exactly one point in the plane is mapped onto itself. Under the mapping $\boldsymbol{g}[\alpha; a, b]$ no point in the plane is mapped onto itself unless the numbers α, a, b are connected by the relation:

$$a \,.\, \cos \frac{1}{2}\,\alpha + b \,.\, \sin \frac{1}{2}\,\alpha = 0;$$

in that case all the points in the plane that are mapped onto themselves form a straight line. For the composition of the mappings $\boldsymbol{f}[\alpha; a, b]$, $\boldsymbol{g}[\alpha; a, b]$ there hold the following formulae:

$$\boldsymbol{f}[\beta; c, d]\ \ \boldsymbol{f}[\alpha; a, b] = \boldsymbol{f}[\alpha + \beta;\ \ a \,.\, \cos\beta + b \,.\, \sin\beta + c,\ -a \,.\, \sin\beta + b \,.\, \cos\beta + d],$$

$$\boldsymbol{g}[\beta; c, d]\ \ \boldsymbol{f}[\alpha; a, b] = \boldsymbol{g}[\alpha + \beta;\ \ a \,.\, \cos\beta + b \,.\, \sin\beta + c,\ a \,.\, \sin\beta - b \,.\, \cos\beta + d],$$

$$\boldsymbol{f}[\beta; c, d]\ \ \boldsymbol{g}[\alpha; a, b] = \boldsymbol{g}[\alpha - \beta;\ \ a \,.\, \cos\beta + b \,.\, \sin\beta + c,\ -a \,.\, \sin\beta + b \,.\, \cos\beta + d],$$

$$\boldsymbol{g}[\beta; c, d]\ \ \boldsymbol{g}[\alpha; a, b] = \boldsymbol{f}[\alpha - \beta;\ \ a \,.\, \cos\beta + b \,.\, \sin\beta + c,\ a \,.\, \sin\beta - b \,.\, \cos\beta + d].$$

Remark. The mappings $\boldsymbol{f}[\alpha; a, b]$ and $\boldsymbol{g}[\alpha; a, b]$ are called *Euclidean motions in a plane.*

6. Every α-membered (infinite) sequence on a set A is the set formed from the images of the elements of the set $\{1, \ldots, \alpha\}$ ($\{1, 2, \ldots\}$) onto A under a convenient mapping of the latter onto the set A (1.7).
7. For the equivalence of nonempty sets A, B, C the following statements are correct: a) $A \simeq A$ (reflexivity); b) from $A \simeq B$ there follows $B \simeq A$ (symmetry); c) from $A \simeq B$, $B \simeq C$ there follows $A \simeq C$ (transitivity) (6.4).
8. Let $\boldsymbol{g}$, $\boldsymbol{h}$ denote mappings of the set G into itself and $\overline{G}_{\boldsymbol{g}}$, $\overline{G}_{\boldsymbol{h}}$, $\overline{G}_{\boldsymbol{hg}}$ be decompositions on G, corresponding to the mappings $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{hg}$. Show that the following relations apply:
   a) $\boldsymbol{hg}G \subset \boldsymbol{h}G$, $\overline{G}_{\boldsymbol{hg}} \geqq \overline{G}_{\boldsymbol{g}}$,
   b) the equality $\boldsymbol{hg}G = \boldsymbol{h}G$ yields $\boldsymbol{g}G \sqsubset \overline{G}_{\boldsymbol{h}} = \overline{G}_{\boldsymbol{h}}$ and vice versa,
   c) the equality $\overline{G}_{\boldsymbol{hg}} = \overline{G}_{\boldsymbol{g}}$ yields $\boldsymbol{g}G \cap \overline{G}_{\boldsymbol{h}} = \overline{(\boldsymbol{g}G)}_{\min}$ and vice versa. ($\overline{(\boldsymbol{g}G)}_{\min}$ is the least decomposition of the set $\boldsymbol{g}G$.)
9. Any two adjoint chains of decompositions in G have a coupled refinement. (Prove it by means of the construction described in 4.2.)

7. Mappings of decompositions

Let $\boldsymbol{g}$ denote a mapping of the set G onto a set G^*. Thus every element $\boldsymbol{a} \in G$ is, under $\boldsymbol{g}$, mapped onto a certain element $\boldsymbol{a}^* \in G^*$; $\boldsymbol{a}^*$ is the image of the element $\boldsymbol{a}$ under the mapping $\boldsymbol{g}$. To the mapping $\boldsymbol{g}$ there corresponds a certain decomposition $\overline{G}$ on G; each element of $\overline{G}$ consists of all $\boldsymbol{g}$-inverse images of the same point in G^*. The decomposition $\overline{G}$ is equivalent to the set G^*.

7.1. Extended mappings

The mapping $\boldsymbol{g}$ determines a mapping $\bar{\boldsymbol{g}}$ of the system of all subsets of G into the system of all subsets of G^*, the so-called *extended mapping*. $\bar{\boldsymbol{g}}$ is defined in the way that, for $\emptyset \neq A \subset G$, $\bar{\boldsymbol{g}}A \subset G^*$ is the set of the $\boldsymbol{g}$-images of all the points lying in A; moreover, we put $\bar{\boldsymbol{g}}\emptyset = \emptyset$. In particular, for $\bar{a} \in \bar{G}$, the set $\bar{\boldsymbol{g}}\bar{a}$ consists of a single point of G^*, namely, of the $\boldsymbol{g}$-image of the points of G lying in $\bar{a}$.

To simplify the notation, we generally write $\boldsymbol{g}$ instead of $\bar{\boldsymbol{g}}$. The symbol $\boldsymbol{g}$ is thus applied to the points of G, e.g. $a \in G$, and then the result $\boldsymbol{g}a$ denotes the image of the point a under the original mapping $\boldsymbol{g}$. The symbol $\boldsymbol{g}$ is also applied to subsets of G, e.g. $A \subset G$, in which case the result $\boldsymbol{g}A$ denotes the image of the subset A under the extended mapping $\bar{\boldsymbol{g}}$.

This rule is observed even for systems of subsets of G: If $\tilde{A}$ is a nonempty system of subsets of G, then we generally denote the system of the $\bar{\boldsymbol{g}}$-images of the indivdual elements of $\tilde{A}$ by the symbol $\boldsymbol{g}\tilde{A}$.

Fori example, if $\bar{A}$ is a decomposition of G, then $\boldsymbol{g}\bar{A}$ denotes the system of the $\bar{\boldsymbol{g}}$-images of the elements of $\bar{A}$. If, in particular, $\boldsymbol{g}\bar{A}$ is a decomposition on G^*, then the extended mapping $\bar{\boldsymbol{g}}$ defines the partial mapping $\boldsymbol{g}_{\bar{A}}$ of the decomposition $\bar{A}$ onto the decomposition $\boldsymbol{g}\bar{A}$ under which there corresponds, to every element $\bar{a} \in \bar{A}$, its image $\boldsymbol{g}\bar{a} \in \boldsymbol{g}\bar{A}$.

Let A and B stand for arbitrary subsets of G.

It is obvious that $A \subset B$ yields $\boldsymbol{g}A \subset \boldsymbol{g}B$.

Let us prove the following theorem:

The equality $\boldsymbol{g}A = \boldsymbol{g}B$ is true if and only if every element of $\bar{G}$, incident with one of the subsets A, B, is also incident with the other.

Proof. a) Suppose $\boldsymbol{g}A = \boldsymbol{g}B$. If an element $\bar{g} \in \bar{G}$ is incident with, for example, A, then there exists an element $a \in A$ such that $\bar{g}$ is the set of all the $\boldsymbol{g}$-inverse images of $\boldsymbol{g}a$. Since $\boldsymbol{g}a \in \boldsymbol{g}A = \boldsymbol{g}B$, there exists an element $b \in B$ such that $\boldsymbol{g}b = \boldsymbol{g}a$, so that $b \in \bar{g}$ and, consequently, $\bar{g}$ is incident with B.

b) Let every element of $\bar{G}$, incident with one of the sets A, B, be also incident with the other. Then, e.g., for $a^* \in \boldsymbol{g}A$, the element $\bar{g} \in \bar{G}$ which consists of all the $\boldsymbol{g}$-inverse images of a^* is incident with A and therefore, by the assumption, even with B. Hence there exists an element $b \in B$ such that $a^* = \boldsymbol{g}b \in \boldsymbol{g}B$ and we have $\boldsymbol{g}A \subset \boldsymbol{g}B$. At the same time there holds, of course, the relation $\boldsymbol{g}B \subset \boldsymbol{g}A$ and we have $\boldsymbol{g}A = \boldsymbol{g}B$.

The above theorem can, naturally, also be expressed by saying that *the equality $\boldsymbol{g}A = \boldsymbol{g}B$ applies if and only if $A \sqsubset \bar{G} = B \sqsubset \bar{G}$.*

Let $\tilde{A}$ stand for a system of subsets of G.

If all the elements of $\tilde{A}$ have, under the extended mapping $\boldsymbol{g}$, the same image $A^* \subset G^*$ so that, for $A \in \tilde{A}$, there holds $\boldsymbol{g}A \subset A^*$, then even the set $\boldsymbol{s}\tilde{A}$ is mapped onto A^*, i.e., $\boldsymbol{g}(\boldsymbol{s}\tilde{A}) = A^*$.

Indeed, first of all, for every element $A \in \tilde{A}$ there holds $A \subset \boldsymbol{s}\tilde{A}$ whence $A^* = \boldsymbol{g}A \subset \boldsymbol{g}(\boldsymbol{s}\tilde{A})$. Moreover, every element $a \in \boldsymbol{s}\tilde{A}$ lies in a certain subset $A \in \tilde{A}$ and we have: $\boldsymbol{g}a \in \boldsymbol{g}A = A^*$ which yields $\boldsymbol{g}(\boldsymbol{s}\tilde{A}) \subset A^*$ and the proof is accomplished.

7.2. Theorems on mappings of decompositions

Let $\bar{A}$ denote a decomposition on G.

The system $\boldsymbol{g}\bar{A}$ of the subsets of G^* evidently covers the set G^*. But this system is not necessarily a decomposition of the set G^* because the $\boldsymbol{g}$-images of two different elements of $\bar{A}$ may be incident without coinciding.

The following theorem states a necessary and sufficient condition under which the decomposition $\bar{A}$ is mapped, under $\boldsymbol{g}$, onto a decomposition of G^*.

$\boldsymbol{g}\bar{A}$ is a decomposition of the set G^* if and only if the decompositions $\bar{A}$, $\bar{G}$ are complementary.

Proof. a) Suppose $\boldsymbol{g}\bar{A}$ is a decomposition on G^*. Let the elements $\bar{a} \in \bar{A}$, $\bar{g} \in \bar{G}$ lie in the same element $\bar{u} \in [\bar{A}, \bar{G}]$. We are to show that $\bar{a} \cap \bar{g} \neq \emptyset$. Let $\bar{b} \in \bar{A}$ stand for an arbitrary element incident with $\bar{g}$. Then $\bar{b} \subset \bar{u}$, hence there exists a binding $\{\bar{A}, \bar{B}\}$ from $\bar{a}$ to $\bar{b}$:

$$(\bar{a} =) \quad \bar{a}_1, \ldots, \bar{a}_\alpha \quad (= \bar{b}).$$

By the definition of a binding, every two of its neighbouring elements $\bar{a}_\beta$, $\bar{a}_{\beta+1}$ ($\beta = 1, \ldots, \alpha - 1$) are incident with an element of the decomposition $\bar{G}$ and thus both images $\boldsymbol{g}\bar{a}_\beta$, $\boldsymbol{g}\bar{a}_{\beta+1}$ are incident. Since $\boldsymbol{g}\bar{A}$ is a decomposition on G^*, we have $\boldsymbol{g}\bar{a}_\beta = \boldsymbol{g}\bar{a}_{\beta+1}$ and thus even $\boldsymbol{g}\bar{a} = \boldsymbol{g}\bar{b}$. Consequently, $\bar{a} \sqsubset \bar{G} = \bar{b} \sqsubset \bar{G}$. As $\bar{g} \in \bar{b} \sqsubset \bar{G}$, we have $\bar{g} \in \bar{a} \sqsubset \bar{G}$ so that $\bar{a} \cap \bar{g} \neq \emptyset$.

b) Let the decompositions $\bar{A}$, $\bar{G}$ be complementary. Our object now is to show that, for $\bar{a}, \bar{b} \in \bar{A}$, the sets $\boldsymbol{g}\bar{a}$, $\boldsymbol{g}\bar{b}$ either are disjoint or coincide. If the sets $\boldsymbol{g}\bar{a}$, $\boldsymbol{g}\bar{b}$ are not disjoint, then there exist points $a \in \bar{a}$, $b \in \bar{b}$ such that $\boldsymbol{g}a = \boldsymbol{g}b \in \boldsymbol{g}\bar{a} \cap \boldsymbol{g}\bar{b}$. Then the element $\bar{g} \in \bar{G}$, consisting of all the $\boldsymbol{g}$-inverse images of the element $\boldsymbol{g}a$, is incident with both the elements $\bar{a}$, $\bar{b}$ and the latter therefore lie in the same element of the decomposition $[\bar{A}, \bar{G}]$. Since the decompositions $\bar{A}$, $\bar{G}$ are complementary, there holds $\bar{a} \sqsubset \bar{G} = \bar{b} \sqsubset \bar{G}$ which yields $\boldsymbol{g}\bar{a} = \boldsymbol{g}\bar{b}$.

Let again $\bar{A}$, $\bar{G}$ be complementary.

By the above theorem, $\boldsymbol{g}\bar{A}$ is a decomposition on G. The extended mapping $\boldsymbol{g}$ determines the partial mapping of the decomposition $\bar{A}$ onto $\boldsymbol{g}\bar{A}$ under which there corresponds, of course, to every element $\bar{a} \in \bar{A}$, its image $\boldsymbol{g}\bar{a} \in \boldsymbol{g}\bar{A}$. By the mapping $\boldsymbol{g}$ of the decomposition $\bar{A}$ onto $\boldsymbol{g}\bar{A}$ we shall, in what follows, understand this partial mapping.

To the mapping $\boldsymbol{g}$ of $\bar{A}$ onto $\boldsymbol{g}\bar{A}$ there naturally corresponds a certain decomposition $\bar{\bar{A}}$ of $\bar{A}$. Its elements consist of all the elements of $\bar{A}$ that have, under the extended mapping $\boldsymbol{g}$, the same image.

We shall show that *the covering of the decomposition* $\bar{A}$ *enforced by* $\bar{\bar{A}}$ *is the least common covering* $[\bar{A}, \bar{G}]$ *of the decompositions* $\bar{A}, \bar{G}$.

Indeed, consider an arbitrary element $\bar{\bar{a}} \in \bar{\bar{A}}$. We are to show that the set $\boldsymbol{s}\bar{\bar{a}}$ is an element of the decomposition $[\bar{A}, \bar{G}]$. Let $\bar{a} \in \bar{\bar{a}}$ be an arbitrary element and $\bar{u} \in [\bar{A}, \bar{G}]$ the element of $[\bar{A}, \bar{G}]$, containing $\bar{a}$; consequently, we have $\bar{a} \subset \boldsymbol{s}\bar{\bar{a}} \cap \bar{u}$. Every element $\bar{x} \in \bar{\bar{a}}$ has, under the extended mapping $\boldsymbol{g}$, the same image as $\bar{a}$, hence $\bar{a} \sqsubset \bar{G} = \bar{x} \sqsubset \bar{G}$; it follows that the element $\bar{x}$ may be connected with the element $\bar{a}$ in the decomposition $\bar{G}$ and therefore lies in the element $\bar{u}$. Thus we have verified that $\boldsymbol{s}\bar{\bar{a}} \subset \bar{u}$. Conversely, for any element $\bar{x} \in \bar{A}$ lying in $\bar{u}$ there holds $\bar{a} \sqsubset \bar{G} = \bar{x} \sqsubset \bar{G}$; consequently, the element $\bar{x}$ has, under the extended mapping $\boldsymbol{g}$, the same image as $\bar{a}$, thus $\bar{x} \subset \bar{u}$ and we have $\bar{x} \subset \boldsymbol{s}\bar{\bar{a}}$. Hence $\bar{u} \subset \boldsymbol{s}\bar{\bar{a}}$ and the proof is accomplished.

Associating, with every element $\bar{u} \in [\bar{A}, \bar{G}]$, the element $\bar{\bar{a}} \in \bar{\bar{A}}$ which contains all the elements of $\bar{A}$ lying in $\bar{u}$, we obtain a simple mapping of the decomposition $[\bar{A}, \bar{G}]$ onto $\bar{\bar{A}}$ (6.8); associating, with every element $\bar{\bar{a}} \in \bar{\bar{A}}$, the element $\bar{a}^* \in \boldsymbol{g}\bar{A}$ which is the image of every element $\bar{a} \in \bar{A}$ lying in $\bar{\bar{a}}$, we obtain a simple mapping of the decomposition $\bar{\bar{A}}$ onto $\boldsymbol{g}\bar{A}$ (6.8). Composing these simple mappings, we get a simple mapping of the decomposition $[\bar{A}, \bar{G}]$ onto $\boldsymbol{g}\bar{A}$ (6.7). Under this mapping there corresponds, to every element $\bar{u} \in [\bar{A}, \bar{G}]$, a certain element $\bar{a}^* \in \boldsymbol{g}\bar{A}$; the element $\bar{a}^*$ is the image, under the extended mapping $\boldsymbol{g}$, of every element of $\bar{A}$ lying in the element $\bar{\bar{a}} \in \bar{\bar{A}}$ which contains all the elements of $\bar{A}$ lying in $\bar{u}$. Since $\bar{u} = \boldsymbol{s}\bar{\bar{a}}$ and for $\bar{a} \in \bar{\bar{a}}$ we have $\boldsymbol{g}\bar{a} = \bar{a}^*$, we conclude, with respect to the last theorem in 7.1, that the element $\bar{u}$ has, under the extended mapping $\boldsymbol{g}$, the image $\bar{a}^*$, i.e., $\boldsymbol{g}\bar{u} = \bar{a}^*$.

Thus we have the following result:

If a decomposition $\bar{A}$ *on* G *is mapped, under* $\boldsymbol{g}$, *onto some decomposition* $\bar{A}^*$ *on* G, *then the decompositions* $[\bar{A}, \bar{G}]$ *and* $\bar{A}^*$ *are equivalent*, i.e., $[\bar{A}, \bar{G}] \simeq \bar{A}^*$; *a simple mapping of the decomposition* $[\bar{A}, \bar{G}]$ *onto* $\bar{A}^*$ *is obtained by associating, with each element of* $[\bar{A}, \bar{G}]$, *its image under the extended mapping* $\boldsymbol{g}$.

Consequently, *every covering of the decomposition* $\bar{G}$ *is equivalent to its image under* $\boldsymbol{g}$; *the mapping under which every element of the covering is associated with its own image is simple.*

7.3. Exercises

1. Let $\boldsymbol{g}$ be a mapping of the set G onto G^* and A, B stand for arbitrary subsets of G. Show that the following relations are true: $\boldsymbol{g}(A \cup B) = \boldsymbol{g}A \cup \boldsymbol{g}B$; $\boldsymbol{g}(A \cap B) \subset \boldsymbol{g}A \cap \boldsymbol{g}B$.

2. Assuming the situation described in exercise 1., let $\overline{G}$ be the decomposition on G corresponding to the mapping $\boldsymbol{g}$. Show that the equality $\boldsymbol{g}(A \cap B) = \boldsymbol{g}A \cap \boldsymbol{g}B$ applies if and only if there holds $(A \cap B) \sqsubset \overline{G} = (A \sqsubset \overline{G}) \cap (B \sqsubset \overline{G})$.

3. Let $\boldsymbol{g}$ be a mapping of the set G onto G^* and $\{\bar{a}, \bar{b}, \ldots\}$ stand for a decomposition on G. Then $\{\boldsymbol{g}\bar{a}, \boldsymbol{g}\bar{b}, \ldots\}$ is a decomposition on G^* if and only if $\{\bar{a}, \bar{b}, \ldots\}$ is a covering of the decomposition corresponding to $\boldsymbol{g}$.

4. Suppose $\boldsymbol{g}$ is a simple mapping of the set G onto G^*. Let, moreover, $A \subset G$ be a nonempty subset and $\bar{A}, \bar{B}$ stand for decompositions in (on) G. In this situation there holds:
   a) the extended mapping $\bar{\boldsymbol{g}}$ of the system of all the nonempty parts of G onto the system of all the nonempty parts of G^* is simple;
   b) the sets A, $\boldsymbol{g}A$ are equivalent, i.e., $A \simeq \boldsymbol{g}A$;
   c) $\boldsymbol{g}\bar{A}$ is a decomposition in (on) the set G^*;
   d) the decompositions $\bar{A}$, $\boldsymbol{g}\bar{A}$ are equivalent, i.e., $\bar{A} \simeq \boldsymbol{g}\bar{A}$;
   e) if the decompositions $\bar{A}, \bar{B}$ are equivalent or loosely coupled or coupled, then the decompositions $\boldsymbol{g}\bar{A}, \boldsymbol{g}\bar{B}$ have, in each case, the same property.

8. Permutations

In this chapter we shall deal with simple mappings of finite sets onto themselves; they play an important role in algebra, particularly, in the theory of groups.

8.1. Definition

By a *permutation of the set* G we mean a simple mapping of the set G onto itself (6.6).

In this section we shall restrict our considerations to permutations of *finite* sets.

Let G denote an arbitrary set consisting of a finite number $n(\geqq 1)$ of elements. From the assumption that G is finite it follows that every simple mapping $\boldsymbol{p}$ of the set G into itself is a permutation of G (6.10.2).

Let the elements of G be denoted by the letters $a, b, \ldots, m$. Then we can uniquely associate, with every permutation $\boldsymbol{p}$ of the set G, a symbol of the form:

$$\begin{pmatrix} a & b & \ldots & m \\ a^* & b^* & \ldots & m^* \end{pmatrix}.$$

where $a^*, b^*, \ldots, m^*$ are the letters denoting the elements $\boldsymbol{pa}, \boldsymbol{pb}, \ldots, \boldsymbol{pm}$. Since $\boldsymbol{p}G = G$, the letters $a^*, b^*, \ldots, m^*$ are again $a, b, \ldots, m$ written in a certain order.