is the $\boldsymbol{n}$-image of the element $pa'^{-1} \in p\mathfrak{A}$ $(a'^{-1} \in \mathfrak{A})$. Thus we have $\mathfrak{A}p^{-1} \subset \boldsymbol{n}(p\mathfrak{A})$ and, consequently, $\boldsymbol{n}(p\mathfrak{A}) = \mathfrak{A}p^{-1}$, which completes the proof.

Remark. Both $p\mathfrak{A}$ and $\mathfrak{A}p^{-1}$ are referred to as *mutually inverse cosets.* If one of them is denoted e.g. by $\bar{a}$, then the other is $\bar{a}^{-1}$.

9. *The left coset $p\mathfrak{A}$ and the right coset $\mathfrak{A}q$ are equivalent sets.*

We are to prove that there exists a simple mapping of the set $p\mathfrak{A}$ onto $\mathfrak{A}q$. In accordance with theorem 8 and 7.3.4, the sets $p\mathfrak{A}$ and $\mathfrak{A}p^{-1}$ are equivalent; by the theorem analogous to theorem 5 and valid for the right cosets, $\mathfrak{A}p^{-1}$ and $\mathfrak{A}q$ have the same property. Consequently, by 6.10.7, the assertion is correct.

20.3. Exercises

1. If $\mathfrak{G}$ is Abelian, then the left coset of an element $p \in \mathfrak{G}$ with regard to a subgroup $\mathfrak{A} \subset \mathfrak{G}$ is, at the same time, the right coset and so $p\mathfrak{A} = \mathfrak{A}p$.
2. Let $\mathfrak{A}, \mathfrak{B}$ denote arbitrary subgroups and C a complex in $\mathfrak{G}$. Prove that there holds: a) the sum of all left (right) cosets with regard to $\mathfrak{A}$ which are incident with C coincides with the complex $C\mathfrak{A}$ $(\mathfrak{A}C)$; b) the sum $\mathfrak{B}p\mathfrak{A}$ of all left cosets with regard to $\mathfrak{A}$ which are incident with some right coset $\mathfrak{B}p$ $(p \in \mathfrak{G})$ coincides with the sum of all right cosets with regard to $\mathfrak{B}$ which are incident with the left coset $p\mathfrak{A}$.
3. Let $p \in \mathfrak{G}$ be an arbitrary element and $\mathring{\mathfrak{G}}$ the (p)-group associated with $\mathfrak{G}$ (19.7.11). Next, let $\mathfrak{A}$ be an arbitrary subgroup of $\mathfrak{G}$. Prove that: a) the left (right) coset $p\mathfrak{A}$ $(\mathfrak{A}p)$ of p with regard to $\mathfrak{A}$ is the field of a subgroup $\mathring{\mathfrak{A}}_l \subset \mathring{\mathfrak{G}}$ $(\mathring{\mathfrak{A}}_r \subset \mathring{\mathfrak{G}})$ of $\mathring{\mathfrak{G}}$; b) the left (right) coset $x \circ \mathring{\mathfrak{A}}_l$ $(\mathring{\mathfrak{A}}_r \circ x)$ coincides, for each element x of $\mathring{\mathfrak{G}}$, with the left (right) coset $x\mathfrak{A}$ $(\mathfrak{A}x)$.

21. Decompositions generated by subgroups

A most remarkable property of groups is that every subgroup of an arbitrary group determines certain decompositions on the latter.

21.1. Left and right decompositions

Consider the system of all the subsets of the group $\mathfrak{G}$ given by the left cosets with regard to $\mathfrak{A}$. By 20.2.1, every element $p \in \mathfrak{G}$ is included in the left coset $p\mathfrak{A}$ which is, of course, an element of the considered system. By 20.2.4, every two elements of the system are disjoint. The system in question is therefore a decomposition of $\mathfrak{G}$, called the *decomposition of* $\mathfrak{G}$ *into left cosets, generated by* $\mathfrak{A}$, briefly, the *left decomposition of* $\mathfrak{G}$ *generated by* $\mathfrak{A}$. Notation: $\mathfrak{G}/_l\mathfrak{A}$.

Analogously, the system of all subsets of $\mathfrak{G}$ given by the right cosets with regard to $\mathfrak{A}$ is the *decomposition of* $\mathfrak{G}$ *into right cosets, generated by* $\mathfrak{A}$, briefly, the *right decomposition generated by* $\mathfrak{A}$. Notation: $\mathfrak{G}/_r\mathfrak{A}$:

We have, for instance, the formulas: $\mathfrak{G}/_l\mathfrak{G} = \mathfrak{G}/_r\mathfrak{G} = \bar{G}_{\max}$, $\mathfrak{G}/_l\{\underline{1}\} = \mathfrak{G}/_r\{\underline{1}\} = \bar{G}_{\min}$; $\bar{G}_{\max}$, $\bar{G}_{\min}$ are, of course, the greatest and the least decomposition of $\mathfrak{G}$, respectively.

In the following theorems we shall describe the properties of the left decompositions of a group. The properties of the right decompositions are analogous and will therefore be omitted. Finally, we shall deal with the relations between the left and the right decompositions of the group $\mathfrak{G}$ with regard to the same subgroup $\mathfrak{A}$.

21.2. Intersections and closures in connection with left decompositions

1. Let $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{C}$ be arbitrary subgroups of $\mathfrak{G}$. Consider the intersection $\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}$ and the closure $\mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B}$. Since $A \cap C \neq \emptyset$, neither of these figures is empty; $A \supset B, C$ denote, of course, the fields of the corresponding subgroups.

We shall prove: *There holds*

$$\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C}). \tag{1}$$

If the subgroups $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{B}$ *are interchangeable, then there also holds*:

$$\mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/_l\mathfrak{B}. \tag{2}$$

Proof. a) We shall show that each element of the decomposition on the right- or the left-hand side of the formula (1) is an element of the decomposition on the left- or the right-hand side, respectively. Every element $\bar{a} \in (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C})$ has the form

$$\bar{a} = a(\mathfrak{B} \cap \mathfrak{C}) = a\mathfrak{B} \cap a\mathfrak{C},$$

where $a \in \mathfrak{A} \cap \mathfrak{C}$. From $a \in \mathfrak{A}$ and $\mathfrak{A} \supset \mathfrak{B}$ there follows $a\mathfrak{B} \in \mathfrak{A}/_l\mathfrak{B}$ and from $a \in \mathfrak{C}$ we have $a\mathfrak{C} = C$. So there holds:

$$\bar{a} = a\mathfrak{B} \cap \mathfrak{C} \in \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}.$$

Now let $\bar{a} \in \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}$ be an arbitrary element and so $\bar{a} = a\mathfrak{B} \cap \mathfrak{C}$ $(\neq \emptyset)$, $a \in \mathfrak{A}$. Moreover, let $x \in \bar{a}$ be an arbitrary element. From $x \in a\mathfrak{B}$ there follows $a\mathfrak{B} = x\mathfrak{B}$ and, since $x \in \mathfrak{C}$, there holds $C = x\mathfrak{C}$ and therefore $\bar{a} = x\mathfrak{B} \cap x\mathfrak{C} = x(\mathfrak{B} \cap \mathfrak{C})$. Since $a \in \mathfrak{A}$, $\mathfrak{A} \supset \mathfrak{B}$ yields $a\mathfrak{B} \subset \mathfrak{A}$, we have $x \in \mathfrak{A} \cap \mathfrak{C}$ so that $\bar{a} \in (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{C})$ and the proof of the formula (1) is complete.

b) Let us now assume that the subgroups $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{B}$ are interchangeable. That occurs if, for example, the subgroups $\mathfrak{B}$, $\mathfrak{C}$ are interchangeable (22.2.1).

To prove the formula (2) we shall proceed analogously as in the case a). Every element $\bar{a} \in (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/_l\mathfrak{B}$ has the form $x\mathfrak{B}$ where $x \in (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}$; we observe that the element x is the product ab of a point $a \in \mathfrak{C} \cap \mathfrak{A}$ and a point $b \in \mathfrak{B}$. Hence $\bar{a} = (ab)\,\mathfrak{B} = a(b\mathfrak{B}) = a\mathfrak{B}$ (the last equality is true with regard to the relation $b\mathfrak{B} = B$, correct by 20.2.2). From $a \in \mathfrak{A}$, $\mathfrak{A} \supset \mathfrak{B}$ we have $a\mathfrak{B} \in \mathfrak{A}/_l\mathfrak{B}$ and, since $a \in \mathfrak{C}$, the left coset $a\mathfrak{B}$ is incident with $\mathfrak{C}$. Thus we have $\bar{a} \in \mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B}$. Let now $\bar{a}$ be an arbitrary element of $\mathfrak{C} \sqsubset \mathfrak{A}/_l\mathfrak{B}$ and so $\bar{a} = a\mathfrak{B}$ where a is a point of $\mathfrak{A}$ and $a\mathfrak{B}$ is incident with $\mathfrak{C}$; furthermore. let $c \in \mathfrak{C} \cap a\mathfrak{B}$ be an arbitrary point. From $c \in a\mathfrak{B}$ there follows, by the theorems 20.2.1 and 20.2.4, $\bar{a} = c\mathfrak{B}$ which yields (since $\bar{a} \subset \mathfrak{A}$) $c \in \mathfrak{A}$. So we have $c \in \mathfrak{C} \cap \mathfrak{A}$ and, consequently, $c = c\,.\,\underline{1} \in (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}$. From this and $\mathfrak{B} \subset (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}$ we have $\bar{a} \in (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/_l\mathfrak{B}$ and the proof is accomplished.

Let us note, in particular, the case *when the subgroup* $\mathfrak{A}$ *coincides with* $\mathfrak{G}$. *Then we have*:

$$\mathfrak{G}/_l\mathfrak{B} \sqcap \mathfrak{C} = \mathfrak{C}/_l(\mathfrak{B} \cap \mathfrak{C}) \tag{1'}$$

and, moreover, if the subgroups $\mathfrak{B}$, $\mathfrak{C}$ *are interchangeable*:

$$\mathfrak{C} \sqsubset \mathfrak{G}/_l\mathfrak{B} = \mathfrak{C}\mathfrak{B}/_l\mathfrak{B}. \tag{2'}$$

2. The above deliberations will now be extended in the sense that the subgroup $\mathfrak{C}$ will be replaced by the left decomposition of a subgroup of $\mathfrak{G}$.

Let $\mathfrak{A} \supset \mathfrak{B}$ and $\mathfrak{C} \supset \mathfrak{D}$ be arbitrary subgroups of $\mathfrak{G}$. Consider the intersection $\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}/_l\mathfrak{D}$ and the closure $\mathfrak{C}/_l\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B}$. Since $A \cap C \neq \emptyset$, neither of these figures is empty. $A \supset B$, $C \supset D$ are, of course, the fields of the corresponding subgroups.

We shall show that *there holds*

$$\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}/_l\mathfrak{D} = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{D}) \tag{3}$$

and, moreover, if the subgroups $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{B}$ *are interchangeable, even*

$$\mathfrak{C}/_l\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{B}/_l\mathfrak{B}. \tag{4}$$

Proof. a) Every element $\bar{a} \in (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{D})$ has the form $\bar{a} = a(\mathfrak{B} \cap \mathfrak{D}) = a\mathfrak{B} \cap a\mathfrak{D}$ where $a \in \mathfrak{A} \cap \mathfrak{C}$. From $a \in \mathfrak{A}$, $\mathfrak{A} \supset \mathfrak{B}$ there follows $a\mathfrak{B} \in \mathfrak{A}/_l\mathfrak{B}$. Analogously, from $a \in \mathfrak{C}$, $\mathfrak{C} \supset \mathfrak{D}$ we have $a\,\mathfrak{D} \subset \mathfrak{C}/_l\mathfrak{D}$. It is easy to see that $\bar{a}$ is the (nonempty) intersection of the elements $a\mathfrak{B}$ and $a\mathfrak{D}$ of the decompositions $\mathfrak{A}/_l\mathfrak{B}$ and $\mathfrak{C}/_l\mathfrak{D}$, respectively, so we have $\bar{a} \in \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}/_l\mathfrak{D}$.

Now let $\bar{a} \in \mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}/_l\mathfrak{D}$ be an arbitrary element, hence

$$\bar{a} = a\mathfrak{B} \cap c\mathfrak{D}\ (\neq \emptyset),\ a \in \mathfrak{A},\ c \in \mathfrak{C};$$

furthermore, let $x \in \bar{a}$ denote an arbitrary point. From $x \in a\mathfrak{B}$ we have $a\mathfrak{B} = x\mathfrak{B}$ and, analogously, $x \in c\mathfrak{D}$ yields $c\mathfrak{D} = x\mathfrak{D}$; hence

$$\bar{a} = a\mathfrak{B} \cap c\mathfrak{D} = x\mathfrak{B} \cap x\mathfrak{D} = x(\mathfrak{B} \cap \mathfrak{D}).$$

Since $a \in \mathfrak{A} \supset \mathfrak{B}$, $c \in \mathfrak{C} \supset \mathfrak{D}$, we have $a\mathfrak{B} \subset \mathfrak{A}$, $c\mathfrak{D} \subset \mathfrak{C}$ and, consequently, $x \in \mathfrak{A} \cap \mathfrak{C}$. Thus we arrive at the result:

$$\bar{a} \in (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{B} \cap \mathfrak{D})$$

and there follows (3).

b) The formula (4) directly follows from

$$\mathfrak{C}/_l\mathfrak{D} \sqsubset \mathfrak{A}/_l\mathfrak{B} = \mathbf{s}(\mathfrak{C}/_l\mathfrak{D}) \sqsubset \mathfrak{A}/_l\mathfrak{B}, \quad \mathbf{s}(\mathfrak{C}/_l\mathfrak{D}) = C$$

and from the formula (2).

In the particular case when the subgroups $\mathfrak{A}, \mathfrak{C}$ coincide with $\mathfrak{G}$ and, consequently, the decompositions $\mathfrak{A}/_l\mathfrak{B}$ $(= \mathfrak{G}/_l\mathfrak{B})$, $\mathfrak{C}/_l\mathfrak{D}$ $(= \mathfrak{G}/_l\mathfrak{D})$ lie on $\mathfrak{G}$, the intersection $\mathfrak{G}/_l\mathfrak{B} \sqcap \mathfrak{G}/_l\mathfrak{D}$ of the latter coincides with the greatest common refinement $(\mathfrak{G}/_l\mathfrak{B}, \mathfrak{G}/_l\mathfrak{D})$ (3.5). Hence

$$(\mathfrak{G}/_l\mathfrak{B}, \mathfrak{G}/_l\mathfrak{D}) = \mathfrak{G}/_l(\mathfrak{B} \cap \mathfrak{D}).$$

21.3. Coverings and refinements of the left decompositions

Given two subgroups $\mathfrak{A}$, $\mathfrak{B}$ in $\mathfrak{G}$, let us ascertain when the left decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$ ($\mathfrak{B}$) is a covering (refinement) of the left decomposition generated by $\mathfrak{B}$ ($\mathfrak{A}$), i.e., $\mathfrak{G}/_l\mathfrak{A} \geqq \mathfrak{G}/_l\mathfrak{B}$.

If the left decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$ is a covering of the left decomposition generated by $\mathfrak{B}$ then, in particular, the field A of $\mathfrak{A}$ is the sum of certain left cosets with regard to $\mathfrak{B}$. Among the latter there is the field B of $\mathfrak{B}$ because both A and B have a common element $\underline{1}$. Consequently, $\mathfrak{A}$ is a supergroup of $\mathfrak{B}$, i.e., $\mathfrak{A} \supset \mathfrak{B}$. Conversely, if $\mathfrak{A}$ is a supergroup of $\mathfrak{B}$, then (by 20.2.7) every left coset with regard to $\mathfrak{A}$ is the sum of all the left cosets with regard to $\mathfrak{B}$ that are incident with it. We observe that the left decomposition of $\mathfrak{G}$ generated by $\mathfrak{A}$ ($\mathfrak{B}$) is a covering (refinement) of the left decomposition generated by $\mathfrak{B}$ ($\mathfrak{A}$).

The result: *The left decomposition of $\mathfrak{G}$ generated by the subgroup $\mathfrak{A}$ ($\mathfrak{B}$) is a covering (refinement) of the left decomposition generated by $\mathfrak{B}$ ($\mathfrak{A}$) if and only if $\mathfrak{A}$ is a supergroup of $\mathfrak{B}$.* In other words: *$\mathfrak{G}/_l\mathfrak{A} \geqq \mathfrak{G}/_l\mathfrak{B}$ holds if and only if $\mathfrak{A} \supset \mathfrak{B}$.*

21.4. The greatest common refinement of two left decompositions

Let $\mathfrak{A}, \mathfrak{B} \supset \mathfrak{G}$ be subgroups of $\mathfrak{G}$.

The greatest common refinement of the left decompositions of $\mathfrak{G}$, *generated by* $\mathfrak{A}$, $\mathfrak{B}$, *is the left decomposition generated by the subgroup* $\mathfrak{A} \cap \mathfrak{B}$, i.e., $(\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}) = \mathfrak{G}/_l(\mathfrak{A} \cap \mathfrak{B})$.

Indeed, the greatest common refinement of the decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B}$ is the system of all nonempty intersections of the left cosets $p\mathfrak{A}$ and the left cosets $q\mathfrak{B}$ (3.5). Every nonempty intersection $p\mathfrak{A} \cap q\mathfrak{B}$ is the left coset of each of its elements with regard to the subgroup $\mathfrak{A} \cap \mathfrak{B}$. Every left coset $c(\mathfrak{A} \cap \mathfrak{B})$ is the intersection of the left cosets $c\mathfrak{A}$ and $c\mathfrak{B}$ (20.2.6), which accomplishes the proof. (Cf. the result in 21.2.)

21.5. The least common covering of two left decompositions

Suppose $\mathfrak{A}$, $\mathfrak{B}$ are two interchangeable subgroups of $\mathfrak{G}$.

Then there exists the product $\mathfrak{A}\mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ which is a subgroup of $\mathfrak{G}$.

The least common covering of the left decompositions of $\mathfrak{G}$, *generated by* $\mathfrak{A}$, $\mathfrak{B}$, *is the left decomposition generated by the subgroup* $\mathfrak{A}\mathfrak{B}$, i.e., $[\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}] = \mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$.

In fact, first, with regard to $\mathfrak{A} \subset \mathfrak{A}\mathfrak{B}$, $\mathfrak{B} \subset \mathfrak{A}\mathfrak{B}$ and to the theorem in 21.3, the decomposition $\mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$ is a common covering of the decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B}$. We are to show that two cosets $c\mathfrak{A}, p\mathfrak{A} \in \mathfrak{G}/_l\mathfrak{A}$ can be connected in $\mathfrak{G}/_l\mathfrak{B}$ if and only if they lie in the same element of $\mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$.

a) If the left cosets $c\mathfrak{A}$, $p\mathfrak{A}$ lie in the same element of $\mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$, then $p = cba$, while $b \in \mathfrak{B}$, $a \in \mathfrak{A}$ denote convenient elements. Both $c\mathfrak{A}$ and $p\mathfrak{A}$ are incident with $c\mathfrak{B} \in \mathfrak{G}/_l\mathfrak{B}$ and so they can be connected in $\mathfrak{G}/_l\mathfrak{B}$.

b) If there exists a binding $\{\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}\}$ from $c\mathfrak{A}$ to $p\mathfrak{A}$,

$$c_1\mathfrak{A}, \ldots, c_\alpha\mathfrak{A} \quad (c_1 = c,\ c_\alpha = p),$$

then every two neighbouring cosets $c_\beta\mathfrak{A}$, $c_{\beta+1}\mathfrak{A}$ are incident with a certain coset $d_\beta\mathfrak{B}$; therefore there exist elements

$$x_\beta \in c_\beta\mathfrak{A} \cap d_\beta\mathfrak{B}, \quad y_\beta \in d_\beta\mathfrak{B} \cap c_{\beta+1}\mathfrak{A} \quad (\beta = 1, \ldots, \alpha - 1).$$

The elements x_γ, $y_{\gamma-1}$ ($\gamma = 1, \ldots, \alpha$; $y_0 = c_1$, $x_\alpha = c_\alpha$) lie in the same coset $c_\gamma\mathfrak{A}$ and, similarly, the elements x_β, y_β lie in the same coset $d_\beta\mathfrak{B}$. Consequently, there holds $x_\gamma = y_{\gamma-1}\ a_\gamma$, $y_\beta = x_\beta b_\beta$ where $a_\gamma \in \mathfrak{A}$, $b_\beta \in \mathfrak{B}$ denote convenient elements. Thus,

$$c_\alpha = c_1 a_1 b_1 \ldots b_{\alpha-1} a_\alpha \in c_1\mathfrak{A}\mathfrak{B}$$

from which it is clear that the left cosets $c\mathfrak{A}$, $p\mathfrak{A}$ lie in the same coset $c\mathfrak{A}\mathfrak{B} \in \mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$.

21.6. Complementary left decompositions

Consider arbitrary subgroups $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{G}$ of $\mathfrak{G}$.

The left decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B}$ of $\mathfrak{G}$ are complementary if and only if the subgroups $\mathfrak{A}$, $\mathfrak{B}$ are interchangeable.

Proof. a) Suppose $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_l\mathfrak{B}$ are complementary. Let $\bar{u} \in [\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_l\mathfrak{B}]$ be the element containing the unit $\underline{1} \in \mathfrak{G}$. From $\underline{1} \in \mathfrak{A} \cap \mathfrak{B}$ it is obvious that the fields of $\mathfrak{A}$ and $\mathfrak{B}$ are parts of $\bar{u}$. Consider arbitrary points $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and the left cosets $b\mathfrak{A} \in \mathfrak{G}/_l\mathfrak{A}$, $a^{-1}\mathfrak{B} \in \mathfrak{G}/_l\mathfrak{B}$. The latter are incident with the subgroups $\mathfrak{B}$ or $\mathfrak{A}$, respectively, hence they are subsets of $\bar{u}$ and we have $b\mathfrak{A} \subset \bar{u}$, $a^{-1}\mathfrak{B} \subset \bar{u}$. But, since $\mathfrak{G}/_l\mathfrak{A}$ and $\mathfrak{G}/_l\mathfrak{B}$ are complementary, there holds $b\mathfrak{A} \cap a^{-1}\mathfrak{B} \neq \emptyset$. Consequently, there exist points $a' \in \mathfrak{A}$, $b' \in \mathfrak{B}$ such that $ba' = a^{-1}b'$. Hence $ab = b'a'^{-1} \in \mathfrak{B}\mathfrak{A}$ and we have $\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}\mathfrak{A}$. Analogously, we may show that $\mathfrak{B}\mathfrak{A} \subset \mathfrak{A}\mathfrak{B}$. Thus $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}$.

b) Suppose the subgroups $\mathfrak{A}$, $\mathfrak{B}$ are interchangeable.

By the above theorem (21.5), the least common covering of $\mathfrak{G}/_l\mathfrak{A}$ and $\mathfrak{G}/_l\mathfrak{B}$ is $\mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$. Let $c\mathfrak{A}\mathfrak{B} \in \mathfrak{G}/_l\mathfrak{A}\mathfrak{B}$ be an arbitrary element. Every element of $\mathfrak{G}/_l\mathfrak{A}$ lying in $c\mathfrak{A}\mathfrak{B}$ is $cb\mathfrak{A}$ where $b \in \mathfrak{B}$ is a convenient element. Similarly, every element of $\mathfrak{G}/_l\mathfrak{B}$ lying in $c\mathfrak{A}\mathfrak{B}$ is $ca\mathfrak{B}$, where $a \in \mathfrak{A}$ is a convenient element. We are to show that every two left cosets $cb\mathfrak{A}$ and $ca\mathfrak{B}$ lying in $c\mathfrak{A}\mathfrak{B}$ are incident, that is to say, that there exist elements $a_1 \in \mathfrak{A}$, $b_1 \in \mathfrak{B}$ such that $ba_1 = ab_1$. That is easy: Since the subgroups $\mathfrak{A}$ and $\mathfrak{B}$ are interchangeable, there exist elements $a_1 \in \mathfrak{A}$, $b_1 \in \mathfrak{B}$ satisfying the equality $a^{-1}b = b_1a_1^{-1}$. Hence $ba_1 = ab_1$ and the proof is complete.

21.7. Relations between the left and the right decompositions

Let $\mathfrak{A}$, $\mathfrak{B}$ stand for arbitrary subgroups of $\mathfrak{G}$.

1. *The left or the right decomposition $\mathfrak{G}/_l\mathfrak{A}$ or $\mathfrak{G}/_r\mathfrak{A}$, respectively, is mapped, under the extended inversion $\boldsymbol{n}$ of $\mathfrak{G}$, onto the right or the left decomposition $\mathfrak{G}/_r\mathfrak{A}$ or $\mathfrak{G}/_l\mathfrak{A}$ and so*

$$\boldsymbol{n}(\mathfrak{G}/_l\mathfrak{A}) = \mathfrak{G}/_r\mathfrak{A}, \quad \boldsymbol{n}(\mathfrak{G}/_r\mathfrak{A}) = \mathfrak{G}/_l\mathfrak{A}.$$

The decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{A}$ are therefore equivalent sets:

$$\mathfrak{G}/_l\mathfrak{A} \simeq \mathfrak{G}/_r\mathfrak{A}.$$

Proof. In accordance with 7.3.4, the set $\boldsymbol{n}(\mathfrak{G}/_l\mathfrak{A})$ is a decomposition of $\mathfrak{G}$ equivalent to $\mathfrak{G}/_l\mathfrak{A}$. By 20.2.8, each element of $\boldsymbol{n}(\mathfrak{G}/_l\mathfrak{A})$ is an element of $\mathfrak{G}/_r\mathfrak{A}$. Hence $\boldsymbol{n}(\mathfrak{G}/_l\mathfrak{A}) = \mathfrak{G}/_r\mathfrak{A}$. Analogously we arrive at $\boldsymbol{n}(\mathfrak{G}/_r\mathfrak{A}) = \mathfrak{G}/_l\mathfrak{A}$.

2. *The least common covering of the left decomposition* $\mathfrak{G}/_l\mathfrak{A}$ *and the right decomposition* $\mathfrak{G}/_r\mathfrak{B}$ *is the set consisting of all the complexes* $\mathfrak{B}p\mathfrak{A} \subset \mathfrak{G}$ $(p \in \mathfrak{G})$. *The decompositions* $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{B}$ *are complementary.*

Proof. Let us associate, with each point $p \in \mathfrak{G}$, the complex $\mathfrak{B}p\mathfrak{A} \subset \mathfrak{G}$ and consider the set $\bar{C}$ consisting of all these complexes. We observe, first, that each point of $\mathfrak{G}$ lies at least in one element of $\bar{C}$. Next, we shall show that two different elements of $\bar{C}$ are disjoint. Indeed, if any elements $\mathfrak{B}p\mathfrak{A}$, $\mathfrak{B}q\mathfrak{A} \in \bar{C}$ are incident, then there exist points $a, a' \in \mathfrak{A}$, $b, b' \in \mathfrak{B}$ such that $bpa = b'qa'$. Hence we have

$$(\mathfrak{B}b)p(a\mathfrak{A}) = (\mathfrak{B}b')q(a'\mathfrak{A})$$

and, moreover (by 20.2.2 and by the analogous theorem on right cosets), $\mathfrak{B}p\mathfrak{A} = \mathfrak{B}q\mathfrak{A}$. Thus the set $\bar{C}$ is a decomposition of $\mathfrak{G}$. Furthermore, by 20.3.2, each element $\mathfrak{B}p\mathfrak{A} \in \bar{C}$ is the sum of all elements of the left decomposition $\mathfrak{G}/_l\mathfrak{A}$ that are incident with the right coset $\mathfrak{B}p$ and, at the same time, the sum of all elements of the right decomposition $\mathfrak{G}/_r\mathfrak{B}$ incident with $p\mathfrak{A}$. We observe that the decomposition $\bar{C}$ is a common covering of the decompositions $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{B}$. Let $\bar{u} = \mathfrak{B}p\mathfrak{A} \in \bar{C}$ be an arbitrary element and $\bar{a} \in \mathfrak{G}/_l\mathfrak{A}$, $\bar{b} \in \mathfrak{G}/_r\mathfrak{B}$ arbitrary cosets lying in $\bar{u}$. Then we have $\bar{a} = bp\mathfrak{A}$, $\bar{b} = \mathfrak{B}pa$ where $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. Since $bpa \in \bar{a} \cap \bar{b}$, the sets $\bar{a}, \bar{b}$ are incident. Consequently, by 5.2, we have:

$$\bar{C} = [\mathfrak{G}/_l\mathfrak{A}, \mathfrak{G}/_r\mathfrak{B}].$$

Hence $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{B}$ are complementary and the proof is accomplished.

For $\mathfrak{B} = \mathfrak{A}$, in particular, there applies:

The system of sets $\mathfrak{A}p\mathfrak{A} \subset \mathfrak{G}$, *where* $p \in \mathfrak{G}$, *is for each subgroup* $\mathfrak{A} \subset \mathfrak{G}$ *the least common covering of the left and the right decompositions* $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{A}$ *of* $\mathfrak{G}$. *The decompositions* $\mathfrak{G}/_l\mathfrak{A}$, $\mathfrak{G}/_r\mathfrak{A}$ *are complementary.*

21.8. Exercises

1. In every Abelian group $\mathfrak{G}$, the left and the right decompositions with regard to any subgroup $\mathfrak{A} \subset \mathfrak{G}$ coincide: $\mathfrak{G}/_l\mathfrak{A} = \mathfrak{G}/_r\mathfrak{A}$.
2. The left (and, simultaneously, the right) decomposition of the group $\mathfrak{Z}$ with regard to the subgroup $\mathfrak{A}$ consisting of all the multiples of some natural number n is the decomposition $\bar{Z}_n$ described in 15.2.
3. Give an example to show that the left decomposition of a group $\mathfrak{G}$ with regard to a given subgroup $\mathfrak{A} \subset \mathfrak{G}$ need not coincide with the right decomposition.

4. Suppose $\mathfrak{A} \supset \mathfrak{B}$ are subgroups of $\mathfrak{G}$. Consider arbitrary left and right cosets $\bar{a}_l$, $\bar{c}_l$ and $\bar{a}_r$, $\bar{c}_r$ with regard to $\mathfrak{A}$, respectively, and denote:

$$\bar{A}_l = \bar{a}_l \sqcap \mathfrak{G}/_l\mathfrak{B}\ (= \bar{a}_l \sqsubset \mathfrak{G}/_l\mathfrak{B}), \quad \bar{C}_l = \bar{c}_l \sqcap \mathfrak{G}/_l\mathfrak{B}\ (= \bar{c}_l \sqsubset \mathfrak{G}/_l\mathfrak{B}),$$
$$\bar{A}_r = \bar{a}_r \sqcap \mathfrak{G}/_r\mathfrak{B}\ (= \bar{a}_r \sqsubset \mathfrak{G}/_r\mathfrak{B}), \quad \bar{C}_r = \bar{c}_r \sqcap \mathfrak{G}/_r\mathfrak{B}\ (= \bar{c}_r \sqsubset \mathfrak{G}/_r\mathfrak{B}).$$

Each element of the decompositions $\bar{A}_l, \bar{C}_l$ or $\bar{A}_r, \bar{C}_r$ is a left or a right coset with regard to $\mathfrak{B}$, respectively. Moreover, there holds: $\bar{A}_l \simeq \bar{C}_l$, $\bar{A}_r \simeq \bar{C}_r$.

5. Let $\mathfrak{A} \supset \mathfrak{B}$ be subgroups of $\mathfrak{G}$. Consider arbitrary cosets $\bar{a} \in \mathfrak{G}/_l\mathfrak{A}$, $\bar{a}^{-1} \in \mathfrak{G}/_r\mathfrak{A}$ inverse of each other and, on the latter, the decompositions set out below:

$$\bar{A}_l = \bar{a} \sqcap \mathfrak{G}/_l\mathfrak{B}\ (= \bar{a} \sqsubset \mathfrak{G}/_l\mathfrak{B}), \quad \bar{A}_r = \bar{a}^{-1} \sqcap \mathfrak{G}/_r\mathfrak{B}\ (= \bar{a}^{-1} \sqsubset \mathfrak{G}/_r\mathfrak{B}).$$

Either of the decompositions $\bar{A}_l$, $\bar{A}_r$ is, under the extended inversion $\boldsymbol{n}$ of $\mathfrak{G}$, mapped onto the other. $\bar{A}_l$, $\bar{A}_r$ are equivalent sets, hence: $\bar{A}_l \simeq \bar{A}_r$.

6. If $\bar{A}_l$ and $\bar{C}_r$ are the same as in exercise 4, there holds $\bar{A}_l \simeq \bar{C}_r$.

7. Let $p \in \mathfrak{G}$ denote an arbitrary element and $\mathring{\mathfrak{G}}$ the p-group associated with $\mathfrak{G}$ (19.7.11). Moreover, let $\mathfrak{A} \subset \mathfrak{G}$ be a subgroup of $\mathfrak{G}$ and $\mathring{\mathfrak{A}}_l \subset \mathring{\mathfrak{G}}$ ($\mathring{\mathfrak{A}}_r \subset \mathring{\mathfrak{G}}$) the subgroup of $\mathring{\mathfrak{G}}$ on the field $p\mathfrak{A}$ ($\mathfrak{A}p$) (20.3.3). Show that the left (right) decomposition of the group $\mathring{\mathfrak{G}}$ with regard to the subgroup $\mathring{\mathfrak{A}}_l$ ($\mathring{\mathfrak{A}}_r$) coincides with the left (right) decomposition of $\mathfrak{G}$ with regard to $\mathfrak{A}$, that is to say:

$$\mathring{\mathfrak{G}}/_l\mathring{\mathfrak{A}}_l = \mathfrak{G}/_l\mathfrak{A}, \quad \mathring{\mathfrak{G}}/_r\mathring{\mathfrak{A}}_r = \mathfrak{G}/_r\mathfrak{A}.$$

22. Consequences of the properties of decompositions generated by subgroups

22.1. Lagrange's theorem

Assuming $\mathfrak{A} \subset \mathfrak{G}$ to be an arbitrary subgroup of $\mathfrak{G}$, we shall now consider the consequences of the properties of the decompositions $\mathfrak{G}/_l\mathfrak{A}$ and $\mathfrak{G}/_r\mathfrak{A}$.

Suppose $\mathfrak{G}$ is finite.

Let us denote by N and n the order of $\mathfrak{G}$ and $\mathfrak{A}$, respectively, so that N is the number of the elements of G and n the number of the elements of $\mathfrak{A}$. One of the elements of $\mathfrak{G}/_l\mathfrak{A}$ is the field A of $\mathfrak{A}$. This element therefore consists of n elements of $\mathfrak{G}$ and, consequently (by 20.2.5), each element of $\mathfrak{G}/_l\mathfrak{A}$ consists of n elements of $\mathfrak{G}$. Hence $N = qn$, q denoting the number of the elements of $\mathfrak{G}/_l\mathfrak{A}$. Thus we have arrived at the following result:

The order of each subgroup $\mathfrak{A}$ of an arbitrary finite group $\mathfrak{G}$ is a divisor of the order of $\mathfrak{G}$.