3. Decompositions on sets

In: Otakar Borůvka (author): Foundations of the Theory of Groupoids and Groups. (English). Berlin: VEB Deutscher Verlag der Wissenschaften, 1974. pp. 27--34.

Persistent URL: http://dml.cz/dmlcz/401542

3. If $B \sqsubset \bar{A} = B \sqcap \bar{A}$, then for every element $\bar{a} \in \bar{A}$ there holds either $\bar{a} \subset B$ or $\bar{a} \cap B = \emptyset$; and conversely.
4. $s(s\bar{A} \sqsubset \bar{C}) \sqcap \bar{A} = s\bar{C} \sqcap \bar{A}$.
5. If $B \supset C$, then there holds: a) $(C \sqsubset \bar{A}) \sqcap B = C \sqsubset (\bar{A} \sqcap B)$. With regard to this equality, the set on either side of the latter may be denoted by $C \sqsubset \bar{A} \sqcap B$. In particular, for $C = B$, we have $(B \sqsubset \bar{A}) \sqcap B = \bar{A} \sqcap B$; b) $(B \sqsubset \bar{A}) \sqcap C = \bar{A} \sqcap C$.
6. If one of the following three statements is true, then the remaining two are true as well: a) Every element of the decomposition $\bar{A}$ is incident with at least one element of the decomposition $\bar{C}$; b) $\bar{A} = \bar{C} \sqsubset \bar{A}$; c) $s\bar{A} = s(s\bar{C} \sqsubset \bar{A})$.
7. Every lengthening of an arbitrary chain of decompositions is simultaneously its refinement.
8. The number p_{n+1} of decompositions of every finite set of order $n + 1$ $(\geqq 1)$ is finite. The numbers p_{n+1} are given by the formula:

$$p_{n+1} = \sum_{\nu=1}^{n} \binom{n}{\nu} p_\nu \qquad (p_0 = 1).$$

So we have, in particular:

$$p_1 = 1, \ p_2 = 2, \ p_3 = 5, \ p_4 = 15, \ p_5 = 52, \ p_6 = 203, \ \ldots$$

3. Decompositions on sets

In this chapter we shall deal with decompositions on sets. The results are often useful (see: 2.2) when we are to describe the properties of decompositions in sets; in fact: a decomposition $\bar{A}$ in the set G is, simultaneously, a decomposition on the set $s\bar{A}$.

3.1. Bindings in decompositions

Let $\bar{A}$, $\bar{B}$ stand for decompositions of G.

Consider two arbitrary elements $\bar{a}, \bar{p} \in \bar{A}$.

A *binding from $\bar{a}$ to $\bar{p}$ in $\bar{A}$ with regard to $\bar{B}$* is a finite sequence of elements of $\bar{A}$:

$$\bar{a}_1, \ldots, \bar{a}_\alpha \quad (\alpha \geqq 2)$$

such that $\bar{a}_1 = \bar{a}$, $\bar{a}_\alpha = \bar{p}$ and that every two neighbouring members $\bar{a}_\beta$, $\bar{a}_{\beta+1}$ $(\beta = 1, \ldots, \alpha - 1)$ are incident with the same element $\bar{b}_\beta \in \bar{B}$. Such a binding is said to be *generated* by the decomposition $\bar{B}$; we speak, briefly, about the *binding* $\{\bar{A}, \bar{B}\}$ *from $\bar{a}$ to $\bar{p}$.*

Note that the individual members of the binding need not be different from one another.

If there exists a binding $\{\overline{A}, \overline{B}\}$ from $\bar{a}$ to $\bar{p}$, then we say that the element $\bar{p}$ can be *connected*, in $\overline{B}$, with the element $\bar{a}$ or, briefly, that $\bar{p}$ can be connected with $\bar{a}$.

Let us now consider the properties of bindings.

First, it is easy to see (the proof may be left to the reader) that, for arbitrary elements $\bar{a}, \bar{b}, \bar{c} \in \overline{A}$, there holds:

a) *The element $\bar{a}$ can be connected with $\bar{a}$.*
b) *If $\bar{b}$ can be connected with $\bar{a}$ and $\bar{c}$ with $\bar{b}$, then $\bar{c}$ can be connected with $\bar{a}$.*
c) *If $\bar{b}$ can be connected with $\bar{a}$, then $\bar{a}$ can be connected with $\bar{b}$.*

Taking account of the statement c), we generally speak about the binding between two elements, or say that two elements can be connected, without stressing which can be connected with which.

If the elements $\bar{a}, \bar{b} \in \overline{A}$ can be connected with an element $\bar{c} \in \overline{A}$, then they can also be connected with each other.

In fact, if the assumption is satisfied, then $\bar{a}$ can be connected with $\bar{c}$, $\bar{c}$ with $\bar{b}$ and, therefore, even $\bar{a}$ with $\bar{b}$.

Let $\bar{a}, \bar{p} \in \overline{A}$; $\bar{b}, \bar{q} \in \overline{B}$ be arbitrary elements and suppose that the elements $\bar{a}, \bar{b}$ as well as $\bar{p}, \bar{q}$ are incident.

If $\bar{a}, \bar{p}$ can be connected in $\overline{B}$, then $\bar{b}, \bar{q}$ can be connected in $\overline{A}$.

Indeed, if there exists a binding $\{\overline{A}, \overline{B}\}$ from $\bar{a}$ to $\bar{p}$ of the form:

$$\bar{a}_1, \ldots, \bar{a}_\alpha \quad (\bar{a}_1 = \bar{a}, \bar{a}_\alpha = \bar{p}),$$

then every two neighbouring elements $\bar{a}_\beta, \bar{a}_{\beta+1}$ are incident with a certain element $\bar{b}_\beta \in \overline{B}$; consequently, $\bar{b}_\beta, \bar{b}_{\beta+1}$ are incident with $\bar{a}_{\beta+1}$. Moreover, $\bar{b}$ is incident with $\bar{a}_1$ and $\bar{q}$ with $\bar{a}_\beta$. Consequently,

$$\bar{b}_0, \ldots, \bar{b}_\alpha \quad (\bar{b}_0 = \bar{b}, \bar{b}_\alpha = \bar{q})$$

is a binding $\{\overline{B}, \overline{A}\}$ from $\bar{b}$ to $\bar{q}$.

3.2. Coverings and refinements of decompositions in sets

Let us, first, introduce once more the notions of a covering and a refinement of a decomposition lying *on* the set G. These notions have already been mentioned in 2.4 and play an important part in the following deliberations.

Let $\bar{A}$, $\bar{B}$ denote arbitrary decompositions on G.

$\bar{A}$ and B are called a covering and a refinement of $\bar{B}$ and $\bar{A}$, respectively, if every element of $\bar{A}$ is the sum of some elements of $\bar{B}$. This relation between $\bar{A}$ and $\bar{B}$ is expressed by $\bar{A} \geqq \bar{B}$ or $\bar{B} \leqq \bar{A}$. In particular ($\bar{A} = \bar{B}$), every decomposition on G is both its own covering and refinement. If $A \geqq \bar{B}$, then the decomposition $\bar{B}$ is obtained, as we have seen, by replacing each element of $\bar{A}$ by its suitable decomposition; we have also noticed that the covering $\bar{A}$ is enforced by a certain decomposition lying on $\bar{B}$.

Let us now proceed to a more detailed study of the above concepts.

Let $\bar{A}$, $\bar{B}$, $\bar{C}$ be arbitrary decompositions on G.

First, we shall show that $\bar{A} \geqq \bar{B}$ *is true if and only if, for any two incident elements* $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$, *there holds* $\bar{a} \supset \bar{b}$.

Suppose $\bar{A} \geqq \bar{B}$. Let $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$ be arbitrary incident elements, hence $\bar{a} \cap \bar{b} \neq \emptyset$. Then $\bar{a}$ is the sum of certain subsets of G that are elements of $\bar{B}$. One of them is $\bar{b}$ because $\bar{a} \cap \bar{b} \neq \emptyset$ and the elements of $\bar{B}$ are disjoint. So we have $\bar{a} \supset \bar{b}$.

Let, conversely, for any two incident elements $\bar{a} \in \bar{A}, \bar{b} \in B$, the relation $\bar{a} \supset \bar{b}$ be true. Then $\bar{a}$ is the sum of those subsets of G that are elements of $\bar{B}$ and are incident with $\bar{a}$ and, consequently, the relation $\bar{A} \geqq \bar{B}$ applies.

Note, furthermore, that the statements set below are correct:

a) $\bar{A} \geqq \bar{A}$.
b) *From* $\bar{A} \geqq \bar{B}$, $\bar{B} \geqq \bar{C}$ *there follows* $\bar{A} \geqq \bar{C}$.
c) *From* $A \geqq B$, $B \geqq \bar{A}$ *there follows* $A = \bar{B}$.

3.3. Common covering and common refinement of two decompositions

Let $\bar{A}$, $\bar{B}$ denote arbitrary decompositions on G.

A *common covering of the decompositions* $\bar{A}$ *and* $\bar{B}$, briefly, a *covering of* $\bar{A}$, $\bar{B}$ is a decomposition on G that is a covering of $\bar{A}$ as well as of $\bar{B}$.

Analogously, by a *common refinement of the decompositions* $\bar{A}$ *and* $\bar{B}$, briefly, a *refinement of* $\bar{A}$, $\bar{B}$ we mean a decomposition on G that is a refinement of $\bar{A}$ as well as of $\bar{B}$.

For example, the greatest decomposition $\bar{G}_{\max}$ is a common covering and the least decomposition $\bar{G}_{\min}$ is a common refinement of $\bar{A}$, $\bar{B}$.

It is easy to understand that *every covering of any common covering of* $\bar{A}$, $\bar{B}$ *is again a covering of* $\bar{A}$, $\bar{B}$; similarly, *every refinement of any common refinement of* $\bar{A}$, $\bar{B}$ *is again a refinement of* $\bar{A}$, $\bar{B}$.

A remarkable progress in the trend of the above considerations leading to a number of important results is due to the notions of the least common covering and the greatest common refinement of two decompositions. We shall deal with them in 3.4; 3.5; 3.6.

3.4. The least common covering of two decompositions

In 3.3 we saw that every covering of any common covering of two decompositions $\bar{A}$, $\bar{B}$ is again a covering of $\bar{A}, \bar{B}$. It is important to note that among all the common coverings of two decompositions $\bar{A}$, $\bar{B}$ there is one least covering, $\bar{X}$; least in the sense that every common covering of $\bar{A}$ and $\bar{B}$ is a covering of $\bar{X}$. This particular covering is called the *least common covering of the decompositions* $\bar{A}$ *and* $\bar{B}$ or, briefly, the *least covering of* $\bar{A}$, $\bar{B}$.

Let $\bar{A}$, $\bar{B}$, $\bar{C}$ be decompositions on G.

We shall now construct a decomposition on G, denoted by $[\bar{A}, \bar{B}]$, and verify that it is the least common covering of $\bar{A}$ and $\bar{B}$.

Let $\bar{\bar{A}}$ be the system of all subsets of $\bar{A}$, characterized by the following property: Every subset $\bar{\bar{a}} \in \bar{\bar{A}}$ consists of all the elements of $\bar{A}$ that can be connected, in the decomposition $\bar{B}$, with some element $\bar{a} \in \bar{A}$.

First of all, we shall show that $\bar{\bar{A}}$ is a decomposition on $\bar{A}$.

In fact, every element $\bar{a} \in \bar{A}$ lies in some subset $\bar{\bar{a}} \in \bar{\bar{A}}$ because $\bar{a}$ can be connected with itself and therefore lies in the subset $\bar{\bar{a}} \in \bar{\bar{A}}$ consisting of all the elements of $\bar{A}$ that can be connected with $\bar{a}$.

Moreover, every two elements of the system $\bar{\bar{A}}$ are either disjoint or identical. To prove this, let us consider two arbitrary elements $\bar{\bar{a}}$, $\bar{\bar{b}} \in \bar{\bar{A}}$, $\bar{\bar{a}}$ consisting of all the elements of $\bar{A}$ that can be connected with an element $\bar{a} \in \bar{A}$ and $\bar{\bar{b}}$ of all the elements of $\bar{A}$ that can be connected with an element $\bar{b} \in \bar{A}$. Suppose the elements $\bar{\bar{a}}$ and $\bar{\bar{b}}$ are incident so that they have a common element $\bar{c} \in \bar{A}$. The latter lies in $\bar{\bar{a}}$ and can therefore be connected with $\bar{a}$; it also lies in $\bar{\bar{b}}$ so that it can be connected with $\bar{b}$. Hence $\bar{a}$ and $\bar{b}$ can be connected with $\bar{c}$ and, consequently, they can be connected with each other (3.1). We observe that any element $\bar{x} \in \bar{\bar{a}}$ can be connected with $\bar{a}$ and the latter again with $\bar{b}$. Thus the element $\bar{x}$ can be connected with $\bar{b}$, and we have $\bar{\bar{a}} \subset \bar{\bar{b}}$. In a similar way we verify that $\bar{\bar{b}} \subset \bar{\bar{a}}$ and we have $\bar{\bar{a}} = \bar{\bar{b}}$.

Consequently, $\bar{\bar{A}}$ is a decomposition on $\bar{A}$.

Note that any two elements of $\bar{A}$ that are in the same element of $\bar{\bar{A}}$ can be connected with each other, whereas two elements that do not lie in the same element of $\bar{\bar{A}}$ cannot be connected.

The decomposition $\bar{\bar{A}}$ enforces a certain covering of the decomposition $\bar{A}$, denoted by $[\bar{A}, \bar{B}]$. So we have

$$[\bar{A}, \bar{B}] \geqq \bar{A}.$$

Let us remark that every element $\bar{u} \in [\bar{A}, \bar{B}]$ is the sum of all the elements of $\bar{A}$ that lie in some element of $\bar{\bar{A}}$. In other words, $\bar{u}$ is the sum of all the elements of A that can be connected, in $\bar{B}$, with some element $\bar{a} \in \bar{A}$ lying in $\bar{u}$: $\bar{a} \subset \bar{u}$.

Now we shall consider the properties of the decomposition $[\bar{A}, \bar{B}]$.

First, we shall show that the *equality* $[\bar{A}, \bar{B}] = \bar{A}$ *and the relation* $\bar{A} \geqq \bar{B}$ *are simultaneously valid.*

Proof. a) Suppose $[\bar{A}, \bar{B}] = \bar{A}$. Let $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$ be arbitrary incident elements. If $\bar{a} \supset \bar{b}$ does not apply, then there exists an element $\bar{p} \in \bar{A}$ incident with $\bar{b}$ and different from $\bar{a}$. The elements $\bar{a}, \bar{p}$, arranged in this order, form a binding $\{\bar{A}, \bar{B}\}$ from $\bar{a}$ to $\bar{p}$, so that the set $\bar{a} \cup \bar{p}$ is a part of a certain element $\bar{u} \in [\bar{A}, \bar{B}]$. Consequently, $\bar{u}$ is the sum of at least two different elements of $\bar{A}$, therefore it is not an element of $\bar{A}$, which contradicts the assumption. Thus $\bar{a} \supset \bar{b}$ and we have $\bar{A} \geqq \bar{B}$.

b) Suppose $\bar{A} \geqq \bar{B}$. In that case every element of $\bar{B}$ is a part of an element of $\bar{A}$. Consequently, no two different elements of $\bar{A}$ can be connected in $\bar{B}$. We observe that the above decomposition $\bar{\bar{A}}$ is the least decomposition on $\bar{A}$, hence $[\bar{A}, \bar{B}] = \bar{A}$.

Furthermore, we are going to prove that *there holds*:

a) $[\bar{A}, \bar{B}] = [\bar{B}, \bar{A}]$;
b) $[\bar{A}, \bar{A}] = \bar{A}$;
c) $[A, [B, \bar{C}]] = [[\bar{A}, \bar{B}], \bar{C}]$.

Proof. a) Let $\bar{u} \in [\bar{A}, \bar{B}]$, $\bar{v} \in [\bar{B}, \bar{A}]$ be arbitrary incident elements. Since $\bar{u}$ and $\bar{v}$ are the sums of certain elements of $\bar{A}$ and $\bar{B}$, respectively, there exist elements $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$ such that $\bar{a} \subset \bar{u}, \bar{b} \subset \bar{v}$ and $\bar{a} \cap \bar{b} \neq \emptyset$. Because $\bar{A}$ covers G, every point $p \in \bar{u}$ lies in an element $\bar{p} \in \bar{A}$. We see that $\bar{u} \supset \bar{p}$ and, as the elements $\bar{a}, \bar{p} \in \bar{A}$ lie in $\bar{u} \in [\bar{A}, \bar{B}]$, $\bar{p}$ can be connected, in $\bar{B}$, with $\bar{a}$. Since $\bar{B}$ covers G, p lies in an element $\bar{q} \in B$ which is, of course, incident with $\bar{p}$. Moreover, in accordance with 3.2, the element $\bar{q}$ can be connected, in $\bar{A}$, with $\bar{b}$. Hence $\bar{v} \supset \bar{q}$ and thus even $\bar{v} \supset \bar{u}$. So we have $[\bar{B}, \bar{A}] \geqq [\bar{A}, \bar{B}]$. Simultaneously, for analogous reasons, there holds the relation $\leqq$ and we have $[\bar{A}, \bar{B}] = [\bar{B}, \bar{A}]$.

b) Since there holds $\bar{A} \geqq \bar{A}$, there also holds $[\bar{A}, \bar{A}] = \bar{A}$.

c) If any two elements $\bar{a}_1, \bar{a}_2 \in \bar{A}$ are incident with some element $\bar{z} \in [\bar{B}, \bar{C}]$, then they lie in the same element of the decomposition $[[\bar{A}, \bar{B}], \bar{C}]$. In fact, in that case there exist elements $\bar{b}, \bar{q} \in \bar{B}$ such that $\bar{b}$, $\bar{q} \subset \bar{z}$, $\bar{b} \cap \bar{a}_1 \neq \emptyset \neq \bar{q} \cap \bar{a}_2$, and a binding $\{\bar{B}, \bar{C}\}$ from $\bar{b}$ to $\bar{q}$:

$$\bar{b}_1, \ldots, \bar{b}_\gamma \quad (\bar{b}_1 = \bar{b}, \quad \bar{b}_\gamma = \bar{q}).$$

Every element $\bar{b}_\delta$ of the latter is a part of a certain element $\bar{u}_\delta \in [\bar{B}, \bar{A}] = [\bar{A}, \bar{B}]$, $\delta = 1, \ldots, \gamma$. Since $\bar{a}_1$ ($\bar{a}_2$) is incident with $\bar{b}_1$ ($\bar{b}_\gamma$) and $\bar{b}_1$ ($\bar{b}_\gamma$) is a part of $\bar{u}_1$ ($\bar{u}_\gamma$), the element $\bar{a}_1$ ($\bar{a}_2$) is incident with $\bar{u}_1$ ($\bar{u}_\gamma$) and therefore $\bar{a}_1 \subset \bar{u}_1, \bar{a}_2 \subset \bar{u}_\gamma$. As every two neighbouring elements $\bar{b}_\delta, \bar{b}_{\delta+1}$ are incident with some element $\bar{c}_\delta \in \bar{C}$, the same holds for every two elements $\bar{u}_\delta, \bar{u}_{\delta+1}$ so that $\bar{u}_1, \ldots, \bar{u}_\gamma$ is a binding $\{[\bar{A}, \bar{B}], \bar{C}\}$ from $\bar{u}_1$ to $\bar{u}_\gamma$. Consequently, the elements $\bar{u}_1$, $\bar{u}_\gamma$, and therefore even $\bar{a}_1$, $\bar{a}_2$, lie in the same element of the decomposition $[[\bar{A}, \bar{B}], \bar{C}]$.

Now, let $\bar{u} \in [\bar{A}, [\bar{B}, \bar{C}]]$, $\bar{v} \in [[\bar{A}, \bar{B}], \bar{C}]$ be arbitrary incident elements. Then there exists an element $\bar{a} \in \bar{A}$, $\bar{a} \subset \bar{u} \cap \bar{v}$; the element $\bar{u}$ is the sum of all the elements $\bar{p} \in \bar{A}$ such that there exists a binding $\{\bar{A}, [\bar{B}, \bar{C}]\}$ from $\bar{a}$ to $\bar{p}$:

$$\bar{a}_1, \ldots, \bar{a}_\alpha \ (\bar{a}_1 = \bar{a}, \bar{a}_\alpha = \bar{p}).$$

Every two neighbouring elements $\bar{a}_\beta$, $\bar{a}_{\beta+1}$ are incident with some element of the decomposition $[\bar{B}, \bar{C}]$ so that they lie, as we have just verified, in the same element of the decomposition $[[\bar{A}, \bar{B}], \bar{C}]$. From this and from $\bar{v} \supset \bar{a}_1$ there follows $\bar{v} \supset \bar{a}_\alpha$ so that $\bar{v} \supset \bar{p}$ and we have $\bar{v} \supset \bar{u}$. Hence, $[[\bar{A}, \bar{B}], \bar{C}] \geqq [\bar{A}, [\bar{B}, \bar{C}]]$. With regard to a), there follows:

$$\begin{aligned}[\bar{A}, [\bar{B}, \bar{C}]] = [[\bar{B}, \bar{C}], \bar{A}] \geqq [\bar{B}, [\bar{C}, \bar{A}]] = [[\bar{C}, \bar{A}], \bar{B}] \\ \geqq [\bar{C}, [\bar{A}, \bar{B}]] = [[\bar{A}, \bar{B}], \bar{C}] \geqq [\bar{A}, [\bar{B}, \bar{C}]],\end{aligned}$$

and then, by 3.2c),

$$[\bar{A}, [\bar{B}, \bar{C}]] = [[\bar{A}, \bar{B}], \bar{C}],$$

which completes the proof.

Now we can show that *the decomposition* $[\bar{A}, \bar{B}]$ *is the least common covering of the decompositions* $\bar{A}$, $\bar{B}$.

Indeed, the decomposition $[\bar{A}, \bar{B}]$ is, by its construction, a covering of $\bar{A}$ and, by a), also a covering of $\bar{B}$. Therefore it is a common covering of $\bar{A}$ and $\bar{B}$. Let, moreover, $\bar{X}$ be an arbitrary common covering of $\bar{A}$ and $\bar{B}$. Then there holds

$$[\bar{X}, \bar{A}] = \bar{X}, \quad [\bar{X}, \bar{B}] = \bar{X},$$

and, by c),

$$[\bar{X}, [\bar{A}, \bar{B}]] = [[\bar{X}, \bar{A}], \bar{B}] = [\bar{X}, \bar{B}] = \bar{X},$$

which proves that $\bar{X}$ is a covering of $[\bar{A}, \bar{B}]$.

Every common covering of $\bar{A}$ and $\bar{B}$ is, therefore, a covering of $[\bar{A}, \bar{B}]$ so that $\bar{A}, \bar{B}]$ is the least common covering of $\bar{A}$ and $\bar{B}$.

3.5. The greatest common refinement of two decompositions

In 3.3 we saw that every refinement of any common refinement of two decompositions $\bar{A}$, $\bar{B}$ on G is again their refinement. It is important to note that among all the common refinements of two decompositions $\bar{A}$, $\bar{B}$ there is one greatest refinement $\bar{Y}$; greatest in the sense that every common refinement of $\bar{A}$ and $\bar{B}$ is a refinement of $\bar{Y}$. This particular refinement is called the *greatest common refinement of the decompositions* $\bar{A}$ *and* $\bar{B}$ or, briefly, the greatest refinement of $\bar{A}, \bar{B}$.

Suppose $\bar{A}$, $\bar{B}$, $\bar{C}$ are decompositions on G.

We shall now construct a decomposition on G, denoted by $(\bar{A}, \bar{B})$, and show that it is the greatest common refinement of $\bar{A}$ and $\bar{B}$.

The construction: If every element $\bar{a} \in \bar{A}$ is replaced by its decomposition $\bar{a} \sqcap \bar{B}$, we obtain a certain decomposition on G, $(\bar{A}, \bar{B})$.

The decomposition $(\bar{A}, \bar{B})$ is therefore the system of all nonempty intersections of the elements $\bar{a} \in \bar{A}$ and the elements $\bar{b} \in \bar{B}$.

$(\bar{A}, \bar{B})$ is evidently a refinement of $\bar{A}$, i.e.,

$$(\bar{A}, \bar{B}) \leqq \bar{A}.$$

Let us now consider the properties of the decomposition $(\bar{A}, \bar{B})$.

First, *the following relations are simultaneously valid*:

$$(\bar{A}, \bar{B}) = \bar{A}, \quad \bar{A} \leqq \bar{B}.$$

Proof. a) Suppose $(\bar{A}, \bar{B}) = \bar{A}$. Let $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$ stand for arbitrary incident elements. Then there holds:

$$\bar{a} \cap \bar{b} \in \bar{a} \sqcap \bar{B} \subset (\bar{A}, \bar{B}) = \bar{A}$$

and, consequently, $\bar{a} \cap \bar{b} = \bar{a}$. Hence $\bar{a} \subset \bar{b}$ and, moreover, $\bar{A} \leqq \bar{B}$, by 3.2.

b) Suppose $\bar{A} \leqq \bar{B}$. Then every element $\bar{a} \in \bar{A}$ is a part of an element of $\bar{B}$ so that $\bar{a} \sqcap \bar{B}$ consists of a single element $\bar{a}$. Hence $(\bar{A}, \bar{B}) \geqq \bar{A}$. Since there simultaneously holds the relation $\leqq$ (as we have seen above), the equality $(\bar{A}, \bar{B}) = \bar{A}$ is correct (3.2c).

Furthermore, *there holds*:

a) $(\bar{A}, \bar{B}) = (\bar{B}, \bar{A})$;
b) $(\bar{A}, \bar{A}) = A$;
c) $\big(\bar{A}, (B, \bar{C})\big) = \big((\bar{A}, \bar{B}), \bar{C}\big)$.

Proof. a) Every element $\bar{v} \in (\bar{A}, \bar{B})$ is an element of the decomposition $\bar{a} \sqcap \bar{B}$ where $\bar{a}$ stands for a convenient element of $\bar{A}$. So we have $\bar{v} = \bar{a} \cap \bar{b}$ where $\bar{b} \in B$ is a convenient element. Hence: $\bar{v} \in \bar{b} \sqcap \bar{A} \subset (\bar{B}, \bar{A})$. There follows $(\bar{A}, \bar{B}) \subset (\bar{B}, \bar{A})$ and, for analogous reasons, there holds the relation $\supset$ and the proof of the equality a) is complete.

Note that the equality in question also follows from the relations $(\bar{A}, \bar{B}) = \bar{A} \sqcap \bar{B}$ and $\bar{A} \sqcap \bar{B} = \bar{B} \sqcap \bar{A}$ (valid by 2.3).

b) Since $\bar{A} \leqq \bar{A}$, there holds $(\bar{A}, \bar{A}) = \bar{A}$.

c) Let $\bar{v} \in \big(\bar{A}, (B, \bar{C})\big)$, so that $\bar{v} = \bar{a} \cap (\bar{b} \cap \bar{c})$ where $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$, $\bar{c} \in \bar{C}$ are convenient elements. Since $\bar{a} \cap (\bar{b} \cap \bar{c}) = (\bar{a} \cap \bar{b}) \cap \bar{c}$ and, moreover, $(\bar{a} \cap \bar{b}) \cap \bar{c} \in \in \big((\bar{A}, \bar{B}), \bar{C}\big)$, we have $\big(\bar{A}, (\bar{B}, \bar{C})\big) \subset \big((\bar{A}, \bar{B}), \bar{C}\big)$. From this and a) it follows that there also holds the relation $\supset$, which completes the proof of c).

By means of these results we can show that *the decomposition* $(\bar{A}, \bar{B})$ *is the greatest common refinement of the decompositions* $\bar{A}$ *and* $\bar{B}$.

Indeed, by its construction, $(\bar{A}, \bar{B})$ is a refinement of $\bar{A}$ and, by the relation a), it is also a refinement of $\bar{B}$. Let, furthermore, $\bar{Y}$ stand for an arbitrary common refinement of $\bar{A}$ and $\bar{B}$. Then we have:

$$(\bar{Y}, \bar{A}) = \bar{Y}, \quad (\bar{Y}, \bar{B}) = \bar{Y}.$$

Hence, by the relation c), there holds:

$$\big(\bar{Y}, (\bar{A}, \bar{B})\big) = \big((\bar{Y}, \bar{A}), \bar{B}\big) = (\bar{Y}, \bar{B}) = \bar{Y}.$$

We see that $\bar{Y}$ is a refinement of the decomposition $(\bar{A}, \bar{B})$.

Every common refinement of the decompositions A and $\bar{B}$ is, therefore, a refinement of their common refinement $(\bar{A}, \bar{B})$. Thus $(\bar{A}, B)$ is the greatest common refinement of $\bar{A}$ and $\bar{B}$.

3.6. Relations between the least common covering and the greatest common refinement of two decompositions

Let $\bar{A}$ and $\bar{B}$ stand for arbitrary decompositions on the set G.

It is easy to show that *between the least common covering* $[\bar{A}, \bar{B}]$ *and the greatest common refinement* $(\bar{A}, \bar{B})$ *of* $\bar{A}$, $\bar{B}$ *there hold the following equalities*:

$$[\bar{A}, (\bar{A}, \bar{B})] = \bar{A}, \quad (\bar{A}, [\bar{A}, \bar{B}]) = \bar{A}.$$

In fact, these equalities express the relations $\bar{A} \geqq (\bar{A}, \bar{B})$ and $[\bar{A}, \bar{B}] \geqq \bar{A}$ (3.4; 3.5).

3.7. Exercises

1. Deduce, for arbitrary decompositions $\bar{A}$, $\bar{B}$ of the set G, on the ground of $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$, $\mathbf{s}(\bar{a} \sqsubset \bar{B}) = \mathbf{s}(\bar{b} \sqsubset \bar{A}) = \bar{u}$, the relation $\bar{u} \in [\bar{A}, \bar{B}]$.
2. For any decompositions $\bar{A}, \bar{B}, \bar{X}$ on G, where $\bar{X} \geqq \bar{A}$, there holds a) $[\bar{X}, \bar{B}] \geqq [\bar{A}, \bar{B}]$, $(\bar{X}, \bar{B}) \geqq (\bar{A}, \bar{B})$; b) $(\bar{X}, [A, \bar{B}]) \geqq [\bar{A}, (\bar{X}, \bar{B})]$.
3. Find an example to show that, under the assumptions of the previous exercise, the equality in formula b) need not be valid.
4. Two decompositions *in* G always have the least common covering but need not have the greatest common refinement. For the least common coverings of the decompositions $\bar{A}$, $\bar{B}$, $\bar{C}$ in G there hold the formulae 3.4 a) b) c).

4. Special decompositions

In this chapter we shall deal with particular kinds of relations between decompositions in or on the set G.