tures are, consequently, formed in the following way: Every element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$ is an α-membered sequence each member $\bar{a}_\gamma$, $(\gamma = 1, \ldots, \alpha)$ of which is a decomposition in $\mathfrak{G}$ and, in fact, a complex in $\overline{\mathfrak{A}}_\gamma$. The multiplication in $\tilde{\mathfrak{A}}$ is such that, for any two elements

$$\bar{\bar{a}} = (\bar{a}_1, \cdots, \bar{a}_\alpha), \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{\mathfrak{A}}$$

and their product

$$\bar{\bar{a}}\bar{\bar{b}} = \bar{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_\alpha) \in \tilde{\mathfrak{A}},$$

there holds:

$$\bar{a}_1 \circ \bar{b}_1 \subset \bar{c}_1, \ldots, \bar{a}_\alpha \circ \bar{b}_\alpha \subset \bar{c}_\alpha.$$

15.6. Exercises

1. Show that the groupoids $\mathfrak{Z}_n$, $\overline{\mathfrak{Z}}_n$ $(n \geqq 1)$ are isomorphic.
2. Let $\mathfrak{A}_m$ stand for the subgroupoid of $\mathfrak{Z}$ whose field consists of all the integer multiples of a certain natural number $m > 1$. Of which elements do the factoroids $\mathfrak{A}_m \sqsubset \overline{\mathfrak{Z}}_n$ and $\overline{\mathfrak{Z}}_n \sqcap \mathfrak{A}_m$ $(n > 1)$ consist if m, n are not relatively prime?
3. Every factoroid on an Abelian (associative) groupoid is Abelian (associative).
4. If a groupoid $\mathfrak{G}$ contains an element a such that $aa = a$, i.e., a so-called *idempotent element* (15.4.2), then the element of any factoroid in $\mathfrak{G}$ comprising a is idempotent as well.

16. Deformations of factoroids

16.1. The isomorphism theorems for groupoids

Let us now proceed to the isomorphism theorems for groupoids. These theorems describe situations occurring under homomorphic mappings of groupoids or factoroids and connected with the concept of isomorphism. The set structure of these theorems is expressed by the equivalence theorems dealt with in 6.8.

1. *The first theorem.* Let $\mathfrak{G}$, $\mathfrak{G}^*$ be groupoids and suppose there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$. In 14.2 we have shown that the decomposition $\bar{D}$ of $\mathfrak{G}$ corresponding to d is generating. Let $\overline{\mathfrak{D}}$ stand for the factoroid corresponding to $\bar{D}$. Associating with each element $\bar{a} \in \overline{\mathfrak{D}}$ that element $a^* \in \mathfrak{G}^*$ of whose $\boldsymbol{d}$-inverse

images the element $\bar{a}$ consists, we obtain a simple mapping of $\bar{\mathfrak{D}}$ onto $\mathfrak{G}^*$; let us denote it $\boldsymbol{i}$. By the definition of $\boldsymbol{i}$, there holds $\boldsymbol{i}\bar{a} = \boldsymbol{d}a$ for every $\bar{a} \in \bar{\mathfrak{D}}$ and $a \in \bar{a}$. Let $\bar{a}, \bar{b}$ stand for arbitrary elements of $\bar{\mathfrak{D}}$, a for an element of $\bar{a}$ and b for an element of $\bar{b}$. Then there holds: $ab \subset \bar{a}\bar{b} \subset \bar{a} \circ \bar{b} \in \bar{\mathfrak{D}}$ and hence: $\boldsymbol{i}(\bar{a} \circ \bar{b}) = \boldsymbol{d}ab = \boldsymbol{d}a \,.\, \boldsymbol{d}b = \boldsymbol{i}\bar{a} \,.\, \boldsymbol{i}\bar{b}$. So we have the equality $\boldsymbol{i}(\bar{a} \circ \bar{b}) = \boldsymbol{i}\bar{a} \,.\, \boldsymbol{i}\bar{b}$ by which $\boldsymbol{i}$ is a deformation and therefore (since it is simple) an isomorphism of $\bar{\mathfrak{D}}$ onto $\mathfrak{G}^*$. Thus we have shown that if there exists a deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$, then there is on $\mathfrak{G}$ a factoroid isomorphic with $\mathfrak{G}^*$, namely, the factoroid $\bar{\mathfrak{D}}$ corresponding to the generating decomposition belonging to $\boldsymbol{d}$ while the mapping $\boldsymbol{i}$ is an isomorphism. $\bar{\mathfrak{D}}$ is said to *belong* or *correspond* to the deformation $\boldsymbol{d}$.

Let now, conversely, $\bar{\mathfrak{D}}$ be an arbitrary factoroid on $\mathfrak{G}$ and $\boldsymbol{d}$ a mapping of $\mathfrak{G}$ onto $\bar{\mathfrak{D}}$ defined as follows: The $\boldsymbol{d}$-image of any element $a \in \mathfrak{G}$ is that element $\bar{a} \in \bar{\mathfrak{D}}$ for which $a \in \bar{a}$. It is easy to show that $\boldsymbol{d}$ is a deformation of $\mathfrak{G}$ onto $\bar{\mathfrak{D}}$. Let us consider any elements $a, b \in \mathfrak{G}$ and the elements $\bar{a}, \bar{b} \in \bar{\mathfrak{D}}$ containing a, b, hence $\bar{a} = \boldsymbol{d}a, \bar{b} = \boldsymbol{d}b$. The relations $ab \in \bar{a}\bar{b} \subset \bar{a} \circ \bar{b} \in \bar{\mathfrak{D}}$ yield $ab \in \bar{a} \circ \bar{b}$ and, moreover, $\boldsymbol{d}ab = \bar{a} \circ \bar{b} = \boldsymbol{d}a \circ \boldsymbol{d}b$ so that the mapping $\boldsymbol{d}$ actually preserves the multiplications in $\mathfrak{G}$ and $\bar{\mathfrak{D}}$. Consequently, $\mathfrak{G}$ may be deformed onto any factoroid $\bar{\mathfrak{D}}$ lying on $\mathfrak{G}$ in the way that every element of $\mathfrak{G}$ is mapped onto that element of $\bar{\mathfrak{D}}$ in which it is contained. Hence, $\mathfrak{G}$ may be deformed onto any groupoid $\mathfrak{G}^*$ isomorphic with some factoroid on $\mathfrak{G}$.

The above results are briefly summed up in *the first isomorphism theorem for groupoids*:

If a groupoid $\mathfrak{G}^*$ is homomorphic with a groupoid $\mathfrak{G}$, then it is isomorphic with a certain factoroid on $\mathfrak{G}$; if $\mathfrak{G}^*$ is isomorphic with some factoroid on $\mathfrak{G}$, then it is homomorphic with $\mathfrak{G}$. The mapping of the factoroid $\bar{\mathfrak{D}}$, belonging to the deformation $\boldsymbol{d}$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$, under which every element $\bar{a} \in \bar{\mathfrak{D}}$ is mapped onto the $\boldsymbol{d}$-image of the points of $\bar{a}$ is an isomorphism.

2. *The second theorem.* Let $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$ stand for coupled factoroids in $\mathfrak{G}$.

Each element of $\bar{\mathfrak{A}}$ is incident with exactly one element of $\bar{\mathfrak{B}}$ and, simultaneously, each element of $\bar{\mathfrak{B}}$ is incident with exactly one element of $\mathfrak{A}$ (15.3.3). Associating, with every element $\bar{a} \in \bar{\mathfrak{A}}$, the element $\bar{b} \in \bar{\mathfrak{B}}$ incident with it, we obtain a simple mapping $\boldsymbol{i}$ of $\bar{\mathfrak{A}}$ onto $\bar{\mathfrak{B}}$. We shall show that $\boldsymbol{i}$ is an isomorphism of $\bar{\mathfrak{A}}$ onto $\bar{\mathfrak{B}}$. To that purpose, let us consider arbitrary elements $\bar{a}_1, \bar{a}_2 \in \bar{\mathfrak{A}}$ and the elements $\bar{b}_1, \bar{b}_2 \in \bar{\mathfrak{B}}$ incident with the former so that $\bar{b}_1 = \boldsymbol{i}\bar{a}_1, \bar{b}_2 = \boldsymbol{i}\bar{a}_2$. Set $\bar{x}_1 = \bar{a}_1 \cap \bar{b}_1 \ (\neq \emptyset)$, $\bar{x}_2 = \bar{a}_2 \cap \bar{b}_2 \ (\neq \emptyset)$. There obviously holds:

$$\bar{x}_1\bar{x}_2 \subset \bar{a}_1\bar{a}_2 \subset \bar{a}_1 \circ \bar{a}_2, \quad \bar{x}_1\bar{x}_2 \subset \bar{b}_1\bar{b}_2 \subset \bar{b}_1 \circ \bar{b}_2$$

where, of course, $\bar{a}_1 \circ \bar{a}_2 \in \bar{\mathfrak{A}}$ $(\bar{b}_1 \circ \bar{b}_2 \in \bar{\mathfrak{B}})$ is the product of $\bar{a}_1$ $(\bar{b}_1)$ and $\bar{a}_2$ $(\bar{b}_2)$. So we have: $\bar{x}_1\bar{x}_2 \subset \bar{a}_1 \circ \bar{a}_2 \cap \bar{b}_1 \circ \bar{b}_2$ and observe that $\bar{b}_1 \circ \bar{b}_2$ is incident with $\bar{a}_1 \circ \bar{a}_2$. Hence $\bar{b}_1 \circ \bar{b}_2 = \boldsymbol{i}(\bar{a}_1 \circ \bar{a}_2)$ and so $\boldsymbol{i}(\bar{a}_1 \circ \bar{a}_2) = \boldsymbol{i}\bar{a}_1 \circ \boldsymbol{i}\bar{a}_2$, which completes the proof.

The result we have arrived at is summed up in *the second isomorphism theorem for groupoids*:

Every two coupled factoroids $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ in $\mathfrak{G}$ are isomorphic, hence $\overline{\mathfrak{A}} \simeq \overline{\mathfrak{B}}$. *The mapping of $\overline{\mathfrak{A}}$ onto $\overline{\mathfrak{B}}$ obtained by associating with every element of $\overline{\mathfrak{A}}$ the element of $\overline{\mathfrak{B}}$ incident with it is an isomorphism.*

A remarkable case (15.3.3) of the second isomorphism theorem concerns the isomorphism of the closure and the intersection of a subgroupoid $\mathfrak{X} \subset \mathfrak{G}$ and a factoroid $\overline{\mathfrak{Y}}$ in $\mathfrak{G}$: *if $X \cap s\overline{Y} \neq \emptyset$, there holds*

$$\mathfrak{X} \sqsubset \overline{\mathfrak{Y}} \simeq \overline{\mathfrak{Y}} \sqcap \mathfrak{X}$$

while the isomorphism is realized by the incidence of elements. X and $\overline{Y}$ denote the fields of $\mathfrak{X}$ and $\overline{\mathfrak{Y}}$, respectively.

3. *The third theorem.* Let $\overline{\mathfrak{B}}$ and $\overline{\overline{\mathfrak{B}}}$ denote arbitrary factoroids on $\mathfrak{G}$ and $\overline{\mathfrak{B}}$, respectively. As we know (15.4.1), $\overline{\overline{\mathfrak{B}}}$ enforces a certain covering $\overline{\mathfrak{A}}$ of $\overline{\mathfrak{B}}$. Note that $\overline{\mathfrak{A}}$ is a factoroid on $\mathfrak{G}$ and each of its elements is the sum of all elements of $\overline{\mathfrak{B}}$ comprised in the same element of $\overline{\overline{\mathfrak{B}}}$. Associating, with every element $\bar{\bar{b}} \in \overline{\overline{\mathfrak{B}}}$, that element $\bar{a} \in \overline{\mathfrak{A}}$ which is the sum of all the elements of $\overline{\mathfrak{B}}$ lying in $\bar{\bar{b}}$, we obtain a mapping of $\overline{\overline{\mathfrak{B}}}$ onto $\overline{\mathfrak{A}}$; let us denote it by $\boldsymbol{i}$. We shall show that $\boldsymbol{i}$ is an isomorphism.

First, it is obvious that $\boldsymbol{i}$ is simple. To prove that it is a deformation, consider arbitrary elements $\bar{\bar{b}}_1, \bar{\bar{b}}_2 \in \overline{\overline{\mathfrak{B}}}$ and the product $\bar{\bar{b}}_3 \in \overline{\overline{\mathfrak{B}}}$ of $\bar{\bar{b}}_1$ and $\bar{\bar{b}}_2$. In accordance with the definition of the multiplication in $\overline{\overline{\mathfrak{B}}}$ there holds, for any $\bar{b}_1 \in \overline{\mathfrak{B}}$ of $\bar{\bar{b}}_1$ and any $\bar{b}_2 \in \overline{\mathfrak{B}}$ of $\bar{\bar{b}}_2$, the relation $\bar{b}_1 \circ \bar{b}_2 \in \bar{\bar{b}}_3$. Now let $\bar{a}_1$ be that element of $\overline{\mathfrak{A}}$ which is the sum of all $\bar{b}_1 \in \overline{\mathfrak{B}}$ contained in $\bar{\bar{b}}_1$, hence $\bar{a}_1 = \bigcup \bar{b}_1$ $(\bar{b}_1 \in \bar{\bar{b}}_1)$ and, analogously, let $\bar{a}_2 = \bigcup \bar{b}_2$ $(\bar{b}_2 \in \bar{\bar{b}}_2)$, $\bar{a}_3 = \bigcup \bar{b}_3$ $(\bar{b}_3 \in \bar{\bar{b}}_3)$ so that $\bar{a}_1 = \boldsymbol{i}\bar{\bar{b}}_1$, $\bar{a}_2 = \boldsymbol{i}\bar{\bar{b}}_2$, $\bar{a}_3 = \boldsymbol{i}\bar{\bar{b}}_3 \in \overline{\mathfrak{A}}$. Then the relation $\bar{b}_1 \circ \bar{b}_2 \in \bar{\bar{b}}_3$ $(\bar{b}_1 \in \bar{\bar{b}}_1, \bar{b}_2 \in \bar{\bar{b}}_2)$ yields $\bar{b}_1 \circ \bar{b}_2 \subset \bar{a}_3$ and, furthermore, $\bar{a}_1\bar{a}_2 = \bigcup \bar{b}_1\bar{b}_2 \subset \bigcup \bar{b}_1 \circ \bar{b}_2 \subset \bar{a}_3$; hence $\bar{a}_3$ is the element of $\overline{\mathfrak{A}}$ comprising $\bar{a}_1\bar{a}_2$ and we have $\bar{a}_3 = \bar{a}_1 \circ \bar{a}_2$. This equality may be written in the form $\boldsymbol{i}\bar{\bar{b}}_3 = \boldsymbol{i}\bar{\bar{b}}_1 \circ \boldsymbol{i}\bar{\bar{b}}_2$ and expresses that $\boldsymbol{i}$ is a deformation and therefore (since it is simple) an isomorphism. Thus we have arrived at the result summed up in *the third isomorphism theorem for groupoids*:

Any factoroid $\overline{\overline{\mathfrak{B}}}$ on a factoroid $\overline{\mathfrak{B}}$ of $\mathfrak{G}$ and the covering $\overline{\mathfrak{A}}$ of $\overline{\mathfrak{B}}$ enforced by $\overline{\overline{\mathfrak{B}}}$ are isomorphic, i.e., $\overline{\overline{\mathfrak{B}}} \simeq \overline{\mathfrak{A}}$. *The mapping of $\overline{\overline{\mathfrak{B}}}$ onto $\overline{\mathfrak{A}}$ under which every element $\bar{\bar{b}} \in \overline{\overline{\mathfrak{B}}}$ is mapped onto the sum of the elements of $\overline{\mathfrak{B}}$ contained in $\bar{\bar{b}}$ is an isomorphism.*

16.2. Extended deformations

Let $\boldsymbol{d}$ be a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$.

From 16.1.1 we know that $\mathfrak{G}^*$ is isomorphic with the factoroid $\overline{\mathfrak{D}}$ corresponding to $\boldsymbol{d}$, i.e., with the factoroid on $\mathfrak{G}$ whose field is the decomposition $\overline{D}$ corresponding to $\boldsymbol{d}$.

In accordance with 7.1, $\boldsymbol{d}$ determines the extended mapping $\bar{\boldsymbol{d}}$ of the system of all subsets of $\mathfrak{G}$ into the system of all subsets of $\mathfrak{G}^*$; the $\boldsymbol{d}$-image of every subset $A \subset \mathfrak{G}$ is the subset $\boldsymbol{d}A \subset \mathfrak{G}^*$ consisting of the $\boldsymbol{d}$-images of the individual elements $a \in A$; moreover, we put $\boldsymbol{d}\emptyset = \emptyset$. Sometimes we write $\boldsymbol{d}$ instead of $\bar{\boldsymbol{d}}$, e.g., $\boldsymbol{d}A$ instead of $\bar{\boldsymbol{d}}A$.

Let us now consider an arbitrary factoroid $\bar{\mathfrak{A}}$ on $\mathfrak{G}$. Its field is a certain generating decomposition $\bar{A}$ of $\mathfrak{G}$.

With respect to the theorem in 7.2, $\boldsymbol{d}\bar{A}$ is a decomposition of $\mathfrak{G}^*$ if and only if $\bar{A}, \bar{D}$ are complementary, that is to say, if the factoroids $\bar{\mathfrak{A}}, \bar{\mathfrak{D}}$ are complementary. Suppose this condition is satisfied.

1. It is easy to show that the *decomposition* $\boldsymbol{d}\bar{A}$ *is generating*. Indeed, let $\bar{a}^*, \bar{b}^* \in \boldsymbol{d}\bar{A}$ be arbitrary elements. Then there exist elements $\bar{a}, \bar{b}, \bar{c} \in \bar{A}$ such that $\boldsymbol{d}\bar{a} = \bar{a}^*, \boldsymbol{d}\bar{b} = \bar{b}^*, \bar{a}\bar{b} \subset \bar{c}$. By the theorem 13. 3. 2, we have $\boldsymbol{d}\bar{a} \,.\, \boldsymbol{d}\bar{b} \subset \boldsymbol{d}\bar{c}$ and observe that there exists an element $(\boldsymbol{d}\bar{c} =)\ \bar{c}^* \in \boldsymbol{d}\bar{A}$ such that $\bar{a}^*\bar{b}^* \subset \bar{c}^*$. Hence, $\boldsymbol{d}\bar{A}$ is generating.

The factoroid on $\mathfrak{G}^*$ whose field is the decomposition $\boldsymbol{d}\bar{A}$ is called the *image of* $\bar{\mathfrak{A}}$ *under the extended mapping* $\boldsymbol{d}$ and denoted by the symbol $\boldsymbol{d}\bar{\mathfrak{A}}$; $\bar{\mathfrak{A}}$ is called the *inverse image of* $\boldsymbol{d}\bar{\mathfrak{A}}$ *under the extended mapping* $\boldsymbol{d}$.

2. The extended mapping $\boldsymbol{d}$ determines a partial mapping of $\bar{\mathfrak{A}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ under which there corresponds, to every element $\bar{a} \in \bar{\mathfrak{A}}$, its image $\boldsymbol{d}\bar{a} \in \boldsymbol{d}\bar{\mathfrak{A}}$. By the mapping $\boldsymbol{d}$ of $\bar{\mathfrak{A}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ we shall, in what follows, mean this partial mapping.

We shall show that *the mapping* $\boldsymbol{d}$ *of* $\bar{\mathfrak{A}}$ *onto* $\boldsymbol{d}\bar{\mathfrak{A}}$ *is a deformation*. Indeed, from $\bar{a}, \bar{b}, \bar{c} \in \bar{\mathfrak{A}},\ \bar{a} \circ \bar{b} = \bar{c}$ we have $\bar{a}\bar{b} \subset \bar{c}$ and, moreover, $\boldsymbol{d}\bar{a} \,.\, \boldsymbol{d}\bar{b} \subset \boldsymbol{d}\bar{c}$, hence $\boldsymbol{d}\bar{a} \circ \boldsymbol{d}\bar{b} = \boldsymbol{d}\bar{c} = \boldsymbol{d}(\bar{a} \circ \bar{b})$ and the proof is complete.

With regard to this result, the mapping $\boldsymbol{d}$ of $\bar{\mathfrak{A}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ is called the *extended deformation* $\boldsymbol{d}$.

3. To the extended deformation $\boldsymbol{d}$ of $\bar{\mathfrak{A}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ there corresponds a certain factoroid $\bar{\bar{\mathfrak{A}}}$ on $\bar{\mathfrak{A}}$. Its individual elements consist of all the elements of $\bar{\mathfrak{A}}$ that have the same image in the extended deformation $\boldsymbol{d}$.

In accordance with the theorem in 7.2, we conclude that the covering of $\bar{\mathfrak{A}}$ enforced by $\bar{\bar{\mathfrak{A}}}$ is the least common covering $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$ of $\bar{\mathfrak{A}}, \bar{\mathfrak{D}}$.

Associating with every element $\bar{u} \in [\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$ the element $\bar{\bar{a}} \in \bar{\bar{\mathfrak{A}}}$ that contains the elements of $\bar{\mathfrak{A}}$ lying in $\bar{u}$, we get an isomorphic mapping of $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$ onto $\bar{\bar{\mathfrak{A}}}$ (16.1.3); associating, on the other hand, with every $\bar{\bar{a}} \in \bar{\bar{\mathfrak{A}}}$ the element $\bar{a}^* \in \boldsymbol{d}\bar{\mathfrak{A}}$ that is the image of every $\bar{a} \in \bar{\mathfrak{A}}$ lying in $\bar{\bar{a}}$, we obtain an isomorphic mapping of $\bar{\bar{\mathfrak{A}}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ (16.1.1). Composing these two mappings, we get an isomorphic mapping of $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}$ onto $\boldsymbol{d}\bar{\mathfrak{A}}$ (13.4). Under this mapping there corresponds, to every element $\bar{u} \in [\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$, a certain element $\bar{a}^* \in \boldsymbol{d}\bar{\mathfrak{A}}$ which is the image of $\bar{u}$ under the extended mapping $\boldsymbol{d}$ (7.2).

The result:

If a factoroid $\bar{\mathfrak{A}}$ on $\mathfrak{G}$ is mapped, under the extended deformation $\boldsymbol{d}$, *onto some factoroid $\bar{\mathfrak{A}}^*$ on $\mathfrak{G}^*$, then the factoroids $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$, $\bar{\mathfrak{A}}^*$ are isomorphic. An isomorphic mapping of $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$ onto $\bar{\mathfrak{A}}^*$ is obtained by associating, with every element of $[\bar{\mathfrak{A}}, \bar{\mathfrak{D}}]$, its image under the extended mapping* $\boldsymbol{d}$.

In particular, *every factoroid which is a covering of the factoroid $\bar{\mathfrak{D}}$ is isomorphic with its own image under the extended deformation* $\boldsymbol{d}$; *an isomorphic mapping is obtained by associating, with every element of the covering, its own image under the extended mapping* $\boldsymbol{d}$.

16.3. Deformations of sequences of groupoids and α-grade groupoidal structures

In this chapter we shall be concerned with some more complicated situations in connection with deformations of sequences of groupoids and α-grade groupoidal structures. Our considerations naturally follow from situations treated in 6.9; we only add the algebraic part based on the multiplication.

1. *Mappings of sequences of groupoids.* Let $\alpha(\geqq 1)$ be an arbitrary positive integer.

Consider two α-membered sequences:

$$(\mathfrak{a}) = (\mathfrak{a}_1, \ldots, \mathfrak{a}_\alpha), \quad (\mathfrak{b}) = (\mathfrak{b}_1, \ldots, \mathfrak{b}_\alpha)$$

whose members $\mathfrak{a}_1, \ldots, \mathfrak{a}_\alpha$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_\alpha$ are groupoids.

a) The sequence $(\mathfrak{b})$ is said to be *isomorphic with* $(\mathfrak{a})$ if the following situation arises: There exists a mapping $\boldsymbol{a}$ of the sequence $(\mathfrak{a})$ onto the sequence $(\mathfrak{b})$ such that to every member $\mathfrak{a}_\gamma$ of $(\mathfrak{a})$ there corresponds an isomorphism $\boldsymbol{i}_\gamma$ of $\mathfrak{a}_\gamma$ onto $\mathfrak{b}_\delta = \boldsymbol{a}\mathfrak{a}_\gamma$ of $(\mathfrak{b})$.

If $(\mathfrak{b})$ is isomorphic with $(\mathfrak{a})$, then obviously $(\mathfrak{a})$ has the same property with respect to $(\mathfrak{b})$. Consequently, we speak about isomorphic sequences $(\mathfrak{a})$, $(\mathfrak{b})$.

b) Let us now assume that the members $\mathfrak{a}_1, \ldots, \mathfrak{a}_\alpha$ of $(\mathfrak{a})$ as well as the members $\mathfrak{b}_1, \ldots, \mathfrak{b}_\alpha$ of $(\mathfrak{b})$ are factoroids in $\mathfrak{G}$.

The sequence $(\mathfrak{b})$ is called *semi-coupled* or *loosely coupled* with the sequence $(\mathfrak{a})$ if the sequence $(b) = (b_1, \ldots, b_\alpha)$ consisting of the fields of the members of the sequence $(\mathfrak{b})$ is semi-coupled with the sequence $(a) = (a_1, \ldots, a_\alpha)$ consisting of the fields of the memberes of the sequence $(\mathfrak{a})$ (6.9.1c); the sequence $(\mathfrak{b})$ is called *coupled* with the sequence $(\mathfrak{a})$ if the sequence $(b) = (b_1, \ldots, b_\alpha)$ is coupled with $(a) = (a_1, \ldots, a_\alpha)$.

If $(\mathfrak{b})$ is loosely coupled (coupled) with $(\mathfrak{a})$, then $(\mathfrak{a})$ has the same property with regard to $(\mathfrak{b})$ and we speak about semi-coupled or loosely coupled (coupled) sequences $(\mathfrak{a})$, $(\mathfrak{b})$.

From the second isomorphism theorem for groupoids (16.1.2) we realize that *very two coupled sequences of factoroids in* $\mathfrak{G}$ *are isomorphic.*

2. *Deformations of* α*-grade groupoidal structures.* Let $\alpha(\geqq 1)$ be a positive integer and

$$\big((\mathfrak{A}) =\big)\ (\mathfrak{A}_1, \ldots, \mathfrak{A}_\alpha),$$
$$\big((\mathfrak{A}^*) =\big)\ (\mathfrak{A}_1^*, \ldots, \mathfrak{A}_\alpha^*)$$

arbitrary sequences of groupoids. Moreover, let $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}^*$ be arbitrary α-grade groupoidal structures with regard to $(\mathfrak{A})$ and $(\mathfrak{A}^*)$, respectively (15.5).

Note that every element $\bar{\bar{a}} \in \tilde{\mathfrak{A}}$ $(\bar{\bar{a}}^* \in \tilde{\mathfrak{A}}^*)$ is an α-membered sequence of sets, $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha)$ $\big(\bar{\bar{a}}^* = (\bar{a}_1^*, \ldots, \bar{a}_\alpha^*)\big)$ each member of which $\bar{a}_\gamma$ $(\bar{a}_\gamma^*)$ is a complex in the groupoid $\mathfrak{A}_\gamma$ $(\mathfrak{A}_\gamma^*)$; $\gamma = 1, \ldots, \alpha$.

Suppose there exists an isomorphism $\boldsymbol{i}$ of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}^*$. Then, for every two $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha)$, $\bar{\bar{b}} = (b_1, \ldots, b_\alpha) \in \tilde{\boldsymbol{A}}$, we have: $\boldsymbol{i}\bar{\bar{a}} \,.\, \boldsymbol{i}\bar{\bar{b}} = \boldsymbol{i}(\bar{\bar{a}} \,.\, \bar{\bar{b}})$.

a) $\boldsymbol{i}$ is said to be a *strong isomorphism of* $\tilde{\mathfrak{A}}$ *onto* $\tilde{\mathfrak{A}}^*$ if the following situation occurs:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect: To every member $\bar{a}_\gamma$ $(\gamma = 1, \ldots, \alpha)$ of an element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$ there corresponds a simple function $\boldsymbol{a}_\gamma$ under which the set $\bar{a}_\gamma$ is mapped onto the set $\bar{a}_\delta^*$ which is the δ-th member of the element $\boldsymbol{i}\bar{\bar{a}} = \bar{\bar{a}}^* = (\bar{a}_1^*, \ldots, \bar{a}_\alpha^*) \in \tilde{\mathfrak{A}}^*$; $\delta = \boldsymbol{p}\gamma$. Moreover, the following "deformation phenomenon" arises: Let

$$\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha),\ \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{\mathfrak{A}}$$

be arbitrary elements and

$$\bar{\bar{a}}\bar{\bar{b}} = \bar{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_\alpha) \in \tilde{\mathfrak{A}}$$

the corresponding product; by the definition of $\tilde{\mathfrak{A}}$, we have:

$$\bar{a}_\gamma \bar{b}_\gamma \subset \bar{c}_\gamma.$$

Now let

$$\boldsymbol{i}\bar{\bar{a}} = \bar{\bar{a}}^* = (\bar{a}_1^*, \ldots, \bar{a}_\alpha^*),\quad \boldsymbol{i}\bar{\bar{b}} = \bar{\bar{b}}^* = (\bar{b}_1^*, \ldots, \bar{b}_\alpha^*)$$

be the $\boldsymbol{i}$-images of the elements $\bar{\bar{a}}$, $\bar{\bar{b}}$ and

$$\bar{\bar{a}}^*\bar{\bar{b}}^* = \bar{\bar{c}}^* = (\bar{c}_1^*, \ldots, \bar{c}_\alpha^*)$$

the corresponding product so that $\bar{a}_\gamma^*\bar{b}_\gamma^* \subset \bar{c}_\gamma^*$. Finally, let $\boldsymbol{a}_\gamma$, $\boldsymbol{b}_\gamma$, $\boldsymbol{c}_\gamma$ be the mentioned simple functions belonging to the members $\bar{a}_\gamma$, $\bar{b}_\gamma$, $\bar{c}_\gamma$; under these functions the sets $\bar{a}_\gamma$, $\bar{b}_\gamma$ are simply mapped onto $\bar{a}_\delta^*$, $\bar{b}_\delta^*$ and (since $\boldsymbol{i}\bar{\bar{c}} = \bar{\bar{c}}^*$) the set $\bar{c}_\gamma$ onto

$\bar{c}_\delta{}^*$ ($\delta = \boldsymbol{p}\gamma$). Then the deformation phenomenon can be described as follows: For every two points $a \in \bar{a}_\gamma$, $b \in \bar{b}_\gamma$ there holds: $\boldsymbol{a}_\gamma a \,.\, \boldsymbol{b}_\gamma b = \boldsymbol{c}_\gamma(ab)$.

We easily realize that the inverse mapping $\boldsymbol{i}^{-1}$ is a strong isomorphism of $\tilde{\mathfrak{A}}^*$ onto $\tilde{\mathfrak{A}}$.

If there exists a strong isomorphism of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}^*$, then $\tilde{\mathfrak{A}}^*$ is said to be *strongly isomorphic with* $\tilde{\mathfrak{A}}$. This notion applies, of course, equally to either $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}^*$; therefore we sometimes speak about strongly isomorphic groupoidal structures $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{A}}^*$.

b) Let us now assume that the sequences of the groupoids $(\mathfrak{A})$ and $(\mathfrak{A}^*)$ consist of factoroids $\overline{\mathfrak{A}}_1, \ldots, \overline{\mathfrak{A}}_\alpha$ and $\overline{\mathfrak{A}}_1{}^*, \ldots, \overline{\mathfrak{A}}_\alpha{}^*$ in $\mathfrak{G}$. In that case every element

$$\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$$
$$(\bar{\bar{a}}^* = (\bar{a}_1{}^*, \ldots, \bar{a}_\alpha{}^*) \in \tilde{\mathfrak{A}}^*)$$

is an α-membered sequence and each member $\bar{a}_\gamma$ ($\bar{a}_\gamma{}^*$) is a decomposition in $\mathfrak{G}$ which is a complex in the factoroid $\overline{\mathfrak{A}}_\gamma$ ($\overline{\mathfrak{A}}_\gamma{}^*$).

The mapping $\boldsymbol{i}$ is said to be an *isomorphism with semi-coupling* or *isomorphism with loose coupling* (*isomorphism with coupling*) of $\tilde{\mathfrak{A}}$ on $\tilde{\mathfrak{A}}^*$ under the following circumstances:

There exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect: Every member $\bar{a}_\gamma$ ($\gamma = 1, \ldots, \alpha$) of an arbitrary element $\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha) \in \tilde{\mathfrak{A}}$ and the member $\bar{a}_\delta{}^*$, $\delta = \boldsymbol{p}\gamma$ of the corresponding element $\boldsymbol{i}\bar{\bar{a}} = \bar{\bar{a}}^* = (\bar{a}_1{}^*, \ldots, \bar{a}_\alpha{}^*) \in \tilde{\mathfrak{A}}^*$ are semi-coupled (coupled) decompositions in $\mathfrak{G}$.

It is easy to see that the inverse mapping $\boldsymbol{i}^{-1}$ is an isomorphism of the same kind but in the opposite direction, i.e., of $\tilde{\mathfrak{A}}^*$ onto $\tilde{\mathfrak{A}}$.

Let $\boldsymbol{i}$ be an isomorphism with loose coupling of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}^*$. Consider arbitrary members $\bar{a}_\gamma$, $\bar{a}_\delta{}^*$ to which the above relation applies so that $\bar{a}_\gamma$ and $\bar{a}_\delta{}^*$ are members of $\bar{\bar{a}}$ and $\boldsymbol{i}\bar{\bar{a}} = \bar{\bar{a}}^*$, respectively, $\delta = \boldsymbol{p}\gamma$. In this situation the closures $\mathrm{H}\bar{a}_\gamma = \bar{a}_\delta{}^* \sqsubset \sqsubset \bar{a}_\gamma$, $\mathrm{H}\bar{a}_\delta{}^* = \bar{a}_\gamma \sqsubset \bar{a}_\delta{}^*$ are nonempty and coupled (4.1).

Let $\boldsymbol{a}_\gamma$ denote the mapping of $\mathrm{H}\bar{a}_\gamma$ onto $\mathrm{H}\bar{a}_\delta{}^*$ defined by incidence of elements. In accordance with the second equivalence theorem (6.8), $\boldsymbol{a}_\gamma$ is simple. We observe that for every element $a \in \mathrm{H}\bar{a}_\gamma$ there holds $\boldsymbol{a}_\gamma a = a^*$ ($\in \mathrm{H}\bar{a}_\delta{}^*$) if and only if $a \cap a^* \neq \emptyset$.

Let us show that, *for the mappings $\boldsymbol{a}_\gamma$ of the closures $\mathrm{H}a_\gamma$ corresponding to the individual members $\bar{a}_\gamma$ ($\gamma = 1, \ldots, \alpha$) of the elements $\bar{\bar{a}} \in \tilde{\mathfrak{A}}$, the above deformation phenomenon arises.*

Indeed, let

$$\bar{\bar{a}} = (\bar{a}_1, \ldots, \bar{a}_\alpha), \quad \bar{\bar{b}} = (\bar{b}_1, \ldots, \bar{b}_\alpha) \in \tilde{\mathfrak{A}}$$

be arbitrary elements and let

$$\bar{\bar{a}}\bar{\bar{b}} = \bar{\bar{c}} = (\bar{c}_1, \ldots, \bar{c}_\alpha) \in \tilde{\mathfrak{A}}$$

be their product; next, let

$$\boldsymbol{i}\bar{\bar{a}} = \bar{\bar{a}}^* = (\bar{a}_1{}^*, \ldots, \bar{a}_\alpha{}^*), \quad \boldsymbol{i}\bar{\bar{b}} = \bar{\bar{b}}^* = (\bar{b}_1{}^*, \ldots, \bar{b}_\alpha{}^*) \in \tilde{\mathfrak{A}}^*$$

stand for the images of $\bar{\bar{a}}$, $\bar{\bar{b}}$ under the isomorphism $\boldsymbol{i}$ and

$$\bar{\bar{a}}^* \bar{\bar{b}}^* = \bar{\bar{c}}^* = (\bar{c}_1^*, \ldots, \bar{c}_\alpha^*) \in \tilde{\mathfrak{A}}$$

for their product; finally, let $\boldsymbol{a}_\gamma$, $\boldsymbol{b}_\gamma$, $\boldsymbol{c}_\gamma$ denote the corresponding simple mappings of $\mathrm{H}\bar{a}_\gamma$, $\mathrm{H}\bar{b}_\gamma$, $\mathrm{H}\bar{c}_\gamma$, respectively.

Consider any two elements $a \in \mathrm{H}\bar{a}_\gamma$, $b \in \mathrm{H}\bar{b}_\gamma$, their images $\boldsymbol{a}_\gamma a = a^* \in \mathrm{H}\bar{a}_\delta^*$, $\boldsymbol{b}_\gamma b = b^* \in \mathrm{H}\bar{b}_\delta^*$ and the corresponding products $c = a \circ b \in \bar{c}_\gamma$, $c^* = a^* \circ b^* \in \bar{c}_\delta^*$. Then we have $a \cap a^* \neq \emptyset \neq b \cap b^*$ and, furthermore,

$$c = a \circ b \supset ab \supset (a \cap a^*)\,(b \cap b^*),$$

$$c^* = a^* \circ b^* \supset a^* b^* \supset (a^* \cap a)\,(b^* \cap b).$$

We see that c and c^* are incident. So we have: $c \in \mathrm{H}\bar{c}_\gamma$, $c^* \in \mathrm{H}\bar{c}_\delta^*$ and, moreover, $c^* = \boldsymbol{c}_\gamma c$. Consequently, $\boldsymbol{a}_\gamma a \circ \boldsymbol{b}_\gamma b = \boldsymbol{c}_\gamma(a \circ b)$, which completes the proof.

If $\boldsymbol{i}$ is, in particular, an isomorphism with coupling, then the considered closures coincide with the corresponding elements. We observe that *every isomorphism with coupling of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}^*$ is a strong isomorphism.*

If there exists an isomorphism with semi-coupling (isomorphism with coupling) of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{A}}^*$, then $\tilde{\mathfrak{A}}^*$ is said to be *isomorphic and semi-coupled* or *isomorphic and loosely coupled* (*isomorphic and coupled*) *with* $\tilde{\mathfrak{A}}$. These notions are symmetric for both $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}^*$ and so we sometimes speak about isomorphic and semi-coupled or isomorphic and loosely coupled (isomorphic and coupled) groupoidal structures $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{A}}^*$. In particular, *every two isomorphic and coupled α-grade groupoidal structures are strongly isomorphic.*

16.4. Exercises

1. Consider the isomorphism theorems in connection with the groupoids $\mathfrak{Z}$, $\mathfrak{A}_m$, $\bar{\mathfrak{Z}}_n$, $\bar{\bar{\mathfrak{Z}}}_d$ dealt with in 15.2, 15.3.2, 15.4.1.
2. Let $\boldsymbol{i}$ be an isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}^*$. The image of any factoroid $\bar{\mathfrak{A}}$ on $\mathfrak{G}$ under the extended mapping $\boldsymbol{i}$ is a factoroid $\boldsymbol{i}\bar{\mathfrak{A}}$ on $\mathfrak{G}^*$ and the partial extended mapping $\boldsymbol{i}$ of $\bar{\mathfrak{A}}$ onto $\boldsymbol{i}\bar{\mathfrak{A}}$ is an isomorphism.
3. Let $\boldsymbol{d}$ be a deformation of $\mathfrak{G}$ onto $\mathfrak{G}^*$. Every factoroid $\bar{\mathfrak{A}}^*$ on $\mathfrak{G}^*$ is the $\boldsymbol{d}$-image of a certain factoroid $\bar{\mathfrak{A}}$ which lies on $\mathfrak{G}$ and is a covering of the factoroid corresponding to $\boldsymbol{d}$.
4. Any two adjoint chains of factoroids in $\mathfrak{G}$ have coupled refinements. (Cf. 15.3.5; 6.10.9.)