, a subgroup $\mathfrak{A}$ is not invariant in $\mathfrak{G}$, it is invariant in $\mathfrak{A}$ but not in $\mathfrak{G}$.

If a subgroup $\mathfrak{A}$ is invariant in $\mathfrak{G}$, then it is interchangeable with every complex $C \subset \mathfrak{G}$. Indeed, in that case we have $x\mathfrak{A} = \mathfrak{A}x$ for every $x \in \mathfrak{G}$ and therefore even for every $x \in C$. Consequently, $C\mathfrak{A} = \mathfrak{A}C$. We observe, in particular, that any two subgroups $\mathfrak{A}, \mathfrak{C} \subset \mathfrak{G}$ one of which is invariant in $\mathfrak{G}$, are interchangeable.

If, vice versa, some subgroups $\mathfrak{A}$, $\mathfrak{C} \subset \mathfrak{G}$ are interchangeable, then neither of them is necessarily invariant in $\mathfrak{G}$. That occurs, for example, if $\mathfrak{A}$ is not invariant in $\mathfrak{G}$ and $\mathfrak{A} = \mathfrak{C}$.

Moreover, there applies the following theorem:

If the subgroups $\mathfrak{A}$, $\mathfrak{C}$ are invariant in $\mathfrak{G}$, then even the intersection $\mathfrak{A} \cap \mathfrak{C}$ and the product $\mathfrak{A}\mathfrak{C}$ are invariant subgroups of $\mathfrak{G}$.

In fact, if the assumption is satisfied, then there hold, for any $x \in \mathfrak{G}$, the equalities $x\mathfrak{A} = \mathfrak{A}x$, $x\mathfrak{C} = \mathfrak{C}x$. So we have, with respect to 20.2.6 and to an analogous theorem for right cosets,

$$x(\mathfrak{A} \cap \mathfrak{C}) = x\mathfrak{A} \cap x\mathfrak{C} = \mathfrak{A}x \cap \mathfrak{C}x = (\mathfrak{A} \cap \mathfrak{C})x;$$

consequently, $\mathfrak{A} \cap \mathfrak{C}$ is invariant in $\mathfrak{G}$. Furthermore we have, with respect to 12.9.8,

$$x(\mathfrak{A}\mathfrak{C}) = (x\mathfrak{A})\mathfrak{C} = (\mathfrak{A}x)\mathfrak{C} = \mathfrak{A}(x\mathfrak{C}) = \mathfrak{A}(\mathfrak{C}x) = (\mathfrak{A}\mathfrak{C})x;$$

consequently, $\mathfrak{A}\mathfrak{C}$ is invariant in $\mathfrak{G}$.

The information that any two invariant subgroups of $\mathfrak{G}$ are interchangeable and the properties of interchangeable subgroups (22.2) yield a number of results as to invariant subgroups. We shall introduce only the following two theorems:

1. *The Dedekind-Ore theorem. For any three invariant subgroups $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{D}$ of $\mathfrak{G}$ there holds:*

$$\mathfrak{A} \cap \mathfrak{D}\mathfrak{B} = (\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}.$$

2. *The system of all invariant subgroups of $\mathfrak{G}$ is closed with regard to the intersections and the products and is, when completed by the multiplications defined by forming intersections and products, a modular lattice with extreme elements.*

24.3. Generating decompositions of groups

1. *First theorem.* Let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$. As we have seen (21.1) $\mathfrak{A}$ generates a left decomposition $\mathfrak{G}/_l\mathfrak{A}$ and a right decomposition $\mathfrak{G}/_r\mathfrak{A}$ of $\mathfrak{G}$. Let us find out whether, for example, the left decomposition $\mathfrak{G}/_l\mathfrak{A}$ can be generating.

Suppose, first, that $\mathfrak{G}/_l\mathfrak{A}$ is generating and consider two elements $p\mathfrak{A}, q\mathfrak{A} \in \mathfrak{G}/_l\mathfrak{A}$, p, q being arbitrary elements of $\mathfrak{G}$. By the definition of a generating decomposition there exists an element $r\mathfrak{A} \in \mathfrak{G}/_l\mathfrak{A}$ such that:

$$p\mathfrak{A} \,.\, q\mathfrak{A} \subset r\mathfrak{A}.$$

Hence, in particular, $pq\mathfrak{A} = (p\underline{1}) \,.\, q\mathfrak{A} \subset r\mathfrak{A}$, thus $pq\mathfrak{A} \subset r\mathfrak{A}$ and, consequently, $pq = pq \,.\, \underline{1} \in r\mathfrak{A}$ whence, by 20.2.1 and 20.2.4, there follows $r\mathfrak{A} = pq\mathfrak{A}$. So we have, first, $p\mathfrak{A} \,.\, q\mathfrak{A} \subset pq\mathfrak{A}$. Each element of the left coset $pq\mathfrak{A}$ is the product $pq \,.\, x$ of the element pq and some element $x \in \mathfrak{A}$. There obviously holds pqx

$= (p\underline{1})\,(qx) \in p\mathfrak{A}\,.\,q\mathfrak{A}$; hence $p\mathfrak{A}\,.\,q\mathfrak{A} \subset pq\mathfrak{A}$. So we have

$$p\mathfrak{A}\,.\,q\mathfrak{A} = pq\mathfrak{A}, \tag{1}$$

i.e., the product of the left cosets $p\mathfrak{A}$ and $q\mathfrak{A}$ is the left coset $pq\mathfrak{A}$.

The equality (1) yields, in particular, for $q = p^{-1}$ the relations:

$$p\mathfrak{A}p^{-1} = p\mathfrak{A}(p^{-1}\underline{1}) \subset p\mathfrak{A}\,.\,p^{-1}\mathfrak{A} = pp^{-1}\mathfrak{A} = \mathfrak{A}$$

so that $p\mathfrak{A}p^{-1} \subset \mathfrak{A}$. Since p is an arbitrary element of $\mathfrak{G}$, the same holds even for p^{-1} and we have $p^{-1}\mathfrak{A}p \subset \mathfrak{A}$. Consequently,

$$\mathfrak{A} = (pp^{-1})\mathfrak{A}(pp^{-1}) = p(p^{-1}\mathfrak{A}p)p^{-1} \subset p\mathfrak{A}p^{-1},$$

i.e., $p\mathfrak{A}p^{-1} \supset \mathfrak{A}$. Hence

$$p\mathfrak{A}p^{-1} = \mathfrak{A}$$

or, which is the same, $p\mathfrak{A} = \mathfrak{A}p$. Therefore the left coset of each element $p \in \mathfrak{G}$ with regard to $\mathfrak{A}$ is, simultaneously, the right coset of p with regard to $\mathfrak{A}$. We see that $\mathfrak{A}$ is invariant in $\mathfrak{G}$.

Now let us assume, conversely, that the subgroup $\mathfrak{A}$ is invariant in $\mathfrak{G}$. Then, by the definition, there first follows that the left coset $p\mathfrak{A}$ of each element $p \in \mathfrak{G}$ with regard to $\mathfrak{A}$ is, simultaneously, the right coset $\mathfrak{A}p$ of p with regard to $\mathfrak{A}$. Then for any two left cosets $p\mathfrak{A}$, $q\mathfrak{A}$ there holds

$$p\mathfrak{A}\,.\,q\mathfrak{A} = p(\mathfrak{A}q)\mathfrak{A} = p(q\mathfrak{A})\mathfrak{A} = pq(\mathfrak{A}\mathfrak{A}) = pq\mathfrak{A}$$

which yields $p\mathfrak{A}\,.\,q\mathfrak{A} = pq\mathfrak{A}$. Hence, if our assumption is true, the product of $p\mathfrak{A}$ and $q\mathfrak{A}$ is $pq\mathfrak{A}$. Thus we have verified that the decomposition $\mathfrak{G}/_l\mathfrak{A}$ of $\mathfrak{G}$ which is, of course, equal to $\mathfrak{G}/_r\mathfrak{A}$ is generating and we may sum up the above results in the following theorem:

The left as well as the right decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$ is generating if and only if the subgroup $\mathfrak{A}$ is invariant in $\mathfrak{G}$. Then the product of any elements $p\mathfrak{A}$ and $q\mathfrak{A}$ of the decomposition generated by $\mathfrak{A}$ is the element $pq\mathfrak{A}$.

2. *Second theorem.* A remarkable property of the groups consists in that each generating decomposition of a group is generated by some of its invariant subgroups.

Consider a generating decomposition $\overline{G}$ of $\mathfrak{G}$. Since each element of $\mathfrak{G}$ is contained in some element of $\overline{G}$, there exists an element $A \in \overline{G}$ comprising the unit $\underline{1}$ of $\mathfrak{G}$. We shall prove that A is the field of an invariant subgroup $\mathfrak{A}$ of $\mathfrak{G}$ and $\overline{G}$ the decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$.

To that end, let us first consider that, since $\overline{G}$ is generating, there exists an element $\bar{a} \in \overline{G}$ such that $AA \subset \bar{a}$. As there holds, on the one hand, $\underline{1} = \underline{1}\,.\,\underline{1} \in AA \subset \bar{a}$ and, on the other hand, $\underline{1} \in A$, we have $\bar{a} = A$. Consequently, A is groupoidal.

The corresponding subgroupoid $\mathfrak{A}$ comprises the unit $\underline{1}$ of $\mathfrak{G}$ and, as we shall see, contains with each element a even its inverse a^{-1}.

Assuming $a \in A$, let $\bar{b}$ denote the element of $\bar{G}$ that includes a^{-1}. Since $\underline{1} = aa^{-1} \in A\bar{b}$, the element $\underline{1}$ is contained in the product $A\bar{b}$ and, of course, also in A. As $\bar{G}$ is generating and both subsets $A\bar{b}$ and A comprise the element $\underline{1}$, we have $A\bar{b} \subset A$. Hence: $\underline{1} \,.\, a^{-1} \in A$, i.e., $a^{-1} \in A$ which proves that $\mathfrak{A}$ is a subgroup of $\mathfrak{G}$.

It remains to be shown that $\mathfrak{A}$ is invariant in $\mathfrak{G}$ and that any element $\bar{a} \in \bar{G}$ is the coset of an arbitrary element $a \in \bar{a}$ with regard to $\mathfrak{A}$. Suppose $a \in \mathfrak{G}$ and let $\bar{a}$ denote the element of $\bar{G}$ containing a so that: $a \in \bar{a} \in \bar{G}$. If $x \in \bar{a}$, then $x = \underline{1} \,.\, x \in A\bar{a}$ whence $\bar{a} \subset A\bar{a}$. As $\bar{G}$ is generating and both subsets $A\bar{a}$, $\bar{a}$ comprise the element a, there holds $A\bar{a} \subset \bar{a}$. So we have $A\bar{a} = \bar{a}$ and, analogously, $\bar{a}A = \bar{a}$. Consequently,

$$\bar{a} = A\bar{a} = \bar{a}A. \tag{2}$$

There obviously holds $aA \subset \bar{a}A$. Let us show that there simultaneously holds $\bar{a}A \subset aA$. Let $\bar{b}$ denote the element of $\bar{G}$ comprising a^{-1}. As $\bar{G}$ is generating and both the subsets $\bar{b}\bar{a}$ and A include the element $\underline{1}$, there holds $\bar{b}\bar{a} \subset A$. Thus the product $a^{-1}x$ of a^{-1} and an element $x \in \bar{a}$ is contained in A. Consequently, $x = a(a^{-1}x) \in aA$ and we have $\bar{a} \subset aA$. Hence $\bar{a}A \subset aAA = aA$. So we have $\bar{a}A = aA$. Analogously we arrive at $A\bar{a} = Aa$. From that and from (2) there follows

$$\bar{a} = a\mathfrak{A} = \mathfrak{A}a.$$

From these equalities we, first, see that the subgroup $\mathfrak{A}$ is invariant in $\mathfrak{G}$. Since they hold for every element $a \in \mathfrak{G}$ and the element $\bar{a} \in \bar{G}$ comprising a, they also apply to any $\bar{a} \in \bar{G}$ and $a \in \bar{a}$; every element $\bar{a} \in \bar{G}$ is the coset of an arbitrary element $a \in \bar{a}$ with regard to $\mathfrak{A}$.

Thus we have determined all the generating decompositions of $\mathfrak{G}$:

All generating decompositions of $\mathfrak{G}$ are precisely those decompositions of $\mathfrak{G}$ that are generated by the individual invariant subgroups of $\mathfrak{G}$.

24.4. Properties of the generating decompositions of a group

On $\mathfrak{G}$ there always exist two generating decompositions, namely, the two extreme decompositions $\bar{G}_{\max}$ and $\bar{G}_{\min}$ (14.1) generated by the extreme invariant subgroups $\mathfrak{G}$, $\{\underline{1}\}$ of $\mathfrak{G}$ (24.2).

Let $\bar{A}$, $\bar{B}$ stand for arbitrary generating decompositions on $\mathfrak{G}$. By the above theorem, $\bar{A}$ and $\bar{B}$ are decompositions generated by appropriate subgroups $\mathfrak{A}$ and $\mathfrak{B}$ invariant in $\mathfrak{G}$, respectively. Consequently, $\mathfrak{A}$ and $\mathfrak{B}$ are interchangeable

From the results obtained in 21.3—6 we observe that $\bar{A}, \bar{B}$ have the properties stated below:

The decomposition $\bar{A}$ ($\bar{B}$) *is a covering (refinement) of* $\bar{B}$ ($\bar{A}$) *if and only if the subgroup* $\mathfrak{A}$ *is a supergroup of* $\mathfrak{B}$, *i.e.,* $\mathfrak{A} \supset B$.

The greatest common refinement $(\bar{A}, \bar{B})$ *of* $\bar{A}$ *and* $\bar{B}$ *is generated by the invariant subgroup* $\mathfrak{A} \cap \mathfrak{B}$.

The least common covering $[\bar{A}, \bar{B}]$ *of* $\bar{A}, \bar{B}$ *is generated by the invariant subgroup* $\mathfrak{A}\mathfrak{B}$.

$\bar{A}$ *and* $\bar{B}$ *are complementary.*

Furthermore, there holds:

The system of all generating decompositions of $\mathfrak{G}$ *is, with regard to the operations* ( ), [ ], *closed and is, together with the multiplications defined by the latter, a modular lattice with extreme elements. This lattice is isomorphic with the lattice consisting of invariant subgroups of* $\mathfrak{G}$ (24.2).

24.5. Further properties of invariant subgroups

The theorems (24.3) on generating decompositions in groups, together with the study of generating decompositions in groupoids and of decompositions of groups generated by subgroups, lead to fresh information about the properties of invariant subgroups.

1. *Let* $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{C}$ *stand for subgroups of* $\mathfrak{G}$, *the subgroup* $\mathfrak{B}$ *being invariant in* $\mathfrak{A}$. *Then* $\mathfrak{B} \cap \mathfrak{C}$ *is invariant in* $\mathfrak{A} \cap \mathfrak{C}$. *Moreover, the subgroups* $\mathfrak{A} \cap \mathfrak{C}, \mathfrak{B}$ *are interchangeable and* $\mathfrak{B}$ *is invariant in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$.

Proof. a) Since $\mathfrak{B}$ is invariant in $\mathfrak{A}$, the decomposition $\mathfrak{A}/_l\mathfrak{B}$ is generating (24.3.1). By 21.2 (1), we have

$$\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C}).$$

Furthermore, from 14.3.2 we know that the left decomposition in question of $\mathfrak{A} \cap \mathfrak{C}$ with regard to $\mathfrak{B} \cap \mathfrak{C}$ is generating. Consequently, $\mathfrak{B} \cap \mathfrak{C}$ is invariant in $\mathfrak{A} \cap \mathfrak{C}$ (24.3.1).

b) By 19.5.1, $\mathfrak{A} \cap \mathfrak{C}$ is a subgroup of $\mathfrak{A}$. As $\mathfrak{B}$ is invariant in $\mathfrak{A}$, the subgroups $\mathfrak{A} \cap \mathfrak{C}, \mathfrak{B}$ are interchangeable (24.2). In accordance with 21.2 (2), we have

$$\mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/_l\mathfrak{B}.$$

Moreover, from 14.3.2 we know that the left decomposition in question of $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}$ with regard to $\mathfrak{B}$ is generating. Hence $\mathfrak{B}$ is invariant in $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ (24.3.1).

In particular, for $\mathfrak{A} = \mathfrak{G}$ we have the following theorem:

If $\mathfrak{B}, \mathfrak{C}$ *are subgroups of* $\mathfrak{G}$ *and* $\mathfrak{B}$ *is invariant in* $\mathfrak{G}$, *then* $\mathfrak{B} \cap \mathfrak{C}$ *is invariant in* $\mathfrak{C}$.

2. *Let* $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{C} \supset \mathfrak{D}$ *be subgroups of* $\mathfrak{G}$ *while* $\mathfrak{B}$ *and* $\mathfrak{D}$ *are invariant in* $\mathfrak{A}$ *and* $\mathfrak{C}$, *respectively. Then* $\mathfrak{A} \cap \mathfrak{D}$ *and* $\mathfrak{B} \cap \mathfrak{C}$ *are invariant in* $\mathfrak{A} \cap \mathfrak{C}$. *Let, furthermore,* $\mathfrak{U}$ *be an invariant subgroup of* $\mathfrak{A} \cap \mathfrak{C}$ *such that*

$$(\mathfrak{A} \cap \mathfrak{C}) \supset \mathfrak{U} \supset (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}). \tag{1}$$

Then $\mathfrak{A} \cap \mathfrak{C}$ *and* $\mathfrak{U}$ *are interchangeable with both* $\mathfrak{B}$ *and* $\mathfrak{D}$ *and* $\mathfrak{U}\mathfrak{B}$ *or* $\mathfrak{U}\mathfrak{D}$ *is invariant in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ *or* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{D}$, *respectively. Moreover,* by 23.2.(1), *there holds*:

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} \cap \mathfrak{U}\mathfrak{D} = \mathfrak{U} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D} \cap \mathfrak{U}\mathfrak{B}.$$

Proof. In accordance with 1, the subgroups $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{B} \cap \mathfrak{C}$ are invariant in $\mathfrak{A} \cap \mathfrak{C}$. Since $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{U}$ are subgroups of $\mathfrak{A}$ and $\mathfrak{C}$, respectively, and $\mathfrak{B}$ and $\mathfrak{D}$ are invariant in $\mathfrak{A}$ and $\mathfrak{C}$, respectively, $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{U}$ are interchangeable with both the subgroups $\mathfrak{B}$ and $\mathfrak{D}$.

By 1., $\mathfrak{B}$ is invariant in $\mathfrak{A}' = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ and $\mathfrak{D}$ in $\mathfrak{C}' = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}$. By 24.3.1, the decompositions $\bar{A} = \mathfrak{A}'/_l\mathfrak{B}$, $\bar{C} = \mathfrak{C}'/_l\mathfrak{D}$ are generating and, by 14.3.2, the same applies to the decompositions

$$\bar{A} \sqcap \mathfrak{C}' = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C}), \quad \bar{C} \sqcap \mathfrak{A}' = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{A} \cap \mathfrak{D}).$$

From (1) we conclude that the decomposition $\bar{B} = (\mathfrak{A} \cap \mathfrak{C})/_l\mathfrak{U}$ is a common covering of $\bar{A} \sqcap \mathfrak{C}'$, $\bar{C} \sqcap \mathfrak{A}'$. Since $\mathfrak{U}$ is invariant in $\mathfrak{A} \cap \mathfrak{C}$, $\bar{B}$ is generating. Consequently, the coverings

$$\mathring{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B}, \quad \mathring{C} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}$$

of the decompositions $\bar{A}$, $\bar{C}$, enforced by $\bar{B}$, are generating (14.3.3).

On taking account of 24.3.1, we observe that $\mathfrak{U}\mathfrak{B}$ or $\mathfrak{U}\mathfrak{D}$ is invariant in $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ or $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{D}$, respectively.

In particular $\big($for $\mathfrak{U} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{B} \cap \mathfrak{C})\big)$, there holds the following theorem:

Let $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{C} \supset \mathfrak{D}$ *be subgroups of* $\mathfrak{G}$, $\mathfrak{B}$ *and* $\mathfrak{D}$ *invariant in* $\mathfrak{A}$ *and* $\mathfrak{C}$, *respectively. Then* $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{B} \cap \mathfrak{C}$ *are invariant in* $\mathfrak{A} \cap \mathfrak{C}$. *Moreover,* $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{A} \cap \mathfrak{D}$ *are interchangeable with* $\mathfrak{B}$ *and, similarly,* $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{B} \cap \mathfrak{C}$ *with* $\mathfrak{D}$. *The subgroup* $(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}$ *is invariant in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$ *and the same holds for* $(\mathfrak{B} \cap \mathfrak{C})\mathfrak{D}$ *in* $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{D}$. *Furthermore* (according to 23.2(2)), *there holds*:

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} \cap (\mathfrak{B} \cap \mathfrak{C})\mathfrak{D} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{B} \cap \mathfrak{C}) = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{D} \cap (\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}.$$

24.6. Series of invariant subgroups

In the classical study of groups, the theory of series of invariant subgroups of $\mathfrak{G}$ is generally based on the assumption that each member of the series, except the first, is an invariant subgroup of the element immediately preceding it. The results are of local character in the sense that they concern only situations in the neighbourhood of the unit of $\mathfrak{G}$. The following study will be restricted, for simplicity, to the special case when each member of the series is an invariant subgroup of $\mathfrak{G}$. On the ground of previous results (23.4), we may immediately proceed to the main part of the theory. Contrary to the classical theory, we shall arrive at results of global character, informative about the situation in the neighbourhood of any point of $\mathfrak{G}$.

Consider two series of subgroups of $\mathfrak{G}$, namely:

$$\big((\mathfrak{A}) =\big) \qquad \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha \qquad (\alpha \geqq 1),$$
$$\big((\mathfrak{B}) =\big) \qquad \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_\beta \qquad (\beta \geqq 1)$$

and suppose that all the subgroups in question are invariant in $\mathfrak{G}$.

Then the following theorem is true:

The series $(\mathfrak{A})$, $(\mathfrak{B})$ *have co-basally joint refinements* $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ *with coinciding initial and final members, all the subgroups of these refinements being invariant in* $\mathfrak{G}$. $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ *are given by the construction described in part* a) *of the proof in* 23.4.5.

Proof. Since the members of the series $(\mathfrak{A})$, $(\mathfrak{B})$ are invariant in $\mathfrak{G}$, the series $(\mathfrak{A})$, $(\mathfrak{B})$ are complementary (23.4.4; 24.4); we can apply to them the construction described in part a) of the proof in 23.4.5. That leads to co-basally joint refinements $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ of $(\mathfrak{A})$, $(\mathfrak{B})$; the refinements have the same initial and final members $\mathfrak{U} = \mathfrak{A}_1\mathfrak{B}_1$ and $\mathfrak{V} = \mathfrak{A}_\alpha \cap \mathfrak{B}_\beta$, respectively. In accordance with the construction in question, $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ consist of the following subgroups of $\mathfrak{G}$:

$$\mathfrak{A}_{\gamma,\nu} = \mathfrak{A}_\gamma(\mathfrak{A}_{\gamma-1} \cap \mathfrak{B}_\nu) = \mathfrak{A}_{\gamma-1} \cap \mathfrak{A}_\gamma\mathfrak{B}_\nu,$$
$$\mathfrak{B}_{\delta,\mu} = \mathfrak{B}_\delta(\mathfrak{B}_{\delta-1} \cap \mathfrak{A}_\mu) = \mathfrak{B}_{\delta-1} \cap \mathfrak{B}_\delta\mathfrak{A}_\mu$$

$(\gamma, \mu = 1, 2, \ldots, \alpha + 1; \delta, \nu = 1, 2, \ldots, \beta + 1; \mathfrak{A}_0 = \mathfrak{B}_0 = \mathfrak{G}, \mathfrak{A}_{\alpha+1} = \mathfrak{B}_{\beta+1} = \mathfrak{V})$. From the results of 24.2 it is obvious that $\mathfrak{A}_{\gamma,\nu}$, $\mathfrak{B}_{\delta,\mu}$ are invariant in $\mathfrak{G}$ and the proof is complete.

24.7. Exercises

1. In the group $\mathfrak{S}_4$ consisting of all permutations of the set $\{a, b, c, d\}$, all the permutations mapping the element d onto itself form a subgroup $\mathfrak{S}_3'$. The permutations which map the elements a, b, c in the same manner as $\boldsymbol{e}, \boldsymbol{a}, \boldsymbol{b}$ in 11.4.2 without changing the element d, form a subgroup of $\mathfrak{S}_4$ which is invariant in $\mathfrak{S}_3'$ but not in $\mathfrak{S}_4$.

2. Let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$. The set of all elements $p \in \mathfrak{G}$ such that $p\mathfrak{A} = \mathfrak{A}p$ is a subgroup $\mathfrak{N}$ of $\mathfrak{G}$, the so-called *normalizer* of $\mathfrak{A}$. The latter is the greatest supergroup of $\mathfrak{A}$ in which $\mathfrak{A}$ is invariant; that is to say, $\mathfrak{A}$ is invariant in $\mathfrak{N}$ and each subgroup of $\mathfrak{G}$ in which $\mathfrak{A}$ is invariant is a subgroup of $\mathfrak{N}$.
3. The center of $\mathfrak{G}$ is an invariant subgroup of $\mathfrak{G}$.
4. If there exists, in a finite group of order N ($\geqq 2$), a subgroup of order $\frac{1}{2}N$, then the latter is invariant in the former. For example, in the diedric permutation group of order $2n$ ($n \geqq 3$) there is an invariant subgroup of order n consisting of all the elements of the group corresponding to the rotations of the vertices of a regular n-gon about its center (19.7.2).
5. Associating, with every element $p \in \mathfrak{G}$, any element $x^{-1}px \in \mathfrak{G}$ with $x \in \mathfrak{G}$ arbitrary, we obtain a symmetric congruence on $\mathfrak{G}$. The decomposition $\overline{G}$ corresponding to the latter is called the *fundamental decomposition of* $\mathfrak{G}$. The field of each invariant subgroup of $\mathfrak{G}$ is the sum of certain elements of $\overline{G}$. $\overline{G}$ is complementary to every generating decomposition of $\mathfrak{G}$.
6. Let $p \in \mathfrak{G}$ be an arbitrary point and $\mathring{\mathfrak{G}}$ the (p)-group associated with $\mathfrak{G}$ (19.7.11). Consider a subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$ and the subgroup $\mathring{\mathfrak{A}}$ of $\mathring{\mathfrak{G}}$, lying on the field $p\mathfrak{A} = \mathfrak{A}p$ (20.3.3; 21.8.7). Show that: a) $\mathring{\mathfrak{A}}$ is invariant in $\mathring{\mathfrak{G}}$; b) all generating decompositions of $\mathring{\mathfrak{G}}$ coincide with the generating decompositions of $\mathfrak{G}$.

25. Factor groups

25.1. Definition

Let us now consider a factoroid $\overline{\overline{\mathfrak{G}}}$ on $\mathfrak{G}$. According to the definition of a factoroid, the field of $\overline{\overline{\mathfrak{G}}}$ is a generating decomposition of $\mathfrak{G}$ and is therefore generated by a suitable subgroup $\mathfrak{A}$ invariant in $\mathfrak{G}$ (24.3.2). The product $p\mathfrak{A} \,.\, q\mathfrak{A}$ of an element $p\mathfrak{A} \in \overline{\overline{\mathfrak{G}}}$ and an element $q\mathfrak{A} \in \overline{\overline{\mathfrak{G}}}$ is, by the definition of multiplication in a factoroid, the element of $\overline{\overline{\mathfrak{G}}}$ that contains the set $p\mathfrak{A} \,.\, q\mathfrak{A}$. Since the latter coincides, as we know, with $pq\mathfrak{A} \in \overline{\overline{\mathfrak{G}}}$, the multiplication in $\overline{\overline{\mathfrak{G}}}$ is given by the following formula:

$$p\mathfrak{A} \circ q\mathfrak{A} = pq\mathfrak{A}. \tag{1}$$

Now we shall show that $\overline{\overline{\mathfrak{G}}}$ *is a group whose unit is the field of the invariant subgroup* $\mathfrak{A}$ *and the element inverse of an arbitrary element* $a\mathfrak{A}$ *is* $a^{-1}\mathfrak{A}$.

In fact, first, by 15.6.3, $\overline{\overline{\mathfrak{G}}}$ is associative. Next, by 18.7.5, the field A of the invariant subgroup $\mathfrak{A}$ is the unit of $\overline{\overline{\mathfrak{G}}}$. Finally we have:

$$p\mathfrak{A} \circ p^{-1}\mathfrak{A} = pp^{-1}\mathfrak{A} = \underline{1}\mathfrak{A} = A$$

and so $p^{-1}\mathfrak{A} \in \overline{\overline{\mathfrak{G}}}$ is the inverse element of $p\mathfrak{A} \in \overline{\overline{\mathfrak{G}}}$.