Magdalena Hykšová

Bolzano’s inheritance research in Bohemia

In: Eduard Fuchs (editor): Mathematics throughout the ages. Contributions from the summer school and seminars on the history of mathematics and from the 10th and 11th Novembertagung on the history and philosophy of mathematics, Holbaek, Denmark, October 28-31, 1999, and Brno, the Czech Republic, November 2-5, 2000. (English). Praha: Prometheus, 2001. pp. 67–91.

Persistent URL: http://dml.cz/dmlcz/401238

BOLZANO'S INHERITANCE RESEARCH IN BOHEMIA

Magdalena Hykšová

1 Bernard Bolzano (1781–1848)

First let us remind some facts on the life of Bernard Bolzano. He was born on October 5, 1781 in Prague, in the family of Bernard Pompeius Bolzano, an educated artwork trader born in Italy, and Maria Cecilia Maurer from a Prague German family. After the education at the piaristic grammar school, Bolzano started to study at the Faculty of Arts of Charles University in Prague (1796). After finishing the basic philosophical studies he devoted the whole year 1799–1800 to further education in higher mathematics, above all with prof. František Josef Gerstner (1756–1832), as well as in philosophy, and was thinking about his future. Finally he decided to study theology, but his interest in mathematics didn't fall away. In 1804 Bolzano took part in the competition for both the professorship of elementary mathematics and the planned post of the teacher of religious science. In both competitions he was assessed the highest, but the professorship of mathematics gained Ladislav Josef Jandera (1776–1857) who had been substituting for diseased Stanislav Vydra (1741–1804), the professor of this subject, for three years, so that it was "convenient" to assign the post to him. And Bolzano became a religion teacher (1805); soon he was graduated and ordained and started lecturing. At the end of the year 1819, in consequence of insidious intrigues, he was suspended for alleged propagation of improper views. Till 1825 he had still been persecuted by clerical dignitaries. Nevertheless, leaving the university helped Bolzano's weak health and allowed him a more intensive scientific research. For examle, in the period 1820 – 1830 an extensive work *Wissenschaftslehre* [19] originated. Since 1825 Bolzano lived outside Prague – in the family of his friend Hoffmann in

This work was supported by the grant LN00A041 of the Ministry of Education of the Czech Republic.

Těchobuz or with the lawyer Pistl in Radič, later with A. Veith in Liběchov or Veith's sister in Jirny near Úvaly. Towards the end of his life he lived with his brother in Celetná street in Prague, where he died of tuberculosis on December 18, 1848.

From Bolzano's mathematical works originated during the period he spent at the university, let us recall [13]-[17] and [21]. Since 1820 Bolzano was working on the mentioned extensive treatise *Wissenschaftslehre* [19] aimed at the foundation and methodology of science in general. It was intended as a basis of an extensive work *Grössenlehre* (theory of quantities), on which Bolzano worked since 1830 and which was rewritten and revised several times but remained unfinished (although some parts were almost ready) and neither during Bolzano's life nor soon after his death it was published. Nowadays we can't than imagine the development of mathematics provided Bolzano didn't dealt with theology so intensively, had more energy for finishing his *Grössenlehre* or, at least, found a continuator who would have understood, finished and published his manuscripts. Bolzano sought such a continuator – finally he invested his hopes to the young Robert Zimmermann (1824–1898) and willed him the mathematical manuscripts. But Zimmermann concentrated only on philosophy and later became a professor of this science (1852 in Prague, 1861 in Vienna). In 1882 he handed Bolzano's mathematical inheritance over to the Vienna Academy of Sciences, which passed it on (1892) to the manuscript department of the Vienna Court Library, later National Library.[1] In this regard, an exception is represented by *Paradoxien des Unendlichen* [22] published only three years after Bolzano's death, thanks to his scholar and collaborator Franz Příhonský (1788–1859). This work is cited for example by George Cantor (1845–1918), the founder of the set theory, in his work [32], and by Richard Dedekind (1831–1916) in the preface to the second edition of his book [36].

2 Bolzano Committee

After Bolzano's death there were various attempts to publish his complete work, but they were not successfull.[2] In the early 1920's Martin Jašek (1879–1945), a secondary school teacher in Pilsen, who had looked into Bolzano's inheritance deposited in Vienna National Library, pointed out some important results concerning the theory of functions contained there, namely in the manuscript *Functionenlehre*. He referred

[1]More details can be found in [101], chap. VII.

[2]More information can be found e.g. in [11], [53], [68].

to it in his papers [54] – [57] and in three lectures presented to the Union of Czech Mathematicians and Physicists.[3] First Jašek turned to Karel Petr (1868–1950), who initiated the lectures, organized by Karel Rychlík (1885–1968) that was soon strongly attracted by this topic.

Jašek's discovery stimulated Czech mathematicians to study and order Bolzano's inheritance. On March 5, 1924 the Bolzano Committee under the Royal Bohemian Society of Sciences (KČSN[4]) was established. Its members were K. Petr – chairman,[5] M. Jašek – secretary, B. Bydžovský, M. Horáček, F. Krejčí, V. Novotný, K. Rychlík, J. Sobotka, J. Vojtěch and K. Vorovka.[6] The aim of the committee was to acquire, unify and publish Bolzano's manuscripts, a part of which was in Prague but the majority in Vienna. It was decided to make photocopies (so called "black snaps" – white writing on the black background) of the manuscripts located in Vienna. The Society supported for this purpose M. Jašek, who stayed in Vienna studying Bolzano's mathematical manuscripts for more than seventeen months and prepared the photocopies of a part of the inheritance, according to his own choice. Nowadays the photocopies are stored in A ASCR in Prague.[7]

At the beginning of the work of Bolzano Committee there was a great optimism. The committee obtained 15 000 crowns from the ministry of

[3]The lectures were read on December 3, 1921, January 14 and Deceber 2, 1922.

[4]In Czech *Královská česká společnost nauk*.

[5]Let us mention that he chose the theme *Bernard Bolzano and His Significance for Mathematics*, later published as [78], for his inaugural lecture on the occasion of ascending to the post of the rector of Charles University for the school year 1925/26; see also [34].

[6]Central Archives of the Academy of Sciences of Czech Republic (below A ASCR), fund KČSN, carton 53, inventory number 292.

[7]Photocopies in A ASCR: *Zu vier besonderen Problemen der Geometrie und Anti–Euklid*: fund KČSN, cart. 92, inv. n. 613, explanatory notes by M. Jašek dated on October 18, 1924, complementary notes by K. Rychlík dated in February, 1951 (in Vienna section VI, volumes 1–5); *Zur Mathematik*: cart. 92, inv. n. 614, undated notes by K. Rychlík (vol. 1 of sec. VII – *Grössenlehre*); *Von der mathematischen Lehrart*: cart. 92, inv. n. 615, notes by M. Jašek dated on October 3, 1924 (sec. VII, second part of vol. 6 that consists of the third version of the manuscript, and several demonstrations of the previous versions contained in vol. 4 and 5); *Zahlenlehre*: cart. 93–94, inv. n. 616–623, notes by M. Jašek dated on October 22 and 29, 1924 and January 29, 1925, and by K. Rychlík dated in March, 1951 (sec. VII, vol. 10 – 3rd version, several demonstrations of the previous versions contained in vol. 8 and 9); *Functionenlehre*: cart. 95, inv. n. 624, notes by M. Jašek dated on September 18, 1924 (the second version and several demonstrations of the first one, both in sec. VII, vol. 12); *Zeit- und Raumlehre*: cart. 95, inv. n. 625, notes by K. Rychlík dated in March 1951 (sec. VII, vol. 14); non-ordered photocopies from the inheritance of M. Jašek (cart. 96, inv. n. 626).

education and asked T. G. Masaryk, the president of Czechoslovakia, for the protectorate – he accepted it, contributed 50 000 crowns and promised a further "material and moral" aid which he keapt.[8] The committee also got "Prioritäts–Herausgeberrechte" from the National Library in Vienna for five years (later it was many times prolonged, till the end of the existence of the committee). The first volume of the series (*Functionenlehre*) was supposed to appear in 1925, the rest in the course of the following five years.[9]

But the initial optimism gradually faded away. A lot of problems emerged, not only financial. For example, the ministry did not allow a further leave to M. Jašek for organizing the Prague inheritance of B. Bolzano, in spite of repeated intercession of KČSN; some dissensions within the committee appeared, too. In short, the publication of Bolzano's manuscripts was delayed. In 1930 KČSN finally started to publish the series *Bernard Bolzano's Schriften*. But till the end of its existence altogether only five volumes were published: 1. *Functionenlehre* [23]; 2. *Zahlentheorie* [24]; 3. *Von dem besten Staate* [25]; 4. *Der Briefwechsel B. Bolzano's mit F. Exner* [26]; 5. *Memoires géométriques* [27].

Towards the end, the constitution of the committee was markedly changed. Its members in 1951 were B. Bydžovský – chairman, J. Vojtěch, K. Rychlík (the only members from the beginning), Q. Vetter, J. B. Kozák, J. Král, V. Laufberger, F. Slavík and V. Vojtíšek.[10] In 1952 the Bolzano Committee was dissolved together with KČSN. At the same time the Czechoslovak Academy of Sciences (CSAS) was established, but the Bolzano Committee was restored only in 1958,[11] under the *First section* (mathematics and physics) of CSAS; in this form it lasted till 1961, then CSAS was reorganized. The members of the committeee were the mathematicians M. Kössler – chairman, O. Borůvka, J. Holubář, V. Kořínek, K. Rychlík and I. Seidlerová.[12] Although the collected edition was not realized, many studies concerning various Bolzano's manuscripts were published and some manuscripts were rewritten, also independently of the existence of the Bolzano Committee. Since 1961 CSAS had been preparing a collected critical edition, notably due

[8] Including the initial amount, the president contributed in total 80 000 crowns and the ministry 32 000 crowns; the account book, A ASCR, fund KČSN, cart. 116, inv. n. 828.

[9] A ASCR, fund KČSN, cart. 53, inv. n. 292.

[10] Ibid.

[11] Nevertheless, for example, in 1955 the department of mathematics and physics of CSAS deputed Karel Rychlík to organize Bolzano's Prague inheritance.

[12] A ASCR, fund I. sekce ČSAV 1952–1961, cart. 15, inv. n. 38.

to the endeavour of K. Večerka, who had rewritten different versions of Vienna mathematical manuscripts (from copies made anew) and started with their comparison and editing. The preserved versions were planned to be summarized in a single critical edition. In 1967 Večerka published Bolzano's *Anti-Euklid* [28] and various studies of various authors appeared again.[13] In 1969 *Bernard Bolzano – Gesamtausgabe* began to be published in *Friedrich Frommann Verlag* in Stuttgart–Bad Cannstatt (editors: Eduard Winter, Jan Berg, Friedrich Kambartel, Jaromír Loužil and Bob van Rootselaar), yet based on simpler edition principles than it was planned by CSAS (see the volume E2/1 of [8]; since the putative last versions are printed without a comparison with the others, it is not such a critical edition as the manuscripts deserve). Till 2000 in total 54 volumes out of about 120 have been published, although the initial intention was to publish the collected papers by 1981 to celebrate Bolzano's bicentenary.[14]

It is beyond the aim of this contribution to describe the whole development of the Bolzano research in Bohemia and to cite all publications concerning Bolzano's mathematical manuscripts. We only mention the jubilee year 1981 when various events devoted to Bernard Bolzano took place in Czechoslovakia, e.g. the international conference *Impact of Bolzano's Epoch on the Development of Science* (Prague, September 7–12, the proceedings [12]), the national conference *Bernard Bolzano – Epoch, Life and Work* (Prague, May 20–21, the proceedings [10]) and the conference of Czech mathematicians *Bernard Bolzano* (Zvíkovské Podhradí, February 9–11, the proceedings [9]). Bolzano was remembered also at two purely scientific conferences with a significant international attendance, namely at Toposym V (Prague, August 24–28, compare [60]) and Equadiff 5 (Bratislava, August 24–28, see [61]) as well as at the statewide congress of the Union of Czechoslovak Mathematicians and Physicists and the Union of Slovak Math. and Phys. (Karlovy Vary, October 12–14, see [87]). Around the year 1981 also a lot of works devoted to Bolzano's life and work were published. Let us cite Czech translations or reprints of [18], [19], [20], [25] and [30], the special issue [11] of *Acta*

[13] As for the above period, we refer e.g. to works of J. Folta [40]–[43], V. Jarník [51], [52], L. Nový [69]–[74], M. Pavlíková [75], K. Rychlík [R50], [R64], [R65], [R66], [R67], [R72], [R83], [R84], [R85] (see the list at pp. 279–283), I. Seidlerová [81]–[86], K. Večerka [92], etc.

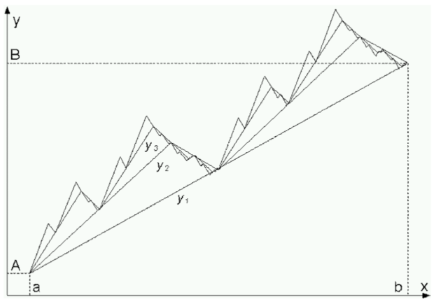
[14] More information including the list of volumes can be found at *The Bernard Bolzano Pages at the FAE*: `http://www.sbg.ac.at/fph/bolzano/` .

On June 28, 1991 the *International Bolzano Society* was established in Salzburg; details can be found at the above internet address.

historiae rerum naturalium necnon technicarum containing Bolzano's mathematical works [13] – [17] together with an interesting introduction by L. Nový and J. Folta, the book [53] containing the English translation of papers [49] – [52] of V. Jarník and an erudite introductory article *Life and Scientific Endeavour of Bernard Bolzano* written by J. Folta, other Folta's papers [45] and [46], the book [7] and the papers (also a little bit older) [2] – [6] of K. Berka, the book [67] of J. Loužil, papers of L. Nový [73] – [74], M. Pavlíková [76], Š. Schwabik [88] and Š. Schwabik together with J. Jarník [47] – [48] and others; also the whole sixth issue of the volume 1981 of the journal *Filosofický časopis* [*Philosophical Journal*] was dedicated to B. Bolzano.

3 *Functionenlehre*

A strong initial stimulus for the mentioned efforts was the discovery of the so-called *Bolzano's function* contained in the manuscript *Functionenlehre*, written before 1834 and intended as a part of the extensive work *Grössenlehre*. First *Bolzano's function* is constructed as an example of a function that is continuous in an interval $[a, b]$, but is not monotone in any subinterval. Later Bolzano shows that the points at which this function has no derivative, are everywhere dense in the interval $[a, b]$. Of course, Bolzano didn't know today terminology and showed that when the function does not have a derivative at two different points, then there is a point between them where again the derivative does not exist. This is equivalent to the density of the mentioned points. Already the fact that it occured to Bolzano at all that such a function might exist, deserves our respect. The fact that he actually succeeded in its construction, is even more admirable.



Bolzano's function is defined as a limit of continuous functions y_1, y_2, $y_3, \ldots$ defined on an interval $[a, b]$. Here y_1 is a function for which $y_1(a) = A$ and $y_1(b) = B$ and which is linear on the interval $[a, b]$:

$$y_1(x) = A + (x - a)\frac{B - A}{b - a}.$$

To define the function y_2, Bolzano divides the interval $[a, b]$ into four subintervals limited by points:

$$a,\ a+\frac{3}{8}(b-a),\ \frac{1}{2}(a+b),\ a+\frac{7}{8}(b-a),\ b.$$

To these points he assigns the values:

$$A,\ A+\frac{5}{8}(B-A),\ A+\frac{1}{2}(A+B),\ B+\frac{1}{8}(B-A),\ B,$$

and y_2 is linear in each of the four subintervals. The function y_3 is defined analogously, besides the fact that each of the four subintervals is considered instead of the interval $[a,b]$, etc. Bolzano's proof of the continuity of the resulting function is not fully correct. It is based on the erroneous assertion that the limit of a sequence of continuous functions is always a continuous function (it becomes true, however, if we require for example uniform convergence).

The first lecture of M. Jašek reporting on *Functionenlehre* was given on December 3, 1921. Already on February 3, 1922 Karel Rychlík presented to KČSN his treatise [R19][15] where the correct proof of the continuity of Bolzano's function was given as well as the proof of the assertion that this function does not have a derivative at any point of the interval (a,b) (finite nor infinite). The same assertion was proved by Vojtěch Jarník (1897 – 1970) at the same time but in a different way in his paper [49]. Both Jarník and Rychlík knew about the work of the other. Giving a reference to Rychlík's paper, Jarník did not prove the continuity of Bolzano's function; on the other hand, Rychlík cited the work of Jarník (an idea of another way to the same partial result).[16]

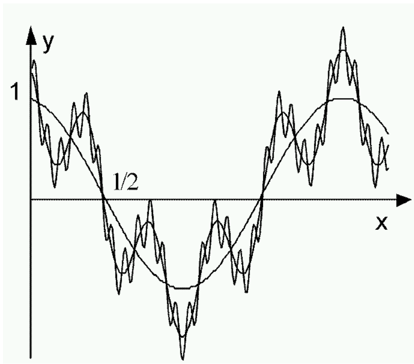
For a deeper understanding the extraordinarity of Bolzano's function let us mention some facts on the history of continuous nowhere differentiable functions. Keep in mind that Bolzano's manuscript had been written before the year 1834.

On July 18, 1872 Karl Weierstrass lectured in the Royal Academy of Sciences in Berlin on a function which is continuous in the domain of all real numbers but has a derivative an no real point. This example is defined as follows:

$$f(x)=\sum_{n=1}^{\infty} a^n \cos(\pi b^n x),\ 0<a<1;\ ab>1+\frac{3}{2}\pi.$$

[15] References marked [R...] refer to the list of publications of K. Rychlík at pp. 279–283.

[16] For Bolzano's function see also papers [58] and [59] of G. Kowalewski and the paper [31] of V. F. Bržečka (born in Volyně; his papers published in Germany are signed Břečka – this fact led to the conjecture, expressed by Rychlík in [R86], that Bržečka might have been of a Czech origin).



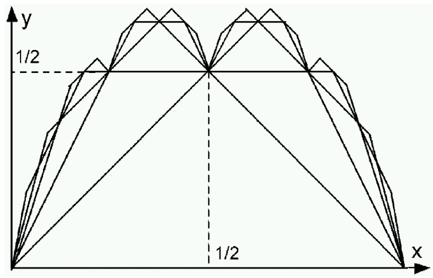
Three approximations of the function for $a = 1/2,\ b = 5$ can be seen on the figure. Weierstrass' function was published in 1875 by P. du Bois-Reymond [79], the student of Karl Weierstrass. Of course, du Bois-Reymond quoted Weierstrass' name. K. Weierstrass himself published his example only in 1880.

For a long time Weierstrass' example was beeing considered as the first example of the continuous nowhere differentiable function. Since then many mathematicians were interested in this topic, for example G. Darboux [35], V. Dini [37], M. Lerch [66] and others.

In 1890 the example constructed by Ch. Cellèrier already in 1860 was posthumously published in the paper [33]. Cellèrier's function is defined alike the Weierstrass' one:

$$f(x) = \sum_{n=0}^{n} b^{-n} \sin(\pi b^n x);\ b > 1000.$$

The fact that it was already written in 1860 caused a real sensation. Hence we can imagine the sensation caused by Jašek's discovery of Bolzano's function, which was constructed before the year 1834.



Let us add one more remark. In 1903 the function constructed by T. Takagi was published [91]:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Delta(2^n x),$$

$\Delta(x) = dist(x, \mathbb{Z})$. One of its modifications is now known as the so–called *van der Waerden's example.*

It was published in 1930 by B. L. van der Waerden [100] and it is generally considered to be the easiest example of a continuous function without a derivative at any point of its domain. In this context, see also the section 2.1 of the paper *Life and Work of Karel Rychlík* in these proceedings, where examples given by K. Petr and K. Rychlík are described.

In 1930 *Functionenlehre* was finally published.[17] The book is provided with careful, detailed notes by Karel Rychlík and with an interesting foreword written by Karel Petr. We shall emphasize that the main significance of the manuscript does not lie in the described example but in a systematic exposition of the theory of continuity and derivative of functions of one variable. Let us close the section devoted to Bolzano's *Functionenlehre* with words of V. Jarník:

> *It is such an extraordinary work that we cannot but regret that, as an unpublished manuscript, it had not the opportunity to influence the development of mathematics in his own time. In Bolzano's days ... the theory of functions was already considerably developed, its main concepts, however, lacked sharp contours and the principal theorems were not upheld by exact proofs. And it is in the very foundations of the theory of functions that Bolzano's Functionenlehre represents a virtual milestone, unfortunately a milestone overgrown with the moss of ignorance.*
>
> *Among Bolzano's contemporaries, only Gauss, Abel and Cauchy manifested the same sense for the proper construction of the foundations of the theory of functions. Two of them, Gauss and Abel, presented masterpieces of exact mathematical methods but did not deal with these fundamental problems systematically. The last of them, Cauchy, in his works "Cours d'Analyse" (1821), "Résumé des leçons ... sur le Calcul Infinitésimal" (1823), "Leçons sur le Calcul différentiel" (1829) based the main branches of the theory of functions ... on firm foundations (or let us say more carefully on firmer foundations) in a systematic way. However, Bolzano goes in his efforts even beyond Cauchy's achievements. Cauchy usually contented himself with building the foundations to a level necessary for his further deductions; unlike him, Bolzano was more of a philosopher, interested in the fundamental problems of mathematics. We shall see later how rigorously Bolzano introduces his definitions, how critically he dissects his concepts, with what deep interest and thoroughness he discusses all logically possible cases regard-*

[17]In addition to the papers mentioned above, let us cite the papers [50]–[52] of V. Jarník (English translation in [53]) and the contribution of K. Rychlík at the *International Congress of Mathematicians* in Bologna, published as [R28].

> *less of their greater or lesser importance for concrete mathematical problems.* [18]

4 *Zahlenlehre*

The second volume of *Bernard Bolzano's Schriften* was published in 1931 under the title *Zahlentheorie* [24] and again it was edited and provided with notes by K. Rychlík. The book contains a part of the manuscript *Zahlenlehre*, another component of *Grössenlehre*. Precisely the part, entitled by Bolzano *Zweyter Abschnitt: Verhältniss der Theilbarkeit unter den Zahlen*, of the section *Hauptstück. Besondere Verhältnisse zwischen den Zahlen*. The manuscript treats elementary properties of integers, being called by Bolzano *wirkliche Zahlen* – true numbers.

Another part of *Zahlenlehre*, called by Bolzano *Unendliche Grössenbegriffe (Grössenausdrücke)*, was published in 1962 in [R84] by K. Rychlík, who had referred to it also in his papers [R50], [R64], [R65] and [R83] and who named this part *Theorie der reellen Zahlen* (TRZ). As it was concluded by E. Winter from the letters written by Bolzano to Michael Josef Fesl (1788 – 1863) and F. Příhonský,[19] Bolzano worked at the said manuscript mainly in 1830–35, in 1840 he came back to it again, but he did not finish it. As for the question, why only this fragment of the whole *Zahlenlehre* was chosen for publication, the answer can be found in Rychlík's foreword:

> *Die bisher erschienen Schriften von B. Bolzano enthalten eine ganze Reihe von Sätzen über reelle Zahlen. Es sind dies seine ersten Arbeiten aus der Analysis: "Der binomische Lehrsatz ... "* [15] *und "Rein analytischer Beweis ... "* [16] *und besonders die "Functionenlehre"* [23] ...
>
> *In der TRZ versucht Bolzano eine Arithmetisierung der Theorie der reellen Zahlen durchzuführen, die viel später auf drei verschiedene Weisen von Weierstrass (1860), Méray (1869) und G. Cantor (1872) und endlich von Dedekind (1872) entwickelt wurde. Bolzano kann mit vollem Recht als Vorläufer dieser Mathematiker betrachtet werden: Der Gedanke der rein arithmetischen Begründung der reellen Zahlen tritt nämlich bei ihm ganz klar hervor, obwohl seine Ausführungen*

[18] [53], pp. 43–44.

[19] See [101], chap. VII (particularly p. 214), [103], letters 15, 41, 43, 44, 107, and Rychlík's introduction to [R84], p. 13.

> *nicht als ganz stichhaltig betrachtet werden können. Dann bringt Bolzano die Entwicklung der reellen Zahlen in die sogenannten "Cantorschen Reihen" und beweist weitere Sätze aus der Theorie der reellen Zahlen: die Trichotomie der Beziehungen "größer als" und "kleiner als", den Satz von Archimedes, den Satz, daß die Menge der reellen Zahlen überall dicht ist, den Satz von Cauchy-Bolzano, den Satz von Bolzano-Weierstraß und endlich einen Satz, der an den Satz von Dedekind erinnert. Diese Entwicklungen könnten ohne wesentliche Veränderungen zu der heute verlangten Schärfe ausgefeilt werden. Tatsächlich hätte diese Handschrift, wäre sie selbst so wie sie ist veröffentlicht worden, den Fortschritt der Mathematik beschleunigen können.*[20]

First we mention the basic concepts of Bolzano's theory. An *infinite number expression* (*unendlicher Größenausdruck*) denotes an expression, where an infinite number of operations (addition, subtraction, multiplication and division) with natural numbers occurs. *Measurable* (*meßbar*) is an expression S, such that for each positive integer q there exists an integer p such that

$$S = \frac{p}{q} + P_1; \qquad S = \frac{p+1}{q} - P_2, \tag{1}$$

where P_1 (resp. P_2) is a non-negative (resp. positive) number expression,[21] i.e.

$$\frac{p}{q} \leq S < \frac{p+1}{q}; \tag{2}$$

the fraction p/q is called a *measuring fraction* (*messender Bruch*). An *infinitely small positive number* (*unendlich kleine positive Zahl*) S has all its measuring fractions equal to zero, $-S$ is called *infinitely small negative number.* Measurable expressions or numbers A, B are identified, if they yield the same results with respect to measuring: for each positive integer q there exists an integer p such that

$$A = \frac{p}{q} + P_1 = \frac{p+1}{q} - P_2; \qquad B = \frac{p}{q} + P_3 = \frac{p+1}{q} - P_4, \tag{3}$$

where $P_1,\ P_3$ ($P_2,\ P_4$) are non-negative (positive) expressions.

[20] [R84], p. 5.

[21] Bolzano writes: ... *ein Paar durchaus positive Zahlenausdrücke oder das erstere zuweilen auch eine blosse Null bedeutet.*

Besides the foreword, the book [R84] is provided with Rychlík's introduction, concluding notes and the survey of the history of real numbers, and it is equipped with a foreword written by Ladislav Rieger (1916–1963). In his notes Rychlík gives a possible interpretation of Bolzano's theory, which is not completely correct, where he tries to preserve as most as possible. He assigns the following concepts, using Cantor's theory of real numbers:

in Bolzano's theory:	in Rychlík's interpretation:
infinite number expression	*sequence of rational numbers*
measurable number expression	*convergent seq. of rational numbers*
infinitely small number	*null sequence*
equality of measurable numbers	*equivalence of convergent sequences*

L. Rieger outlined in his foreword another possible interpretation of Bolzano's infinite number expression: as symbols for effectively described, infinite computational procedures on rational numbers.

The publication of the book [R84] stirred up a discussion on several levels, which is worth a brief note. First, the published Bolzano's manuscript is not complete. This rebuke was expressed e.g. by J. Berg in the preface to *Reine Zahlenlehre* ([29]; it includes also TRZ), J. Folta in the review of [R84][22] or B. van Rootselaar in the paper [80]. Although TRZ gives sense to many concepts and assertions used in various Bolzano's works (to the ones cited above we can also add e.g. *Paradoxien des Unendlichen* [22]), there are still references to the previous part (first 77 sheets) of *Zahlenlehre*. As it has been mentioned, Rychlík chose just TRZ, because it was so interesting, showing how strikingly Bolzano was ahead of his time – as in many other cases. And compared with TRZ, the previous sheets treating rational numbers are not so "revolutionary".[23]

Nevertheless, still there remained some gaps. As Rychlík himself writes in the introduction, he omitted some comments in margins and several pages for a bad legibility (although he was very well experienced in reading Bolzano's scratchy writing). Similarly Rychlík's notes were

[22] ČPM **89**(1964), pp. 115–116.

[23] It should be added that the publication of Bolzano's manuscripts was strongly influenced by the way in which Jašek had organized and sorted the photocopies. Specifically *Zahlenlehre* was divided into eight separate segments I–VIII (TRZ is the second of them, Zahlentheorie [24] the fourth). The view that TRZ was not chosen only accidentaly can be also supported by the fact, that by 1958 Rychlík had already rewritten both parts I and II and was working on III (according to the record of the meeting of the Bolzano Committee held in October 1958; A ASCR, fund I. sekce ČSAV 1952–1961, cart. 15, inv. n. 38).

regarded somewhat incomplete for they did not give a precise reference to Bolzano's failures mentioned in the epilogue, although they sometimes supported Bolzano's reasoning.

The second respect was a general one: unsystematic publication of the inheritance (see e.g. Folta's review, here footnote 22). Undoubtedly this had been the most serious problem since the twenties. Nevertheless, in this case and from Rychlík's point of view, the systematic and critical publication of the whole inheritance was beyond the scope of a single person, even an experienced one.

The third aspect of the discussion concerned Rychlík's interpretation. In 1963 B. van Rootselaar handed in his paper [80] for publication in *Archive for History of Exact Sciences.* In the introduction we can read:

> *First of all I should like to emphasize that I completely agree with Rychlík when he says that it is justified to consider Bolzano as a forerunner of Weierstrass, Méray, Cantor and Dedekind because the idea of a purely arithmetical foundation of the theory is not quite correct ... Concerning the last statement, however, I strongly differ, and I should say that Bolzano's elaboration is quite incorrect.* [24]

Van Rootselaar regards Rychlík's interpretation as too broad and narrows the exposition of a measurable number:

> *A measurable number expression S is an infinite sequence of rational numbers $S = \{s_n\}$ such that to any natural number q there exists an integer $p_q(S)$ such that for all n we have $s_n = p_q(S)/q + P_{q,1,n} = (p_q(S) + 1)/q - P_{q,2,n}$ where either $P_{q,1,n} = 0$ for all n, or there exists an n_0 such that $P_{q,1,n} > 0$ for $n > n_0$, and there exists an n_1 such that $P_{q,2,n} > 0$ for $n > n_1$.* [25]

He remarks that it may be weakened by requiring only $P_{q,1,n} \geq 0$ for $n > n_0$. Under this interpretation e.g. Bolzano's assertion, that the sum of two measurable numbers is again a measurable number, fails. Van Rootselaar gives an example (used in a little bit different context in Rychlík's note in [R84], p. 99):

$$a_n = \frac{1}{n}; \qquad b_{2n-1} = -\frac{1}{2n}, \quad b_{2n} = -\frac{1}{2n-1}; \tag{4}$$

[24] [80], p. 168.
[25] Ibid, p. 173.

the sequences $A = \{a_n\},\ B = \{b_n\}$ represent infinite expressions

$$A = \frac{1}{1+1+1+\cdots \text{ in inf.}}, \qquad B = \frac{1}{-2+1-3+1-3+\cdots \text{ in inf.}}.$$

The sequence $\{c_n\} = \{a_n + b_n\}$, where

$$c_{2n-1} = \frac{1}{2n(2n-1)}, \qquad c_{2n} = \frac{-1}{2n(2n-1)}$$

is not a measurable number under the interpretation considered.

Another contradiction can be found in the assertion that if A and J are measurable and J infinitely small, then $A \pm J$ is measurable with the same measuring fractions as A. It suffices to consider

$$A = 1, \qquad J = \frac{1}{1+1+1+\cdots \text{ in inf.}}. \tag{5}$$

In the conclusion of the detailed analysis of the theory van Rootselaar writes:

> *Our interpretation permits us to represent all of Bolzano's notions and all his theorems. Some of these theorems are converted into incorrect ones, and these are precisely those to which counterexamples can be given within Bolzano's own theory. From this property of the interpretation may be judged its adequacy.*
>
> *Those theorems of Bolzano's theory which are converted into incorrect theorems by the interpretation are his most interesting and indispensable theorems. From this may be judged the value of Bolzano's theory.*
>
> *Rychlík proposed a corrected version of Bolzano's theory (viz Cantor's theory) which converts Bolzano's incorrect theorems into correct ones but does not account for most of the correct theorems of Bolzano's theory, in particular those on measuring fractions.* [26]

As a reaction to van Rootselaar's paper, the article [64] of D. Laugwitz appeared in the same journal.

> *Ich werde zeigen, daß Bolzanos Fehler im wesentlichen auf eine einzige unzulängliche Definition zurückgehen, nämlich*

[26]Ibid, p. 179.

> *auf seine Definition der unendlich kleinen Zahlen, welche zu eng ist. Nach einer vorsichtigen Abänderung dieser Definition, welche in Übereinstimmung mit Bolzano's anderweitig geäußerten Meinungen stehen dürfte, läßt sich dann Bolzano's Theorie widerspruchsfrei aufbauen, wenn man die auch von Rychlík und besonders von van Rootselaar zugrundgelegte Interpretation der unendlichen Größenausdrücke als Folgen rationaler Zahlen verwendet. Bolzano's Theorie geht dann in die von C. Schmieden und dem Verfasser vor Bekanntwerden des Bolzano-Manuskripts* [TRZ] *angegebene erweiterte Analysis über* [62], *welche sich neuerdings auch für die Bewältigung moderner Begriffsbildungen der Analysis (Distributionen) als brauchbar erwiesen hat* [63].[27]

In short, the point is that Laugwitz defines the infinitely small number as an expression C such that for each natural q we have

$$-\frac{1}{q} < C < \frac{1}{q}, \tag{6}$$

i.e. in the sequence interpretation: the corresponding sequence is a null sequence, and the inequality (2) is slightly modified:

$$\frac{p}{q} < S \; < \frac{p+2}{q} \tag{7}$$

(it is necessary for the case that – in a present sense – the corresponding sequence converges to a rational number; for the uniqueness the greatest possible p is chosen). Now all the incorrect assertions become true. Laugwitz also points out the passage of *Paradoxien des Unendlichen* [22] (see pp. 59–60), which shows that Bolzano himself was later aware of the failure of the assertion about $A \pm J$ mentioned above.

Now we leap to 1981 and mention the lecture of D. R. Kurepa at the conference on topology *Toposym V* held in Prague, which was published one year later as [60]. This detailed analysis discusses various aspects showing how fruitful and farreaching Bolzano's theory was. It is concluded with the following words.

> *So, on this day August 24, 1981 when we are commemorating the 200-th anniversary of birth of Bernard Bolzano in his birth town Praha we can frankly say that Bolzano's contribution around his approach to real numbers was tremendously*

[27] [64], p. 399.

> *fruitful and that standard mathematics, non standard mathematics, constructive mathematics and applications are firmly established, greatly in the spirit forecasted by Bolzano; Bolzano's critical minds would surely agree with such results.* [28]

The paper [60] is followed by the article [65] written by D. Laugwitz, which contains some supplements to Kurepa's lecture. While Kurepa comes out of Rychlík's book [R84], Laugwitz cites the new Berg's edition [29] from 1976, which brings a great surprise to us. Laugwitz writes:

> *In* [64] *I indicated modifications of Bolzano's definitions, regarding the partial publication* [R84]. *It was a surprise to see from* [29] *that Bolzano himself had discovered the difficulties, and he proposed modifications on sheets in his own shorthand writing which was deciphered by Jan Berg, who reads* [[29], p. 130]*: "A und B heißen hier einander gleich in der Hinsicht, daß beide dieselben Beschaffenheiten haben, daß ihr Unterschied … absolut betrachtet die gleichen Merkmale bei dem Geschäfte des Messens darbietet wie Null." … In other words, $A \approx B$ iff $|A - B|$ is an infinitesimal. All of Bolzano's theorems become true with this definition. He proves that the equivalence classes of measurable expressions, which are called measurable numbers, have the properties of an ordered field. He also gives a proof of what we now call completeness …*
>
> *At the end of the manuscript* [[29], p. 168] *there is a remark which has been read by Berg as follows: "Zur Lehre von den meßbaren Zahlen. Sollte die Lehre von den meßbaren Zahlen nicht vielleicht vereinfacht werden können, wenn man die Erklärung derselben so erreicht, daß A meßbar heißt, wenn man 2 Gleichungen von der Form*
>
> $$A = \frac{p}{q} + P = \frac{p+n}{q} - P \tag{8}$$
>
> *hat, wo bei einerlei n, q ins Unendliche zunehmen kann?" Actually, the capital P is always standing for a positive number, such that the equations can be translated into*
>
> $$\frac{p}{q} < A < \frac{p+n}{q}. \tag{9}$$

[28] [60], pp. 664–665.

> *As was shown in [64], $n = 1$ will suffice if the "limit" of the sequence belonging to A is irrational, and $n = 2$ in the rational case.* [29]

Although one can regret that the above mentioned notes of Bolzano were not reproduced in Rychlík's book [R84], still it is necessary to keep in mind that it declassified Bolzano's theory of real numbers much sooner than the more comprehensive Berg's edition, and by stirring up a fertile discussion it stimulated a strong interest in Bolzano's manuscripts – not only in TRZ.

5 Bolzano and Cauchy

We will not continue in the discussion of particular manuscripts. Our last remark concerns the possibility of a personal meeting of Bernard Bolzano and Augustin-Louis Cauchy, who was appointed tutor in mathematics to the young duke of Bordeaux (later Henry of Chambord) by the banished king of France, Charles X., and stayed in Prague in 1833–36. Bolzano was living with Mr and Mrs Hoffmann in Těchobuz at that time.

In 1928 Ruth (born Rammler, comming from Prague) and Dirk J. Struiks published their conjecture in the paper [90]. They get to the inference that the meeting was implausible. The following citation illustrates their main argument.

> *It is also highly improbable that Cauchy, compelled by his position to be extremely careful not to offend the imperial and royal authorities of Austria, would have sought a personal connection with a man like the compromised Bolzano.*
>
> *Besides this Cauchy had already completed long before, as had Bolzano, his works on the exact foundation of the theory of real functions . . . Bolzano did not publish any pure mathematics after 1817, and was, about 1835, probably occupied by philosophical questions concerning theology, or perhaps with axiomatic problems in mechanics . . .* [30]

On the other hand, in 1957 P. Funk emphasizes in his review of E. Winter's book *Der böhmische Vormärz in Briefen B. Bolzanos an F. Příhonský (1924–1848)* [103] the passage of Bolzano's letter to Příhonský that shows, how much Bolzano respected Cauchy and how much he desired to meet him personally:

[29] [65], pp. 669–670.
[30] [90], p. 365.

> *Die Nachricht von der Anwesenheit Cauchys in Prag ist für mich ungemein interessant. Er ist unter allen jetzt lebenden Mathematikern derjenige, den ich am meisten schätze und dem ich mich am verwandtesten fühle; seinem bestens zu empfehlen und zu sagen, daß ich jetzt gleich nach Prag gereist wäre, um seine persönliche Bekanntschaft zu machen, wenn ich – nach dem, was Sie mir von seiner Anstellung sagen, nicht sicher hoffen könnte, daß ich ihn Ende September, wo ich Sie begleiten will, noch antreffen werde ...* [31]

Obviously, this argument is not completely satisfactory. But in 1962 I. Seidlerová pointed out in [83] and [85] an interesting document: a letter of Bolzano to Fesl in Vienna dated on December 18, 1843, which was together with the rest of their correspondence deposited in the Literary Archives in Prague. From this letter it is possible to conclude that Bolzano really met Cauchy; the same opinion was held by E. Winter, who was working on the publication of the mentioned correspondence [104], and K. Rychlík, who dealt with this question in the paper [R85]. Let us close this contribution with the citation of the considered letter.

> *Cauchy, der Mathematiker, war – wie Ihnen vielleicht bekannt sein dürfte – in den Jahren 1834 und 35, im Gefolge des 10. Karls oder des 5. Heinrichs in Prag, wo wir uns einigemal besuchten während der wenigen Tage, die ich in jener Zeit (zu Ostern und im Herbste) in Prag zuzubringen pflegte ...* [32]

It seems to be clear that Bolzano himself gives an answer to the "problem" of his personal meeting with A. L. Cauchy.

6 References

The abbreviations of magazines and edition series used bellow:

Acta = *Acta historiae rerum naturalium necnon technicarum*; **Archive** = *Archive for History of Exact Sciences*; **ČMŽ** = *Čechoslovackij matematičeskij žurnal – Czechoslovak Mathematical Journal*; **ČPM(F)** = *Časopis pro pěstování mathematiky (a fysiky)*; **DVT** = *Dějiny věd a techniky*; **Pokroky** = *Pokroky matematiky, fyziky a astronomie*; **Sborník** = *Sborník pro dějiny přírodních věd a techniky.*

[31] Monatshefte für Mathematik 61, 1957, p. 251.
[32] [83], p. 225.

[1] Berg, J., *Bolzano's Logic*, Almqvist and Wiksell, Stockholm, 1962.

[2] Berka, K., *K současnému stavu Bolzanovského bádání* [*On the Present State of Bolzano Research*], Filosofický časopis **24**(1976), 705–720 [Czech].

[3] Berka, K., *Bernard Bolzano*, Filosofický časopis **26**(1978), 742–760 [Czech].

[4] Berka, K., Prokešová, B., *Bolzanovy boje o vydání a uznání Vědosloví* [*Bolzano's Fights for Publication and Appreciation of Wissenschaftslehre*], Filosofický časopis **27**(1979), 697–725 [Czech].

[5] Berka, K., *Bolzanova filozofie matematiky* [*Bolzano's Philosophy of Mathematics*], Filosofický časopis **28**(1980), 559–589 [Czech].

[6] Berka, K., *Bernard Bolzano – předchůdce moderní logiky* [*B. B. – Precursor of Modern Logic*], DVT **81**(1981), 205–216 [Czech, English and Russian summary].

[7] Berka, K., *Bernard Bolzano*, Horizont, Prague, 1981.

[8] *Bernard Bolzano – Gesamtausgabe*, Friedrich Frommann Verlag, Stuttgart-Bad Cannstatt, since 1969.

[9] *Bernard Bolzano – konference českých matematiků*, Proceedings of the conference of Czech mathematicians in Zvíkovské Podhradí, Prague, 1981.

[10] *Bernard Bolzano 1781 – 1848*, Proceedings of the conference organized by Charles University, Prague, 1981 [edited by M. Jauris].

[11] *Bernard Bolzano – Early Mathematical Works*, Acta **12**(1981) (special issue) [edited by J. Folta and L. Nový].

[12] *Bernard Bolzano – Impact of Bolzano's Epoch on the Development of Science*, Acta **13**(1982) (special issue) [edited by L. Nový and col.].

[13] Bolzano, B., *Betrachtungen über einige Gegenstände der Elementar-Geometrie*, Prague, 1804; also in [11].

[14] Bolzano, B., *Beyträge zu einer begründeteren Darstellung der Mathematik. Erste Lieferung*, Prague, 1810; also in [11].

[15] Bolzano, B., *Der binomische Lehrsatz, und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrössen dienen, genauer als bisher erwiesen*, Prague, 1816; also in [11].

[16] Bolzano, B., *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*, Prague, 1817; also in [11]; Czech translation and notes by F. J. Studnička: *Ryze analytický důkaz poučky, že mezi dvěma hodnotami, jež poskytují opačně označené výsledky, leží nejméně jeden reálný kořen rovnice*, ČPMF **11**(1881–82), pp. 1–38; also separately at the expense of the translator, Prague, 1881.

[17] Bolzano, B., *Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahme des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt*, Leipzig, 1817; also in [11].

[18] Bolzano, B., *Lebensbeschreibung des Dr. B. Bolzano*, Sulzbach, 1836 [edited by J. M. Fesl]; Vienna, 1875; Czech translations according to the first edition: *Autobiografie* by V. Stoklasa, Prague, 1913; *Vlastní životopis* by M. Pavlíková, Odeon, Prague, 1981 [provided also with explanatory notes, editorial note and a closing study by M. Pavlíková].

[19] Bolzano, B., *O logice* [*Über die Logik*], the Czech translation of the German manuscript by F. Šír, Krok **2**(1831); Památník národního písemnictví, Prague, 1981.

[20] Bolzano, B., *Wissenschaftslehre. Versuch einer ausführlichen und grösstentheils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter*, Sulzbach, 1837; an anthology in Czech: *Vědosloví*, edited by K. Berka, Academia, Prague, 1981.

[21] Bolzano, B., *Versuch einer Objectiven Begründung der Lehre von den drei Dimensionen des Raumes*, Prague, 1843 [written about 1815].

[22] Bolzano, B., *Paradoxien des Unendlichen*, Leipzig, 1851 [edited by Fr. Příhonský]; Czech translation by O. Zich: *Paradoxy nekonečna*, Nakladatelství ČSAV, Prague, 1963 [notes by O. Zich, the foreword by A. Kolman].

[23] Bolzano, B., *Functionenlehre*, the series *Spisy Bernarda Bolzana – Bernard Bolzano's Schriften*, vol. 1, KČSN, Prague, 1930 [edited and provided with notes by K. Rychlík, the foreword written by K. Petr].

[24] Bolzano, B., *Zahlentheorie*, ibid, vol. 2, KČSN, Prague, 1931 [edited and provided with notes by K. Rychlík].

[25] Bolzano, B., *Von dem besten Staate*, ibid, vol. 3, KČSN, Prague, 1932 [edited and *Einführende Betrachtungen* by A. Kowalewski]; Czech translations named *O nejlepším státě*: M. Jašek, on translator's expense, Prague, 1934 and Melantrich, Prague, 1949 [the foreword by L. Svoboda]; V. Bláha, Vyšehrad, Prague, 1952 [the foreword by J. Plojhar] and Mladá Fronta, Prague, 1981 [the epilogue by J. Loužil].

[26] Bolzano, B., *Der Briefwechsel B. Bolzano's mit F. Exner*, ibid, vol. 4, KČSN, Prague, 1935 [edited and provided with the introduction and notes by E. Winter].

[27] Bolzano, B., *Memoires géométriques*, ibid, vol. 5, KČSN, Prague, 1948 [edited and provided with notes by J. Vojtěch].

[28] Bolzano, B., *Anti-Euklid*, Sborník **11**(1967) [edited and provided with the introduction by K. Večerka].

[29] Bolzano, B., *Reine Zahlenlehre*, vol. 2A/7 of [8] 1976 [edited and provided with the introduction by J. Berg].

[30] Bolzano, B., *Výbor z filozofických spisů* [*An Anthology of Philosophical Treatises*], Svoboda, Prague, 1981 [edited by J. Černý and J. Loužil, translation from German by J. Loužil].

[31] Bržečka, V. F., *O funkcii Bol'cano*, Uspechi matematičeskich nauk **4(30)**(1949), no. 2, 15–21 [Russian].

[32] Cantor, G., *Über unendliche lineare Punktmannigfaltigkeiten IV*, Mathematische Annalen **21**(1883), 51–58, 545–591.

[33] Cellérier, Ch., *Note sur les principes fondamentaux de l'analyse*, Bulletin des sciences mathématiques **14**(1890), 142–160.

[34] Crkalová, Z., *Karel Petr a Bernard Bolzano*, in: *IX. Seminář o filozofických otázkách matematiky a fyziky*, Prometheus, Velké Meziříčí, 2000 [Czech].

[35] Darboux, G., *Mémoire sur les fonctions discontinues*, Annales de l'Ecole normale (2) IV(1875) 57–112.

[36] Dedekind, R., *Was sind und was sollen die Zahlen?*, Harzburg, 1887; second ed. 1893; third ed. Braunschweig, 1911.

[37] Dini, V., *Fondamenti per la teorica delle funzioni di variabili reali*, Pisa, 1878.

[38] Folta, J., Nový, L., *Vytváření předpokladů širšího vědeckého rozvoje (Od devadesátých let 18. století do šedesátých let 19. století* [*Formation of Conditions of a Wider Scientific Development (From the Nineties of the 18th Century to the Sixties of the 19th Century*], in: *Dějiny exaktních věd v českých zemích do konce 19. století*, Prague, 1961, 133–161 [Czech].

[39] Folta, J., *N. I. Lobačevskij a B. Bolzano*, Pokroky **6**(1961), 283–284 [Czech].

[40] Folta, J., *Základy geometrie na rozhraní 18. a 19. století a Bolzanovo dílo* [The Foundations of Geometry at the Turn of the 18th and 19th Centuries], Zprávy **3**(1966), 14–18 [Czech].

[41] Folta, J., *Bernard Bolzano and the Foundations of Geometry*, Acta, special issue **2**(1966), 75–104.

[42] Folta, J., *Základy geometrie v pracích českých matematiků 19. století* [*The Foundations of Geometry in Works of Czech Mathematicians in the 19th Century*], Sborník **11**(1967), 169–202 [Czech, English summary].

[43] Folta, J., *The Foundations of Geometry at the Turn of the 18th and 19th Centuries and Bolzano's Contribution*, Actes du XI^{e} Congrès International d'Histoire des Sciences 3, Warszawa, 1968, 225–228.

[44] Folta, J., *Zamyšlení nad bolzanovskými výročími* [*A Reflection on Bolzano Anniversaries*], Pokroky **26**(1981), 241–248 [Czech].

[45] Folta, J., *Bolzanova prvotina ve vývoji elementární geometrie počátku 19. století* [*Bolzano's First Fruit in the Development of Elementary Geometry at the Beginning of 19th Century*], DVT **81**(1981), 228–236 [Czech, English and Russian summary].

[46] Folta, J., *Život a vědecké snahy Bernarda Bolzana* [*Life and Scientific Endeavour of B. B.*], Matematika a fyzika ve škole **12**(1981–82), 85–104 [Czech].

[47] Jarník, J., Schwabik, Š., *Skončil jubilejní rok Bernarda Bolzana* [*The jubilee year of B. B. has ended*], ČPM **107**(1982), 196–202 [Czech].

[48] Jarník, J., Schwabik, Š., *1981 – The Bicentenary of Bernard Bolzano*, ČMŽ **32(107)**(1982), 329–333 [the English variant of the most of [47]].

[49] Jarník, V., *O funkci Bolzanově* [*On Bolzano's Function*], ČPMF, **51**(1922), 248–264 [Czech].

[50] Jarník, V., *Bolzanova "Functionenlehre"* [*Bolzano's ...*], ČPMF, **60**(1931), 240–262 [Czech].

[51] Jarník, V., *Bernard Bolzano a základy matematické analýzy* [*B. B. and the Foundations of Mathematical Analysis*], in: *Zdeňku Nejedlému Československá akademie věd*, Nakladatelství ČSAV, Prague, 1953, 450–458 [Czech].

[52] Jarník, V., *Bernard Bolzano (October 5, 1781 – December 18, 1848)*, ČMŽ, **2(86)**(1961), 485–489.

[53] Jarník, V., *Bolzano and the Foundations of Mathematical Analysis*, Society of Czechoslovak Mathematicians and Physicists, Prague, 1981 [English, the same publication also in Czech under the tittle *Bolzano a základy matematické analýzy*].

[54] Jašek, M., *Aus dem handschriftlichen Nachlass B. Bolzano's*, Věstník KČSN 1920–21, Nr. 1, 1–32.

[55] Jašek, M., *Funkce Bolzanova* [*Bolzano's Function*], ČPMF **51**(1922), 69–76 [Czech].

[56] Jašek, M., *Über den wissenschaftlichen Nachlaß Bernard Bolzanos*, Jahresbericht der Deutschen Mathematiker-Vereinigung **31**(1922), 2. Abteilung, 109–110.

[57] Jašek, M., *O funkcích s nekonečným počtem oscilací v rukopisech Bernarda Bolzana* [*On Functions with an Infinite Number of Oscilations in Manuscripts of B. B.*], ČPMF **53**(1923–24), 102–109 [Czech].

[58] Kowalewski, G., *Bolzanos Verfahren zur Herstellung einer nirgends differenzierbaren stetigen Funktion*, Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften, math.-phys. Klasse **74**(1922), 91–95.

[59] Kowalewski, G., *Über Bolzanos nichtdifferenzierbare stetige Funktion*, Acta mathematica **44**(1923), 315–319.

[60] Kurepa, D., *Around Bolzano's Approach to Real Numbers*, ČMŽ **32(107)**(1982), 655–666.

[61] Kurzweil, J., *Tribute to Bernard Bolzano*, in: Equadiff 5, Teubner, Leipzig, 1982, pp. 212–217.

[62] Laugwitz, D., Schmieden, C., *Eine Erweiterung der Infinitesimalrechnung*, Mathematische Zeitschrift **69**(1958), 1–39.

[63] Laugwitz, D., *Anwendungen unendlich kleiner Zahlen*, Crelle **207**(1961), 53–60, **208**(1961), pp. 22–34.

[64] Laugwitz, D., *Bemerkungen zu Bolzanos Größenlehre*, Archive **2**(1962–66), 398–409 [handed in: 1965].

[65] Laugwitz, D., *Bolzano's Infinitesimal Numbers*, ČMŽ **32(107)**(1982), 667–670.

[66] Lerch, M., *Über die Nichtdifferentiirbarkeit gewisser Function*, Crelle **92**(1888), 126–138.

[67] Loužil, J., *Bernard Bolzano*, Melantrich, Prague, 1978 [Czech].

[68] Němcová – Bečvářová, M., *František Josef Studnička a Bernard Bolzano*, in *Matematika v 19. století*, series *Dějiny matematiky*, vol. 3, Prometheus, Prague, 1996, pp. 115–119 [Czech].

[69] Nový, L., *Matematika v Čechách v 2. polovině 18. století, část 1* [*Mathematics in Bohemia in the Second Half of the 18th Century, Part 1*], Sborník **5**(1960), 9–113 [Czech, German summary].

[70] Nový, L., *K otázce Bolzanovy profesury matematiky v r. 1821* [*To a Question of Bolzano's Professorship of Mathematics in yr. 1821*], Zprávy **7**(1961), 28–30 [Czech].

[71] Nový, L., *Základy matematické analysy u Bolzanových pražských současníků* [*The Foundations of Mathematical Analysis by Bolzano's Prague contemporaries*], Sborník **6**(1961), 28–43 [Czech, German summary].

[72] Nový, L., *K rozsahu znalostí zahraniční matematické literatury v českých zemích v prvé polovině 19. století* [*To the Extent of Knowledges of the Foreign Mathematical Literature in Czech Countries in the First Half of the 19th Century*], Zprávy **9**(1961), 30–32 [Czech].

[73] Nový, L., *Zamyšlení nad některými metodologickými problémy bolzanovského bádání* [Remarks about Certain Methodological Problems of Bolzano's Research], DVT **81**(1981), 199–204 [Czech, English and Russian summary].

[74] Nový, L., *Poznámky o "stylu" Bolzanova matematického myšlení* [*Remarks on the "Style" of Bolzano's Mathematical Thinking*], DVT **81**(1981), 217–227 [Czech, English and Russian summary].

[75] Pavlíková, M., *Zahraniční bolzanovské studie* [*Foreign Bolzano Treatises*], Filosofický Časopis **16**(1968), 149–154 [Czech].

[76] Pavlíková, M., *Působení Bernarda Bolzana na pražské univerzitě* [*The Activity of B. B. at the Prague University*], Zprávy archivu UK **3**(1980), 5–31 [Czech].

[77] Pavlíková, M., *Bolzanovo působení na prašké univerzitě* [*Bolzano's Activities at the Prague University*], UK, Prague, 1985 [Czech].

[78] Petr, K., *Bernard Bolzano a jeho význam v matematice* [*B. B. and His Significance in Mathematics*], Prague, 1926 [Czech].

[79] du Bois–Reymond, P., *Versuch einer Classification der willkürlichen Functionen reeler Argumente nach ihren Aenderungen in den kleinsten Intervallen*, Crelle **79**(1875), 21–37.

[80] van Rootselaar, B., *Bolzano's Theory of Real Numbers*, Archive **2**(1962–66), 168–180 [handed in: 1963].

[81] Seidlerová, I., *Politické a sociální názory B. Bolzana* [*Political and Social Opinions of B. B.*], ČPMF **81**(1956), 388–390 [Czech].

[82] Seidlerová, I., *Fysikální práce Bernarda Bolzana* [*Physical Works of B. B.*], Zprávy **2**(1960), 14–16 [Czech].

[83] Seidlerová, I., *Bemerkung zu den Umgängen zwischen B. Bolzano und A. Cauchy*, ČPMF **87**(1962), 225–226 [German, Czech and Russian summary].

[84] Seidlerová, I., *Ještě několik poznámek o Bolzanově vztahu k českému vědeckému prostředí* [*Several More Remarks on Bolzano's Relation to the Czech Mathematical Surroundings*], Zprávy **11**(1962), 42–45 [Czech].

[85] Seidlerová, I., *Poznámka k otázce styků mezi Bolzanem a Cauchym* [*A Remark on the Question of Relations between Bolzano and Cauchy*], Zprávy **12**(1962), 30–31 [Czech].

[86] Seidlerová, I., *Politické a sociální názory B. Bolzana* [*Political and Social Opinions of B. B.*], Nakladatelství ČSAV, Praha, 1963 [Czech, German summary].

[87] Schwabik, Š., *Matematik Bernard Bolzano* [*The Mathematician B. B.*], Vesmír **60**(1981), 293–296 [Czech].

[88] Schwabik, Š., *Jednota československých matematiků a fyziků k uctění památky Bernarda Bolzana* [*The Union of Czechoslovak Mathematicians and Physicists in Memory of B. B.*], in: Sjezdový sborník 1981, Karlovy Vary, pp. 5–9 [Czech].

[89] Simon, P., *Bernard Bolzano a teorie dimenze* [*B. B. and the Diminsion Theory*], Pokroky **26**(1981), 259–261 [Czech; the same paper also in [10]].

[90] Struik, R. and D. J., *Cauchy and Bolzano in Prague*, ISIS **11**(1928), 364–366.

[91] Takagi, T., *A simple example of the continuous function without derivative*, Tokyo sugaku butsurigaku kwai kiji **1**(1903), 176–177 [English and Japan].

[92] Večerka, K., *Poznámka k Bolzanovu pojetí množiny* [*A Remark on Bolzano's Conception of a Set*], Zprávy **10**(1962), 37–38 [Czech].

[93] Veselý, F., *Život a dílo B. Bolzana* [*Life and Work of B. B.*], Matematika ve škole **6**(1956), 449–464 [Czech].

[94] Veselý, F., *Život Bernarda Bolzana a jeho matematicko-přírodovědné práce* [*Life of B. B. and His Works in Mathematics and Exact Sciences*], Pokroky **2**(1957), 119–127, 234–243 [Czech].

[95] Vetter, Q., *Bernard Bolzano, filosof a matematik* [*B. B., Philosopher and Mathematician*], pp. 56–57 of [102].

[96] Vlček, E., *Fyzická osobnost Bernarda Bolzana* [*Physical Personality of B. B.*], Pokroky **26**(1981), 259–261 [Czech].

[97] Vojtěch, J., *O geometrických pojednáních Bolzanových* [*On Geometrical Treatises of B. B.*], ČPMF **64**(1935), 264–265 [Czech].

[98] Vondruška, P., *Spojité funkce bez derivace* [*Continuous Functions without a Derivative*], Matematické Obzory **29**(1987), 43–54 [Czech].

[99] Vopěnka, P., *Nekonečno, množiny a možnost v Bolzanově pojetí* [*Infinity, Sets and Possibility in Bolzano's Conception*], Ann. Soc. Math. Pol., Ser. II. Wiadomoscy Matematyczne **26**(1985), 171–204 [also in [9]; Czech].

[100] van der Waerden, B. L., *Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion*, Mathematische Zeitschrift **32**(1930), 474–475.

[101] Winter, E., *Bernard Bolzano und sein Kreis*, Verlag von Jakob Hegner, Leipzig, 1933; Czech translation by Z. Kalista: *Bernard Bolzano a jeho kruh*, edice Akordu, spolek katolických akademiků Moravan, Brno [the foreword written by A. Novák, notes by Z. Kalista].

[102] Winter, E., *Bernard Bolzano, filosof a matematik* [*B. B., Philosopher and Mathematician*], in: *Co daly naše země Evropě a lidstvu 2* Praha, 1940, 54–57 [compare [95]; Czech].

[103] Winter, E., *Der böhmische Vormärz in Briefen B. Bolzanos an F. Příhonský (1824–1848)*, Deutsche Akad. d. Wiss., Berlin, 1956.

[104] Winter, E., *Wissenschaft und Religion im Vormärz. Der Briefwechsel Bernard Bolzanos mit Michael Josef Fesl 1822–1848*, Berlin, 1965 [edited by E. Winter and W. Zeil in the cooperation with L. Zeil, the introduction by E. Winter].

Magdalena Hykšová
Department of Applied Mathematics
Faculty of Transportation Sciences
Czech Technical University
Na Florenci 25,
110 00 Prague,
Czech Republic
e-mail: hyksova@fd.cvut.cz