Witold Więsław

Squaring the circle in XVI - XVIII centuries

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PIV ELEVATA, ET SPECVLATIVA PARTE DELLA PRATICA
Arithmetica, laqual è tutte le regole, & operationi praticali
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CON LI SVOI PRIVILEGII.

In Vinegia per Curtio Troiano dei Nauò.
M D LVI.

N. Tartaglia: *General trattato* ...

SQUARING THE CIRCLE IN XVI–XVIII CENTURIES

Witold Więsław

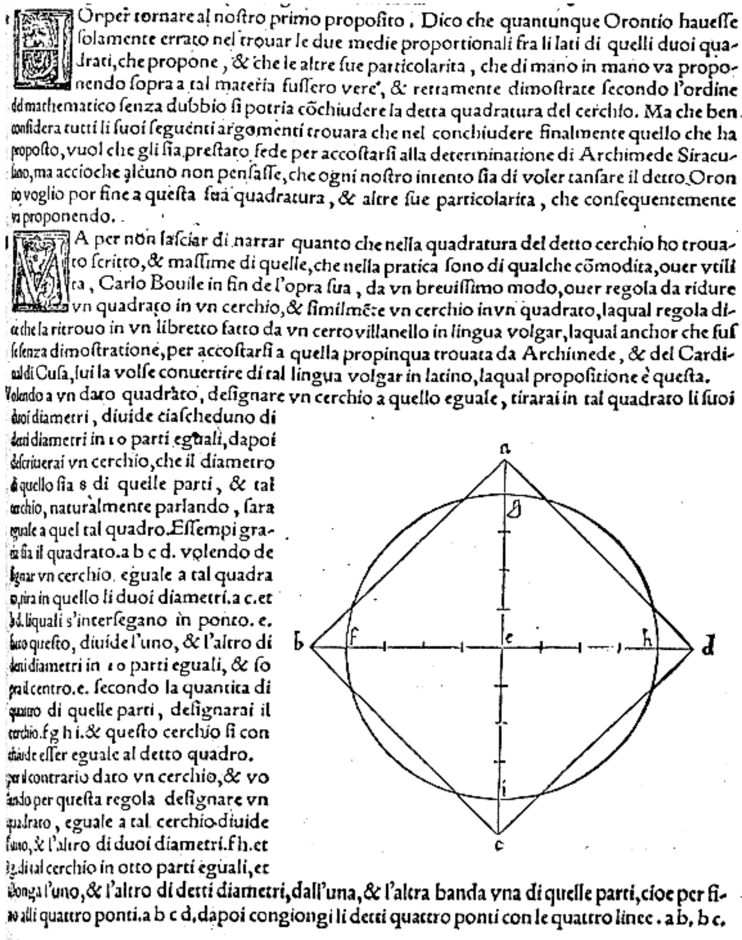
Squaring the circle, traditionally called *Quadratura Circuli* in Latin, was one of the most fascinating problems in the history of mathematics. Nowadays it is formulated as the problem of constructing the side of a square with area equal to the given circle by ruler and compass. Evidently, the problem is equivalent to the *rectification of the circle*, i.e. to the problem of constructing in the same way, by ruler and compass, a segment of the length equal to the perimeter of the circle. In the first case the problem leads to the construction of a segment of lenght $\sqrt{\pi}$, in the second one to the construction of a segment of the length π.

I shall mention only that the first essential result in this direction goes back to Archimedes, who found the connections between plane and linear measures of a circle: the area of the circle equals to the area of rectangular triangle with teh legs equal, respectively, to its radius and the perimeter.

The history of the problem is long and I am not going to give it here completely. I would like to present here only some examples of the efforts in this direction from the period XVI–XVIII century. Let us also remark that for centuries, the problem meant rather to measure the circle than to construct its perimeter by ruler and compass. Since from the Greek antiquity geometrical constructions by ruler and compass were mathematical instruments, we now have a much more restricted formulation of the problem.

Niccolo Tartaglia (1500–1557) presents in [2] the following approximate squaring the circle. He transforms a square into the circle dividing its diagonals into ten equal parts and taking as a diameter of the circle eight parts (see the original picture from [2]). A simple calculation shows that the construction leads to the Babylonian approximation $\pi = \frac{25}{8}$.

Jean de Buteo (c.1492–1572) in [1] and [3] presents a construction leading to Ptolemy's approximation of π, namely $\frac{327}{120}$, i.e. to 3;8,30 in the sexagesimal system of numeration.



Or per tornare al nostro primo proposito. Dico che quantunque Orontio hauesse solamente errato nel trouar le due medie proportionali fra li lati di quelli duoi quadrati, che propone, & che le altre sue particolarita, che di mano in mano va proponendo sopra a tal materia fussero vere, & rettamente dimostrate secondo l'ordine del mathematico senza dubbio si potria cōchiudere la detta quadratura del cerchio. Ma che ben considera tutti li suoi seguenti argomenti trouara che nel conchiudere finalmente quello che ha proposto, vuol che gli sia prestato fede per accostarsi alla determinatione di Archimede Siracusano, ma accioche alcuno non pensasse, che ogni nostro intento sia di voler tansare il detto Orontio voglio por fine a questa sua quadratura, & altre sue particolarita, che consequentemente va proponendo.

Ma per non lasciar di narrar quanto che nella quadratura del detto cerchio ho trouato scritto, & massime di quelle, che nella pratica sono di qualche cōmodita, ouer vtilita, Carlo Bouile in fin de l'opra sua, da vn breuissimo modo, ouer regola da ridure vn quadrato in vn cerchio, & similmēte vn cerchio in vn quadrato, laqual regola dice che la ritrouo in vn libretto fatto da vn certo villanello in lingua volgar, laqual anchor che fusse senza dimostratione, per accostarsi a quella propinqua trouata da Archimede, & del Cardinal di Cusa, lui la volse conuertire di tal lingua volgar in latino, laqual propositione è questa.

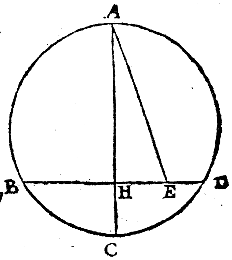
Volendo a vn dato quadrato, designare vn cerchio a quello eguale, tirarai in tal quadrato li suoi duoi diametri, diuide ciascheduno di detti diametri in 10 parti eguali, dapoi descriuerai vn cerchio, che il diametro di quello sia 8 di quelle parti, & tal cerchio, naturalmente parlando, sara eguale a quel tal quadro. Essempi gratia sia il quadrato. a b c d. volendo designar vn cerchio eguale a tal quadrato, tira in quello li duoi diametri. a c. et b d. liquali s'intersegano in ponto. e. fatto questo, diuide l'uno, & l'altro di detti diametri in 10 parti eguali, & sopra il centro. e. secondo la quantita di quattro di quelle parti, designarai il cerchio. f g h i. & questo cerchio si conchiude esser eguale al detto quadro. Per il contrario dato vn cerchio, & volendo per questa regola designare vn quadrato, eguale a tal cerchio diuide l'uno, & l'altro di duoi diametri. f h. et g i. di tal cerchio in otto parti eguali, et slonga l'uno, & l'altro di detti diametri, dall'una, & l'altra banda vna di quelle parti, cioe per fino alli quattro ponti. a b c d, dapoi congiongi li detti quattro ponti con le quattro linee. a b, b c,

A page from the *General trattato* ...
by N. Tartaglia

Another one, IOSEPH SCALIGER in his beautiful book [6], in which mathematical symbols are printed in red, takes $\sqrt{10}$ for π in his construction. Indeed, he draws diameter $d = 2r$ in a circle, next the middle point of its radius and constructs rectangular triangle with legs $\frac{3}{2}r$ and $\frac{1}{2}r$. Its hypotenuse gives, in his opinion, an approximate squaring of the circle.

PROPOSITIO IIII. Problema.

Dato circulo rectam æqualem eius perimetro inuenire.



Circuli dati ABCD *perimetro ſit inuenienda recta æqualis. Accommodetur ei* BD *latus trianguli iſopleuri eidem circulo inſcribendi. Per* XII *tertij decimi, recta* HC *erit quarta pars Diametri* AC. *Ex recta* HD *abſcindatur recta* HE *æqualis ipſi* HC. *Connectatur recta* EA. *Quia igitur longitudo rectæ* AH *ex conſtructione eſt tripla longitudinis* HC, *id eſt* HE: *quadratum* AH *erit nonuplum quadrati* HE. *&* *propterea quadratum rectæ* AE *erit decuplum quadrati* HE, *vel* HC, *per* XLVII *primi. Sed* HC *eſt quarta pars longitudinis diametri. Ergo quadratum rectæ* AE *eſt decuplum quadrati* HC. *Et quia peripheria ad ſuum quadrantem habet eam rationem, quam diametrus ad* HC: *recta igitur* AE *erit quarta pars peripheriæ. &c.*

FRANÇOIS VIÈTE (1540–1603) is well-known as the author of literal notations consequently used in the algebra. He used the Latin letters $A, B, C, D, \ldots$ to denote the known quantities and letters $\ldots, W, X, Y, Z$ to denote indeterminates. He introduced such notation in [4]. His achievements in geometry are less known. VIÈTE presents some approximate constructions of squaring and rectification the circle in [5]. We show one of them. On page 26 (loc. cit.) we can find the following exercise: *quadrant circumferentiae dati circuli invenire proxime lineam rectam aequalem,* i.e. *find the segment approximately equal to the quarter of the circle.* The figure below is the same as in [5].

In the figure: $AB = AD = a$, $DF = FA$, $EI = BZ$, GH is orthogonal to BC, and EK is parallel to IH. VIÈTE claims that EK is approximately equal to the quater of the circle $BDCE$. Assume that he is right, i.e. $AK = \frac{1}{2}k$. The similarity of the triangles AIH and AEK implies that $\frac{AL}{AE} = \frac{AH}{AK}$. Since $AK = \frac{AH \times AE}{AI} = \frac{AH \times a}{AI}$, thus $\pi = 2\frac{AH}{AI}$. Now we calculate AH and AI. In $\triangle ABF$ we have: $BF^2 = AF^2 + AB^2 = \frac{1}{4}a^2 + a^2 = \frac{5}{4}a^2$, so $BF = \frac{1}{2}\sqrt{5}$. Since $BZ = BF - ZF = \frac{1}{2}a\sqrt{5} - \frac{1}{2}a = \frac{1}{2}(\sqrt{5}-1)a$, so $AI = a - EI = a - BZ$, and $AI = \frac{1}{2}(3-\sqrt{5})a$. Now we find AH. In $\triangle AGH$: $AH^2 + GH^2 = a^2$. Since the triangles $\triangle BAF$ and $\triangle BHG$ are similar, hence $\frac{BH}{BA} = \frac{GH}{FA}$, i.e. $\frac{BH}{GH} = \frac{BA}{FA} = 2$. The equality $BH = a + AH$ implies that $2GH = BH = a + AH$, thus $GH = \frac{1}{2}(a + AH)$. Substituting it in

FRANCISCI VIETÆ

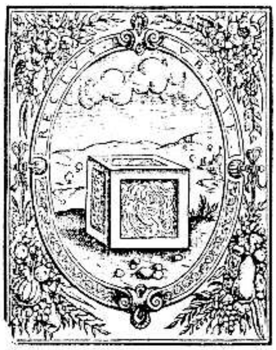
VARIORVM DE REBVS MATHEMATICIS RESPONSORVM, LIBER VIII.

Cuius præcipua capita sunt,

De duplicatione Cubi, & Quadratione Circuli.

Quæ claudit

Πρόχειρον, seu *Ad usum Mathematici Canonis* METHODICA.



TVRONIS,

Apud IAMETTIVM METTAYER,
Typographum Regium.

1593.

F. Viète: *Variorum de Rebus Mathematicis* …

$AH^2 + GH^2 = a^2$, we obtain a quadratic equation with respect to AH: $5AH^2 + 2a \times AH - 3a^2 = 0$, implying that $AH = \frac{3}{5}a$. Consequently, substituting for AH and AI in the above formulae, we have $\pi = 2\frac{AH}{AI}$, i.e. $\pi = \frac{3}{5}(3+\sqrt{5})$ approximately. Consequently VIETE's approximation equals $\pi = 3.1416406\ldots$. KEPLER [7] used ARCHIMEDES' result: $\pi = \frac{22}{7}$.

Sometimes the word *ludolphinum* is used instead of *pi*. This word goes back to LUDOLPH VAN CEULEN (1540–1610). Some epitaphs were found in 1712 in Leyden during rebilding the Church of Sanctus Petrus. Among them there was the epitaph of LUDOLPH VAN CEULEN. We read there:

> *Qui in vita sua multo labore circumferentiae circuli proximam rationem diametram invenit sequentem [which in his life was working much under calulation of an approximate proportion of the circle perimeter to its diameter.*

In the epitaph we find an approximation of π up to 35 digits. At first, VAN CEULEN found 20 digits (*Van den Circkel*, Delf 1596), and next 32 digits (*Fvndamenta Arithmetica et Geometrica*, 1615). The book *De Circvlo et adscriptis Liber* (1619), published by WILLEBRORD SNELL (SNELLIUS) after VAN CEULEN's death, presents his method in the case of 20 digists. In 1621, W. SNELL wrote *Cyclometricus* [10], where he presented VAN CEULEN's algorithm for finding 35 digits. In [8], VAN CEULEN proves many theorems dealing with the equivalence of polygonals by finite division into smaller figures. He evolves there an arithmetic of quadratic irrationals, i.e. he studies numbers of the form $a+b\sqrt{d}$ with a, b, d rational. He states that if d is fixed, then arithmetic operations do not lead out of the set. He proves it on examples, but his arguments are quite general. He considers also the numbers obtained from the above ones by extracting square roots. He uses it intensively in [9]. His method runs as follows. LUDOLPH VAN CEULEN calculates the length of the side of the regular N-gon inscribed in the circle with radius 1, writing the results in tables. He successively determines the side of the regular N-gon for $N = 2^n$, where $2 \leq n \leq 21$, i.e. up to $N = 2.097.152$. Next he makes the same for $N = 3 \times 2^n$, taking $1 \leq n \leq 120$,i.e. until $N = 3.145.728$. Finally, he puts $N = 60 \times 2^n$, with $1 \leq n \leq 20$, i.e. up to $N = 491.520$. For example, in the case considered by ARCHIMEDES (and also by LEONHARDO PISANO, AL-KASCHI and others), i.e. for regular 96-gon inscribed in the circle with radius 1, the length of the side is equal to

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}},$$

which VAN CEULEN writes as

$$\sqrt{.2}-\sqrt{.2}+\sqrt{.2}+\sqrt{.2}+\sqrt{.2}+\sqrt{3}.$$

Next, for all tabulated regular N-gons, he calculates the perimeters and their decimal expansions, taking as the final aproximation of π the last common value from the tables. It gives twenty digits of decimal expansion of π.

The approximation to π by $\frac{355}{113}$, i.e. by the third convergent of the expansion of π into a continued fraction, was attributed to ADRIANUS METIUS already at the end of the XVII century. (The first convergent of π is the Archimedean result $\frac{22}{7}$, and the second one equals $\frac{333}{106}$). JOHN WALLIS attributed the result to ADRIANUS METIUS in *De Algebra Tractatus* (see [18, p. 49]). The truth, however, is quite different. ADRIANUS METIUS ALCMARIANUS writes in [12, p. 89]:

> *Confoederatarum Belgiae Provintiarum Geometra* [...] *Simonis a Quercu demonstravit proportionem peripheriae ad Suam diametrum esse minorem* $3\frac{17}{120}$, *hoc est* $\frac{377}{120}$, *majorem* $3\frac{15}{106}$, *hoc est* $\frac{333}{106}$, *quarum proportionum intermedia existit* $3\frac{16}{113}$, *sive* $\frac{355}{113}$, ...

which means that

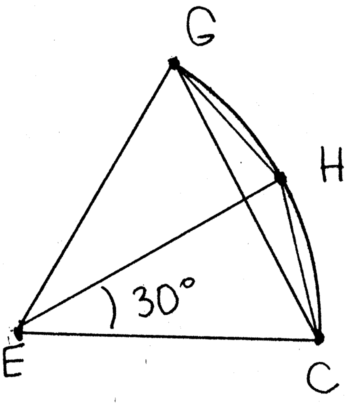
> *Geometra from confederated province of Belgium, Simonis from Quercu, had proved, that the ratio of the perimeter to its diameter is smaller than* $3\frac{17}{120}$, *i.e.* $\frac{377}{120}$, *and greater than* $3\frac{15}{106}$, *i.e. than* $\frac{333}{106}$. *The mean proportion of the fractions is* $3\frac{16}{113}$, *that is* $\frac{355}{113}$, [...]

The mean proportion of fractions $\frac{a}{b}$ and $\frac{c}{d}$ was called the fraction $\frac{a+c}{c+d}$. The result goes back to PTOLEMY. The work [12] is very interesting for another reason. ADRIANUS METIUS describes there an approximate construction changing a circle into equilateral triangle. We present below his construction with the original figure of ADRIANUS.

From the intersection E of two orthogonal lines we draw a circle with radius a. Thus $AE = CE = BE = EG = EF = a$. Next we construct two equilateral triangles: $\triangle CEG$ and $\triangle CEF$. The bisetrix of the angle

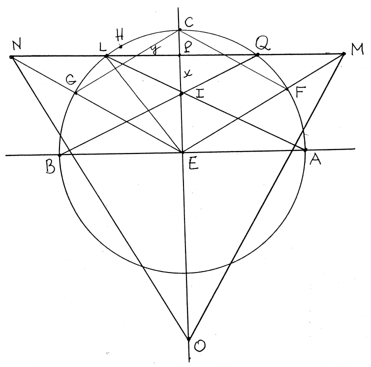
CEG determines the point H. From point C, one constructs $CI = CH$. Let the lines through A and I, B and I meet the circle in points L and Q, respectively. The intersection of the line LQ with the lines EF and EG, defines the points M and N of the constructed equilateral triangle. The third point can be found immediately.

Lemma. In the figure below: $HC = a\sqrt{2-\sqrt{3}}$. Indeed, the Cosine Theorem applied to CEH gives $HC^2 = EC^2 + EH^2 - 2 \times EC \times EH cos\frac{\pi}{6} = a^2(2-\sqrt{3})$.



We calculate the surface of $\triangle MNO$. Let P be the meet of the line EC with MN. Put $PI = x$, $LP = y$. The Lemma implies that $EI = a - CI = a - a\sqrt{2-\sqrt{3}}$, where $\lambda = 1 - \sqrt{2-\sqrt{3}}$. In the rectangular triangle AEI: $IA^2 = EI^2 + EA^2 = EI^2 + a^2$, thus $IA = a\sqrt{1+\lambda^2}$. Similarity of triangles $\triangle LPI$ and $\triangle AEI$ gives $\frac{LI}{AI} = \frac{PI}{EI}$, $\frac{x}{y} = \frac{EI}{EA}$ i.e. $LI = \frac{AI}{EI} PI = \frac{\sqrt{1+\lambda^2}}{\lambda} x$, $x = \lambda y$. In the rectangular triangle $\triangle LPE$: $PE^2 + LP^2 = LE^2$, hence $(x+EI)^2 + y^2 = a^2$, $(x+\lambda a)^2 + y^2 = a^2$, and since $x = \lambda y$, thus $\lambda^2(y+a)^2 + y^2 = a^2$, implying $\lambda^2(y+a)^2 = (a+y)(a-y)$, i.e. $\lambda^2(y+a) = (a-y)$, thus $y = a\frac{1-\lambda^2}{1+\lambda^2}$ and $x = \lambda a \frac{1-\lambda^2}{1+\lambda^2}$.

Since E is the median of the equilateral triangle $\triangle MNO$, so $EP = x + IE$ is half of EO, i.e. $x + EI = \frac{1}{2}EM$, since $EO = EM$, i.e. $EM = 2(x+IE) = 2a\lambda\frac{1-\lambda^2}{1+\lambda^2} + 2a\lambda = \frac{4a\lambda}{1+\lambda^2}$. It implies that the height h in the $\triangle MNO$ equals $h = \frac{3}{2}EM = \frac{6a\lambda}{1+\lambda^2}$.



If z is a side of $\triangle MNO$, then from $\triangle OPM$: $h^2 + (\frac{z}{2})^2 = z^2$, i.e. $z = \frac{2}{\sqrt{3}}h = 4a\sqrt{3}\frac{\lambda}{1+\lambda^2}$. Since, according to ADRIANUS METIUS, the area of $\triangle MNO$ is approximately equal to the area of the circle with the centrum E and radius $EA = a$, hence $\pi a^2 = \frac{1}{2}hz = \frac{1}{2}h\frac{2}{\sqrt{3}}h = \frac{1}{\sqrt{3}}\frac{36a^2\lambda^2}{(1+\lambda)^2}$, i.e. $\pi = 12\sqrt{3}\frac{\lambda^2}{(1+\lambda^2)^2}$. It gives an approximate value for π as $3.1826734\ldots$. Since $\pi = 3.141592\ldots$, hence the error is about 1.3%.

Among the many authors who kept busy in the XVII century with measuring the circle, CHRISTIAN HUYGENS (1629–1695), one of the most famous mathematicians of the century, has a special place. In a short time, he learnt and extended the coordinate methods of Descartes, showing its many applications in mathematics and elsewhere. His known achievements are published in many great volumes. I describe here only a part of his scientific activity. In *Theoremata de Quadratura Hyperboles, Ellipsis et Circuli* from 1651, HUYGENS describes geometrical methods for finding lenghts of their parts. In the treatise *De Circuli Magnitudine Inventa (A study of the circle magnitude)* from the year 1654, he describes different geometrical methods of approximating the perimeter of the circle. HUYGENS in [14] leads to absolute perfection the methods of ARCHIMEDES of approximation of the perimeter of the circle by suitably chosen n-gons. He proves geometrically many inequalities between the lengths of the sides of n-gons, $2n$-gons and $3n$-gons inscribed and described on a circle. In particular, he deduces from them an approximate rectification of an arc. Already in his time, analytical arguments like the ones presented below were known and applied.

Let AOB be a sector of a circle with radius r and angle α. Let OC bisect the angle AOB. We put aside $CD = AC$ on the line through A and C. The circle with centrum A and radius AD meets the line through A and B in G. Finally we put $DE = \frac{1}{3}BG$. Then, as HUYGENS claims, the length of the arc AB is approximately equal to the segment AE. Indeed,

$$AE = AD + DE = AD + \frac{1}{3}BG = AD + \frac{1}{3}(AD - AB) = \frac{4}{3}AD - \frac{1}{3}AB.$$

Since $AD = 2AC$, by the construction, $AB = 2AF = 2r\sin\frac{a}{2}$ from the triangle $\triangle AFO$ and similarly, $AC = 2r\sin\frac{a}{4}$, thus

$$AE = \frac{4}{5}AD - \frac{1}{3}AB = \frac{4}{3}2AC - \frac{1}{3}AB =$$
$$= \frac{8}{3}2r\sin\frac{\alpha}{4} - \frac{1}{3}2r\sin\frac{\alpha}{2} = \frac{2r}{3}(8\sin\frac{\alpha}{4} - \sin\frac{\alpha}{2}).$$

Since the sine function has the expansion $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$, then taking x equal $\frac{1}{4}\alpha$ and $\frac{1}{2}\alpha$, we have

$$8\sin\frac{\alpha}{4} - \sin\frac{\alpha}{2} == 8\left(\frac{\alpha}{4} - \left(\frac{\alpha}{4}\right)^3\frac{1}{3!} + + \left(\frac{\alpha}{4}\right)^5\frac{1}{5!} - \cdots\right) -$$
$$-\left(\frac{\alpha}{2} - \left(\frac{\alpha}{2}\right)^3\frac{1}{3!} + \left(\frac{\alpha}{2}\right)^5\frac{1}{5!} - \cdots\right) =$$

$$= \alpha\left(2 - \frac{1}{2}\right) + \alpha^3\left(\frac{1}{6\times 8} - \frac{8}{6\times 4^3}\right) + \alpha^5\left(\frac{8}{4^5\times 120} - \frac{1}{2^5\times 120}\right) +$$

$$+\alpha^7\left(\frac{1}{2^7\times 7!} - \frac{8}{4^7\times 7!}\right) + \cdots =$$

$$= \frac{3}{2}\alpha + \frac{1}{2^5\times 5!}\left(\frac{1}{2^2} - 1\right)\alpha^5 + \frac{1}{2^7\times 7!}\left(1 - \frac{1}{2^4}\right)\alpha^7 + \frac{1}{2^9\times 9!}\left(\frac{1}{2^6} - 1\right)\alpha^9.$$

Consequently,

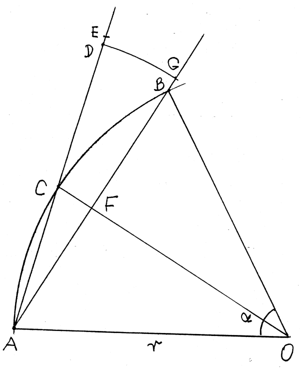
$$\left|-\frac{3}{2}\alpha + 8\sin\frac{\alpha}{2} - \sin\alpha 2\right| \leq$$

$$\leq \frac{3}{4}\frac{1}{2^5\times 5!}\alpha^5\left(1 + \frac{\alpha^2}{2^2\times 6\times 7} + \frac{\alpha^4}{2^4\times 6\times 7\times 8\times 9}\right.$$

$$\left. + \frac{\alpha^6}{2^6\times 6\times 7\times 8\times 9\times 10\times 11} + \cdots\right) \leq$$

$$\leq \frac{3}{4}\frac{1}{2^5\times 5!}\alpha^5\left(1 + \left(\frac{\alpha}{12}\right)^2 + \left(\frac{\alpha}{12}\right)^4 + \left(\frac{\alpha}{12}\right)^6 + \cdots\right) =$$

$$= \frac{3}{4}\frac{\alpha^5}{2^5\times 5!}\frac{1}{1 - \left(\frac{\alpha}{12}\right)^2}.$$



Thus $AE = \frac{2r}{3}\left(\frac{3}{2}\alpha - \frac{3}{4}\frac{1}{2^5\cdot 5!}\alpha^5 + \cdots\right) = r\alpha - \frac{r}{7680}\alpha^5$. Since $AE = r\alpha + rest$, hence our arguments show that

$$|rest| \leq \frac{r}{7680}\frac{\alpha^5}{1 - (\alpha/12)^2}\ .$$

It is interesting, that in HUYGENS' book [14] there is also the constant 7680. The obtained result gives the possibily of rectifying the circle with a given error. Indeed, it is necessary to divide the circle into n equal arcs and next rectify each of them. For example, if $\alpha = \frac{\pi}{2}$, then $|rest| \leq 0.0012636$, which by multiplying by 4 gives an error not greater than 0.00506.

Another *Quadratura circuli* was given by MARCUS MARCI [16]. It was described in [26] by ALENA ŠOLCOVÁ.

MADHAVA (*Yukti-Bhasha*, XIV century) found 3.14159265359... for π. It could be not surprising but he used some calculations equivalent to the series expansion of arcus tangens:

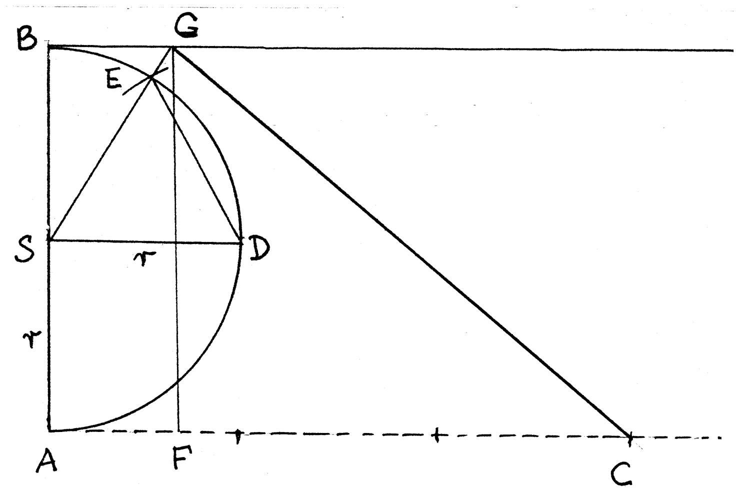
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots ,$$

called now *Gregory's series* (1671). In particular, MADHAVA used the equality $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$, proved in Europe by G. W. LEIBNITZ [17].

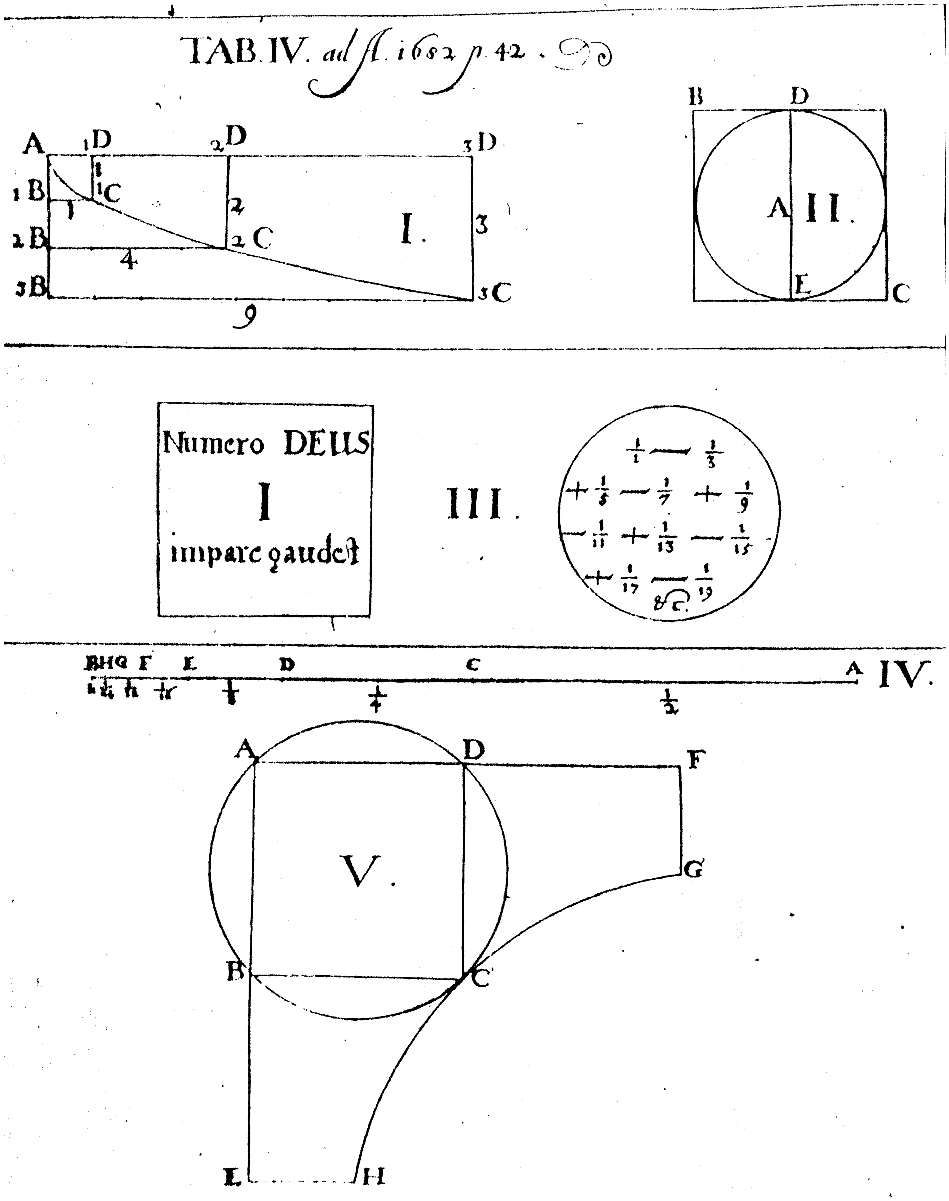
Ancient Indian mathematicians of Madhava times knew much more exact approximations of π. For example *Karana Paddhati* gives 17 digits of π (see [25]).

Now recall an approximate rectification of the circle of ADAM ADAMANDY KOCHAŃSKI (see [21]). The Jesuit KOCHAŃSKI was at first professor of mathematics in Mainz in 1659. In 1667 he was teaching at Jesuits Collegium in Florence, in 1670 he was in Prague, then in Olomouc. Since he was not content with his stay there, he decided in 1677 to ask for his transfer to another place, to Wratislavia (Wrocław), where he observed and described a comet. Later he was a librarian of Polish king JAN III SOBIESKI. He died at the end of XVII century. He entered the history of mathematics as the author of a very simple (approximate) rectification the circle.

We draw two orthogonals to diameter of the semi-circle ADB with centrum S and radius $AS = r$. Next we put $AC = 3r$.



Then we take the parallel SD to AC and construct equilateral triangle SDE. Let the line through S and E meet in G the line from B



A page from Leibniz's paper

parallel to the base line AC. KOCHAŃSKI claims that GC equals approximately to semi-circle ADB. Indeed, since $FC = AC - GB = 3r - r\tan\frac{\pi}{6}$ and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, then from there put aside $CD = AC$ rectangle FCG we obtain successively

$$GC^2 = (2r)^2 + (3r - r\tan\frac{\pi}{6}) = r^2\left(\frac{40}{3} - 2\sqrt{3}\right),$$

thus

$$GC = r\sqrt{\frac{40}{3} - 2\sqrt{3}},$$

which means that approximately

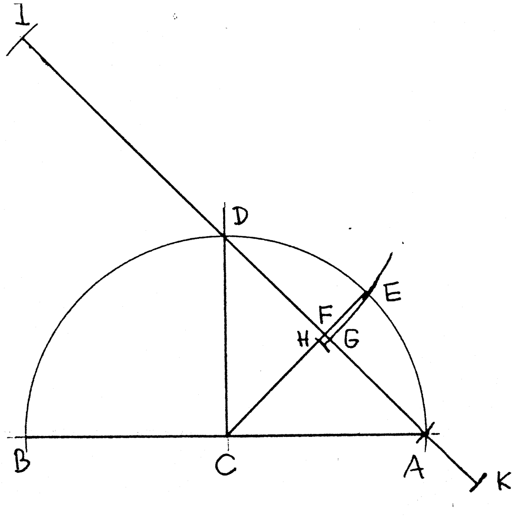
$$\pi = \sqrt{\frac{40}{3} - 2\sqrt{3}} = \frac{1}{3}\sqrt{6(20 - 3\sqrt{3})} = 3.141533\ldots$$

The error equals approximately $3.14159265 - 3.1415333 = 0.00005932$.

The problem of squaring the circle appears in seven EULER's papers and in his correspondence with CHRISTIAN GOLDBACH in years 1729–1730.

We describe one of EULER's approximate rectifications of the circle.

ISAAC BRUCKNER (1686–1762) gave a not very exact rectification of the circle. EULER proposed the following modification of BRUCKNER's construction.



Let CE be bisectrix of the right angle ACD. Let $DI = AD$, $IG = IE$, $FH = FG$, and $AK = EH$. Assume moreover that $AC = 1$. Then

$IA = 2\sqrt{2}$, $CF = \frac{1}{2}\sqrt{2}$, $EF = 1 - \frac{1}{2}\sqrt{2}$, $IF = \frac{3}{2}\sqrt{2}$. Thus $IG^2 = IE^2 = IF^2 + EF^2 = 6 - \sqrt{2}$, implying that $IG = \sqrt{6-\sqrt{2}}$. Consequently, $FH = FG = IG - IF$, i.e. $AK = EH = EF + FH = \sqrt{6-\sqrt{2}}$, and finally $IK = IA + AK = 1 + \sqrt{6-\sqrt{2}} = 3.1414449\ldots$.

LEONHARD EULER improved also the above-described HUYGENS's construction, following his ideas, but obtaining for the approximate length $L(\alpha, r)$ of an arc with the radius r and the angle α, the formula

$$L(\alpha, r) = \frac{r}{45}(256 \sin\frac{\alpha}{4} - 40 \sin\frac{\alpha}{2} + \sin\alpha),$$

much more exact than HUYGENS's. Namely,

$$L(\alpha, r) = \alpha r - \frac{r}{322.560}\alpha^7 + \cdots ,$$

which is slightly better than in HUYGENS' construction.

The bibliography below contains only selected papers and books concerning squaring the circle. The complete bibliography is much more extensive.

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