Štěpánka Bilová

Lattice theory in Czech and Slovak mathematics until 1963

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MATHEMATICS THROUGHOUT THE AGES II

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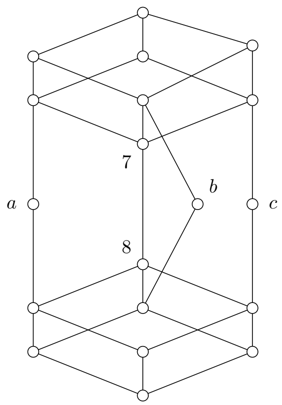

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Chapter 1

Introduction

1.1 Objectives of the paper

Lattice theory has become an important, and we could say a traditional, part of research of Czech and Slovak mathematicians, it is therefore interesting to follow the process of establishing its position within at that time Czechoslovak mathematics. This work, based to a large extend on the author's Ph.D. thesis, describes the context in which lattice theory entered Czech mathematics at the end of 1930's, Slovak mathematics after WWII and analyzes the papers and activities connected with this field appearing in both Czech and Slovak mathematics until the year 1963 when an international conference on ordered sets, which included contributions on lattice theory, was held in Czechoslovakia.

The primary materials for the work consisted of original papers, the ones which were analyzed as well as the ones which the authors referred to so that the previous results could be checked. As the literature contained works in several languages, the author provides a free translation into English where no English version of the quotation exists. As far as the notions are concerned, either a standard English terminology is used (based on [Grä2]), or a suitable English translation is given.

1.2 From the history of lattice theory

The origin of lattice theory and Garrett Birkhoff

The creation and development of lattice theory has a lot of interesting and specific features. Although the roots of this theory can be found in the second half of 19th century, within the areas of algebraic logic and number theory, those early attempts to study abstract mathematical structures called lattices today remained largely unnoticed. It was only in the 1930's that we can see a vivid interest in this field which resulted in a fast establishment of lattice theory during a surprisingly short period of 1933–1940.

The first ideas related to lattice theory can be found in the work of G. Boole on algebra of logic, *The Mathematical Analysis of Logic*, 1847. They were further improved by C. S. Peirce, and later incorporated into *Vorlesungen über die Algebra der Logik* (1890–1905) by E. Schröder. Independently of the investigation in algebraic logic, lattice structures were studied in the field of number theory, namely in the work of R. Dedekind from the last decade of 19th century, who arrived at a number of concepts, examples, and properties of lattice theory. Neither of the results, however, found strong interests to be followed.

At the end of 1920's and the beginning of 1930's several mathematicians came to new formulations of the notion lattice. This happened in more areas: projective geometry, logic and algebra. Those new beginnings did not seem to attract a lot of attention either, however, in the course of 1930's the situation was changing in the favor of studying abstract algebraic structures.

An environment suitable for the development of lattice theory was created by Van der Waerden's book *Moderne algebra* which set a model and provided methods and concepts for the new type of investigation. The idea of universal algebra enabling the treatment of general algebraic structures became a crucial concept of the period and several mathematicians believed the role of universal algebra would be played by lattices.

The first description of lattice theory is considered to be given in two papers from the first half of 1930's: G. Birkhoff's [Bir1] and O. Ore's [Ore1]. In the years 1935 to 1939 a number of articles were published which showed wide applications of this theory; the concept of lattice appeared in investigations within universal algebras, the foundation of geometry, continuous geometry, partly ordered linear spaces, group theory, topology, functional analysis, probability; Boolean algebras found applications to metamathematics and measure theory.

The name which is connected with the origin of lattice theory most closely is G. Birkhoff, who is sometimes called Father of Lattice Theory. He deserved this attribute not only for his significant contributions to the subject, but mainly for providing a unifying framework for the emerging lattice theoretic results, trends and applications. This framework was achieved through the first publication of his famous monograph *Lattice theory* [LT–40] in 1940, which, in a way, marks the formal establishing of this theory. Although this first edition was more a collection of contemporary results than a self–contained study of the subject, it was the first treatment of lattices presented as a theory.

H. Mehrtens [Meh] identifies several factors which played role in a fast establishing of lattice theory as an autonomous mathematical field during the 1930's. Abstract algebraic structures became a new trend in algebra after van der Waerden's book. The adoption of set theoretical language and axiomatic methods resulted in the fact that lattice theory became apparent in nearly all fields of mathematics and thus showed its various forms of applications of the theory. The next factor which helped to move this subject to the centre of attention was that several outstanding mathematicians (e. g. J. von Neumann, M. H. Stone, O. Ore, A. Tarski) produced papers in which they built respective concepts upon lattice theoretical basis. The integrating role was played by the young mathematician G. Birkhoff whose talents and enthusiasm resulted in consolidating the obtained results and presenting the theory as a more or less unified aggregate. G. Birkhoff's role in the development of lattice theory is quite unique, he did not belong to the mathematicians who were attracted to lattices because of motivations from other fields, he played the role of an "organizer" ([Meh], p. 295) of the theory.

Thus, we can witness a very quick development of a new mathematical theory during the 1930's, the theory which is rich in material, recognized and promising. From 1939 more and more mathematicians started their research in pure lattice theory without being justified by its applications in traditional areas and we can, therefore, speak about the beginning of autonomous position of this mathematical field.

The following development of the new theory was, to a large extent, set in the directions put forward by the book *Lattice Theory*, which confirmed its world–wide success and the second, enlarged, edition of the book ([LT–48], 1948) continued to inspire new generations of mathematicians. The monograph was published in another enlarged and revised edition in 1967 [LT–67].

Lattice theory has continued to grow significantly especially since 1960's and by now many of its parts have become research areas on their own. We can say that the mentioned third edition of *Lattice Theory* is the last of works which deal with the theory in a broad sense, discussing also its many applications and abundant relations to other fields of mathematics. The series of books on lattice theory that have been published since that time treat the theory in its restricted sense, i. e. lattice theory proper, or concentrate on its specific branches.

Another important monograph on lattice theory was written by G. GRÄTZER and published for the first time in 1978: *General Lattice Theory* [Grä1]. This book includes what the author considered to be the most important results and research methods of lattice theory proper. As it treats the basics of the theory in depth, it does not include chapters on applications or the areas that had become separate fields of study, the author, however, provides appropriate further references to allied investigations throughout the book. G. GRÄTZER's monograph was published again in 1998 ([Grä2]). The 20 years between the two editions meant a tremendous progress in lattice theory, however, "the change is in the superstructure not in the foundation" ([Grä2], p. xv), so the author managed to keep the content of the first edition unchanged and to present new results and developments in the form of appendices which include essays on several topics written by various mathematicians.

Chapter 2

Mathematical background

In this chapter basic concepts and their properties concerning lattice theory and topology are presented in order to facilitate further reading. The choice of concepts was guided solely by their appearance in the following chapters. We suppose basic knowledge of binary relations, partially ordered sets (posets) and algebras. In the whole of the work the abbreviation "iff" stands for "if and only if".

2.1 Basic concepts in lattice theory

2.1.1 Two definitions of a lattice

A lattice can be viewed as a special type of poset, or as an algebra with two binary operations. These descriptions enable us to apply the concepts and methods of both the theory of posets and universal algebra depending on a particular situation. First, we state a definition of a lattice as an ordered set, for which reason we use the usual symbol "$\leq$" to denote a relation of partial ordering.

> Let L be a nonempty set with partial ordering $\leq$. $(L, \leq)$ is called a *lattice* iff there exist the infimum (greatest lower bound, called *meet*), $\inf\{a, b\}$, and the supremum (least upper bound, called *join*), $\sup\{a, b\}$, for any two elements $a, b \in L$. L is called a *complete lattice* iff there exist the infimum and the supremum for every subset of L.

We shall use the notations

$$a \wedge b = \inf\{a, b\},$$
$$a \vee b = \sup\{a, b\}.$$

The supremum (infimum) of a set H will be denoted by $\bigvee H (\bigwedge H)$. A lattice in which every countable lattice subset has an infimum and supremum is called a *σ–lattice*. The join and meet have the following properties:

(L1) Idempotent identity: $a \wedge a = a,\ a \vee a = a,$

(L2) Commutative identity: $a \wedge b = b \wedge a,\ a \vee b = b \vee a,$

(L3) Associative identity: $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ (a \vee b) \vee c = a \vee (b \vee c),$

(L4) Absorption identity: $a \wedge (a \vee b) = a,\ a \vee (a \wedge b) = a.$

We can define lattices as algebras in the following way: an algebra $(L, \wedge, \vee)$ is called a lattice iff L is a nonempty set, $\wedge$ and $\vee$ are binary operations which satisfy (L1)–(L4). The following theorem states that a lattice as an algebra and a lattice as a poset are equivalent concepts and shows how we obtain an algebra from a poset and inverse:

(i) *Let the poset $\mathcal{L} = (L, \leq)$ be a lattice. Set*

$$a \wedge b = inf\{a, b\},$$
$$a \vee b = sup\{a, b\}.$$

Then the algebra $\mathcal{L}^a = (L, \wedge, \vee)$ is a lattice.

(ii) *Let the algebra $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. Set*

$$a \leq b \quad \text{iff} \quad a \wedge b = a.$$

Then $\mathcal{L}^p = (L, \leq)$ is a poset which is a lattice.

(iii) *Let the poset $\mathcal{L} = (L, \leq)$ be a lattice. Then $(\mathcal{L}^a)^p = \mathcal{L}$.*

(iv) *Let the algebra $\mathcal{L} = (L, \wedge, \vee)$ be a lattice. Then $(\mathcal{L}^p)^a = \mathcal{L}$.*

The starting definition could have been preceded by a more general concepts of meet–semilattice and join–semilattice: a poset is called a *meet–semilattice*, dually *join–semilattice*, iff $\inf\{a, b\}$, dually $\sup\{a, b\}$, exists for any two element a and b of the poset. A lattice would then be defined as a poset which is both a meet– and join–semilattice.

2.1.2 Some order–theoretic notions

If a lattice has a greatest element, we shall denote it by 1, if it has a least element, it will be denoted by 0. A *bounded lattice* means a lattice with 0 and 1. We say that *a covers b* (in notation $a \succ b$, or $b \prec a$) iff $b < a$ and there exists no x such that $b < x < a$. An element a is called an *atom* iff $a \succ 0$ and a *dual atom* iff $a \prec 1$.

A poset is said to satisfy the *ascending chain condition* iff all increasing chains terminate; dually it is said to satisfy the *descending chain condition.* A chain $C(a,b)$, or denoted just C, between elements a and b of a poset P:

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

is called a *maximal chain* between a and b iff a_i covers a_{i-1} for all $i = 1, \ldots, n$. The *length* $l(C)$ of a finite chain C is $|C| - 1$. By the length of a poset P we mean the supremum length of a chain in P. A poset is said to satisfy the *finite chain condition* (or is of *finite length*) iff all chains are finite.

One of important properties a finite poset S may have, and a property which will be also discussed in the papers analyzed in this work, is the Jordan–Dedekind chain condition:

> **(JD)** If $a, b \in S, a < b$, and $C_1(a,b), C_2(a,b)$ are maximal chains with the least element a and the greatest element b, then these chains have the same length.

Let $C_1(a,b) : a = a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n = b$ be a chain between a and b. We call a chain $C_2(a,b) : a = b_0 \leq b_1 \leq \cdots \leq b_{m-1} \leq b_m = b$ a *refinement* of $C_1(a,b)$ iff for each $a_i, i = 0, \ldots, n$ there exists $b_j, j = 0, \ldots, m$ such that $a_i = b_j$. The refinement is called *proper* iff there exists b_k such that $b_k \neq a_i$ for all a_i.

By a *quotient* a/b of L we mean an ordered pair of elements $a, b \in L$ satisfying $b \leq a$. We call a/b a *prime quotient* iff $b \prec a$. By the *interval* $[a,b], a,\ b \in L,\ a \leq b$, we mean the set: $[a,b] = \{x | a \leq x \leq b\}$.

A lattice L is said to satisfy the *Upper Covering Condition* iff $a \preceq b$ implies $a \vee c \preceq b \vee c$, for all $a, b, c \in L$. The *Lower Covering Condition* is the dual.

2.1.3 Some algebraic concepts

We can introduce some common algebraic concepts for lattices in their usual meaning. There are two isomorphism concepts which coincide:

- lattices $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ are *isomorphic*, denoted by $L_1 \cong L_2$, and the mapping $\varphi : L_1 \to L_2$ is an *isomorphism* iff φ is one–to–one and onto and $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$;
- lattices $(L_1, \leq)$ and $(L_2, \leq)$ are *isomorphic*, denoted by $L_1 \cong L_2$, and the mapping $\varphi : L_1 \to L_2$ is an *isomorphism* iff φ is one–to–one and onto and $a \leq b$ in L_1 iff $\varphi(a) \leq \varphi(b)$ in L_2.

An isomorphism of a lattice into itself is called an *automorphism.* A mapping $\varphi : P_1 \to P_2$ is an *isotone mapping* of a poset P_1 into a poset P_2 iff $a \leq b$ in P_1 implies that $\varphi(a) \leq \varphi(b)$ in P_2. A *(lattice) homomorphism* is a mapping of a lattice L_1 into a lattice L_2 satisfying $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ (i. e. it is a *meet–homomorphism* and *join–homomorphism*). A homomorphism of a lattice into itself is called an *endomorphism*, a one–to–one homomorphism is also called an *embedding.*

We also apply the notion of subalgebra: a nonempty subset K of a lattice $(L, \wedge, \vee)$ is called a *sublattice* of L iff $a \wedge b, a \vee b \in K$ for each $a, b \in K$.[1] The subset K of a lattice L is called *convex* iff $a, b \in K, c \in L$ such that $a \leq c \leq b$ imply $c \in K$. An important example of a convex sublattice is an interval. A sublattice I of L is called an *ideal* iff $i \in I$ and $a \in L$ imply that $a \wedge i \in I$. An ideal I is proper iff $I \neq L$. A proper ideal I is *prime* iff $a, b \in L$ and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$. An ideal generated by a one–element set is called a *principal ideal.* Let $\mathrm{Id}L$ denote the set of all ideals of L, then $\mathrm{Id}L$ is a poset under the set inclusion which is a lattice ($I \wedge J = I \cap J, I \vee J = [I \cup J]$, where $[I \cup J]$ means the ideal generated by $I \cup J$). By dualizing the concept of ideal, we obtain the notions of *dual ideal*, *principal dual ideal, prime dual ideal.*

2.1.4 Congruence relations

An equivalence relation Θ on L is called a *congruence relation* L iff

$$a_1 \equiv b_1 (\mathrm{mod}\Theta),\ a_2 \equiv b_2 (\mathrm{mod}\Theta)$$

imply that

$$a_1 \wedge a_2 \equiv b_1 \wedge b_2 (\mathrm{mod}\Theta),\ a_1 \vee a_2 \equiv b_1 \vee b_2 (\mathrm{mod}\Theta).^2$$

Let $\mathrm{Con}L$ denote the set of all congruence relations on L partially ordered by set inclusion. Then $\mathrm{Con}L$ is a lattice, called the *congruence*

[1] We remind that a subset of a lattice L may be a lattice with respect to the inclusion relation $\leq$ of L, without being a sublattice of L.

[2] We can leave out the symbol "mod" in denoting a congruence equation.

lattice of L ($\Theta \wedge \Phi = \Theta \cap \Phi$ and $x \equiv y(\Theta \vee \Phi)$ iff there is a sequence $z_0 = x \wedge y, z_1, \ldots, z_{n-1} = x \vee y$ of elements of L such that $z_0 \leq z_1 \leq \cdots \leq z_{n-1}$ and for each $i, 0 \leq i \leq n-1, z_i \equiv z_{i+1}(\Theta)$ or $z_i \equiv z_{i+1}(\Phi)$. We say that two congruences Θ and Φ of L *permute* (or are *permutable*) iff $\Theta \vee \Phi = \Theta \cdot \Phi$, where $\Theta \cdot \Phi$ is the binary relation defined by

$$a \equiv b(\Theta \cdot \Phi) \text{ iff there exists a } c \in L \text{ such that } a \equiv c(\Theta) \text{ and } c \equiv b(\Phi).$$

An equivalent definition is that Θ and Φ permute iff $\Theta \cdot \Phi = \Phi \cdot \Theta$, which is equivalent to $\Theta \cdot \Phi$ being a congruence relation. A lattice L is called a *simple* iff it has only the two trivial congruences ι and ω ($x \equiv x(\iota)$ iff $x = y$ and $x \equiv y(\omega)$ for all $x, y \in L$).

Another algebraic concept is a quotient lattice: Let L be a lattice, Θ a congruence relation on L. Let L/Θ denote the set of blocks of the partition of L induced by Θ, i. e. $L/\Theta = \{\Theta_a | a \in$L$\}$. We define $\Theta_a \wedge \Theta_b = \Theta_{a \wedge b}, \Theta_a \vee \Theta_b = \Theta_{a \vee b}$, and obtain a lattice, called the *quotient lattice* (or *factor lattice*) of L (modulo Θ).

Congruence relations can be used for defining perspectivity and projectivity of quotients. Let Θ be a congruence relation on a lattice L, let $a/b, c/d$ be quotients in L. We say that the quotient a/b is *perspective* to the quotient c/d iff the following holds:

$$a \equiv b(\Theta) \text{ iff } c \equiv d(\Theta).$$

We say that a/b is *projective* to c/d iff for some natural number n, there exist quotients e_i/f_i:

$$a/b = e_0/f_0, e_1/f_1, \ldots, e_n/f_n = c/d$$

such that e_i/f_i is perspective to e_{i+1}/f_{i+1}, for each $i = 0, \ldots, n-1$.

2.1.5 Direct and subdirect products

A number of papers analyzed in this work deals with the problem of representation of lattices as direct product of indecomposable lattices. Such representation, or factorization, theorems present important tools in investigating the properties of any algebraic structure, and also methods of constructing new algebras.

Let $L_i, i \in I$ be a family of lattices. By the Cartesian product of the sets L_i, we mean the set of all functions

$$f : I \to \bigcup_{i \in I} L_i$$

such that $f(i) \in L_i$ for all $i \in I$. We define $\wedge$ and $\vee$ "componentwise" in the Cartesian product, i. e. $f \wedge g = h, f \vee g = k$ mean:

$$f(i) \wedge g(i) = h(i),\ f(i) \vee g(i) = k(i),$$

for all $i \in I$. The resulting lattice is the *direct product*, denoted by $\prod_{i \in I} L_i$, or $L_1 \times L_2 \times \dots$. The lattices L_i are called *factors*. If $L_i = L$ for all $i \in I$, we get the *direct power* L^I. A lattice A is called *(directly) indecomposable* iff it has no representation in the form $A = A_1 \times A_2$, where both A_1, A_2 have more than one element. If such representation exists, we call A *(directly) decomposable*.

A weakened form of direct product is the concept of subdirect product. Let us define a mapping e_i^I of $\prod_{i \in I} L_i$ into L_i by $e_i^I : f \to f(i)$ for $i \in I$ and call e_i^I a projection. A sublattice S of a direct product $\prod_{i \in I} L_i$ is called a *subdirect product* of $\prod_{i \in I} L_i$ iff $e_i^I(S) = L_i$ for all $i \in I$, where e_i^I is the i-th projection.

2.1.6 Types of lattices and their elements

Complements

Let L be a bounded lattice, $a \in L$ is called a *complement* of $b \in L$ iff $a \wedge b = 0$, $a \vee b = 1$. Let $a \in [x, y]$; b is a *relative complement of* a *in* $[x, y]$ iff $a \wedge b = x$, $a \vee b = y$. A *complemented lattice* is a bounded lattice in which every element has a complement. A *relatively complemented lattice* is a lattice in which every element has a relative complement in any interval containing it. Let L be a lattice with 0; an element a^* is a *pseudocomplement* of $a \in L$ iff $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. A *pseudocomplemented lattice* is a lattice in which every element has a pseudocomplement.

Modularity and semimodularity

A lattice L is called *modular* iff any $a, b, c, \in L, a \leq c$ satisfy the modular identity, i. e.

$$a \vee (b \wedge c) = (a \vee b) \wedge c.$$

There are various conditions describing modular lattices: A lattice L is modular iff it does not contain a sublattice isomorphic to the lattice in Figure 2.1 (we shall call a lattice isomorphic to this lattice a *pentagon*).

A modular lattice satisfies both the Upper Covering Condition and the Lower Covering Condition. If L is a lattice of finite length, it is modular iff it satisfies the Upper and Lower Covering Conditions.

A lattice L is called *(upper) semimodular* iff it satisfies the Upper Covering Condition, i. e. for $a, b \in L$:

$$a \prec b \text{ implies that } a \vee c \prec b \vee c \text{ or } a \vee c = b \vee c.$$

If L is a semimodular lattice of finite length, then any two maximal chains of L are of the same length.

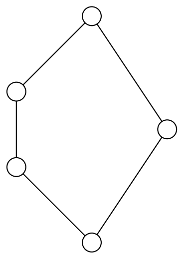

Figure 2.1: A pentagon

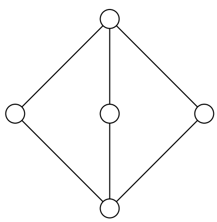

Figure 2.2: A diamond

Distributivity

A lattice L is called *distributive* iff all elements $a, b, c \in L$ satisfy the distributive identity:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

The stated distributivity identity is equivalent to its dual, and we could present a number of other equivalent conditions for a lattice to be distributive, the most typical is: A lattice L is distributive iff it does not contain a sublattice isomorphic to a pentagon or a *diamond*, i. e. the lattice in Figure 2.2. As a consequence we get that L is distributive iff every element has at most one relative complement in any interval, or that L is distributive iff for any two ideals I, J of L: $I \vee J = \{i \vee j | i \in I, j \in J\}$.

In any distributive lattice and any finite index–set S, we have by induction:

$$x \wedge \bigvee_S y_\sigma = \bigvee_S (x \wedge y_\sigma), \sigma \in S, \tag{2.1}$$

$$x \vee \bigwedge_S y_\sigma = \bigwedge_S (x \wedge y_\sigma), \sigma \in S. \tag{2.2}$$

However, the formulas (2.1) and (2.2) do not hold generally for any arbitrary set S, not even in every complete distributive lattice. A lattice in which (2.1) and (2.2) hold for any set S is called an *infinitely distributive lattice*. The distributive identities can be further generalized, we call a (complete) lattice *completely distributive* iff it satisfies the *extended distributive identities*:

$$\bigwedge_C [\bigvee_{A_\gamma} x_{\gamma,\alpha}] = \bigvee_F [\bigwedge_C x_{\gamma,\phi(\gamma)}], \tag{2.3}$$

$$\bigvee_C [\bigwedge_{A_\gamma} x_{\gamma,\alpha}] = \bigwedge_F [\bigvee_C x_{\gamma,\phi(\gamma)}], \tag{2.4}$$

for any nonvoid family of index–sets A_γ, one for each $\gamma \in C$, provided F is the set of all functions ϕ with domain C and $\phi(\gamma) \in A_\gamma$.

Boolean lattices and Boolean algebras

A complemented distributive lattice L is called a *Boolean lattice*. In a Boolean lattice, each element x has one and only one complement, denoted by x', and for each $x, y \in L$: *1.* $(x')' = x$, and *2.* $(x \wedge y)' = x' \vee y'; (x \vee y)' = x' \wedge y'$. Every Boolean lattice is dually isomorphic with itself (i. e. it is self–dual). Since the complements are unique in Boolean lattices, we can regard the latter as algebras with two binary operations $\wedge, \vee$, and one unary operation $'$. When considered in this way, we call them *Boolean algebras*. A distributive lattice in which every element has a relative complement is called a *generalized Boolean algebra*.

All complete Boolean lattices satisfy the distributive identities (2.1) and (2.2) for any index–set S, however, they do not generally fulfill the extended distributive identities (2.3) and (2.4). The class of all completely distributive (complete) lattices contains many non–Boolean lattices, e. g. every complete chain or any closed sublattice of a direct product of complete chains.

Neutral elements and the centre

An element a is called *neutral* iff for all $x, y \in L$:

$$(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a).$$

An ideal I of a lattice L is called neutral iff I is neutral as an element of $\mathrm{Id}L$. The *centre* of a bounded lattice is the sublattice of complemented neutral elements.

2.1.7 Examples of lattices

Examples of lattices include:

- the subsets of a given set, ordered by inclusion (the supremum is given by the union and the infimum by the intersection of the subsets), which form a Boolean lattice;

- the unit interval $[0, 1]$ and the extended real number line with the familiar total order and the ordinary suprema and infima, which are distributive lattices;

- the subgroups of a group, ordered by inclusion (the supremum is given by a subgroup generated by the union of the groups and the infimum is given by their intersection); the lattice of normal subgroups of a group is modular;

- the set of natural numbers, ordered by divisibility (the supremum is given by the least common multiple and the infimum by the greatest common divisor), which is a distributive lattice;

- the submodules of a module, ordered by inclusion (the supremum is given by the sum of submodules and the infimum by the intersection), which form a modular lattice;

- the open sets of a topological space, ordered by inclusion (the supremum is given by the union of open sets and the infimum by the intersection);

- the topologies on a set, ordered by inclusion (the infimum is given by the intersection of the topologies, the supremum by the topology generated by the union of topologies);

- the convex subsets of a real or complex vector space, ordered by inclusion (the infimum is given by the intersection, the supremum by the convex hull of the union).

2.2 Basic concepts in topology

A *topological space* can be defined as a set T together with a family of "closed" subsets of T having the following three properties:

(1) The sum of any two closed sets is closed;

(2) Any intersection of closed sets is closed;

(3) T and empty set $\emptyset$ are closed.

Thus in any topological space there is defined a *closure operation* satisfying:

(C1) $X \subset \overline{X}$;

(C2) $\overline{X} = \overline{\overline{X}}$;

(C3) If $X \subset Y$, then $\overline{X} \subset \overline{Y}$;

(C3') $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

A set is called *open* iff its complement is closed. Let G be a subset of a topological space T, we say that G is dense in T iff $\overline{G} = T$. A one–element subset $\{x\}$ of T will be called a *point* and denoted by x. A *neighborhood* $U(x)$ of a point x is an open set containing x. The *system of neighborhoods* of a point x in the space T is the collection of all neighborhoods of the point x.

A topological space is called a T_1*–space* iff it satisfies:

(C4) If p is a point, then $\overline{p} = p$.

A topological space is called a T_0*–space* iff it satisfies:

(C4') For two points p and q: $\overline{p} = \overline{q}$ implies $p = q$.

A topological space is called a *Hausdorff space* iff it satisfies:

If p and q are two distinct points, then there exist disjoint open sets U and V such that $p \in U$ and $q \in V$ (Hausdorff condition).

A topological space is called *normal* iff it satisfies:

If A and B are disjoint closed sets, then there exist open sets $A \subset U$ and $B \subset V$ such that $U \cap V = \emptyset$ (Normality condition).

A family R of closed sets in a T_0–space T is called a *basis of closed sets* (closed basis) iff every closed subset of T is an intersection of members of R. Dually, a family S of open sets is called a *basis of open sets* (open basis) iff every open subset of T is a sum of sets of R.

A *homeomorphism* between topological spaces X, Y is a continuous, one–to–one and onto mapping $f : X \to Y$ such that the inverse $f^{-1} : Y \to X$ is also continuous (a mapping $f : X \to Y$ is *continuous* iff the completely inverse image $f^{-1}(U)$ of every open set in Y is an open set in X). The continuity of the inverse function does not in general follow from the other; it does, however, if X is compact and Y is Hausdorff.

A topological space X is called *compact* (in early literature referred to as *bicompact*) iff from any family R of open sets S_r such that $\bigcup_R S_r = X$, we can extract a finite subfamily P from R such that $\bigcup_P S_p = X$. X is called *totally disconnected* iff any two distinct points lie in complementary closed sets. A totally disconnected compact Hausdorff space is called a *Boolean space.*

Metric spaces

The theory of metric spaces is a part of general topology although it is built upon the notion of a metric, which is not a topological concept. A *metric space* is a collection M of elements (points) with a defined real–valued distance function (a metric) $\delta(x, y)$ which satisfies:

(M1) $\delta(x, x) = 0$, while $\delta(x, y) > 0$ iff $x \neq y$,

(M2) $\delta(x, y) = \delta(y, x)$,

(M3) $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$.

Every metric induces in a natural way a metric topology on an nonempty set: a subset S of a metric space is called open iff for any $a \in S$, a constant $\varepsilon > 0$ can be found such that $|x - a| < \varepsilon$ implies $x \in S$; the system of all open sets generates a topology called a metric topology. An infinite sequence $x_1, x_2, \ldots$ of points of a metric space is said to *converge* to the limit a iff $\lim_{n \to \infty} \delta(x_n, a) = 0$.

Chapter 3

Lattice theory in Czech mathematics until 1963

3.1 Introduction

3.1.1 The situation in Czech algebra from 1930's until the beginning of 1960's

The implementation of ideas of modern algebra in Czech mathematics, in particular the tendency to study general algebraic structures, is closely connected with the names of V. Kořínek in Prague and O. Borůvka in Brno. In the mid–1930's V. Kořínek started to be interested in problems of the theory of algebras and group theory, which also led him to an investigation concerning questions of lattice theory a few years later. At the end of 1930's O. Borůvka's began his work on set partition theory and the theory of groupoids and groups and his theory of partitions rates among the first on this topic in the world.

The end of 1930's was the time when Czech algebraic research was advancing in the directions initiated by Van der Waerden's book *Modern Algebra*, however, the life of the whole country was soon to be irretrievably disturbed by the political situation. The closure of Czech universities on November 17, 1939 meant a disruption of the official research centers, and led to considerable limitations in scientific life. The mathematicians who continued working found themselves in isolation not being able to maintain adequate contacts with the latest results in their fields.

The situation after WWII was improving slowly, it was necessary to rebuilt both pedagogical and scientific activities. Several years after the war can, in many respects, be labeled as a transitional period, although the professors at the re–opened universities did their best to provide the coming generation with scientific education and guidance in research activities. In the following years the number of mathematicians, as well as mathematics students and teachers increased, new institutions, schools, and research work places were created and the range of investigated areas expanded. To characterize the period 1945–1960, we can say that algebra was developing in the topical directions, however, it was to a certain extent isolated, which resulted from the continuing lack of necessary contacts and literature. The international contacts were very scarce, and they were restricted mainly to the countries with suitable political systems: Poland, Hungary, the German Democratic Republic, the U.S.S.R., Romania and Bulgaria. The overall situation improved in many ways with the 1960's, mainly as far as the availability of literature and possibilities to publish were concerned, which especially the young, coming generations of mathematicians could profit from.[1]

Both V. KOŘÍNEK and O. BORŮVKA also played an important role in educating new generations of mathematicians, dissemination of new algebraic trends and encouraging further research. During several years after WWII V. KOŘÍNEK's "advanced" seminar was the only place in Prague where algebra was systematically studied. The work in the seminar focused on up–to–date problems, mainly abelean groups, universal algebras and lattice theory. A series of theses supervised by V. KOŘÍNEK were inspired by the problems discussed there. The textbook *Foundations of algebra* (1953, 1956) written by V. KOŘÍNEK became the basic algebraic study material for many years to come.

Brno algebraic research is characteristics for studying abstract algebraic structures initiated by the work of O. BORŮVKA. His textbook *An introduction to group theory* (1944, enlarged edition 1952) set a model for the formation of theories of general algebras and provided readers not only with the author's original exposition of the topic but also with an inspiration for further investigation. This textbook became the basis for his renowned monograph *Foundations of the theory of groupoids and groups* (1960, 1962, 1975).

Ordered sets (both totally and partially) became another traditional Brno area of research, which was originated by JOSEF NOVÁK after WWII. He had been led to this field by topological problems and his investigation attracted the attention of young scientists, among them the outstanding mathematician M. NOVOTNÝ, who was particularly inter-

[1] DRBOHLAV, K., "Algebra, logika a teorie množin" in [CM], pp. 54–69 and [Kořš5].

ested in studying cardinal operations within different types of ordered sets in his early period of research. In 1963 M. NOVOTNÝ founded a successful seminar on ordered sets and general algebraic structures, and at the beginning of 1960's he initiated organizing regular summer schools on the theory of ordered sets which played an important role in communicating ideas and results in the field and whose early development is described in the following subsection. In the late 1950's F. ŠIK started the investigation of ordered groups and l–groups which also belong to a traditional field of Brno algebraic research.[2]

As the beginning of 1960's marked a change in the development of mathematical research in the sense that the efforts for improvements from the previous years brought their results, and the coming years became characteristic for fast growth of intensive work with significant contributions of the young generation, it is natural to end our analysis of initial activities connected with lattice theory at that time. The ending year was set to be 1963 because that year an international conference on ordered sets, the first in Czechoslovakia, was organized in Brno.

3.1.2 Summer schools and the international Conference on Ordered Sets

The roots of summer schools on ordered sets and general algebras are attributed to J. NOVÁK. He was appointed Professor at Masaryk University in 1945 and, in a certain way, continued in the investigation of topological problems which had been started by E. ČECH in his Brno topological seminar 1936–39. J. NOVÁK commenced systematic studies of topologies on ordered sets and introduced this topic also to young researchers interested in the field. He organized summer seminars on ordered sets in 1950 and 1951, which can be considered the first stage in the tradition of the later summer schools on partially ordered sets and general algebras. These two seminars were characteristic for a small number of participants (the first three, the second four), which enabled an individual care for the young participants and resulted in intensive and fruitful work. The main topic was ordered continua and the problems discussed there became an inspiration for the participants further research. J. NOVÁK's departure to Prague ended the sessions. However, 11 years later he inspired a resumption of the seminars.

The summer school (still called a seminar) of 1962 was on the one hand a continuation of the previous seminars, on the other hand its

[2] [UB], pp. 207–209.

new form and content reflected the new period: areas of investigation shifted from totally ordered sets to partially ordered sets (an influence of G. BIRKHOFF's *Lattice Theory*) and the number of mathematicians interested in the research increased. The system of work took form of lectures, which focused mainly on partially ordered sets and partially ordered sets with algebraic operations (lattices, l–groups) or with topology. The following year, 1963, brought another new feature which became a tradition of summer schools: participation of young talented students who had an opportunity to present their own results there. The two seminars proved the usefulness of such sessions, they were seen as a highly beneficial contribution in many directions, generally improving the quality of research work. Thus, the summer schools on ordered sets and general algebras became an annual event. Though the initial organization came from Brno departments of mathematics, the participants included also mathematicians from Slovakia.

The success of the two summer schools convinced the organizers that they could prepare an international conference on partially ordered sets. It was held in Brno in November 1963 as a part of the celebration of 50 years of Brno branch of the Union of Czechoslovak Mathematicians and Physicists. The conference enabled personal meetings of Czechoslovak mathematicians with algebraists from several other countries (Hungary, Romania, Bulgaria, Yugoslavia, the Soviet Union, the German Democratic Republic, the choice being limited for obvious political reasons), which was at that time a rare event. Sharing the results showed that the research on ordered sets in the Czechoslovak Republic could be compared internationally.

The topics of the conference talks can be divided into four groups: general questions concerning partially ordered sets, lattice theory, ordered algebraic structures and the relations between partial order and topology. The following Czech and Slovak mathematicians actively participated in the conference: O. BORŮVKA, J. JAKUBÍK, M. KATĚTOV, M. KOLIBIAR, K. KOUTSKÝ, V. NOVÁK, M. NOVOTNÝ, B. RIEČAN, M. SEKANINA, L. SKULA. The success of the conference brought the idea of organizing summer schools with foreign participation, however, this was realized only several years later.[3]

[3] This subsection is based upon the information from [Nov7], [Nov8], [N–S].

3.1.3 An outline in the development connected to lattice theory until 1963

Although the emergence of lattices in Czech literature is reflected in the work of O. Borůvka as well as V. Kořínek, the two mathematicians approached the new theory differently. While O. Borůvka recognized it as a theory which provided a structural framework for his partition theory, V. Kořínek became interested in one of the problems within this theory, namely the Jordan–Hölder–Schreier–Zassenhaus theorem. O. Borůvka's first reference to lattice theory is recorded in his paper from 1939, V. Kořínek's first paper in this field was published in 1941. Another mathematician whose work was in a way connected to lattice theory in the early 1940's was B. Pospíšil. Although he was involved in the problems of general topological spaces, his results produce important consequences for Boolean rings as well.

Both O. Borůvka and V. Kořínek deserve a great merit for encouraging, supporting and providing suitable topics for young mathematicians' research, including topics from lattice theory. In the first half of 1950's the students of V. Kořínek's seminar: L. Janoš, Č. Vitner, V. Vilhelm, and V. Havel wrote theses which continued in V. Kořínek's study on lattice theory. O. Borůvka, apart from influencing Brno algebra, "exported" lattice theory to Bratislava after WWII when recommending G. Birkhoff's monograph as a possible source of inspiration for young Slovak researches.

Brno mathematics is typical for the treatment of lattice theory in its applications, or working on the border–areas dealing with questions related to lattice theory as well. K. Koutský, who was engaged in topology, created an original theory of topological lattices. In the 1950's several researchers working on ordered sets and general algebraic structures produced results connected to lattice theory, of which the following should be mentioned: M. Novotný's studies of cardinal arithmetic and isotone functionals on ordered sets, F. Šik's research on l–groups, K. Čulík's works concerning lexicographic sum of partially ordered sets. In the first half of 1960's other two mathematicians, V. Novák and L. Skula, joined the group of Brno mathematicians working on abstract algebraic structures and started their research activities.

A very specific contribution to lattice theory is attributed to L. Rieger, who was the first Czech mathematician to work systematically in mathematical logic. As he was interested in problems concerning algebraic methods in this field, his results concern to a great extent Boolean algebras. He dealt mainly with special kinds of Boolean algebras, applied his results to mathematical logic, and his conclusions also solve some problems stated by G. Birkhoff in [LT–48].

The following sections present the works of Czech mathematicians concerning lattice theory, they are divided mainly according to individual mathematicians and we analyze their particular papers. The first one is devoted to O. Borůvka. We follow the way in which he related the theory of partitions of sets and partitions in sets to lattice theory, when preparing them as a starting point for the theory of groupoids and groups. V. Kořínek's papers are analyzed in the next section. As the topic he was working on, the Jordan–Hölder–Schreier–Zassenhaus theorem, belonged to very popular themes, various approaches to the theorem are discussed in more detail there. The papers of L. Janoš, Č. Vitner, and V. Vilhelm are dealt with in one section as they include a continuation of V. Kořínek's results. Next five sections describe the contributions of L. Rieger, K. Koutský, M. Mikulík, O. Hájek and V. Havel. The following section is devoted to mathematicians not dealing directly with lattice theory, it consists of short overviews of results related to this field. The last section briefly comments on the development of Czech lattice theoretical terminology.

3.2 Otakar Borůvka and lattices

Otakar Borůvka (1899–1995) was, for decades, one of the leading mathematician in Czechoslovakia. His mathematical work included the areas of differential geometry, algebra and differential equations, and he became truly a legend among Brno mathematicians.

In 1918–1922 Otakar Borůvka studied at the Czech Technical College in Brno, and from 1920 also at Masaryk University (where he was also working as an assistant to Professor M. Lerch) from which he graduated in 1922 studying mathematics and physics. He finished his doctoral studies there the following year. He spent two years (1926–1927, 1930–1931) at Professor E. Cartane in Paris, and six months (1930–1931) in a course with W. Blaschke in Hamburk. In 1934 O. Borůvka was appointed Assistant Professor, in 1946 (with the effect from 1940) Professor at Masaryk University in Brno. The first works (1923–1925), which were inspired by his teacher M. Lerch, deal with classical mathematical analysis, in 1925–1935 he became engaged in differential geometry. In the middle of 1930's he started the studies concerning modern algebra, formed the theory of set decomposition, the theory of groupoids and laid the foundation of a theory of scientific classification.

In the 1950's he purposefully turned to an investigation of differential equations and created a theory of global transformations of linear differential equations of the second order. O. Borůvka's mathematical achievements met with favorable recognition even abroad and the world literature was affected by a number of his methods and results.[4]

O. Borůvka was the first Czech mathematician who turned his attention to the emerging lattice theory in his papers, which happened while he was developing his theory of partitions of and in a set. He founded the theory around 1939 independently of equivalence theory of P. Dubreil. These two theories are equal, the differences are in methods of work: partitions are more suitable for some applications because they are based on a notion of set, compared to a more complicated concept of equivalence (O. Borůvka, the preface of [Bor7a]). When forming his set partition theory O. Borůvka from the very beginning makes clear its connection to lattice theory, and even stated the aim to present the general theory of partitions in a set as a realization of lattice theory.

O. Borůvka was creating his partition theory with the view of its application to groupoid theory which he saw as a basis for building group theory. We can follow the process of shaping his unique exposition of this topic as it is recorded in several publications started by a paper on groupoid theory in 1939 and completed by the famous monograph *Foundations of the theory of groupoids and groups* first published in 1960. In his earliest paper concerning partition theory [Bor1] the author outlines the first ideas of this theory with regard to its application to groupoid theory, then he develops the theory of partitions of a set [Bor3a], and the theory of partitions in a set [Bor5] in more detail. Group theory is presented in [Bor4, Bor6]. As our objective here is not to analyze these works thoroughly, we shall only show in the following paragraphs how O. Borůvka relates his results to lattice theory.

3.2.1 *Groupoid theory* [Bor1] (1939)

The paper presents an investigation of groupoids defined in the usual sense as a pair consisting of a non–void set G and a multiplication in G. In the previous literature there had already existed a number of publications dealing with groupoids, though, usually with disconnected results. O. Borůvka mentions the papers treating the subject more

[4]We refer the reader to the monograph *Otakar Borůvka*, Universitas Masarykiana, Brno 1996 for a detailed description of the life and work of O. Borůvka.

systematically: B. H. HAUSMANN and O. ORE[5] used the term groupoids in the same sense as we have mentioned and G. BIRKHOFF ([Bir3]) in a somewhat restricted sense. O. BORŮVKA became interested in groupoids when recognizing there areas parallel to group theory. The main concepts of this paper are a homomorphism and factoroid (as a generalization of factor group).

The first part of the paper is devoted to set partition theory because it is considered to be a theory that finds an immediate application in groupoid theory. The author distinguishes a partition of a set and a partition in a set. A *partition of a set* G is defined in the usual way as a set $\overline{G}$ of non–empty pairwise disjoint subsets of G (we shall use the term classes) whose union is G. A *partition in a set* G need not satisfy the condition that the union of the classes is the whole set G. In the case of partitions of a set, the author partially orders a system of partitions in the following way:

Definition 3.1. Let $\overline{G}_1, \overline{G}_2$ be two partitions of a set $G \neq \emptyset$, then $\overline{G}_1 \geq \overline{G}_2$ iff $\overline{G}_1$ consists only of classes which are unions (or equal to) classes of $\overline{G}_2$. In that case we say that $\overline{G}_1(\overline{G}_2)$ is a *covering (refinement)* of $\overline{G}_2(\overline{G}_1)$.

O. BORŮVKA defines the least (common) covering and the greatest (common) refinement of a system of partitions of a set (which are the realizations of join and meet of a non–void set of partitions in a partition lattice) and makes a remark that the set of all partitions of a set with the given partial ordering presents an example of a complete lattice. He gives the definition of a lattice only in terms of the partial ordering and refers the reader for more details to G. BIRKHOFF's paper [Bir3]. There is no specific denotation either for lattice operations, or for partition operations in this paper.

O. BORŮVKA continues in the investigation of groupoids and factoroids in the next paper [Bor2] in which he concentrates on chains in factoroids.

3.2.2 *On partitions of sets* [Bor3a] (1943)

This paper continues to develop the ideas of [Bor1] concerning set partitions. O. BORŮVKA views a systematic investigation of partition theory as meaningful because a number of mathematical theorems (especially in set theory, topology and algebra) are closely connected to the concept

[5]HAUSMANN, B. A., ORE, O., *Theory of quasi–groups*, Amer. J. Math. 59 (1937), 983–1004.

of set partition. He stresses the facts that, although on the one hand partition theory belongs to lattice theory, as it is one of its realization, on the other hand lattice theory cannot provide an exhaustive picture of partition theory because it only describes the properties of partitions as the lattice elements and leaves the relationship between the elements of the partitions aside.

Compared to [Bor1] O. BORŮVKA presents a far more detailed exposition of the basic notions of partitions, which is the content of the first part of the article. He develops the properties of set partitions partially ordered in the usual sense, proving the properties corresponding to lattice elements properties: the commutativity, idempotency and associativity. While discussing the modularity law he distinguishes three types of modular elements (α–, β–, γ– modular) according to V. KOŘÍNEK (see the analysis of [Koř1]). This time O. BORŮVKA provides a definition of a lattice in terms of lattice operations. He denotes "$a \vee b$" by "$[a, b]$" and "$a \wedge b$" by "(a, b)," which corresponds to the notation used by O. ORE. O. BORŮVKA chooses the same denotation also for partition operations (the least covering and the greatest refinement).

The second part of the paper concentrates on permutable partitions which the author defines as follows:

Definition 3.2. Partitions $\overline{G}_1, \overline{G}_2$ are called permutable iff $X \in \overline{G}_1, Y \in \overline{G}_2$ such that $X, Y \subseteq Z \in \overline{G}_1 \vee \overline{G}_2$ imply that $X = Y$.

O. BORŮVKA proves various properties of permutable partitions and their elements, shows some applications to group theory and investigates the properties of partition mappings. The following relation to modularity is proved:

Theorem 3.1. *Let $\overline{G}_1, \overline{G}_2$ be two permutable partitions of $G \neq \emptyset$. Then $\overline{G}_1$ is α–modular with respect to $\overline{G}_2$ and $\overline{G}_3$, where $\overline{G}_3 \geq \overline{G}_1$.*

The third part of the paper deals with partition sequences, i. e. chains, in a partition lattice. The author shows that all permutable maximal chains[6] in a partition lattice have the same length.

This paper written in Czech has also its German version [Bor3b] which consists of the same results, it is, however, shortened by leaving out details of the proofs.

[6]Two chains of partitions are called permutable iff every partition included in one chain is permutable with every partition of the other chain.

3.2.3 *The theory of partitions in a set* [Bor5] (1946)

The paper contains an exposition of the theory of partitions in a set, which is meant to form the basis of the general groupoid theory. The theory is again presented as a realization of lattice theory. The first chapter deals with congruences and lattices. As O. BORŮVKA points out the content is not new, however, the subject is treated in a way formally different from the other authors, with the view of the following applications. The second chapter investigates the notions of the least covering and the greatest refinement of a system of partitions in a set with regard to lattice theory, and studies also some special systems of partitions. The third, last, chapter, includes the theory of associated partitions which play an important role in groupoid theory.

O. BORŮVKA defines a congruence relation as a binary relation on a non–empty set which is reflexive and transitive and distinguishes symmetric and antisymmetric congruences. He develops the concepts of a lower (upper) bound and a lattice in relation with the antisymmetric congruence. He defines a (complete) lattice in terms of two binary operations, and shows the connection between lattices and antisymmetric congruences. The author also speaks of lattice homomorphism and isomorphism.

An important step in the theory of partitions in a set is made by introducing the notions of a least (common) covering and a greatest (common) refinement of a system of partitions. Let $\mathcal{A}$ be a non–void system of partitions in a set $\Gamma, \Gamma \neq \emptyset$. $\bigcup \mathcal{A}$ denotes the sum of all subsets in Γ which are elements of the partitions included in the system $\mathcal{A}$. A finite sequence of subsets $A_1, \ldots, A_n$ of Γ, $A_i \in \bigcup \mathcal{A}$, will be called a chain in $\bigcup \mathcal{A}$ from A_1 to A_n iff $A_i \cap A_{i+1} \neq \emptyset$ for each $i = 1, \ldots, n-1$. The least covering of the system $\mathcal{A}$ is defined by the following way: we construct a partition P on the set $\bigcup \mathcal{A}$ in such a way that two subsets belong to the same class of P iff there exists a chain in $\bigcup \mathcal{A}$ from one to the other; the least covering of the system $\mathcal{A}$ is then the system of all subsets of Γ which are the sum of all elements of $\bigcup \mathcal{A}$ belonging to the same class of the partition P. O. BORŮVKA shows that the least covering of the system $\mathcal{A}$ is the least upper bound of this system in terms of a naturally defined congruence.

The concept of the greatest refinement of a system $\mathcal{A}$ is connected to the notion of the center of a beam[7] in $\mathcal{A}$. The author defines a beam in $\mathcal{A}$ as a subset of $\bigcup \mathcal{A}$ including one and only one element of each partition belonging to $\mathcal{A}$ if their intersection is not void. The centre of a beam is

[7] "Beam" in Czech: "trs".

defined as the intersection of all elements of the beam. Then we define the greatest refinement of the system $\mathcal{A}$ as the system of the centers of all beams in the system $\mathcal{A}$ providing it is not void. In case it is void we say that the system $\mathcal{A}$ does not have a greatest refinement. Thus, while a least covering exists for all non–void systems of refinements in Γ, a greatest refinement exists only in special cases. The author also shows that the greatest refinement of the system $\mathcal{A}$ is the greatest lower bound of this system in terms of a naturally defined congruence.

O. Borůvka investigates behaviour of special types of partitions in a set and applies his results to group theory. A more detailed study is devoted to systems which always have the greatest refinement. An important group of partitions are associated partitions:

Definition 3.3. Let $\overline{A}$ be a partition in a $\Gamma, \Gamma \neq \emptyset$. The *partition associated to* $\overline{A}$ (denoted by $soc\overline{A}$) is defined as the partition in the set $\Gamma \times \Gamma$ consisting of the products $X \times Y$ formed by all pairs (X, Y), $X, Y \in \overline{A}$.

We define the *Cartesian mapping* as a mapping $f : \overline{A} = soc\overline{A}$ for every $\overline{A}$.

The author shows which properties of partitions are invariant under the Cartesian mapping and proves that if we consider the partitions of Γ, the Cartesian mapping is a lattice isomorphism.

3.2.4 *An introduction to group theory* [Bor6] (1952)

The textbook *An introduction to group theory* was first published in 1944 [Bor4], although it had been prepared for printing already in 1941. We shall be interested in its second, enlarged, edition from 1952 because it concerns also lattice theory.

The book is meant to give an elementary introduction to group theory, however, there are two important remarks to be made: O. Borůvka consciously presents the topic in a way which differs from other contemporary writers and he also includes his own original results. The properties of groups are based on the properties of groupoids, and the part concerning groupoid theory consists to a great extent of O. Borůvka's own investigation. The revised edition [Bor6] is enlarged by including his results about partitions of a set from [Bor3a] and by an exposition on congruences and lattices. For the purpose of the book a lattice is here introduced in a part dealing with special types of groupoids; it is defined algebraically as a pair of two groupoids (defined on the same set) whose operations (called upper and lower multiplications) satisfy the commutative, associative, idempotent and absorption laws. The author shows the relation between the two operations and the partial ordering defined by them, and devote some space also to modular lattices. This time (compared to the previous works) O. Borůvka uses the more standard notation $\smile, \frown$ for the lattice operations, however, preserving the symbols [ ], ( ) for the partition operations.

3.2.5 *Foundations of the theory of groupoids and groups* [Bor7a]

This monograph is based on the *Introduction to group theory*, its content is, however, substantially enlarged and includes a number of O. Borůvka's original results concerning the topic. The first publication of *Foundations of the theory of groupoids and groups* was in German in 1960 [Bor7a], the second in Czech in 1962 [Bor7b], and the third in English in 1975 [Bor7c]. The development of the topic follows the principles of [Bor6] but as we have mentioned the extent is far greater.

In the part dealing with chains of partitions O. Borůvka proves a theorem on refinements of two chains which found its generalization in lattice theoretic form in a paper by V. Havel (for details see the analysis of [Hav3]). The introduction of lattice concepts corresponds to the one in [Bor6], however, in the chapter concerning group theory O. Borůvka points out to several examples of modular lattices, e. g.:

Theorem 3.2. *Let $S(G)$ be a nonempty system of subgroups of a group G, let every two elements of $S(G)$ be interchangeable, and let $S(G)$ be closed with respect to the intersections and the products of the pairs of subgroups. Then $S(G)$ is a modular lattice (with the operations of intersections and products). The system of the left (right) decompositions of G, generated by the individual elements of this lattice, forms a modular lattice (with respect to the operations of the least common covering and the greatest common refinement of the decompositions) which is isomorphic to the former one.*

3.3 Vladimír Kořínek and lattices

Vladimír Kořínek (1899–1981) studied mathematics and physics at Charles University in Prague where he graduated in 1923. In 1925–1931 he worked as an assistant at the Czech Technical University in Prague, the first two years for physics, then for mathematics. He was the assistant to Professor K. Rychlík who encouraged him in research work. In

1931 V. KOŘÍNEK made his "habilitation" in mathematics at the Faculty of Science of Charles University, however, due to the economic situation in the country the chances for pursuing academic career were scarce. It was only in 1935 that he was appointed Associate Professor of mathematics (at the Faculty of Science of Charles University). Immediately after WWII he returned to Charles University, where he was appointed Professor with the effect from 1940. He remained in this position (from 1952 at the newly established Faculty of Mathematics and Physics) until his retirement in 1970, though he continued in working part–time at the faculty up to 1975/76.[8]

V. KOŘÍNEK's research activities can be divided into five groups: arithmetic theory of quadratic forms, theory of algebras, group theory, lattice theory and the Frattini subgroups. His most famous work from group theory[9] deals with a decomposition of groups in a direct product of subgroups. This paper was cited in the distinguished books on group theory of the period and it is interesting to notice that its results were re–formulated for the case of completely modular lattice by A. G. KUROŠ in his *Teoria grupp* [Kur2]. V. KOŘÍNEK's work in lattice theory was commenced by the paper on the Zassenhaus refinements [Koř1] published in 1941 and shows a clear influence of O. ORE. Due to the complications in obtaining literature at that time, he obviously did not have access to G. BIRKHOFF's works; he did not get into the possession of the monograph *Lattice Theory* until autumn 1947.[10] Inspired by this book (which was soon to be replaced by its enlarged edition) he produced the second paper on lattice theory [Koř2a]. The work deals with the problem of the Jordan–Hölder theorem, which means a topic closely related to his first paper. Another influence which might have contributed to the choice of the theme for the second paper was again O. ORE. He visited Prague at the end of May in 1947 to give three lectures. One of the topics was the development of the Jordan–Hölder theorem.[11]

In this section we shall present the content of V. KOŘÍNEK's articles on lattice theory in more details.

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[8]For a detailed description of the life and work of V. KOŘÍNEK we refer the reader to [Koh].

[9]KOŘÍNEK, V., *Sur la décomposition d'un groupe en produit direct des sousgroupes*, Časopis pro pěstování matematiky 66 (1937), 261–286.

[10]O. Borůvka's archive, the correspondence between V. KOŘÍNEK and O. BORŮVKA: V. KOŘÍNEK obtained a copy of this book from O. BORŮVKA.

[11]O. BORŮVKA's archive, the correspondence between V. KOŘÍNEK and O. BORŮVKA.

3.3.1 *The Schreier Theorem and the Zassenhaus refinement in lattices* [Kоř1] (1941)

This paper is the first work of a Czech mathematician dealing exclusively with a problem of lattice theory. V. KOŘÍNEK follows the ideas of O. ORE [Ore1, Ore2, Ore3, Ore4] and A. I. UZKOV[12] in which he found the very impulse for his own investigation. His other sources of lattice theoretic concepts were [Köt] and [H–K].

V. KOŘÍNEK treats the so called Jordan–Hölder–Schreier–Zassenhaus theorem (JHSZ theorem) in lattices. This theorem describes an important property in group theory. The first result concerning the theorem belongs to C. JORDAN[13] (with numerical interpretation) and O. HÖLDER.[14] A generalization (from which the Jordan–Hölder theorem comes as a consequence) was produced by O. SCHREIER[15] and further improved by H. ZASSENHAUS.[16] The original Jordan–Hölder theorem says that two composition series[17] of a finite group have the same length and their factor groups are isomorphic (the proof made by induction). O. SCHREIER showed that to every two subnormal series of a group there exist their refinements which are factor isomorphic, and these refinements and isomorphic factor groups were then constructed by H. ZASSENHAUS.

The JHSZ theorem attracted a great deal of attention in lattice theory as well. A number of papers investigated possibilities of producing its lattice theoretic analogue, the first one being R. DEDEKIND,[18], we shall recall mainly the results connected with V. KOŘÍNEK's work.

O. ORE treated the theorem for groups [Ore4], and made attempts to extend it for lattices [Ore1, Ore3]. First he investigated the theorem in modular lattices [Ore1]: he calls a chain *principal* iff each of its elements covers the preceding element (i.e. there is no element between them) and

[12] UZKOV, A, I., *O teoreme Jordana-Höldera*, Matematičeskij sbornik n.s. 4 (1938), 31-43.

[13] JORDAN, C., *Commentaire sur Galois*, Mathematische Annalen 1 (1869), 141–160.

[14] HÖLDER, O., *Zurückführung einer beliebigen Gleichung ...* Math. Annalen 34 (1889), 26–56.

[15] SCHREIER, O., *Über den Jordan–Hölderschen Satz*, Abh. Hamb. 6 (1928), 300–302.

[16] ZASSENHAUS, H., *Zum Satz von Jordan–Hölder–Schreier*, Abh. Hamb. 10 (1934), 106–108.

[17] A composition series in a group is defined as a normal series without a proper refinement.

[18] DEDEKIND, R., *Über die von drei Moduln erzeugte Dualgruppe*, Math. Annalen 53 (1900), 371–403.

states the theorems:

Theorem 3.3. *Let L be a modular lattice, $a, b \in L$. If there exists a finite principal chain between a and b, then all principal chains between a and b have the same length and can be obtained from another by successive prime transpositions.*[19]

Theorem 3.4. *Let L be a modular lattice, $a, b \in L, a \leq b$. Let there exist two finite chains between a and b:*

$$a = a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n = b,$$

$$a = b_0 \leq b_1 \leq \cdots \leq b_{m-1} \leq b_m = b.$$

Then each of the chains has such a refinement that the refined chains have the same length and their quotients are similar[20] *in pairs.*

The terms of the refined chains have the following form (the so called Zassenhaus refinement, or construction):

$$a_{ij} = a_i \vee (a_{i+1} \wedge b_j),\ i = 0, 1, \ldots, n-1,\ j = 0, 1, \ldots, m,$$

$$b_{ji} = b_j \vee (b_{j+1} \wedge a_i),\ i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m-1,$$

the similar quotients being $a_{i,j+1}/a_{ij} \sim b_{j,i+1}/b_{ji}$.

This application of the JHSZ Theorem to lattices, however, does not fully satisfy O. ORE, he asks about its generalization to arbitrary lattices, not only modular. The problem is to find an analogue to the concept of normal subgroup and subnormal series. In [Ore3] he introduces more types of normal elements:

Definition 3.4. Let L be a lattice, $a_0, m, b, c \in L$.

An element $a_0 \leq m$ shall be called *α–normal in m* iff for any $b \geq c$; $b, c \leq m$ it holds:

$$b \wedge (a_0 \vee c) = (b \wedge a_0) \vee c.$$

An element $a_0 \leq m$ shall be called *β–normal in m* iff for any $b, c \leq m, b \geq a_0$ it holds:

$$b \wedge (c \vee a_0) = (b \wedge c) \vee a_0.$$

An element a_0 shall be called *seminormal in m* iff it is both α–normal and β–normal in m.

[19] By prime transpositions in modular lattices O. ORE means transpositions in the form $[a \wedge b, a] \to [b, a \vee b]$.

[20] Similar quotients are used in the sense of projective quotients by O. ORE.

With the defined concepts O. Ore proves the following version of the JHSZ Theorem:

Theorem 3.5. *Let L be a lattice, $a, b \in L$. Let there exist two chains between a and b such that each term is seminormal in the preceding one. Then there exist such refinements of the two chains that the refined chains are quotient isomorphic.*[21]

Comparing this theorem to its original version in group theory, O. Ore still finds one deficiency: we cannot prove that the terms of the refined chains are seminormal in the preceding ones.

Another approach to the problem of defining normal elements in lattices can be found in the mentioned paper by A. I. Uzkov in which the author assigns to each element a set of normal elements and looks for the properties which these sets must satisfy so that the Zassenhaus refinements of two normal chains would be also normal and quotient isomorphic. His definition is the following:

Definition 3.5. Let L be a lattice. Let a set $N_a \subseteq L$ be assigned to each $a \in L$. For elements $x \in N_a$ holds $x \leq a$ and we call them *normal in a*.

A chain in L will be called *normal* iff each term is normal in the preceding term.

The construction of the Zassenhaus refinements is the same as given by O. Ore. A. I.Uzkov places the following condition **(U)** on the elements of L:

$$\textbf{(U)}\ N_a \bigcap N_b \neq \emptyset \ \text{ for every } \ a, b \in L.$$

After analyzing the properties of normal chains A. I. Uzkov states five conditions which are necessary and sufficient for the validity of the JHSZ Theorem:

Theorem 3.6. *Let L be a lattice with the defined normality satisfying* **(U)**. *Let $a, b \in L$ and let there exist two finite chains between a and b:*

$$a = a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n = b,$$

$$a = b_0 \leq b_1 \leq \cdots \leq b_{m-1} \leq b_m = b.$$

[21] The correspondence determining the isomorphism is the following:

$$a' \to b_j \vee (b_{j-1} \wedge a'),\ b' \to a_i \vee (a_{i-1} \wedge b'),$$

for $a_i \vee (a_{i-1} \wedge b_{j-1}) \geq a' \geq a_i \vee (a_{i-1} \wedge b_j)$, $b_j \vee (b_{j-1} \wedge a_{i-1}) \geq b' \geq b_j \vee (b_{j-1} \wedge a_i)$.

Then the Zassenhaus refinements of these two chains are normal and quotient isomorphic iff the following conditions are satisfied:
(i) for each $a \in L: \ a \in N_a$*,*
(ii) if $a, b \in N_x$*, then* $a \vee b \in N_x$*,*
(iii) if $a, b \in N_x$*, then* $a \wedge b \in N_x$*,*
(iv) in $b \in N_a, \ b \wedge c \leq d \leq c, \ d \in N_c$*, then* $b \vee d \in N_{b \vee c}$*,*
(v) every element of N_a *is* M_a*–Dedekindean.*[22]

It was this paper of A. I. UZKOV's that inspired V. KOŘÍNEK to produce an article improving A. I. UZKOV's investigation. He intended to make the results more general and more elegant: he formulated normal elements in a lattice differently, generalized the initial condition **(U)**, simplified other conditions, distinguished upper and lower Zassenhaus refinements, and upper and lower similar quotients, provided shorter proofs and generalized the Zassenhaus method for chains which do not have the same start and endpoints.

V. KOŘÍNEK was not fully satisfied with the definition of normal elements in term of sets, especially because of their complicated treatment in proving some properties. He replaces A. I. UZKOV's formulation by a different one, again abstract, in terms of a relation (which need not be transitive):

Definition 3.6. Let L be a lattice with a relation $\mathbf{N}$ which satisfies:
$a \ \mathbf{N} \ b$ (we say b *is normal in* a) $\Rightarrow a \geq b$.

A *normal chain* between a_0 and a_n is defined as a finite sequence of elements $a_0 \ \mathbf{N} \ a_1 \ \mathbf{N} \dots \mathbf{N} \ a_n$.

Apart from the defined normality, V. KOŘÍNEK introduces other types of elements which are similar to O. ORE's normal elements:

Definition 3.7. Let L be a lattice, $b, c \in L$. An element $a \in L$ will be called *α–modular* with respect to b and $c, b \geq c$ iff

$$b \wedge (a \vee c) = (b \wedge a) \vee c.$$

An element $a \in L$ will be called *β–modular* with respect to b and $c, b \geq a$ iff

$$b \wedge (c \vee a) = (b \wedge c) \vee a.$$

[22] $x \in L$ is called M–Dedekindean iff $(a, c \in M \subseteq L, a \geq b$ or $a \geq c) \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

M_a is the set of all elements $y \leq a$ such that there exists a normal chain between y and a.

An element $a \in L$ will be called γ*–modular* with respect to b and $c, a \geq b$ iff

$$a \wedge (c \vee b) = (a \wedge c) \vee b.$$

V. KOŘÍNEK uses ORE's concepts of similar quotients, however, he finds it effective to make some alterations to his definitions[23] and distinguish lower and upper similarity:

Definition 3.8. Let L be a lattice, $a, b, c, d, u, v \in L$. The quotients $a/b, c/d$ will be called *directly similar* iff either $a = b \vee c$ and $d = b \wedge c$, or $c = d \vee a$ and $b = d \wedge a$.

The quotients $a/b,\ c/d$ will be called *upper simply similar* iff there exist a quotient u/v such that

$$u = a \vee v,\ u = c \vee v,\ b = a \wedge v,\ d = c \wedge v.$$

The quotients $a/b,\ c/d$ will be called *lower simply similar* iff there exist a quotient u/v such that

$$a = b \vee u,\ c = d \vee u,\ v = b \wedge u,\ v = d \wedge u.$$

The quotient u/v will be called a *middle quotient.*

V. KOŘÍNEK applies the Zassenhaus method of refinement to his normal chains in the common way, however he distinguishes the refinement as lower:

Definition 3.9. Let

$$a_0 \mathbf{N}\, a_1 \mathbf{N} \dots \mathbf{N}\, a_r \tag{3.1}$$

$$b_0 \mathbf{N}\, b_1 \mathbf{N} \dots \mathbf{N}\, b_s \tag{3.2}$$

be two normal chains in a lattice L. Then the following r chains will be called the *(lower) Zassenhaus chains of* (3.1) *with respect to* (3.2):

$$a_i \vee (a_{i-1} \wedge b_0) \geq a_i \vee (a_{i-1} \wedge b_0) \geq \cdots \geq a_i \vee (a_{i-1} \wedge b_s),\ i = 1, 2, \dots, r.$$

Analogously the following s chains will be called *(lower) Zassenhaus chains of* (3.2) *with respect to* (3.1):

$$b_j \vee (b_{j-1} \wedge a_0) \geq b_j \vee (b_{j-1} \wedge a_0) \geq \cdots \geq b_j \vee (b_{j-1} \wedge a_r),\ j = 1, 2, \dots, s.$$

If $a_0 = b_0$ and $a_r = b_s$ the chains will merge into one, and thus we obtain a (lower) Zassenhaus refinement of the chain (3.1) and a (lower) Zassenhaus refinement of the chain (3.2).

[23] Direct similarity is called expansion, contraction and similarity transposition by O. ORE.

After rewriting A. I.UZKOV's results in terms of the relation **N**, V. KOŘÍNEK investigates various properties of the defined concepts and looks for solutions of the following problems:

- what are necessary and sufficient conditions for the relation **N** so that the Zassenhaus chains created from two normal chains (3.1) and (3.2) be also normal,
- under which conditions are the quotients of Zassenhaus chains from (3.1) and (3.2) lower simple similar, with the given middle quotient,
- when is the regular correspondence of $[a, a \vee b] \to [a \wedge b, b]$ a lattice isomorphism?

V. KOŘÍNEK argues that all these problems cannot be solved for the relation **N** generally, therefore in some theorems he places one more condition upon the relation (which is weaker than A. I. UZKOV's condition **(U)**):

> **(A)** to each arbitrary elements $a, b \in L$ there exists at least one element $v \in L$ such that $a \mathbf{N} v$ and $b \geq v$.

In his paper, V. KOŘÍNEK analyzes 10 conditions relating to the individual problems. His conditions for solving all the problems are the following:

Theorem 3.7. *Let L be a lattice with a relation* **N** *satisfying the condition* **(A)**. *Let* (3.1) *and* (3.2) *be two normal chains in L. Then the corresponding Zassenhaus chains are normal, the quotients $a_{i,j-1}/a_{i,j}$ and $b_{j,i-1}/b_{j,i}, i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$ are lower simply similar with the middle quotient*

$$(a_{i-1} \wedge b_{j-1})/((a_{i-1} \wedge b_j) \vee (a_i \wedge b_{j-1}))$$

and the chains are quotient isomorphic iff the following conditions are satisfied:

(i) *for each $c_1, c_2, d_1, d_2 \in L$ satisfying $c_1 \mathbf{N} c_2, c_1 \leq d_1 \mathbf{N} d_2$ holds: $(c_2 \vee d_1) \mathbf{N} (c_2 \vee d_2)$,*

(ii) *for each $c_1, c_2, d_1, d_2 \in L$ satisfying $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2 \geq c_2$ holds: $(c_1 \wedge d_1) \mathbf{N} (c_1 \wedge d_2)$,*

(iii) *for each $c_1, c_2, d_1, d_2 \in L$ satisfying $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2$, c_1/c_2 and d_1/d_2 directly similar holds: every element x such that $c_1 \geq x \geq c_2$ is γ–modular with respect to c_2 and d_1,*

(iv) *for each* $c_1, c_2, d_1, d_2 \in L$ *satisfying* $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2$, c_1/c_2 *and* d_1/d_2 *directly similar holds: every element* y *such that* $d_1 \geq x \geq d_2$ *is* β*–modular with respect to* d_1 *and* c_2.

V. KOŘÍNEK's results found further generalization in a paper by A. CH. LIVŠIC[24] who points out that the condition **(A)** is not necessary for the validity of the JHSZ Theorem. He gives an example of a four-element lattice: $a, b, a \wedge b, a \vee b$ in which every element is normal in itself and in the very preceding one. Then the JHSZ Theorem is satisfied, however there does not exist an element which is normal in $a \vee b$ and is smaller than or equal to $a \wedge b$. He, therefore, finds conditions for V. KOŘÍNEK's problems without the relation $\mathbf{N}$ satisfying the assumption **(A)**:

Theorem 3.8. *Let* L *be a lattice with a relation* $\mathbf{N}$. *Let* (3.1) *and* (3.2) *be two normal chains in* L. *Then the corresponding Zassenhaus chains are normal, the quotients* $a_{i,j-1}/a_{i,j}$ *and* $b_{j,i-1}/b_{j,i}, i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$ *are lower simply similar with the middle quotient*

$$(a_{i-1} \wedge b_{j-1})/((a_{i-1} \wedge b_j) \vee (a_i \wedge b_{j-1}))$$

and the chains are quotient isomorphic iff the following conditions are satisfied:

(i) *for each* $c_1, c_2, d_1, d_2 \in L$ *satisfying* $c_1 \mathbf{N} c_2, c_1 \leq d_1 \mathbf{N} d_2$ *holds:* $(c_2 \vee d_1) \mathbf{N} (c_2 \vee d_2)$,

(ii) *for each* $c_1, c_2, d_1, d_2 \in L$ *satisfying* $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2$ *holds:* $(c_1 \wedge d_1) \mathbf{N} ((c_1 \wedge d_2) \vee (c_2 \wedge d_1))$,

(iii) *for each* $c_1, c_2, d_1, d_2 \in L$ *satisfying* $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2$ *holds: every element* x *such that* $(c_2 \vee (c_1 \wedge d_1) \geq x \geq (c_2 \vee (c_1 \wedge d_2))$ *is* γ*–modular with respect to* c_2 *and* $c_1 \wedge d_1$,

(iv) *for each* $c_1, c_2, d_1, d_2 \in L$ *satisfying* $c_1 \mathbf{N} c_2, d_1 \mathbf{N} d_2$ *holds: every element* y *such that* $c_1 \wedge d_1 \geq x \geq ((c_1 \wedge d_2) \vee (c_2 \wedge d_1))$ *is* β*–modular with respect to* $c_1 \wedge d_1$ *and* c_2.

V. KOŘÍNEK's paper is cited by several Romanian mathematicians who based some of their results on [Koř1]: "D. Barbilian[25] and his students generalized the notion of normality; the paper was, however, published in Romanian" [Koh]. V. KOŘÍNEK's results are often referred to in contemporary papers by M. BENADO (e. g. [Ben1, Ben2]).

[24] A. CH. LIVŠIC, *O teoreme Jordana-Höldera v strukturach*, Matematičeskij sbornik n. s. 24 (1949), 227-235.

[25] BARBILIAN, D., *Normalités localement ou intégralement involutives*, Acad. Repub. Pop. Romane Stud. Cerc. Mat. 4 (1953), 29–67.

It is interesting that although V. KOŘÍNEK uses the notation for lattice operation based upon O. ORE, the symbols are applied exactly the other way round compared to ORE. He denotes "$a \wedge b$" by "$[a, b]$" and "$a \vee b$" by "(a, b)". V. KOŘÍNEK gives an explanation that he follows the notation from ideal theory. As the paper is written in German, V. KOŘÍNEK uses the current German lattice theoretic terminology of [Köt, H–K] and, with some alterations, German translations of O. ORE's concepts.

3.3.2 *Lattices in which the theorem of Jordan–Hölder is generally true* [Koř2a] (1949)

The second paper of V. KOŘÍNEK concerning lattices also deals with the JHSZ Theorem, though, this time the author does not investigate it from the point of view of the Zassenhaus refinements. He shows the relationship between the validity of this theorem and the covering conditions. The inspiration for this investigation came from [LT–40], in particular from a search for the meaning of covering conditions.

V. KOŘÍNEK investigates how the covering conditions are related to the similarity of quotients, for which purpose he distinquishes two types similar quotients in a lattice:

Definition 3.10. We say that a quotient a/b is *downwards directly similar* to a quotient c/d by (denoted $a/b \searrow c/d$) iff $a = b \vee c,\ d = b \wedge c$.

We say that a quotient a/b is *upwards directly similar* to a quotient c/d (denoted by $a/b \nearrow c/d$) iff $b = a \wedge d,\ c = a \vee d$.

We say that a quotient a/b is *similar* to a quotient c/d iff there is a finite number of quotients $a_i/b_i, i = 1, 2, \dots, r$ such that in the sequence $a/b, a_1/b_1, \dots,$
$a_r/b_r = c/d$ each two subsequent quotients are directly similar.

The notions of upper and lower simple similarity is used in the same meaning as defined in [Koř1]. V. KOŘÍNEK points out that G. BIRKHOFF [LT–40] uses the terms "transposes" for "directly similar quotients" and "projective quotients" for "similar quotients", however, he himself prefers ORE's terminology which he finds more convenient in the case of general lattices.

In [LT–40] the Jordan–Hölder Theorem (JH Theorem) was proved for modular lattices, and the modularity in lattices of finite chains was shown to be equivalent to the covering conditions. V. KOŘÍNEK calls them (in accordance with O. ORE) the Birkhoff conditions:

Definition 3.11. We say that a lattice L satisfies the *lower Birkhoff condition* (l.B.c.) iff (a covers b and c, $b \neq c$) implies (a and b cover $a \wedge b$).

Dually, we say that a lattice L satisfies the *upper Birkhoff condition* (u.B.c.) iff (b and c cover a, $b \neq c$) implies ($b \vee c$ covers b and c).

V. KOŘÍNEK shows the meaning of l.B.c. and u.B.c. in the relation to the direct similarity of prime quotients:

Definition 3.12. We say that a lattice L satisfies the *lower prime quotient condition* (l.p.q.c) iff $a/b \searrow c/d$ and a/b is a prime quotient imply that c/d is also prime.

Dually, we define *upper prime quotient condition* (u.p.q.c.).

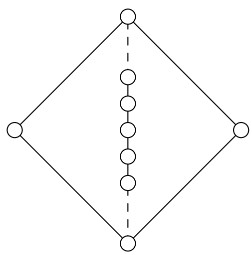

Figure 3.1: A lattice in which the l.B.c. and u.B.c. are true, but neither the l.p.q.c. nor the u.p.q.c.

It is obvious that the l.p.q.c. implies the l.B.c. and the u.p.q.c. implies the u.B.c. V. KOŘÍNEK shows that the converse is not generally true (only in lattices with finite chains) because of the example in Figure 3.1 in which both l.B.c. and u.B.c. are true, but neither the l.p.q.c. nor the u.p.q.c. He proves various other properties concerning these conditions and then looks for their relation to the JH Theorem, for the investigation of which he finds useful to distinguish two types of this theorem:

Definition 3.13. Let L be a lattice with finite chains. Let $a, b \in L, a > b$, let the chains (3.3), (3.4) be two maximal chains between a and b:

$$a = a_0 > a_1 > \cdots > a_r = b, \tag{3.3}$$

$$a = b_0 > b_1 > \cdots > b_s = b. \tag{3.4}$$

If for every two elements $a > b$ of L and every two maximal chains (3.3), (3.4) between them holds 1. $r = s$, and 2. there exists a one–to–one mapping between the quotients $a_i/a_{i+1}, i = 0, 1, \ldots, r-1$ and the quotients $b_j/b_{j+1}, j = 0, 1, \ldots, r-1$ such that the corresponding quotients are lower simply similar to each other, we shall say that L satisfies the *Jordan–Hölder theorem with lower simple similarity of quotients.* Dually we define the *Jordan–Hölder theorem with upper simple similarity of quotients.*

By induction V. Kořínek proves the main theorem of the paper:

Theorem 3.9. *Let L be a lattice with finite chains. The Jordan–Hölder theorem with lower simple similarity of quotients holds in L iff L satisfies the l.p.q.c. And dually.*

V. Kořínek also shows that the corresponding quotients of the chains (3.3) and (3.4) are determined uniquely in a lattice with the l.p.q.c.:

Theorem 3.10. *Let L be a lattice with finite chains satisfying the l.p.q.c. Let* (3.3), (3.4) *be two maximal chains between a and b, $a, b \in L, a > b$. There exists one and only one mapping of the quotients of the chain* (3.3) *onto the quotients of the chain* (3.4) *such that the corresponding quotients are lower simply similar. If a quotient a_i/a_{i+1} is lower simply similar to several quotients b_j/b_{j+1}, then it corresponds to the quotient b_j/b_{j+1} with the greatest index j in this mapping. And dually.*

The final part of the paper analyzes the JH Theorem in modular and distributive lattices. V. Kořínek proves the following theorem by applying the lower and upper Zassenhaus refinements of the chains:

Theorem 3.11. *Let L be a modular lattice with finite chains, $a \geq b, a, b \in L$ and let* (3.3), (3.4) *be two maximal chains between a and b. Then the lower simply similar mapping of the quotients of* (3.3) *onto the quotients of* (3.4) *is identical with the upper simply similar mapping of the quotients of* (3.3) *onto the quotients of* (3.4)*. Therefore there exists only one mapping of the quotients of* (3.3) *onto the quotients of* (3.4) *such that the corresponding quotients are at the same time upper and lower simply similar.*

As this paper is written in Czech, V. Kořínek devotes the first part to presenting the basic lattice theoretic concepts in Czech terminology (mentioning also expressions used in other languages) as there had been written little about lattices in Czech before his paper. He was criticized by O. BORŮVKA's review in *Mathematical Reviews* 12 (1951), pp. 667, 668 for not mentioning his papers [Bor1, Bor3a, Bor5] in which "the fundamental Czech terminology had been introduced". As for the notation, V. KOŘÍNEK chooses the symbols "$\wedge$" and "$\vee$" in this work.

The results of this paper were cited or followed by several mathematicians. Apart from the Czech ones whose work is described in the next section, let us mention W. FELSCHER[26] who generalized V. KOŘÍNEK's results to partially ordered sets and M. BENADO who cited both this and the previous work of V. KOŘÍNEK.

V. KOŘÍNEK produced also an English version of this paper (with some amendments and simplifications): [Koř2b]. A summary of his investigations concerning the JHSZ Theorem from [Koř1] and [Koř2a] is the content of his other two papers: [Koř3], [Koř4].

3.4 Mathematicians influenced by V. Kořínek

Several papers of young Czech mathematicians of 1950's either continued in V. KOŘÍNEK's investigations on lattice theory, or were inspired by discussions in his seminar. His students L. JANOŠ, Č. VITNER and V. VILHELM refer in their papers to some unpublished results of V. KOŘÍNEK presented in the seminar "Talks on group theory and related subjects" in the Mathematical Institute of Czech Academy of Sciences and Arts in the years 1948/49 and 1949/50.

This section analyzes the papers written by the mentioned young mathematicians whose topics were mainly based on the investigations from their RNDr. theses supervised by V. KOŘÍNEK. Another mathematician whose early works were influenced by V. KOŘÍNEK and who wrote a thesis supervised by him was V. HAVEL, he will be, however, dealt with in a separate section.

LUDVÍK JANOŠ defended his thesis *Properties of the Zassenhaus refinement*, the results of which are presented in [Jan1], in 1949/1950, then changed his field of investigation to an area concerning mainly functional analysis, and later topology. After 1963 he deals with lattices only in one paper.[27]

ČESTMÍR VITNER started his mathematical research in algebra, but he soon turned his interest to geometry, in particular differential geom-

[26] FELSCHER, W., *Jordan–Hölder Sätze und modular geordenete Mengen*, Mathem. Zeit. 75 (1960/61), 83–114.

[27] JANOŠ, L., *A minimal property of the Zassenhaus refinement*, J. Nat. Sci. Math. 7 (1967), 101–111.

etry of curves.[28] He defended his thesis called *Semimodular conditions in lattices* in 1951/52 and its results are the content of the paper [Vit].

VÁCLAV VILHELM devoted more papers to lattice theory. [Vil1] includes the results of his thesis textitThe Jordan–Hölder theorem in lattices without finite chains which was defended in 1951/52 and by 1963 he published two more papers from lattice theory: [Vil2] deals with the Birkhoff conditions and [Vil3] with the representation of complete lattices by sets. Later he returned to a lattice theoretic topic[29] after having written several papers from other areas.[30]

3.4.1 *Properties of the Zassenhaus refinement* [Jan1] (1953)

This paper written by L. JANOŠ deals with the contruction of the Zassenhaus refinement in lattices and groups. A detailed study of the first part related to similar quotients in chains of lattices (as introduced in [Koř2a]) leads to the following result:

Theorem 3.12. *Let L be a modular lattice. The Zassenhaus refinement of two given chains in L is not a proper refinement iff the given chains are lower simply similar.*

Then the author investigates other features of the Zassenhaus refinement and proves:

Theorem 3.13. *Let L be a modular lattice. Then the Zassenhaus refinement of two given chains in L is the only Schreier refinement which lies in the sublattice of L generated by the elements of the given chain.*

The last part of the paper shows that analogous theorems (to the presented lattice–theoretic) also hold for groups with composition series.

3.4.2 *The semimodular conditions in lattices* [Vit] (1953)

In his seminar V. KOŘÍNEK raised the following question:

[28] The reader can find more information about ČESTMÍR VITNER e. g. in NÁDENÍK, Z., VILHELM, V., *Šedesát let doc. RNDr. Čestmíra Vitnera, CSc.*, Časopis pro pěstování matematiky 110 (1985), 442–445.

[29] VILHELM, V., *Über die Charakterisierung der Verbände durch ihre c–Teilverbände*, Časopis pro pěstování matematiky 103 (1978), 291–296.

[30] The reader can find more information about V. VILHELM in DRÁBEK, K., *Docent Václav Vilhelm šedesátníkem*, Pokroky matematiky, fyziky a astronomie 31 (1986), 59.

> Is it possible to formulate two properties π_1 (π_2) of a lattice such that the following conditions I–IV are fulfilled?
>
> I. The conditions π_1 and π_2 are dual.
>
> II. In any lattice L the following implications are true: if the condition π_1 (π_2) holds, then L satisfies the lower (upper) prime quotient condition.
>
> III. If all chains of a lattice L are finite, then the following equivalences hold: the condition π_1 (π_2) holds iff L satisfies the lower (upper) prime quotient condition.
>
> IV. A lattice L is modular iff L satisfies both conditions π_1 and π_2.

In the paper Č. VITNER employs the concepts from [Kořl, Koř2a] and introduces the following new notions:

Definition 3.14. We say that a lattice L satisfies the *lower condition of maximal chains* iff for any lower directly similar quotients $a/b, c/d$ and any maximal chain $\{a_\iota\}$ between a, b in L the chain $\{c_\iota\}$ formed by $c_\iota = a_\iota \wedge c$ is a maximal chain between c, d. Dually we define the *upper condition of maximal chains.*

Definition 3.15. Let L be a lattice, $a/b, c/d$ be any lower directly similar quotients in L, $\{a_\iota\}$ be a maximal chain between a and b. Let us denote by K_l the set of all elements of $\{a_\iota\}$ which are not γ–modular with respect to b and c. We say that L satisfies the *γ–condition* iff the set K_l is either empty, or it has a maximal element. By means of β–modularity we dually define the set $K_u \subset \{d_\iota\}$ (where $\{d_\iota\}$ is a maximal chain between c and d) and *β–condition.*

The definitions of the conditions π_1 and π_2 which solve the problem of V. KOŘÍNEK are given as follows:

Definition 3.16. We say that a lattice satisfies the *lower semimodular condition* (which is the condition π_1) iff it satisfies both the γ–condition and the lower condition of maximal chains.

We say that a lattice satisfies the *upper semimodular condition* (which is the condition π_2) iff it satisfies both the β–condition and the upper condition of maximal chains.

Č. VITNER proves that these conditions satisfy I – IV and shows that the lower/upper condition of maximal chains is independent of γ/β–condition. He also gives some other properties of lattices related to the problem and shows the connection of the lower and upper conditions for maximal chains to the Schreier Theorem.

3.4.3 *The Jordan–Hölder theorem in lattices without the finite chain condition* [Vil1] (1954)

V. VILHELM generalizes V. KOŘÍNEK's investigation of the JH theorem with lower or upper simple similarity of quotients, particularly, he shows that it is possible to replace the finite chain condition from [Koř2a] by weaker ones. For infinite chains the JH theorem with simple quotient similarity is defined as follows:

Definition 3.17. Let L be a lattice. We say that in L the *Jordan–Hölder theorem with lower simple similarity of quotients* holds iff for any two elements $a, b \in L, a < b$ and any two maximal chains $C_1(a,b)$ and $C_2(a,b)$ between a and b there exists a one–to–one mapping of the set of all prime quotients of the chain $C_1(a,b)$ onto the set of all prime quotients of the chain $C_2(a,b)$ such that the corresponding quotients are lower simply similar. This mapping will be called the *Jordan–Hölder mapping.*

V. VILHELM denotes an arbitrary (also infinite) chain between elements a and b of a lattice $L, a \leq b$ by $\{a_\iota\}_0^\rho$, where ι goes through a set M ordered by a relation $\prec$ with the first element 0 and the last element $\rho, a = a_0, b = a_\rho$ such that

$$\iota, \kappa \in M, \iota \prec \kappa \ \Rightarrow \ a_\iota \leq a_\kappa.$$

He introduces the following properties of a lattice L:

Property I: Every quotient of L is a complete lattice (as a sublattice of L).

Property II: For any chain $\{a_\iota\}_0^\rho, \iota \in M$ in L and any $c \in L$ holds

$$\lambda \in M, \lambda \neq 0 \ \Rightarrow \bigvee_{i \in M, \iota \prec \lambda} (a_\iota \wedge c) = (\bigvee_{\iota \in M, \iota \prec \lambda} a_\iota) \wedge c.$$

Property III: For any maximal chain $\{a_\iota\}_0^\rho, \iota \in M$ in L and any two elements $a_\alpha, a_\beta \in \{a_\iota\}_0^\rho, a_\alpha < a_\beta$ there exist elements $a_\kappa, a_\lambda \in \{a_\iota\}_0^\rho$ such that $a_\alpha \leq a_\kappa < a_\lambda \leq a_\beta$, and a_λ / a_κ is a prime quotient in L.

Property IV: For every maximal chain (without repetitions) in L $\{a_\iota\}_0^\rho$ and every its quotient a_ρ / b holds:

$$\lambda \in M, \lambda \neq \rho \ \Rightarrow \bigwedge_{\iota \in M, \lambda \prec \iota} (a_\iota \vee b) = (\bigwedge_{\iota \in M, \lambda \prec \iota} a_\iota) \vee b.$$

V. VILHELM proves some consequences of the properties I, II, III and IV and observes that the lattice of subgroups of a group satisfies I, II and III and so does a lattice in which every non–void subset of a quotient has at least one maximal element. He describes a way how to make the Zassenhaus construction for an arbitrary maximal chain: if $\{a_\iota\}_0^\rho, \iota \in M$ is a maximal chain (without repetitions) in a lattice L, $a_\kappa, a_\lambda \in \{a_\iota\}_0^\rho$, and a_κ/a_λ is a prime quotient, we shall write $\kappa = \lambda + 1$. Let $a, b \in L, a < b$ and let

$$\{a_\iota\}_0^\rho, \iota \in M, a_0 = a, a_\rho = b, \tag{3.5}$$

$$\{b_\kappa\}_0^\sigma, \kappa \in N, b_0 = a, b_\sigma = b, \tag{3.6}$$

be two maximal chains without repetitions between a and b. We shall denote by A the set of all $\iota \in M$ for which there exists $a_\lambda \in \{a_\iota\}_0^\rho$ such that a_λ/a_ι is a prime quotient (i.e. $\lambda = \iota + 1$). Analogously we obtain the set B from $\{b_\kappa\}_0^\sigma$. Now we can construct the lower Zassenhaus refinement of the chains (3.5) and (3.6):

$$a_{\iota,\kappa} = a_\iota \vee (b_\kappa \wedge a_{\iota+1}), \text{ for all } \iota \in A, \kappa \in N,$$

$$b_{\kappa,\iota} = b_\kappa \vee (a_\iota \wedge b_{\kappa+1}), \text{ for all } \kappa \in B, \iota \in M.$$

Thus we obtained the refined chains:

$$\{\{a_\iota \vee (b_\kappa \wedge a_{\iota+1})\}_{\kappa=0}^{\kappa=\sigma}\}_{\iota\in A}, \tag{3.7}$$

$$\{\{b_\kappa \vee (a_\iota \wedge b_{\kappa+1})\}_{\iota=0}^{\iota=\rho}\}_{\kappa\in B}. \tag{3.8}$$

By investigating properties of the quotients of the chains (3.5), (3.6), (3.7) and (3.8) and the Jordan–Hölder mapping of the prime quotients of (3.5) and (3.6) V. VILHELM proves the main theorem of the paper:

Theorem 3.14. *Let a lattice L have the properties I, II and III. Then the JH theorem with lower simple similarity of quotients holds iff 1. L satisfies the lower prime quotient condition, and 2. L has the property IV.*

The author also shows that the condition 2. from the preceding theorem can be replaced by another one: *2'. in L there does not exist a sublattice isomorphic to the lattice in Figure 3.2.* The lattice in Figure 3.2 consists of a maximal chain without repetitions $\{a_\iota\}_0^\rho, \iota \in M$ ($M - \{0\}$ does not have a first element) and a chain $\{b_\iota\}_0^\rho, b_0 = a_0$ in L, a_ρ/b_ρ is a prime quotient in L and

$$\iota \in A \Rightarrow a_{\iota+1}/a_\iota \sim_d b_{\iota+1}/b_\iota, a_\iota \vee b_\rho = a_\rho, a_\iota \wedge b_\rho = b_\iota,$$

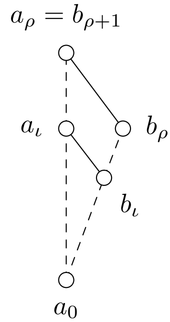

Figure 3.2: A lattice from the condition 2' of Theorem 3.14

where A is the set of all $\iota \in M$ for which there exist $\iota + 1$.

Further V. Vilhelm generalizes V. Kořínek's results by proving that Theorem 3.9 remains valid even if we replace the precondition that L is a lattice satisfying the finite chain condition by a weaker precondition that there exists one maximal chain of a length r. The theorem still holds even if we replace the other precondition, i. e. that L satisfies the lower prime quotient condition, by the lower Birkhoff condition.

3.4.4 *The selfdual kernel of the Birkhoff conditions in lattices with finite chains* [Vil2] (1955)

In this paper V. Vilhelm investigates the Birkhoff conditions in lattices with finite chains building upon the work of V. Kořínek [Koř2a] (the notions of direct and simple similarity of quotients, the Jordan–Hölder theorem with lower/upper simple similarity of quotients) and O. Ore [Ore5], from which he adopts the notion of a cyclic lattice:

Definition 3.18. A lattice L is called *cyclic* iff it is the sum (in the sense of set theory) of such two chains $C_1(a,b)$ and $C_2(a,b)$ that have only the elements a, b in common.

Definition 3.19. A sublattice L_1 of a lattice L is called *saturated* (in L) iff any maximal (=saturated) chain in L_1 is maximal in L too.

V. Vilhelm proves the following theorem:

Theorem 3.15. *Let L be a lattice with finite chains. L satisfies the lower Birkhoff condition iff it contains neither a saturated cyclic sublattice of the length ≥ 3 nor a saturated sublattice isomorphic to the lattice in Figure 3.3.*

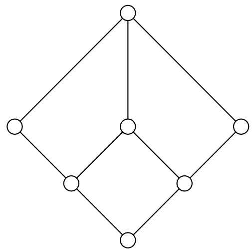

Figure 3.3: A lattice from Theorem 3.15

The author pays attention to the first condition (the non–existence of a saturated sublattice of the length ≥ 3) which is selfdual and is necessary for both the lower and upper Birkhoff conditions. This investigation leads to answering the problem of weakening KOŘÍNEK's theorem 3.8: replacing the Jordan–Hölder theorem with lower/upper similarity of quotients by the Jordan–Hölder theorem with similarity of prime quotients:

Definition 3.20. We shall say that a lattice L with finite chains satisfies the *Jordan–Hölder theorem with similarity of prime quotients* iff for any two maximal chains in L there exists a one–to–one mapping of the set of the prime quotients of the one chain onto the set of the prime quotients of the other chain such that the corresponding prime quotients are similar to each other.

V. VILHELM obtains the following theorem:

Theorem 3.16. *Let L be a lattice with finite chains. The Jordan–Hölder theorem with similarity of prime quotients holds in L iff this theorem is true for any two maximal chains which form a cyclic sublattice of L of the length ≥ 3.*

The last part of the paper presents another modified version of the Jordan–Hölder theorem which enables us to replace the previous theorem with a more effective one.

3.4.5 *A note on complete lattices represented by sets* [Vil3] (1962)

I. V. STELLECKIJ[31] gave necessary and sufficient conditions for a complete lattice to be representable by sets.[32] In this note V. VILHELM proves that STELLECKIJ's conditions can be formulated in another form, which shows a relation of complete lattices to compactly generated lattices (this notion is taken from P. CRAWLEY[33]):

Definition 3.21. An element a of a complete lattice L is called *chain–compact* iff for each chain $\{a_\alpha\}, \alpha \in A$ in L such that $a \leq \bigvee\limits_{\alpha\in A} a_\alpha$ there exists an $\alpha_0 \in A$ such that $a \leq a_{\alpha_0}$. L is called *chain–compactly generated* iff each of its elements is a join of chain–compact elements.

Theorem 3.17. *A complete lattice L can be represented by sets iff L is chain–compactly generated.*

V. VILHELM shows a characterization of compactly generated lattices as a representation by sets (which is related to the studies of [B–F]). The investigation makes it possible to generalize a result from the mentioned paper by P. CRAWLEY by proving:

Theorem 3.18. *Let L be a complete lattice representable by sets. If every two quotients $a\vee b/a$ and $b/a\wedge b$ are isomorphic, then L is modular.*

In the final part of the paper V. VILHELM applies the results to the validity of the Jordan–Hölder Theorem with lower simple similarity of quotients in complete lattices.

3.5 Ladislav Rieger and lattices

LADISLAV SVANTE RIEGER (1916–1963) devoted to the field of lattice theory several early papers which concerned mainly Boolean algebras. He successfully applied the results he obtained to mathematical logic and axiomatic set theory which became his main research interests.

[31] STELLECKIJ, I. V., *O polnych strukturach predstavimych množestvami*, Usp. matem. nauk 12 (1957), 177–180.

[32] I. V.STELLECTIJ's result: a complete lattice L can be represented by sets iff 1. for any $x \in L$ and any chain $\{y_\alpha\}$ in L: $x \wedge \bigvee\limits_\alpha y_\alpha = \bigvee\limits_\alpha (x \wedge y_\alpha)$ and 2. every $z \in L$ exists in the form $z = \bigvee\limits_\gamma z_\gamma$, where for each z_γ holds: $(z_\gamma = \bigvee\limits_\delta t_\delta, \{t_\delta\}$ is a chain in $L)$ $\Rightarrow z_\gamma = t_{\delta_i}$.

[33] CRAWLEY, P., *The Isomorphism Theorem in compactly generated lattices*, Bulletin of the American Mathematical Society 65 (1959), 377–379.

L. RIEGER's first papers [Rie1, Rie2, Rie3] (which form in fact three parts of a single work, based on his doctoral thesis supervised by V. KOŘÍNEK and defended in 1945/46) deal with ordered groups. The aim of the papers is to investigate how the fact that a group can be ordered determines the algebraic structure of the group, and in which manner this structure is determined if the group can be ordered in some specific way. The first two parts use only algebraic methods of proofs, however, in the third one the author also makes a full use of topological means. The work was written in the difficult period at the end of WWII and after it (autumn 1944 and a year later) which resulted not only in a delay in publishing it, but obstacles in obtaining up–to–date literature caused that L. RIEGER was not aware of the fact that his "magnitude subgroups" are identical to G. BIRKHOFF's concept of "l–ideals"[34] introduced in [Bir4], p. 310, thus he rediscovered the term and repeated its relations to congruences and lexicographic products and unions.

After WWII L. RIEGER turned his attention to lattice theory, Boolean algebras, which mathematical logic and axiomatic set theory. His work was influenced to a great extent by Polish mathematicians–logicians, especially after he had had an opportunity to take part in the seminar of Professor L. MOSTOWSKI in Warsaw during April 1950 ([Rie7], p. 29).[35]

The following subsections contain the main results of L. RIEGER's works concerning lattice theoretic questions, however, we shall not present a very detailed analysis as we would often need a much wider background of mathematical logic to describe all the applications. The paper [Rie4] deals with topological representation of distributive lattices, the work [Rie5] presents a lattice theoretic characterization of Heyting formulation of Browerian propositional logic, the papers [Rie6, Rie7, Rie10, Rie11] investigate special kinds of Boolean algebras, often with a view of their application to mathematical logic: the generalized σ–algebras from [Rie7] are applied for a new poof of K. GÖDEL's completeness theorem, a more general Suslin algebras from [Rie10] are used for the description of predicate variables. Some of the mentioned papers, namely [Rie6, Rie8, Rie10], solve several G. BIRKHOFF's problems stated in [LT–48], or questions connected with them. The book [Rie9] is interesting for its special position and role in the list of L. RIEGER's work, it is aimed at the audience of non–mathematicians.

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[34] By an l–ideal of an l–group G is meant a normal subgroup of G which contains with any a also all x such that $|x| \leq |a|$, where the absolute $|a|$ is $a \vee -a$.

[35] The reader who is interested in L. RIEGER's life and work is referred to ČULÍK, K., *O životě, díle a osobnosti L. Riegera*, Časopis pro pěstování matematiky 89 (1964), 492–495.

Before presenting L. RIEGER's results, let us recall some notions from mathematical logic which will be used further.

A *Browerian logic* is a (two–valued) propositional calculus in which the validity of proof by contradiction is not assumed. Although the implication $P \to (P')'$ is admitted, $(P')' \to P$ is not. There is a close analogy between Browerian logic and the distributive lattice of open sets of a topological space, which was recognized by M. H. STONE [Sto2] and A. TARSKI [Tar2], their specific correlation with Browerian lattices[36] was accomplished by G. BIRKHOFF in [LT–40] ([LT–67], p. 281).

The *Lindenbaum–Tarski algebra* of a propositional/lower predicate calculus is the set of all formulas of the (two–valued) propositional/lower predicate calculus in which equivalent formulas became identified.

The fundamental *completeness theorem* (K. GÖDEL) on the propositional calculus states that the class of all derivable formulas coincides with the class of all tautologies. Using the notion of the Lindenbaum–Tarski algebra this theorem can be translated into the language of the theory of Boolean algebras.

3.5.1 ***A note on topological representations of distributive lattices*** **[Rie4] (1949)**

This paper is based on the work [Sto2] where M. H. STONE introduced topological representations of distributive lattices. L. RIEGER uses M. H. STONE's terminology: a μ–ideal meaning an ideal of a lattice, and an α–ideal for a dual ideal of a lattice. Other sources of lattice theoretic notions he drew from were [LT–40] and [H–K].

We say that a distributive lattice L is topologically represented in a topological T_0–space $S(L)$ iff there exists an isomorphism between L and a set ring R of certain open subsets of $S(L)$, where R forms an open basis of $S(L)$.[37] M. H. STONE [Sto2] presented a description of a "universal" T_0–space $\bar{S}(L)$ which contains every representation T_0–space $S(L)$ of a given distributive lattice L as a dense subspace. $\bar{S}(L)$ is the space of all prime dual ideals of L. L. RIEGER introduces another characterization of $\bar{S}(L)$ for distributive lattices with 1.

By developing M. H. STONE's ideas [Sto2] concerning the theory of prime ideals and using E. ČECH's concept of a pseudocomplete system

[36] A Browerian lattice is a lattice in which every two elements have a relative pseudo–complement, the largest element x with $a \wedge x \leq b$.

[37] We can also speak of representations by a ring of closed sets of $S(L)$.

of neighborhoods of a point[38] L. RIEGER proves the main theorem of the paper:

Theorem 3.19. *The space $\bar{S}(L)$ of all prime dual ideals of a distributive lattice L with 0 and 1 is a bicompact T_0–space having an open basis R' with the following properties:*

(i) any system of open sets of R' has a non–void intersection if every finite subsystem thereof does,

(ii) any pseudocomplemented system Q' of neighborhoods of a point $p \in \bar{S}$ which contains with A_1 and A_2 also $A_3 \subset A_1 \cap A_2$ is a complete system of neighborhoods of p.

Conversely, if a bicompact T_0–space $\bar{S}$ has an open basis R' satisfying (i) and (ii), then $\bar{S}$ can be taken as the space of all prime dual ideals of the distributive lattice R generated by R' with 0 and 1 adjoined.

A bicompact T_1–space $\bar{S}$ satisfying (i) and (ii) is a totally disconnected bicompact Hausdorff space, i.e. a Boolean space.[39]

L. RIEGER deduces the following consequences from the stated theorem:

Theorem 3.20. *Any distributive lattice with 0 and 1 in which all prime dual ideals are maximal is a Boolean algebra.*

Theorem 3.21. *Any distributive lattice with 0 in which all prime dual ideals are maximal is a generalized Boolean algebra.*

3.5.2 *On the lattice theory of Brouwerian propositional logic* [Rie5] (1949)

The purpose of this paper as stated by L. RIEGER is to show that by using the notion of certain special free distributive and residuated lattice, lattice theory can be made an efficient mathematical tool for both the syntax and the semantics of a language using Browerian logic.

The timing of the paper was rather unfortunate as it was published a year after [LT–48] and after the paper of J. C. C. MCKINSEY and A. TARSKI [M–T] which deal with similar problems, however, neither of them was available for L. RIEGER when working on this paper. Thus, he independently gives a definition of a lattice with a free set of generators analogous to G. BIRKHOFF's general definition of free algebras,

[38] A system Q of open sets A of the open basis R of any T_0 space S' is called a pseudocomplete system of neighborhoods of a point $p \in S'$ iff $\prod_{p \in A \in R'} A = \prod_{A \in Q'} A$.

[39] The term Boolean space was introduced by M. H. STONE in [Sto1].

and also proves several results established by J. C. C. McKinsey and A. Tarski.[40]

L. Rieger uses the notion of residuated lattice in accordance with M. Ward and R. P. Dilworth[41] and calls its special type sdruz–lattices:

Definition 3.22. A distributive lattice L is said to be a *residuated lattice* iff there is a binary operation $a : b \in L$ defined for each $a, b \in L$ which fulfills the following conditions:

(1) $(a : b) \wedge b \leq a$,

(2) if $x \wedge b \leq a$, then $x \leq (a : b)$.

A countable distributive residuated lattices with 0 and 1 will be called a *sdruz–lattice.*

The author proves the following isomorphism theorem for sdruz–lattices:

Theorem 3.22. *Let I be an ideal of a sdruz–lattice L, let $x \equiv y$ iff there exists an element $c \in I$ such that $x \wedge c = y \wedge c$. Then the relation $\equiv$ is a congruence on L; and thus L/I is a homomorphic image of L.*

Conversely, every homomorphic image of L can be obtained in this way from some ideal I.

L. Rieger also shows that if I is the ideal consisting of the elements $x \in L$ such that $n : x = n$, where n is a non–zero element of L, then L/I is a Boolean algebra.

The central notion of the paper are free sdruz–lattices:

Definition 3.23. A sdruz–lattice L is called a *free sdruz–lattice generated by its subset* G iff for every lattice L' with generators G' every mapping of G into G' can be extended to a homomorphic mapping of L into L'.

L. Rieger shows possible realizations of the defined concept; the same free sdruz–lattice can be constructed in various ways within four areas: in the semantical theories of extensional logical evaluation, of meaning, of a material implication of sentences and in the syntactical theory of the Heytig calculus of propositions. It is proved that, by considering well–formed formulas of the Heyting calculus as equivalent

[40] L. Rieger mentions that he had not seen their paper [M–T] earlier than when his own work was ready for press.

[41] Ward, M., Dilworth, R. P., *Residuated lattices*, Transactions of the American Mathematical Society 45 (1939), 335–354.

when they give the same value for every finite sdruz–lattice, we obtain a free sdruz–lattice with a countable infinity of generators. This sdruz–lattice is also a characteristic matrix for the Heyting calculus because it can be obtained by considering well–formed formulas α and β of the Heyting calculus as equivalent if both $\alpha \to \beta$ and $\beta \to \alpha$ are provable.

Further, his results enable L. RIEGER to give a decision method for the Heyting calculus which produces a method for proving several identities in general topology and in abstract algebra. Compared to similar results of J. MCKINSEY and A. TARSKI this decision method is "more explicit, though equally impracticable to apply".[42] A number theoretic realization is given for a free sdruz–lattice with one generator. L. RIEGER also obtains some known theorems of K. GÖDEL and V. GLIVENKO as consequences of his results. L. RIEGER presents a number of other considerations about applying the concept of sdruz–lattices to the Heyting calculus.

3.5.3 *On free $\aleph_\xi$–complete Boolean algebras* [Rie6] (1951)

In this paper L. RIEGER deals with some properties of free $\aleph_\xi$–complete Boolean algebras in general and concentrates more on σ–complete Boolean algebras.

Definition 3.24. A Boolean algebra A is called $\aleph_\xi$*–complete* iff any subset of A whose power does not exceed $\aleph_\xi$ has a greatest lower bound and a least upper bound in A. Let m be any cardinal number. An $\aleph_\xi$–complete Boolean algebra $A_m^{\aleph_\xi}$ is said to be *free with m free $\aleph_\xi$ generators* iff there exists a subset $G \subset A_m^{\aleph_\xi}$ whose power is m satisfying the following properties:

(i) the only $\aleph_\xi$–complete subalgebra of $A_m^{\aleph_\xi}$ containing G is $A_m^{\aleph_\xi}$ itself, i. e. the elements of G $\aleph_\xi$–generate $A_m^{\aleph_\xi}$,

(ii) if φ is any mapping of G into another $\aleph_\xi$–complete algebra B, then φ can be extended to a $\aleph_\xi$–complete homomorphic mapping of the whole algebra $A_m^{\aleph_\xi}$ into B.

If $\aleph_\xi = \aleph_0$ we speak about *σ–complete Boolean algebras*, in short *σ–algebras.*

L. RIEGER begins the paper by proving some theorems on the existence of free $\aleph_\xi$–complete Boolean algebras (he constructs $A_m^{\aleph_\xi}$), on

[42]The review by J. C. C. MCKINSEY, *Mathematical Reviews* 12 (1951), pp. 663–664.

the uniqueness of $A_m^{\aleph_\xi}$ and the universality of $A_m^{\aleph_\xi}$. Then he is interested almost exclusively in free σ-algebras because of their possibilities of application, which he justifies by the following theorem:

Theorem 3.23. *For any $\aleph_\xi \geq 2^{\aleph_0}$ (and hence for any uncountable $\aleph_\xi$ by the Continuum Hypothesis) the free $\aleph_\xi$–complete Boolean algebra $A_m^{\aleph_\xi}$ with $m \geq \aleph_0$ cannot be $\aleph_\xi$–isomorphically represented by an $\aleph_\xi$–additive field of sets.*

L. RIEGER proves this theorem by using a result of R. SIKORSKI[43] and points out that it in fact leads to the solution of **Problem 80** of [LT–48], p. 168 which asks for a generalization (to cardinal numbers other than countable infinity) of L. H. LOOMIS' Theorem:[44]

Theorem 3.24. *Any σ–complete Boolean algebra is a σ–homomorphic image of a σ–field of sets.*

The mentioned results of R. SIKORSKI states that for $\aleph_\xi \geq 2^{\aleph_0}$, an $\aleph_\xi$–complete Boolean algebra cannot be isomorphic to a quotient algebra X/I, where X is an $\aleph_\xi$–additive field and I is an $\aleph_\xi$–additive ideal of sets. Thus it yields the impossibility of a positive solution to **Problem 80**.

L. RIEGER himself presents two strengthened forms of L. H. LOOMIS' Theorem, the first gives details of the isomorphism, the second one, involving topological strengthening, is the following:

Theorem 3.25. *Any free σ–algebra $A_m^{\aleph_\xi}$ with m generators is σ–isomorphically represented by the minimal σ–field of Borel subsets of the Cantor discontinuum C_m.*

This theorem produces a positive solution, and in fact a generalization, of another problem from [LT–48], **Problem 79**, p. 168:

> Prove (or disprove) that the free Boolean σ–algebra with countably many generators is isomorphic with the field of all Borel subsets of Cantor discontinuum.

Apart from the solution of G. BIRKHOFF's problem L. RIEGER's theorem is also a generalization of some results of R. SIKORSKI.[45]

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[43] SIKORSKI, R., *On the representation of Boolean algebras as fields of sets*, Fund. Math. 35 (1948), 247–258.

[44] LOOMIS, L. H., *On the respresentation of σ–complete algebras*, Bulletin of the American Mathematical Society 53 (1947), 757–760.

[45] SIKORSKI, R., *On inducing of homomorphisms by mappings*, Fund. Math. 36 (1949), 7–22.

L. RIEGER's attention is naturally drawn also to another problem of G. BIRKHOFF concerning the discussed topic, to **Problem 78**, [LT–48], p. 168:

> Prove (or disprove) that if a Boolean σ–algebra A is generated by a subset G, then every $a > 0$ in A contains some finite or infinite countable meet $\wedge g_i > 0$ of elemets of G.

L. RIEGER shows that disproving can be accomplished by an easy example, and therefore he looks for a modification of the problem: "Does there exist, for any $a \neq 0$ in any free σ–algebra a set of free generators G such that $0 \neq \wedge g_i \leq a$, $g_i \in G$?" The affirmative answer to the question is obtained by Theorem 3.26.

The last part of the paper deals with an application of the previous results to logic, specifically to the Lindenbaum–Tarski algebra of the lower predicate calculus.

This paper found a response in a paper by R. SIKORSKI[46] where the author presents a theorem whose consequences are L. RIEGER's results concerning the strengthening of L. H. LOOMIS' Theorem.

3.5.4 *On countable generalised σ–algebras, with a new proof of Gödel's completeness theorem* [Rie7] (1951)

This paper treats the same subject as the previous one: σ–algebras and their application to logic. L. RIEGER presents a generalization of the notion of σ–algebras: he considers certain families Φ of multiple sequences (called *marked sequences*) in a Boolean algebra B which satisfy several conditions (the rule of complement, the rule of joins and meets, the rule of identification of chosen indices - forming "diagonal" sequences, the rule of fixation of indices - forming "cylindric"sequences, the rule of trivial sequences, the rule of the lowest upper bound and the greatest lower bound and the rule of a partial lowest upper bound and partial greatest lower bound). The author calls this generalized σ–algebra a *Φ_σ–algebra.* After defining an appropriate generalizations of the basic notions of σ–homo(iso)morphisms, σ–ideals and the corresponding quotient algebras, L. RIEGER proves that if Φ is countable, then Φ_σ–algebra can be represented by a field of sets, which, however, does not hold generally.

[46] SIKORSKI, R., *A note to Rieger's paper "On free $\aleph_\xi$–complete Boolean algebras"*, Fund. Math. 38 (1951), 53–54.

Since the Lindenbaum–Tarski algebra of the lower predicated calculus provides a typical example of a countable Φ_σ–algebra with a countable family Φ, we immediately obtain a new proof of K. GÖDEL's completeness theorem.

3.5.5 *Some remarks on automorphisms in Boolean algebras* [Rie8] (1951)

The aim of the paper is to construct a Boolean algebra admitting no proper homomorphic mapping onto itself. L. RIEGER's construction of B is topological: he solves an equivalent topological problem (thus disproving the hypothesis that every zero–dimensional bicompact space should admit some proper homeomorphic transformation onto a suitable subspace of it):

Theorem 3.26. *There exists a zero dimensional bicompact ordered (consequently hereditary normal) space Q without proper homoemorphic transformations onto any subspace Q_1 of Q.*

A consequence of this theorem is the negative answer to **Problem 74** of [LT–48], p. 162: "Does every infinite Boolean algebra A admit a proper automorphism?" L. RIEGER remarks that this problem is solved also by M. KATĚTOV in [Katě] who used the method of ČECH's bicompactification and has priority over his result.[47] Another independent solution to **Problem 74** was presented by B. JÓNSSON,[48] whose method is similar to L. RIEGER's (constructing a compact zero–dimensional Hausdorff space admitting no proper homeomorphism onto itself).

L. RIEGER concludes the paper with some remarks on **Problem 75** of [LT–48], p. 162 which is as follows:

> Does there exist a (finite) lattice, not a Boolean algebra, which has a dual automorphism σ of period 2, permutable with every lattice automorphism? Must σ be unique? What about non–Desarguesian projective geometries?

L. RIEGER shows on examples that the answers to the first two questions are positive (the first being a trivial example of a finite or a suitable

[47] The paper [Katě] gives also a negative answer to **Problem 76** of [LT–48], p. 166: "Do the order topology and interval topology coincide for a complete Boolean algebra?"

[48] JÓNSSON, B., *A Boolean algebra without proper automorphisms*, Proceedings of the American Mathematical Society 5 (1951), 766–770.

infinite chain, for the second, more examples of finite and infinite lattices are given).

3.5.6 *On groups and lattices* [Rie9] (1952)

This book was published in Czech as one of the volumes of the Series called "A Road To Knowledge" whose aim was to introduce specialized topics of natural sciences to a reader who is not a professional in the field. The books are not supposed to play the role of textbooks, they should inform the reader about the content of a particular area.

The task of L. Rieger was to present the basic notions of group theory and lattice theory, which did not involve an easy job. First, we must consider how abstract the topic is, and also the period when the book was written. The year of publishing is 1952, which means the time when group theory is in Czech language described only in O. Borůvka's textbook [Bor3a] and lattice theory such as only in the introduction to V. Kořínek's paper [Koř2a]. The author bears in mind that it is necessary to combine accessibility and comprehensibility with mathematical precision. L. Rieger shows on various examples how general the topic is and how many specific shapes both groups and lattices may appear in. He points out their applications in natural and technical sciences and introduces several typical methods of proof. He also outlines some resent results as well as some topical problems emerging in the theories.

The part about lattice theory consists of the following chapters: 1. Introduction, 2. Partial order and semiorder, the concept of lattice in terms of a semiorder, examples of lattices, 3. The notion of lattice in terms of two operations, basic axioms of lattice theory, the principal of duality, lattice isomorphism and homomorphism, isomorphic representation, 4. Axioms of distributivity and a complement, the notion of Boolean algebra, 5. The theory of finite Boolean algebras, 6. "Rational functions" on a Boolean algebra (Boolean functions), complete normal forms, 7. The principle of application of Boolean functions to the algebra $(0, 1)$ in electrical engineering, 8. An application of Boolean algebras to propositional (theoretical) logic, 9. Modular lattices, modular and complemented lattices, projective geometries as lattices, continuously dimensional projective geometries, 10. Conclusion.

3.5.7 *On Suslin–algebras and their representations* [Rie10] (1955)

The paper presents a continuation of some investigation from [Rie6], dealing with a special type of σ–algebras. L. RIEGER introduces algebraic analogs of various notions of descriptive theory with the aim to "bring certain preparative considerations to a planned new theory of the quantification of predicate variables of mathematical logic" (p. 142), however, his results can be viewed independently of mathematical logic.

Definition 3.25. Let B be a Boolean algebra and $\{b_{k_1,k_2,\ldots,k_n}\}$ be a system of elements of B where $k_1, k_2, \ldots, k_n = 1, 2, \ldots; n = 1, 2, \ldots$ (the so called *Suslin–system*). We say that B is a *Suslin–algebra*, or *S–algebra* iff

$$\sup_{\{k_n\}_{n=1}^{\infty}} \inf_{n=1,2,\ldots} b_{k_1,\ldots,k_n} \in B,$$

where sup and inf are meant in the sense of the lattice ordering of B.

If its elements are sets, the S–algebra is called an *S–field.*

Definition 3.26. Let $\{b^j_{k_1,\ldots,k_n}\}$ for $j = 1, 2, \ldots$ be a sequence of Suslin systems in a Boolean algebra B. Then the Suslin system $\{a_{l_1,\ldots,l_m}\}$ given by

$$\begin{gathered} a_k = b^1_k, \\ a_{k,l} = b^1_k \wedge b^1_{k,l}, \\ a_{k,l,m} = b^1_k \wedge b^1_{k,l} \wedge b^2_m, \\ a_{k,l,m,n} = b^1_k \wedge b^1_{k,l} \wedge b^2_m \wedge b^1_{k,l,n}, \\ \ldots \end{gathered}$$

is called the *diagonal system of the sequence* $\{b^j_{k_1,\ldots,k_n}\}_{j=1}^{\infty}$.

The author defines other algebraic notions applied to S–algebras: S–homomorphism, S–ideal, factor S–algebra, free S–algebra and two types of distributivity laws in S–algebras: let $A(b_{k_1,\ldots,k_n})$ denote the so called *Suslin A–operation*: $A(b_{k_1,\ldots,k_n}) = \sup_z \bigwedge_{n=1}^{\infty} b_{z_1,\ldots,z_n}$, $z = z_1, \ldots, z_n$; an S–algebra B is called *weakly distributive* iff

$$\bigwedge_{i=k}^{\infty} \bigvee_{k=1}^{\infty} a_{i,k} = A(b_{z_1,\ldots,z_n}),$$

where $b_{z_1,\ldots,z_n} = a_{1,z_1} \wedge a_{2,z_2} \wedge \cdots \wedge a_{n,z_n}$; B is called *strongly distributive* iff for any infinite sequence of Suslin system $\{a^i_{z_1,\ldots,z_n}\}$:

$$\bigwedge_i A(a^i_{z_1,\ldots,z_n}) = A(b_{t_1}, \ldots, b_{t_n}).$$

L. RIEGER shows that the weak distributivity is not implied by the strong distributivity, and that there exists an S–algebra which is strongly distributive and satisfies the strong zero–condition,[49] e. g. the S–field of all subsets of a set. Every S–field of sets is strongly distributive, but there exists strongly distributive S–algebras not isomorphic to a set–field. The author proves the following existence and uniqueness theorems:

Theorem 3.27. *To each cardinal $m > 0$ there exists a free strongly distributive S–algebra with m free generators. This algebra is unique to within S–isomorphism.*

Theorem 3.28. *To each cardinal $m > 0$ there exists a free weakly distributive S–algebra satisfying the strong zero–condition with m free generators. This algebra is unique to within S–isomorphism.*

The main result of the paper is the following:

Theorem 3.29. *The set–field of the C–subsets of the Cantor discontinuum (i. e. the set field obtained by starting from open–and–closed subsets and repeating the Suslin A–operation and complementation) is a free strongly distributive S–algebra with countable many generators.*

This theorem yields a positive solution to a very restricted form of **Problem 80** of [LT–48] (see the analysis of [Rie6]): restricted to distributive S–algebras, while this problem has a negative solution even for general S–algebras.

The author also shows an algebraic extension of the Kolmogorov–Sierpiński process and the consequences of this extension. At the end of the paper he suggests the possibilities of applying the obtained results to mathematical logic.

3.5.8 *A remark on free closure algebras* [Rie11] (1957)

This note was written to correct one theorem which appeared in [LT–48] (p. 189), its Russian translation, and was thus distributed further. The theorem states that the free closure algebra[50] with one generator has exactly sixteen elements. L. RIEGER gives an example of an infinite

[49] B satisfies the strong zero–condition iff for each $b_k^\alpha \in B(0 < \omega_0, \alpha < \omega_1)$ such that $b_k^\alpha \wedge b_k^\beta = 0(\alpha < \beta < \omega_1)$ holds: $\inf_\alpha \bigvee_k b_k^\alpha = 0$.

[50] A closure algebra is defined as a Boolean algebra with a closure operation satisfying: 1. $\overline{x \vee y} = \overline{x} \vee \overline{y}$, 2. $\overline{0} = 0$, 3. $x \subseteq \overline{x}$, 4. $\overline{\overline{x}} = \overline{x}$.

closure algebra with one generator which proves that such a free algebra must be countably infinite. He also remarks that this result can be deduced from [M–T] or [Rie5]. The incorrect theorem was ascribed to K. KURATOWSKI, also wrongly, as L. RIEGER points out. He sees a probable source of both mistakes in a wrong interpretation of K. KURATOWSKI's survey[51] of 14 sets which are possible to obtain from a given set by means of the closure operation and the operation of complements.

3.6 Karel Koutský and lattices

KAREL KOUTSKÝ (1897–1964) was active in several areas of mathematical research: geometry, number theory, topology (he was one of the first participants in E. ČECH's Brno topological seminar), history of mathematics and lattice theory.[52] His list of publications includes three paper on lattices: [Kou1], [Kou2] and [K–K–N]. The first one, "Sur les lattices topologique" consists of a summary of the results of the second one, [Kou2], which is his "habilitation" work (written in 1947, however, published in 1952) in which he builds an extensive theory of topological lattices. The third paper [K–K–N], of which he is a co–author, deals with irreducible elements and bases in general lattices.

3.6.1 *The theory of topological lattices* [Kou2] (1952)

K. KOUTSKÝ puts forward an idea to generalize the concept of topology on a set by considering a partially ordered set (in the special form of a lattice with 0 and 1, which does not particularly weaken the results) instead of a system of subsets of a set. He presents the study of topology without points and axioms, considering only a general closure operation φ, and investigates the properties of this topology if some more requirements are imposed upon this closure operation.

At the beginning of his paper the author outlines a survey of the existing research in the theory of topological lattices. The very idea of studying topologies on a lattice was not completely new at that time. The Japanese mathematicians H. TERASAKA[53] and M. NAKA-

[51] KURATOWSKI, K., *Topologie*, Warsaw 1948.

[52] The reader can find more information about his life and work in e. g. BORŮVKA, O., *Šedesátiny profesora Karla Koutského*, Časopis pro pěstování matematiky 82 (1957), 493–497.

[53] TERASAKA, H., *Theorie der topologischen Verbände: Ein Versuch zur Normalisierung der allgemeinen Topologie und der Theorie der reellen Funktionen*, Proc. Imp. Acad. Jap. 13 (1937), 401–405.

mura[54] had considered special topologies on Boolean algebras, distributive lattices and complete lattices with 0 (drawing upon the work of M. H. Stone), placing four axioms to be satisfied by the closure operation φ. Independently, Portugese mathematicians A. Monteiro and H. Ribeiro[55] investigated a closure operation on partially ordered sets. Compared to the mentioned papers K. Koutský's novelty lies in studying the most general types of topologies on a lattice. Using the terminology defined below, H. Terasaka investigated **DIMU**–topologies, M. Nakamura **ADIU**–topologies and A. Monteiro and H. Ribeiro **IMU**–topologies. The notion of a topology without axioms is attributed (by K. Koutský) to E. W. Chittenden[56] whose ideas, however, remained without a response. As far as the concept of a space without points is concerned it can be traced in the work of E. Foradori.[57]

Definition 3.27. Let L be a lattice with 0 and 1. By a topology in L we mean any mapping φ of L into itself. The ordered pair (L, φ) is called a *topological lattice* and the image $\varphi(x)$ of an element $x \in L$ is called the *closure* of x. If $\varphi(x) = x$, we say that x is *closed*.

We can require φ to satisfy some axioms:

axiom **M** (monotone): $x, y \in L,\ x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$,

axiom **A** (additive): $x, y \in L \Rightarrow \varphi(x \vee y) = \varphi(x) \vee \varphi(y)$,

axiom **I** (incidental): $x \in L \Rightarrow x \leq \varphi(x)$,

axiom **U** (idempotent): $x \in L \Rightarrow \varphi(\varphi(x)) = \varphi(x)$.

If a topological lattice satisfies the axiom **M** we speak about an **M**–topology, and analogously an **A**–topology, **I**–topology and **U**–topology. The axiom **A** implies the axiom **M**. The author investigates each type of topology in detail in individual chapters of the paper.

Definition 3.28. Let (L, φ) be a topological lattice. We say that an element $x \in L$ is a *D–element* iff there exists at least one element $d \in L$ such that $x \wedge \varphi(d) = 0$. We call d an *anathema*. We say that a system of

[54] Nakamura, M., *Closure in general lattices*, Proc. Imp. Acad. Jap. 17 (1941), 5–6.

[55] Monteiro, A., Ribeiro, H., *L'operation de fermeture et ses invariants dans les systčmes partiellement ordonnés*, Portugaliae Math. 3 (1942), 171–183.

[56] Chittenden, E. W., *On general topology and the relation of the properties of the class of all continuous functions to the properties of the space,* Transactions of the American Mathematical Society 29 (1929), 290–321.

[57] Foradori, E., *Stetigkeit und Kontinuität als Teilbarkeitseigenschaften*, Monatshefte Math. Physik 40 (1933), 161–180.

anathemas D_x of an element x is a *complete system of anathemas* of x iff for any anathema w of x there exists an anathema $d \in D_x$ such that $w \leq d$.

In a normal topology on a set, anathemas are identical to complements of a neighborhood of sets, while this is not generally true in lattices. K. KOUTSKÝ studies the axioms in relation to various properties of anathemas of D–elements of L, of their complete systems and of closed elements. A topology φ can satisfy some of the following properties **(P1)**–**(P6)**:

> **(P1)** if d is an anathema of an element x and $d_1 \leq d$, then d_1 is also an anathema of x,
>
> **(P2)** if d is an anathema of x, then $x \wedge d = 0$,
>
> **(P3)** if d is an anathema of x, then $\varphi(d)$ is also an anathema of x,
>
> **(P4)** for every $x \in L$ there exists such a complete system D_x of its anathemas that every $d \in D_x$ is closed in (L, φ),
>
> **(P5)** if d_1, d_2 are anathemas of x, then $d_1 \vee d_2$ is also an anathema of x,
>
> **(P6)** if d_0 is an anathema of x and D_x is a complete system of its anathemas in (L, φ) and if D_x^* is the system of all $d \in D_x$ such that $d_0 \leq d$, then D_x^* is a complete system of anathemas of x.

K. KOUTSKÝ proves the following theorems:

Theorem 3.30. *If φ is an* $\mathbf{M}$*–topology, then φ has the property* **(P1)**.

Theorem 3.31. *If φ is an* $\mathbf{I}$*–topology, then φ has the property* **(P2)**.

Theorem 3.32. *If φ is an* $\mathbf{U}$*–topology, then φ has the property* **(P3)**.

Theorem 3.33. *If φ is an* $\mathbf{IU}$*–topology, then φ has the property* **(P4)**.

Theorem 3.34. *If φ is an* $\mathbf{A}$*–topology and L is a distributive lattice, then φ has the properties* **(P5)** *and* **(P6)**.

When looking into the validity of the reverse implications K. KOUTSKÝ shows that we need to presuppose modularity and complementarity of a lattice:

Theorem 3.35. *Let (L, φ) be a complemented modular topological lattice. If φ has the property* **(P1)**, *then φ is an* $\mathbf{M}$*–topology.*

Theorem 3.36. *Let (L, φ) be a complemented modular topological lattice. If φ has the property* **(P2)**, *then φ is an* **I**–*topology.*

Theorem 3.37. *Let (L, φ) be a complemented modular topological lattice. If φ has the property* **(P3)**, *then for each $x \in L$ it holds: $\varphi(\varphi(x)) \leq \varphi(x)$ (a weakened form of the axiom* **U**); *thus if φ is an* **I**–*topology and at the same time has the property* **(P3)**, *then φ is a* **U**–*topology.*

Theorem 3.38. *Let (L, φ) be a complemented modular topological lattice. If φ has the property* **(P5)** *or* **(P6)**, *then for each pair $x, y \in L$: $\varphi(x \vee y) \leq \varphi(x) \vee \varphi(y)$ (a weakened form of the axiom* **A**); *if φ is an* **M**–*topology and at the same time has the property* **(P5)** *or* **(P6)**, *then φ is an* **A**–*topology.*

Other results of K. KOUTSKÝ concern the behaviour of closed elements, e. g.:

Theorem 3.39. *Let (L, φ) be a topological lattice. If φ is an* **IM**–*topology, then the intersection of a non–void finite number of closed elements in (L, φ) is again a closed element in (L, φ), and the validity of both axioms is necessary.*

Theorem 3.40. *Let (L, φ) be a topological lattice. If φ is an* **A**–*topology, then the union of a non–void finite number of closed elements in (L, φ) is also a closed element in (L, φ). However, the reverse implication does not generally hold.*

He deduces also a number of other results concerning the character of elements of topological lattices, D–elements, systems of topologies in a given lattice and a relativization of a given topology in sublattices of the lattice. The main theorem of the paper is the following:

Theorem 3.41. *The theory of* **IMU**–*topologies can be built on the concept of a system $F \subset L$ of closed elements of L satisfying:*

(F1) *for each $x \in L$ there exists at least one $f \in F$ such that $x \leq f$,*

(F2) *if $F(x)$ denotes the system of all $f \in F$: $x \leq f$, where $x \in L$, then there exists a lower bound $\bigwedge_x f$ of elements of $F(x)$,*

(F3) *for each $x \in L$: $\bigwedge_x f \in F(x)$.*

The theory of **AIU**–*topologies can be built upon the notion of a system $F \subset L$ of closed elements in L satisfying* **(F1), (F2), (F3)** *and* **(F4)**:

(F4) $f_1, f_2 \in F \Rightarrow f_1 \vee f_2 \in F$.

Generally, however, the concept of closed elements is not enough to built a topology on.

3.6.2 *On additively irreducible elements and additively bases in a lattice* [K–K–N] (1959)

This paper written by three Brno mathematicians K. KOUTSKÝ, L. KOSMÁK and M. NOVOTNÝ discusses the properties of join (they use the term "additively") irreducible elements in lattices. The authors mention previous papers which investigate such elements or which deal with analogs to irreducible elements in lattices. Apart from G. BIRKHOFF's *Lattice Theory* they mainly draw upon [B–F] and the paper of J. R. BUCHI[58] which also deals with bases in lattices.

Definition 3.29. Let L be a lattice with 0, m a cardinal number. An element $x \in L$ is called *(join) m–irreducible* iff $x \in M(x)$ for each subset $M(x) \subseteq L$ such that card$M(x) < m,\ x = \bigvee\limits_{t \in M(x)} t$.

A set $B \subseteq L$ is called a *(join) m–basis* of L iff for each $x \in L$ there exists a non–empty set $B(x) \subseteq B$ such that card $B(x) < m,\ x = \bigvee\limits_{t \in B(x)} t$.

The authors show how it is possible to construct another m–basis from the given two and study the relationship between m–irreducible elements and m–bases:

Theorem 3.42. *Let m be a regular cardinal number and A, B two m–bases of a lattice L with 0. Then the sets $A \cup B, A \vee B$ and $A \wedge B$[59] are also m–bases of L.*

Theorem 3.43. *Let m be a cardinal number, L a lattice with 0. Then $x \in L$ is m–irreducible iff x is an element of every m–basis of L.*

The concept of m–basis is also applied to ordered sets and it is shown that a lattice L has only one m–basis iff each of its elements is m–irreducible, and in this case L is an ordered set with certain special properties.

In the final part of the paper the authors study lattices with particular types of bases:

Theorem 3.44. *Let m be a cardinal number, L a lattice with 0. Then L has a least m–basis[60] iff the set of all m–irreducible elements in L is an m–basis.*

[58] BUCHI, J. R., *Representation of complete lattices by sets*, Portugaliae Math. 11 (1952), 151–167.

[59] $A \wedge B$ denotes the set of all $a \wedge b$, where $a \in A, b \in B$, and analogously $A \vee B$.

[60] An m–basis B_0 of L is called the least m–basis iff $B_0 \subseteq B$ for any m–basis B of L.

Theorem 3.45. *Let m be a cardinal number, L a lattice with 0. Then the following properties are equivalent:*

(i) *there exists an m–basis B in L satisfying: for each $x \in L, x \neq 0$ there exists only one non–empty set $B(x) \subseteq B - \{0\}$, card $B(x) < m$ such that $x = \bigvee_{t \in B(x)} t$.*

(ii) L is isomorphic to a system $\mathcal{L}$ of subsets of a set M satisfying:

1. the empty set and one–element–sets belong to $\mathcal{L}$,

2. for each $T \in \mathcal{L}$: card $T < m$,

3. $\mathcal{L}$ is a lattice in which $R = \bigvee_{T \in \mathcal{R} \subseteq \mathcal{L}} T \Rightarrow R = \bigcup_{T \in \mathcal{R} \subseteq \mathcal{L}} T$.

3.7 Miloslav Mikulík and lattices

M. Mikulík investigated the relationship between different types of convergence in metric lattices in several papers. The impulse for studying metric lattices came from O. Borůvka who turned M. Mikulík's attention to the fact that a suitable system of solutions of a differential equation $x' = f(t, x)$ is realized by a metric lattice satisfying the properties (U1) – (U3) from [Mik1] (see below). The investigation of such lattices became the topic of his RNDr. thesis supervised by O. Borůvka.

In his papers M. Mikulík presents several sufficient conditions in order that a metric convergence, o–convergence and/or $\star$–convergence be identical in a lattice. He always gives an example of a lattice satisfying the particular conditions. Let us recall the definitions of each type of convergence:

Definition 3.30. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a lattice L *metric converges* to an element x (denoted by $x_n \stackrel{\rho}{\to} x$) iff $\lim_{n\to\infty} \rho(x_n, x) = 0$.

Definition 3.31. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a complete lattice L *o–converges* to an element x (denoted by $x_n \stackrel{o}{\to} x$) iff $\bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k = x$.[61]

Definition 3.32. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a complete lattice L *$\star$–converges* to an element x iff every subsequence of $\{x_n\}_{n=1}^{\infty}$ contains a subsubsequence o–converging to x.[62]

[61] A definition of o–convergence applicable to σ–lattices was introduced independently by G. Birkhoff and L. Kantorovič ([LT–40], p. 29.)

[62] $\star$–convergence in lattices was introduced independently by L. Kantorovič, von Neumann and G. Birkhoff ([LT–40], p. 30.)

3.7.1 *Metric lattices* [Mik1] (1954)

MILOSLAV MIKULÍK investigates non–void sets L with the following properties:

(U1) L is partially ordered and it is a complete lattice with respect to this ordering,

(U2) A metric ρ is defined on L such that L is compact with respect to ρ,

(U3) Let $A \neq \emptyset, A \subset L,\ a = \bigwedge_{t \in A} t,\ b = \bigvee_{t \in A} t$ be its infimum and supremum with respect to the partial ordering. Let $d(A)$ be its diameter with respect to ρ. Then $d(A) = \rho(a, b)$.

The author is interested in how different types of convergence are related to one another in such a lattice. He proves the following:

Theorem 3.46. *Let L be a metric lattice with the properties* (U1), (U2) *and* (U3). *Then metric convergence, o–convergence and $\star$–convergence are identical.*

Theorem 3.47. *Let $A \subset L$ be a non–void convex sublattice of L. Then A is closed iff A is compact.*

3.7.2 *A note on $\star$–convergence* [Mik2] (1955)

In this paper M. MIKULÍK generalizes one result stated in [LT–48] and shows that in a lattice with a metric satisfying certain conditions, metric convergence and $\star$-convergence are equivalent.

Let L be a lattice with a metric ρ. The author makes use of the following properties:

(A) if $x, y \in L$, then $\rho(x, y) = \rho(v \wedge y, x \vee y)$,

(B) if $x, y, z \in L, x < y < z$, then $\rho(x, y) \leq \rho(x, z),\ \rho(y, z) \leq \rho(x, z)$,

(C) we can choose from any bounded (in terms of the metric ρ) non–increasing (non–decreasing) sequence of elements of L a subsequence which metric converges,

(D) for $\{x_n\}_{n=1}^{\infty},\ x_n \in L$: if $x_n \xrightarrow{o} x$, then $x_n \xrightarrow{\rho} x$.

M. MIKULÍK's investigations of the properties yield the theorem:

Theorem 3.48. *Let (L, ρ) be a lattice with a metric ρ which satisfies the properties* (A) *and* (B). *If (L, ρ) has the property* (C), *then it is a σ–complete lattice with the property* (D). *Conversely, if a σ–complete lattice with a metric satisfies the properties* (A), (B) *and* (D), *then it has also the property* (C) *and is metric complete. In a lattice (L, ρ) having the properties* (A), (B) *and* (C) *metric convergence and $\star$–convergence are equivalent.*

The author also points out that a lattice (L, ρ) with the properties (A) and (B) need not be modular.

3.7.3 *A note on topological lattices* [Mik3] (1955)

The author investigates the order topology in complete lattices. He shows sufficient conditions for a lattice to be a topological lattice in this topology.

Theorem 3.49. *Let L be a complete lattice with a metric ρ satisfying:*

1. if $x, y \in L$, then $\rho(x, y) = \rho(x \wedge y, x \vee y)$,

2. if $x, y, z \in L, x < y < z$, then $\rho(x, y) < \rho(x, z), \rho(y, z) < \rho(x, z)$,

3. we can choose from any bounded (in terms of lattice ordering) increasing (decreasing) sequence of elements of L a sequence which metric converges in terms of ρ.

Then L is a topological lattice in terms of its order topology.

3.7.4 *Notes on lattices with a metric* [Mik4] (1959)

In this paper M. MIKULÍK deals once more with the relations between a metric convergence and o–convergence in metric lattices. He shows a generalization of his results from [Mik1] which was recommended by J. NOVÁK to the author.

In the first part of the paper the following result is shown:

Theorem 3.50. *Let L be a lattice with a metric ρ such that:*

1. we can choose from any bounded (in terms of lattice ordering) non–increasing (non–decreasing) sequence of elements of L a sequence which metric converges, and

2. if $A \subset L$ is countable infinite or finite and has a supremum and infimum, then the distance of the supremum and the infimum equals to the diameter of A.

Then the metric convergence and the o–convergence are identical.

In the next part the convergence in σ–complete lattices is investigated:

Theorem 3.51. *Let L be a σ–complete lattice with a metric ρ in terms of which the set L is compact. Then the following conditions are equivalent:*

(i) *the metric convergence and the o–convergence are identical,*

(ii) *if $\{x_n\}_{n=1}^{\infty}$, $x_n \in L$ metric converges, then*

$$\lim_{n\to\infty} \rho(\bigvee_{k=n}^{\infty} x_k, \bigwedge_{k=n}^{\infty} x_k) = 0,$$

(iii) *if $\{x_n\}_{n=1}^{\infty}$, $x_n \in L$ metric converges, then*

$$\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k.$$

3.8 Otomar Hájek and lattices

OTOMAR HÁJEK's early works include four papers on lattice theory although he focused on questions of topology. We shall describe the results of his two papers dealing with direct decompositions of lattices [Háj1, Háj2]. In 1965 his other two papers [Háj3, Háj] devoted to lattice theory were published. In [Háj3] the author gives a canonical representation of modular lattices of finite length in terms of simple nondistributive and finite distributive lattice, and introduces a characteristics, called a defect, of a lattice which is put into the relation with lattice decompositions. The other paper is closely connected to the first one, describing two integer–valued characteristics of modular lattices of finite length.

3.8.1 *Direct decompositions of lattices, I* [Háj1] (1957)

O. HÁJEK investigates algebraic properties of central and neutral elements of a lattice in connection with direct and subdirect decompositions of lattices. He also shows an analogy of the obtained results for rings. A decomposition of a lattice L into the direct product of L_a's will be denoted by $L \cong \prod_a L_a$. If L is (decomposable into) the subdirect product of L_a's we shall write $L \leq \prod_a L_a$. The source of lattice theoretic concepts is primarily [LT–48] for O. HÁJEK.

The author first gives a detailed analysis of the notions of central and neutral elements in a lattice and shows some consequences of their definitions. Then he studies the relation of two subdirect products of a lattice and proves the following unicity theorem by deducing it from a more general form:

Theorem 3.52. *Let $L_1 \times L_2 \geq L \leq M_1 \times M_2$ and let a neutral element $e \in L$ be carried to $[1, 0]$ in both mappings. Then $L_1 = M_1$ and $L_2 = M_2$.*

O. HÁJEK investigates the complements of central and neutral elements and studies direct and subdirect products of a lattice in relation to homomorphic images of the lattice. He proves the following condition for a subdirect product to be direct:

Theorem 3.53. *If for three different a's: $L \leq (\prod_{b \in A, a \neq b} L_b) \times L_a$ under the same isomorphism, then $L \cong \prod_A L_a$.*

A generalization of a factor-theorem from G. BIRKHOFF's [LT–48], p. 26, is shown, and the investigated notions are extended to ring theory. In the final part specific examples of rings or lattices and their centers are described.

3.8.2 *Direct decompositions of lattices, II* [Háj2] (1962)

O. HÁJEK investigates cut–completions (denoted by $\sim$) of direct products of partially ordered sets in this paper and his results lead to the Glivenko–Stone theorem.

In the first part the following theorems are proved:

Theorem 3.54. *Let P be a partially ordered set with 0 and 1. Let $P \cong P_A \times P_a$ under the isomorphism f. Then $\tilde{P} \cong P_A \times \tilde{P}_a$ under an extension of f.*

Theorem 3.55. *Let $P, P_a (a \in A)$ be partially ordered sets, where A and all P_a contain more than one element. If $P \cong P_A \times P_a$ and $\tilde{P} \cong P_A \times \tilde{P}_a$, then P (and consequently all P_a) contains both 0 and 1.*

In the second part of the paper the author applies his results to Boolean algebra, and shows how they imply that if P is a Boolean algebra, then its every element is central and $\tilde{P}$ is a distributive lattice which is orthocomplemented and which has unique complements. O. HÁJEK's results and the fact that an orthocomplemented lattice with unique elements is a Boolean algebra yield a new proof of the famous Glivenko–Stone Theorem:[63]

Theorem 3.56. *If B is a Boolean algebra, then so is $\tilde{B}$.*

[63] GLIVENKO, V., *Sur quelques points de la logique de M. Brouwer*, Bull. Acad. Sci. Belgique 15 (1929), 183–188 and [Sto2].

3.9 Václav Havel and lattices

The research work of VÁCLAV HAVEL concerns the study of algebraic structures, emerging on the borders of algebra and foundations of geometry. In 1952/53 he defended his RNDr. thesis on decompositions of elements in lattices satisfying the descending chain condition, which was supervised by and reflects an influence of V. KOŘÍNEK. The results of the thesis were published in V. HAVEL's first paper on lattice theory [Hav1]. The problem of direct decompositions of the unity in finite lattices is the content of the next paper [Hav2]. The third paper [Hav3] of V. HAVEL concerning lattices we analyze shows an influence of O. BORŮVKA. V. HAVEL generalizes a theorem of O. BORŮVKA dealing with an isomorphism between the Zassenhaus refinements of two chains in an equivalence lattice. After 1963 he published several other papers treating lattice theoretic questions.

3.9.1 *Decompositions of elements of a lattice with the minimal condition* [Hav1] (1955)

Let L be a lattice satisfying the descending chain condition. The author introduces a decomposition in L by means of a symmetric binary relation ρ on L. For $c \in L$ we shall call

$$c = a_1 \vee a_2 \vee \cdots \vee a_n \ (n \geq 2) \tag{3.9}$$

a decomposition of c. V. HAVEL distinguishes four types of decompositions in L (and compares them to the decompositions presented by O. ORE in [Ore1, Ore2] and by G. BIRKHOFF in [LT–48]):

Definition 3.33. We shall call (3.9) a *ρ–decomposition* iff $a_i \rho (a_{k_1} \vee a_{k_2} \vee \cdots \vee a_{k_j}$ holds for any $i = 1, \ldots, n \geq 2$ and for any choice of various $k_1, \ldots, k_j$ from $\{1, \ldots, i-1, i+1, \ldots, n\}$.

A ρ–decomposition will be called *proper* iff ($x \rho y \Leftrightarrow x, y$ are incomparable), it will be called *direct* iff ($x \rho y \Leftrightarrow x, y$ are incomparable and $x \wedge y = 0$) and it will be called *strong* iff ($x \rho y \Leftrightarrow x, y$ are incomparable and $(x \vee p) \wedge (y \vee p) = p$ for every $p \in L$).

Definition 3.34. An element $c \in L$ will be called *ρ–(in)decomposable* iff there exists (does not exist) a ρ–decomposition (3.9).

After a detailed study of general properties of ρ–decompositions and relations between the introduced types of ρ–decompositions (e. g. giving a sufficient condition for every ρ–decomposable element of L to have a ρ–decomposition with ρ–indecomposable factors, and a sufficient condition for the uniqueness of this decomposition), the author investigates the Zassenhaus chain construction in modular lattices. He gives necessary and sufficient conditions for this construction to yield no proper refinements of two given chains. He uses results from [Koř2a] and [Jan1], and also applies his notion of ρ–composition to their investigation.

3.9.2 *A note on the uniqueness of direct decompositions in modular lattice of finite length* [Hav2] (1955)

V. HAVEL investigates the validity of the following condition (C) in a modular lattice of finite length:

> (C) There exists only one (up to the order of factors) decomposition if 1 into indecomposable factors.

The following results are presented:

Theorem 3.57. *Let L be a modular lattice of finite length. Let 1 be decomposable. The condition* (C) *is satisfied in L iff every element in L has at most one complement.*

Theorem 3.58. *Let L be a modular lattice of finite length. Let 1 be decomposable. The condition* (C) *holds in L iff for any two decompositions*

$$1 = a_1 \times \cdots \times a_m = b_1 \times \cdots \times b_n$$

the following equations are satisfied for $i = 1, \ldots, m$:

$$a_i = \bigvee_{j=1}^{n} (a_i \wedge b_j).$$

Theorem 3.59. *Let L be a modular lattice of finite length. Let 1 be decomposable. The condition* (C) *is satisfied in L iff no sublattice of L is isomorphic to any of four lattices in Figure 3.4, where a and b are indecomposable elements.*

3.9.3 *On semichained refinements of chains in equivalence lattice* [Hav3] (1963)

The author presents a new proof and a generalization of a theorem discovered by O. BORŮVKA ([Bor7a], pp. 65–68) on semichained refinements of two chains in the equivalence lattice of a given set. O. BORŮVKA used the language of set partition theory when stating and proving this theorem, V. HAVEL employs the methods of lattice theory, where

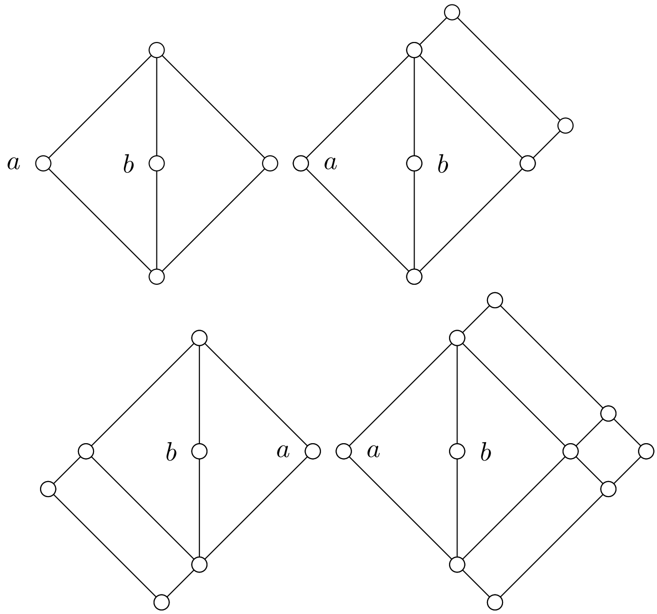

Figure 3.4: To Theorem 3.62

he follows the study of different types of Zassenhaus refinements of two chains in a given lattice (as introduced by V. KOŘÍNEK [Kořl]), and he also applies some results and concepts concerning equivalences of P. DUBREIL and M. L. DUBREIL–JACOTIN.[64]

V. HAVEL uses the following lattice theoretic notions. Let L be a lattice, $a, b \in L, a \geq b$, let $A = \{a_i\}, i = 0, \ldots, r, B = \{b_j\}, j = 0, \ldots, s, a_0 = b_0 = a, a_r = b_s = b$ be two finite chains between a and b. Their Zassenhaus refinements will be denoted by A^* and B^*, where

$$A^* = \{a_{k,j}\},\ a_{k,j} = a_{k+1} \vee (a_k \wedge b_j),\ k = 0, \ldots, r-1;\ j = 0, \ldots, s,$$
$$B^* = \{b_{l,i}\},\ b_{l,i} = b_{l+1} \vee (b_l \wedge a_i),\ l = 0, \ldots, s-1;\ i = 0, \ldots, r.$$

The chains A, B will be called similar iff there exists a one–to–one mapping $f : A \to B : a_i \to b_{f(i)}$ such that $a_k/a_{k+1}, b_{f(k)}/b_{f(k)+1}$ are similar quotients (two quotients $a/b, c/d$ are similar iff there exists the so called middle quotient x/y such that $a = b \vee x, c = d \vee x, y = b \wedge x = d \wedge x$) for every k.

Let $E(S)$ be the lattice of all equivalence relations on the given set S, for $a \in E(S)$, S/a will denote the set of all corresponding a–blocks.

[64]DUBREIL, P., DUBREIL–JACOTIN, M., L., *Théorie algébrique des relations d'équivalence*, Jour. de math. pure et appl. 18 (1939), 63–95.

If $S' \neq \emptyset, S' \subseteq S, a \in E(S)$ we shall consider the one–to–one mapping of the set S'_a consisting of all a–blocks overlapping S' onto the set $S'_{(a)}$ of intersection of these a–blocks with S'. This mapping $S'_a \to S'_{(a)} : A \to A \cap S'$ for every $A \in S/a, A \cap S' \neq \emptyset$ can be extended to a mapping $S/a \to S'_{(a)} \cup \{\emptyset\} : A \to A \cap S'$ for every $A \in S/a$, which we call the *semicontractions*. If $S/a = S'_a$, we speak about a *contraction*. If $a \geq b, a, b \in E(S)$ and the effect of the relation b is reduced to an a–block, we obtain a relation $a//b$ called the relation–quotient of a, b. If $a \geq b, c \geq d, a, b, c, d \in E(S)$ and $a \geq c$, we say that the quotient $c//d$ is deduced from $a//b$ by (semi)contraction iff every $A \in S/a$ is transferred into any $C \in S/c, C \subset A$ by some (semi)contraction $A/(a//b) \to C/(c//d) \cup \{\emptyset\}$. If $a_1/a_2, b_1/b_2$ are similar with the middle quotient c_1/c_2 and $c_1//c_2$ is deduced by some (semi)contraction from $a_1//a_2$ and by some (semi)contraction from $b_1//b_2$, then we call $a_1//a_2, b_1//b_2$ *(semi)chained*. Chains A, B of $E(S)$ are called (semi)chained iff there exists a one–to–one mapping $A \to B : a_i \to b_{f(i)}$ such that $a_k//a_{k+1}, b_{f(k)}//b_{f(k)+1}$ are (semi)chained for every k.

V. Havel proves the following theorem by constructions:

Theorem 3.60. *Let $E(S)$ be the lattice of all equivalence relations on a given set S, A, B two chains in $E(S)$ satisfying the following condition for every k, l:*

$$a_{k,l+1} \wedge (a_k \wedge b_l) = b_{l,k+1} \wedge (b_l \wedge a_k).$$

Then A^* and B^* are semichained by the mapping $a_{k,l} \to b_{l,k}$. The semichaining becomes a chaining iff the relations $a_{k+1} \vee (a_k \wedge b_{l+1}$ and $a_k \wedge b_l$, as well as the relations $b_{l+1} \vee (b_l \wedge a_{k+1})$ and $b_l \wedge a_k$ are permutable[65] *for all k, l.*

3.10 Other papers related to lattice theory

In this section we shall describe the results of works whose authors did not do their research primarily in lattice theory, however, their investigation touches lattice theoretic problems. We have already seen some applications of lattices in the previous sections (O. Borůvka and partition theory, L. Rieger and mathematical logic, K. Koutský and topology, M. Mikulík and functional analysis), but we have to name also some others.

[65] Relations $a, b \in E(S)$ are called permutable iff every a–block contained in a arbitrary $(a \vee b)$–block overlaps every b–block contained in the same $(a \vee b)$–block.

B. POSPÍŠIL (1912–1944) was interested in the field of topology. In the years before and during WWII he studied the problems concerning general topological spaces, his results have, however, important consequences for Boolean rings. After WWII J. NOVÁK led a seminar in Brno whose starting activity involved linearly ordered sets in terms of the theory of topological spaces. This investigation attracted M. NOVOTNÝ who produced several papers from the field of ordered sets which are to be recalled here. In the middle of 1950's F. ŠIK started focusing on partially ordered groups and l–groups. K. ČULÍK's papers of the late 1950's also touch the problems of partially ordered sets and lattices. The above mentioned works will be treated in this section. However, as the papers deal with lattice theoretic problems only partially, we would need more space in order to set them in the specific mathematical contexts and to introduce other important aspects, which is not possible to attain in the scope of this work, the following paragraphs will present only a brief outline of the results without a more detailed analysis.

Another mathematician who can be named here is V. KUDLÁČEK because of his paper on lattice ordered groupoids,[66] in which he generalizes some results obtained for l–groups. His early interest in partially ordered groupoids and rings soon moved to other areas.[67] In the first half of 1960's the group of Brno mathematicians working in the area of general algebraic structures was joined by VÍTĚZSLAV NOVÁK and LADISLAV SKULA, however, as the beginnings of their academic careers coincide with the end of our analyzed period, we shall not include a description of their activities.

3.10.1 Bedřich Pospíšil and Boolean rings

The results presented in B. POSPÍŠIL's papers usually developed within the work in the famous Brno topological seminar of E. ČECH which came into existence in June 1936 and had to be closed in November 1939. An important role in B. POSPÍŠIL's research was played by solving the problem put forward by E. ČECH in January 1937: determine the cardinal number of a bicompact Hausdorff space $\beta(S)$, where S is an infinite countable isolated Hausdorff space, such that (i) S is dense in $\beta(S)$, (ii) any bounded continuous real function defined in the domain

[66] KUDLÁČEK, V., *O svazově uspořádaných grupoidech*, Časopis pro pěstování matematiky 80 (1955), 44–50.

[67] RÁB, M., *K šedesátinám docenta Václava Kudláčka*, Pokroky matematiky, fyziky a astronomie 33 (1988), 345–346.

S admits a continuous extension to the domain $\beta(S)$.[68] B. POSPÍŠIL showed in [Pos1] that if S is an infinite isolated Hausdorff space of cardinal number m, the cardinal number of $\beta(S)$ is exp exp m.[69]

The author develops consequences of the mentioned result in his paper [Pos2] in which he investigates various topological spaces, particularly Boolean spaces as introduced by M. H. STONE in [Sto1]. In this paper M. H. STONE also proved that the theory of Boolean spaces is equivalent to the theory of Boolean rings,[70] which gives B. POSPÍŠIL's results algebraic interpretation. Apart from other important properties, B. POSPÍŠIL proved:

Theorem 3.61. *Let A be the Boolean ring of all subsets of an infinite set T of a cardinal number m, let I be the ideal of all subsets of T of cardinal numbers $< m$, let I^* be an ideal in $A, I^* \subset I$. Then A/I^* has exp m elements, exp exp m ideals and exp exp m prime ideals with characters exp m.*[71]

B. POSPÍŠIL's method of determining the number of ideals enables us to analyze many important Boolean rings and spaces from topology and measure theory. His results concerning ideals in Boolean rings are in a close connection with those of A. TARSKI [Tar1, Tar3], however, the method of obtaining them is different.

B. POSPÍŠIL's research on the theory of Boolean rings aroused attention of mathematicians involved in mathematical logic where this theory presents one of the basic chapters. The editors of *Fundamenta Mathematicae* asked him for a paper which would describe an algebraic interpretation of his topological results, and thus made them available for a wider circle of readers. B. POSPÍŠIL produced the paper [Pos6], which includes not only his previous results in the language of algebra, but also some new ones. Although the paper was prepared for press already in 1939, the whole series of the journal was published in 1945 [Čech].

M. H. STONE's concepts are also used by B. POSPÍŠIL in other papers, in which he develops the theory of the so called continuous distri-

[68]The fact that to every given completely regular space S there exists a bicompact Hausdorff space $\beta(S)$ satisfying the properties (i) and (ii) was proved by A. TICHONOV, *Über die topologische Erweiterung von Räumen*, Mathematische Annalen 102 (1930).

[69]We keep B. POSPÍŠIL's denotation exp $m = 2^m$ for a cardinal number m.

[70]The concept of Boolean rings was also introduced by M. H. STONE: rings in which every element is idempotent.

[71]A character of an ideal in a Boolean ring is meant a minimal cardinal number of a system of generators of this ideal.

butions and an abstract–algebraic theory of measurable functions [Pos3]. Let f be a real function and A the Boolean ring of measurable subsets of the domain of f. The function f is called A–measurable iff for each interval j, $\varphi(j)$ is the set S of x with $f(x) \in j$. By a distribution in A the author means an abstract homomorphic function φ of the ring of intervals to a Boolean ring. He defines a character on a set (ring) of abstract functions φ, satisfying some specific properties, to a fixed A in a way resembling the notion of the character of an abelian group. The continuation of this paper, [Pos4], presents four equivalent conditions for a real–valued function $h(\varphi)$ defined on a Boolean ring A of elements φ to be "continuous". He defines A to be A "separable" when its representation by open–and–closed subsets of a totally disconnected bicompact Hausdorff space is separable and presents various results. B. POSPÍŠIL's research from [Pos3, Pos4] is completed by [Pos5] in which he correlates the bicompact Hausdorff space e associated with a general Boolean algebra (ring) A, the quotient algebra obtained from A by ignoring the sets of first category in e, and the space (Banach lattice) of all functions continuous on e. The discussed ideas relate to the representation theory of vector lattices.[72]

B. POSPÍŠIL's papers include a great number of original results, which is always pointed out by the reviewers. His research was highly prized by his teacher E. ČECH who makes a comment that Czech mathematics suffered a truly irretrievable loss by B. POSPÍŠIL early death caused by a fatal treatment of Gestapo [Čech].

3.10.2 Miroslav Novotný and ordered sets

M. NOVOTNÝ's research, characteristic of its wide range, includes the theory of ordered sets, algebra, topology, mathematical linguistics, information systems, constructions of grammars, monounary algebras and relation structures. His papers written by the beginning of 1960's were influenced by his teachers: O. BORŮVKA (groupoid operations, set partition theory) and J. NOVÁK (the theory of ordered continua). The paper on irreducible elements and additive base in lattices [K–K–N], which he was a co-writer of, has already been analyzed in the section dealing with K. KOUTSKÝ. An initial source of inspiration for M. NOVOTNÝ's interest in the field of ordered sets was G. BIRKHOFF's book *Lattice Theory*. He was attracted by the study of the cardinal powers of the type 2^G (G being a linearly ordered set) which led him to the investigation of

[72] A vector lattice is called a partially ordered linear space which is a lattice.

cardinal powers $\mathbb{R}^G$, where $\mathbb{R}$ is the linearly ordered set of reals. The elements of $\mathbb{R}^G$ are isotone functionals, i. e. isotone functions from a partially ordered set to the set of reals. M. NOVOTNÝ's investigation was motivated by an analogy with linear functionals on a vector space.

The study of problems related to isotone functionals was commenced by M. NOVOTNÝ in the paper [Nov1] in which he gives e.g. necessary and sufficient conditions in order that an isotone functional on a set G be extendible to any superset, and in order that each isotone fuctional on each subset of a set G be extendible to all of G. The author also obtains relations between $\mathbb{R}^G$ and $\mathbb{R}^H$ from the knowledge of ordered sets G and H, and conversely. The paper [Nov2] continues in the investigation of isotone functionals. M. NOVOTNÝ characterizes e. g. subsystems H of ordered system G such that every isotone mapping of H into a gapless chain K can be extended to an isotone mapping of G into K. Cardinal numbers are investigated also in the papers [Nov3, Nov5, Nov6] in which he deals with cardinal arithmetic.[73]

3.10.3 *Convergence in Boolean σ–algebras* [N–N] (1953)

The paper of J. NOVÁK and M. NOVOTNÝ *On the convergence in σ–algebras of point–sets* studies a relationship between metric and topological convergence[74] in a σ–algebra of point-sets, i. e. a class of subsets of an abstract space X which contains X and which is closed under the formation of countable unions and differences. The main result is the following:

Theorem 3.62. *Let A be a σ–algebra of point–set. The following conditions are equivalent:*

(i) there exists a metric in A such that the metric and topological convergences are identical,

(ii) there exists a probability function P defined on A satisfying for every event $a \in A$: if $P(a) = 0$, then $a = 0$,

(iii) A is isomorphic to the system of all subsets of a set which is at most countable.

[73] NOVÁK, V., *Sixty years of Professor Miroslav Novotný*, Czechoslovak Mathematical Journal 43 (1982), 338–343, NOVÁK, V., PŮŽA, B., Seventy years of Professor Novotný, Czechoslovak Mathematical Journal 42 (1992), 379–382.

[74] We say that the sequence of sets $\{A_n\}$ converges topologically iff $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$.

3.10.4 Karel Čulík and partially ordered sets

KAREL ČULÍK (1926–2002) is famous for his work in often interdisciplinary fields: graph theory, algorithms, formal languages and grammars, Boolean equations, and many others.[75] Although his early works deal mostly with graph theory, he also studied problems of partially ordered sets. His paper [Čul1] is devoted to the lexicographic sum of partially ordered sets, the author studies basic properties of the so called inserted partially ordered sets, partitions in inserted and lexicographically irreducible partially ordered subset and factor partially ordered sets which are lexicographically irreducible. He shows that not every partially ordered set is a lexicographic sum of a system of partially ordered and lexicographically irreducible sets over partially ordered and lexicographically irreducible set. In [Čul2] K. ČULÍK studies three types of homomorphism on partially ordered sets, which are isotone, however, they need not be lattice homomorphisms. Analogously to group theory those homomorphisms enable us to study various notions and relations within the theory of partially ordered sets. The author also point out their connection to the previously investigated lexicographic sum.

3.10.5 František Šik and l–groups

FRANTIŠEK ŠIK began his active research career at the beginning of 1950's. His first works were influenced by O. BORŮVKA, they deal with congruence relations and set partition theory. Soon he was also attracted by the fields of algebra and topology because of his postgraduate supervisors E. ČECH and V. KOŘÍNEK. The essential part of FRANTIŠEK ŠIK work belong, however, to the area of partially ordered and lattice ordered groups in which he started working in the middle of the 1950's.

F. ŠIK introduced several notions which became important tools for studying the structure of l–groups: e. g. a polar (the original term was "component") and completely subdirect product. The concept of polar was defined by means of the notion of orthogonality. The orthogonality on an l–group G is defined by the relation $x\delta y \Leftrightarrow |x| \wedge |y| = 0$. For $A \subseteq G$ we denote $A' = \{x \in A : x\delta y \text{ for each } a \in A\}$, the set A being called a polar in G. For $a \in G$, the set $\{\{a\}'\}'$ is called a principal polar and the set $\{a\}'$ a dual principal polar. The systems of all polars, all principal polars and all dual principal polars of G are denoted by $\Gamma(G), \Pi(G)$ and $\Pi'(G)$, respectively. F. ŠIK proved in [Šik1] that the system $\Gamma(G)$

[75] HÁJEK, P., *Zemřel Karel Čulík*, Pokroky matematiky, fyziky a astronomie 47 (2002), 344–348.

partially ordered by inclusion is a complete Boolean algebra and that $\bigwedge_{i\in I} X_i = \bigcap_{i\in I} X_i$ for each $\emptyset \neq \{X_i\}_{i\in I} \subseteq \Gamma(G)$. The conditions for $\Pi(G)$ and $\Pi'(G)$ to be Boolean algebras can be found in the paper [Šik2].[76]

3.11 A note on the Czech terminology concerning lattices

Early lattice theoretic terminology varied in individual languages, in fact, there existed three versions naming the entity "lattice". The word "lattice" was introduced by G. BIRKHOFF, O. ORE called the same object "structure", which was adopted in French and some Slavic languages for a certain period (before the English version prevailed), and "Verband" is the expression used in German.

The materials including some information on the development of Czech lattice theoretic terminology consisted of O. BORŮVKA's works [Bor1, Bor3a, Bor5, Bor6, Bor7b], V. KOŘÍNEK's paper [Koř2a], L. RIEGER's book [Rie9], and also the summaries of the papers [Sto2, Koř1, Rie4].

The first use of the expression "lattice" in the Czech literature appears in the paper of O. BORŮVKA [Bor1] in 1939, and it was in the form we know it today "svaz". This term is clearly the translation of the German "Verband", however, the reasons for choosing this version from the three varieties which were in use in other languages of that time remain hidden. The word "svaz" is used also by V. KOŘÍNEK in the paper [Koř2a] published in 1949. He does not mention O. BORŮVKA's terminology, however, he explains the reasons why he decided to adopt this expression. He does not follow O. ORE's term "structure" because this word can be often used in other, more general, meanings. He expresses some reservations to a possible translation of "lattice" into Czech: "mříž". He sees its slight disadvantage in the standard meaning in geometry. Nevertheless, it is necessary and interesting to point out that the word "mříž" in the sense "lattice" has really appeared as well. In 1938 M. H. STONE published his paper [Sto2] in the Czech mathematical journal *Časopis pro pěstování matematiky a fysiky*. The work was written in English, however, there was a Czech summary in-

[76] F. ŠIK's works in JAKUBÍK, J., SEKANINA, M., *The 60th anniversary of Professor František Šik*, Czechoslovak Mathematical Journal 29 (1979), 494–450, JAKUBÍK, J., ŠMARDA, B., *Seventy years of Professor František Šik*, Czechoslovak Mathematical Journal 42 (1992), 181–185.

cluded in which the expression "distributive lattice" was translated as "distributivní mříž".

While the expression "svaz" became accepted and established easily, designating the lattice operations have a little longer development. The first use of "průsek" and "spojení", i. e. the standard terms for "meet" and "join" nowadays, is to be found again in the mentioned O. Borůvka's paper [Bor1], however, not for the lattice operations. "Průsek" is used in the sense of the "intersection of two decompositions in a set" (or as the "intersection of a non–empty subset of a set G and a decomposition in G"), and the term "spojení" appears in the collocation "spojení" of two groupoids, meaning the groupoid generated by these two groupoids. The lattice operations are not designated in that paper, yet, however, implicitly O. Borůvka's terms "průsek" and "spojení" are realizations of the lattice operations.

O. Borůvka devotes more time to lattice theoretic concepts in the following papers [Bor3a, Bor5], in which he states the aim to present his decomposition theory as a realization of a more general lattice theory. He pays attention to the properties of the greatest common refinement and the least common covering of a system of decompositions in relation to the lattice operation for which he chose the terms "průnik" ("intersection") and "spojení". "Průsek" keeps the same meaning as in [Bor1]. This terminology is preserved by O. Borůvka in his following works ([Bor6, Bor7b]) as well.

V. Kořínek opted for the terms "průsek" for "meet", and "spojení" for "join" in the paper [Koř2a]. He explains his choice by the fact that he wanted to distinguish the lattice operations "průsek" and "spojení" from the set operations "průnik" (= intersection) and "sjednocení" (= union).

The disunity which accompanied the early Czech terminology for the lattice operations is also reflected in L. Rieger's book [Rie9]. He chose the expressions "protínání" and "spojování" for the first encounter of a reader (who is a non–mathematician) with lattice theory since he presents the lattice operations as a generalization of intersecting and joining in geometry. When giving a precise definition of a lattice, L. Rieger applies O. Borůvka's terms "průnik" and "spojení", in the following text, however, he uses both expressions "průnik" and "průsek" interchangeably.

Chapter 4

Lattice theory in Slovak mathematics until 1963

4.1 The situation in Slovak mathematics before and after WWII

Due to specific historical circumstances Slovakia does not have a long tradition of mathematical research. The development of scientific studies in general started after WWI (during the Austro–Hungarian monarchy there had existed almost none) when the first Slovak university, Comenius University, was founded in Bratislava in 1919. The planned faculty of science (which was to provide mathematical education) was not, however, established until 1940. Slovak mathematicians had to study and work at other, Czech or foreign, institutions before WWII. The first Chair of Mathematics was founded in the newly established Technical University of M. R. Štefánik in Košice in 1938. This college was moved to Bratislava and was renamed the Slovak Technical University in 1939. Although the two Slovak universities did not interrupt their activities during the war, we can speak about the beginning of an intensive scientific research only after WWII.[1]

An important role in achieving a high standard of mathematical university education and raising a new generation of Slovak mathematicians was played by professors teaching at the universities: Professors J. Hronec, Š. Schwarz, J. Kaucký. They also invited mathematicians from Brno and Prague to help with educating mathematics students. O. Borůvka organized lectures and seminars in Bratislava for

[1] http://www.uniba.sk/webuk/uk, http://www.km.sjf.stuba.sk/english/origin.htm

many years, occasionally E. ČECH, F. VYČICHLO, M. KATĚTOV, and several others came to lecture. The Czech professors also participated in recommending suitable topics and literature for postgraduate studies. Thus in a relatively short period a group of young Slovak mathematicians started their successful research and one of the algebraic fields they became interested in was lattice theory.[2]

4.2 The road to lattices

The first Slovak mathematicians engaged in the problems of lattice theory were J. JAKUBÍK and M. KOLIBIAR to whom the majority of this chapter is devoted. We shall mention also the papers of B. RIEČAN, Z. RIEČANOVÁ and T. KATRIŇÁK which deal with lattice theoretical problems and were published at the end of the analyzed period. B. RIEČAN and Z. RIEČANOVÁ specialized on other mathematical branches later; T. KATRIŇÁK's field of research remained within lattice theory: he has achieved outstanding results in the area of pseudocomplemented lattices and semilattices.

4.2.1 Ján Jakubík and Milan Kolibiar

JÁN JAKUBÍK (born in 1923) finished his university studies at the Faculty of Sciences (studying mathematics and physics) of Comenius University in 1949, and then became an assistant to Professor Š. SCHWARZ in the Mathematical Institute of the Slovak Technical University where he worked until 1952. That year he moved to the newly established Technical University in Košice. He was appointed Associated Professor there in 1956 and Full Professor in 1963. At the university he began organizing a seminar on ordered algebraic structures which he was leading for many years and which is still active in research. Since 1952 J. JAKUBÍK has been working also in the Mathematical Institute of Slovak Academy of Sciences in Košice. The scientific work of J. JAKUBÍK is very extensive (the number of papers exceeds 200), and at the same time, rich in profound results. Apart from several papers, his work concerns mostly algebraic disciplines: partially ordered sets, lattices and mainly partially ordered groups and lattice ordered groups, in recent years also MV–algebras.[3]

[2] [Koř5] and GREGUŠ, M., "Vzťahy českej a slovenskej matematiky", in [CM], pp. 30–37.

[3] For more details about life and work of J. JAKUBÍK we refer the reader e. g. to the papers: KOLIBIAR, M., *Professor Ján Jakubík Sexagenarian*, Czechoslovak Mathe-

Milan Kolibiar (1922–1994), a fellow student and a good friend of J. Jakubík, graduated from the Faculty of Sciences (in mathematics and physics) of Comenius University in 1947. Then he became an assistant to Professor J. Kaucký in the Mathematical Institute of the Slovak Technical University and in 1951 he came to the Department of Mathematics of the Faculty of Sciences at Comenius University where he stayed until his retirement in 1987.[4] In 1956 he was appointed Associated Professor and in 1965 Full Professor. The research activities of M. Kolibiar lie mainly in the areas of partially ordered sets, lattices and universal algebra. He was particularly interested in the connections of algebra and topology. Apart from his own scientific work, M. Kolibiar is also highly praised for his educational achievements: encouraging and inspiring many mathematics students, organizing the Mathematical Olympiad, being engaged in activities and organizations both inside and outside the University.[5]

4.2.2 O. Borůvka's influence

The beginnings of J. Jakubík's and M. Kolibiar's mathematical research were greatly affected by their teacher O. Borůvka. He recommended them to study *Lattice Theory* by G. Birkhoff, supervised their RNDr. theses and advised them in the course of finding their first results.

O. Borůvka considered lattice theory to be a young, intensively developing field with a number of open topical problems, and G. Birkhoff's monograph to provide many areas of interest. Although this book might not be very suitable for beginners as it refers the reader to various papers and works of other scientists, it did not take a beginner too long to understand its methods, results, and problems as lattice the-

matical Journal 33 (1983), 657–664, Černák, Š., Kolibiar, M., *Životné jubileum akademika Jána Jakubíka*, Mathematica Slovaca 33 (1983), 321–326, or Kolibiar, M., *K životnému jubileu akademika Jakubíka*, Časopis pro pěstování matematiky 108 (1983), 425–429, and especially to the diploma work [Fab] dealing with Š.Schwarz, J. Jakubík, M. Kolibiar (however, probably due to the large extend of the work there appear some mistakes in accuracy when describing their results).

[4]Expanding the University resulted in establishing the Department of Algebra and Number Theory in 1965, and the Faculty of Mathematics and Physics in 1980, which became his new working places within the University.

[5]For those interested in more information about the life and work of M. Kolibiar we recommend e. g. Jakubík, J., Katriňák, T., *The sixtieth anniversary of Professor Milan Kolibiar*, Czechoslovak Mathematical Journal 32 (1982), 498–503, Katriňák, T., *Milan Kolibiar (1922–1994)*, Math. Slovaca 46 (1996), 297–304., the diploma work [Fab], or his own book of memories [Kol11].

ory was a relatively new, not extensively developed field. This choice of literature proved to be truly advantageous and fruitful. J. JAKUBÍK and M. KOLIBIAR succeeded not only in understanding the topic, but also very soon in solving some of its problems. J. JAKUBÍK, M. KOLIBIAR, L. MIŠÍK, M. ŠVEC and several others even started an informal seminar in which they were reading and discussing G. BIRKHOFF's monograph. The seminar, however, ended with J. JAKUBÍK's departure to Košice in 1952.

Another feature of J. JAKUBÍK's and M. KOLIBIAR's work which shows an influence of O. BORŮVKA can be found in their ways of approaching problems: they successfully applied O. BORŮVKA's algebraic concepts, mainly determining partition. In their early papers they often cite his paper on set partition [Bor3a] and his textbook on group theory [Bor6].

4.2.3 Contacts with other mathematicians and results

We have pointed out in the preceding chapter that the situation in Czech mathematical research after WWII was difficult for the lack of literature and contacts. The same was true even on a larger scale for Slovak research. The mathematicians in Bratislava did not have all necessary literature available and making trips to universities abroad was hardly possible, even journeys to Prague or Brno were very scarce at that time. During 1950's M. KOLIBIAR could still discuss the problems he was working on with O. BORŮVKA, who continued to lecture in Bratislava until 1958, J. JAKUBÍK in Košice, however, was in a more difficult situation. He is therefore highly praised by O. BORŮVKA for his abilities to present remarkable results in spite of his isolation from mathematical environment.[6]

Because of the lack of a well–equipped library it is not very surprising that J. JAKUBÍK and M. KOLIBIAR could not avoid repeating some already published results, however, without knowing about them. It is obvious that they were following the referative journals the *Mathematical Reviews* and *Referativnyj žurnal*, but, if they did not have access to the original paper itself, they could easily miss some important sources.

We also witness another typical feature of scientific research in several of their papers: the same problem is solved independently by two (or more) mathematicians. This fact is little surprising if we realize that the problems of G. BIRKHOFF's widely available monograph inspired a

[6] O. Borůvka's archive: O. BORŮVKA's reference to J. JAKUBÍK's appointment to the Slovak Technical University in 1959.

number of mathematicians all over the world. The cases of previous or parallel solutions of problems will be pointed out in the analysis.

The contact with Czech mathematicians engaged in lattice theory at that time (V. KOŘÍNEK, L. RIEGER, V. VILHELM, M. NOVOTNÝ) was kept especially in the situation when they were to asked to review J. JAKUBÍK's and M. KOLIBIAR's papers. Then the Czech mathematicians provided some suggestions or referred to related works.

An important step forward in communicating their results was the beginning of the Summer schools on partially ordered sets and universal algebras in 1962, first as an opportunity to discuss topics with Moravian and Czech mathematicians, later also with participants from abroad. Another significant event helping getting into contact with foreign mathematicians was the international Conference on Ordered Sets in 1963 where J. JAKUBÍK and M. KOLIBIAR had a possibility to learn more especially about the results of young Hungarian mathematicians G. GRÄTZER and E. T. SCHMIDT who were working on similar problems. The mutual reactions to each other's results in the 1950's also put J. JAKUBÍK into contact (corresponding) with M. BENADO and G. SZÁSZ.

4.3 The analysis of Ján Jakubík's works

4.3.1 An introduction to the analysis

This section is devoted to the analysis of J. JAKUBÍK's papers dealing with problems of lattice theory published by the year 1963. The papers are discussed one by one, in chronological order of their publishing, with references to the sources the author drew upon and to other information concerning the topic. The main inspiration for the papers was naturally *Lattice Theory* by G. BIRKHOFF ([LT–48] and its Russian translation 1952), and often also contemporary papers to which J. JAKUBÍK reacted.

A short description of the papers

As the subjects and methods of the papers are often closely related to one another, we can identify several areas in which J. JAKUBÍK worked in this period. He frequently treats the same, or a similar problem in more papers, usually generalizing his results for a larger class of lattices. A question he investigated in great detail in his first works was the relationship between graphical and lattice isomorphisms of a pair of finite lattices: distributive [J–K], modular [Jak2] and semimodular lattices

[Jak3], and also multilattices [Jak12], he even looked for an equivalent of graphical isomorphism in the case of infinite lattices [Jak9]. Several papers concern a decomposition into a direct product: the question of the uniqueness of a decomposition of lattices is treated in [Jak1], the existence of a decomposition of complete lattices is discussed in [Jak8], the investigation of [Jak14] produced results about a decomposition of infinitely distributive lattices and [Jak7] deals with a decomposition of 1 into a direct product. The properties of congruence relations present a strong tool for solving problems in many papers, however, they are primarily studied in [Jak4], which is a systematic investigation of the solvability of a system of two congruence relations (defined in terms of ideals or convex sublattices) in modular and distributive lattices, in [Jak6], in which J. JAKUBÍK as one of the first mathematicians used the notion of weak projectivity of intervals for studying congruences, and in [Jak16], where he investigated in which case any two congruences on a lattice are permutable. The Jordan–Dedekind chain condition in infinite lattices is the topic of [Jak10] for complete and completely distributive lattices, [Jak13] and [Jak17] deal with the Jordan–Dedekind condition in Boolean algebras; the results from these papers are used in further investigation of the Jordan–Dedekind chain condition in direct product of partially ordered sets [Jak19]. J. JAKUBÍK also reacted to the new concept of multilattices introduced by M. BENADO [Jak11, Jak12]. Many of the mentioned papers, [Jak1, J–K, Jak2, Jak3, Jak6, Jak15], also present a partial or a full answer to some problems stated in [LT–48].

Jakubík's other papers of this period

In the investigated period J. JAKUBÍK worked apart from the analyzed papers on other problems as well. In the area of abstract algebras, partially ordered groups and l–groups he published papers dealing with similar problems to those studying in lattices: ideals, chains, congruences, direct products.

Let us mention the paper *On congruence relations in abstract algebras* [Jak5] in which he constructs an example which gives the affirmative answer to G. BIRKHOFF's **Problem 33** ([LT–48], p. 90): "Let A be an algebra with a one element subalgebra and permutable congruence relations. Can A have distinct congruence relations $\theta \neq \theta'$ such that $S(\theta) = S(\theta')$?"[7] This problem was solved independently and using a

[7] $S(\theta)$ is defined as the subalgebra of $x \equiv 1(\theta)$ in A, where 1 denotes a selected one–element algebra.

different method by A. I. Mal'cev.[8]

An important question J. Jakubík solved in several papers dealing with partially ordered groups was which properties of a partially ordered group, or an l–group depend only on the partial order, i. e. which properties remain if we define the group operation in a different way. The paper *O glavnych idealach v strukturno usporjadočennych gruppach* [Jak18], apart from other results, solved another problem of G. Birkhoff (**Problem 99**, [LT–48], p. 224): "Is it true that any l–ideal of an l–ideal of a free l–group G with a finite number of generators is an l–ideal of G? Is it true that if the lattice of all l–ideals of a free l–group G with a finite number of generators has a finite length, then every l–ideal of G is principal?" J. Jakubík constructs such an l–group which provides negative answers to both questions.

A note on the language and notation of the analysis

J. Jakubík's papers are written in Slovak, Russian, English or German, the language of the analysis follows the standard English terminology (based mainly on G. Grätzer [Grä2]), sometimes preserving the original terms (or their translations). The notation is chosen to facilitate clear orientation in the text. Where it does not disturb, or where there does not exist any standard notation, the original notation is preserved. The symbol which is strictly copied from the original papers is "R" (and possible indices) for a congruence relation. This relation plays an important role in proving a number of properties, however, it is important to point out that the author works with the term of "determining partition" which presents an equal concept to a congruence.

The notion of determining partition was introduced and deeply investigated by O. Borůvka within his theory of partitions on sets and in sets. J. Jakubík uses this concept in lattices analogously: a determining partition is a partition imposed by a congruence relation on a lattice, i. e. if R is a congruence relation on a lattice L, then $a, b \in L$ belong to the same class of a determining partition iff $a \equiv b (\mathrm{mod}\ R)$. Since determining partition and congruence relation are "equivalent" concepts (each enabling us to approach a problem from a different side), we can use them interchangeably. It is true, however, that most mathematicians have preferred working with congruences, therefore J. Jakubík's presentation is quite unique. He himself, after the publication of the first papers, started using the language of congruences in the foreign

[8] A. I. Mal'cev, *K obščej teorii algebraičeskich sistem*, Matem. Sbornik 35 (1954), 3–20.

language summaries (although Slovak full papers are written in the language of determining partitions, e. g. [Jak6, Jak16]). In our analysis we shall preserve the speech of determining partition only in the first papers [J–K, Jak2], then we shall start using the language of congruences (from [Jak4]). However, we shall always use the symbol "R", no matter if we speak about a congruence, or a determining partition.

4.3.2 *Uniqueness of decomposition of a lattice into a direct product* [Jak1] (1951)

Direct product and factorization theorems have an important position in group theory and they can also be applied to partially ordered systems:

Theorem 4.1. *Let P be a partially ordered set with 0 and 1. If P can be decomposed into direct product of indecomposable factors, this decomposition is unique.*

The limitation concerning the validity of the theorem is justified: there exist partially ordered sets without greatest and least elements whose factorisation into indecomposable factors is not unique. Examples were given by T. NAKAYAMA,[9] or J. HASHIMOTO.[10] For lattices the first unique factorization theorem was proven for distributive lattices with 0 and 1 by G. BIRKHOFF in [Bir1], and later the same author [Bir2] showed its validity for general lattices with 0 and 1. In [LT–48], p. 27 G. BIRKHOFF states **Problem 11**:

> Is the unique factorisation theorem valid for general lattices (without the presumption of the existence of 0 and 1)?

J. JAKUBÍK was already interested in this question in his RNDr. Thesis (1951) in which he provided a positive answer to this problem. In this paper he even proves a more generalized case: the number of factors can be infinite. We shall outline the method of his proof.

J. JAKUBÍK uses for the isomorphism $L \cong \prod_{\iota} L_\iota, \iota \in \mathcal{M}$ the notation $(i) \quad x \longleftrightarrow \{x^\iota\}$ where $x \in L, \{x^\iota\} \in \prod_{\iota} L_\iota, x^\iota \in L_\iota$, and calls the element x^ι the projection of x into lattice L_ι and writes $x^\iota = [x]_{L_\iota}$. The symbol $[M]_{L_\iota} (M \subset L)$ means the set of all projections of the elements $x \in M$ into L_ι. For $u \in L, M_\alpha \subset L_\alpha$ the set $M_\alpha(u)$ is defined:

$$x \in M_\alpha(u) \quad \text{iff} \quad 1.\ [x]_{L_\alpha} \in M_\alpha, \quad 2.\ [x]_{L_\iota} = [u]_{L_\iota}, \text{ for } \quad \alpha \neq \iota.$$

[9] NAKAYAMA, T., *Algebraic theory of lattices*, Tokyo 1944.

[10] HASHIMOTO, J., *On the product decomposition of partially ordered sets*, Math. Japonicae 1 (1948), 120–123.

The definition implies that the partially ordered sets M_α and $M_\alpha(u)$ are isomorphic. It also holds that if X is a convex sublattice of $L = \prod_\iota L_\iota$, then $X \cong \prod_\iota [X]_{L_\iota}$. J. Jakubík then proves the following lemma:

Lemma 4.2. *Let $L \cong \prod_\iota A_\iota$ (isomorphism (i_1)), $\iota \in \mathcal{M}$ and at the same time $L \cong \prod_\nu B_\nu$ (isomorphism (i_2)), $\nu \in \mathcal{N}$, $u \in L$. Let us construct the set $A_\alpha(u)$ (with respect to the isomorphism (i_1)). We will denote the projection of $A_\alpha(u)$ into B_β (with respect to (i_2)) by A_α^β. If we construct the sets $A_\alpha^\beta(u), B_\beta(u)$ (with respect to (i_2)), then*

$$A_\alpha^\beta(u) = A_\alpha(u) \wedge B_\beta(u).$$

By this lemma and the reasoning before we get $A_\alpha \cong \prod_\nu A_\alpha^\nu(u)$. If we define $B_\beta^\alpha(u)$ analogously to A_α^β, we obtain $B_\beta \cong \prod_\iota B_\beta^\iota(u)$. Then $B_\beta^\iota(u) = A_\alpha^\beta(u)$. Thus we get the main theorem of the paper:

Theorem 4.3. *Let $L \cong \prod_\iota A_\iota, \iota \in \mathcal{M}, L \cong \prod_\nu B_\nu, \nu \in \mathcal{N}$. Let every factor A_ι, B_ν have more than one element and let them be indecomposable. Then there exists one-to-one mapping of $\mathcal{M}$ onto $\mathcal{N}$ which has the following property: if $\beta \in \mathcal{N}$ is the image of $\alpha \in \mathcal{M}$, then lattices A_α, B_β are isomorphic.*

The same result was independently reached by a Japanese mathematician F. Maeda.[11] J. Jakubík learnt about his paper only later from *Mathematical Reviews* 15 (1954).

In some of his later papers [Jak8, Jak14] J. Jakubík comes back to the problem of direct decomposition of lattices and investigates under which conditions a lattice can be decomposed into irreducible factors.

4.3.3 *On some properties of a pair of lattices* [J–K] (1954)

This paper was written by both J. Jakubík and M. Kolibiar and it analyzes various properties of a pair of lattices defined on the same set. The authors provide a number of results, mainly for distributive lattices, which lead to a partial solution of **Problem 8** stated by G. Birkhoff in [LT–48].

[11] Maeda, F., *Direct and subdirect factorizations of lattices*, Journ. Sci. Hiroshima Univ. Ser. A 15 (1951), 97–102.

The content of the paper is based on the results obtained in the authors' doctoral theses. The part written by J. JAKUBÍK concerns mainly the investigation of distributive lattices with the property **A** (see below) and the solution of **Problem 8**. M. KOLIBIAR's thesis investigated a ternary operation (a, b, c) in terms of which we can define another lattice from a given one and the author describes relations between such two pairs of lattices.

Let L be a distributive lattice. A congruence relation Θ on L may be defined by means of the ternary operation $(a, b, c) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ in the following way:

$$x \Theta y \Leftrightarrow (a, t, x) = (a, t, y);\ a, t \in L.$$

If L has 0 and 1, and the element t has a complement, the determining partition defined by the congruence relation Θ is called the *principal determining partition*. Let two lattices L_1, L_2 be defined on the same set M and let their lattice operations and ordering relations be denoted by $\cap, \cup, \subseteq$ in L_1 and $\wedge, \vee, \leq$ in L_2. The authors describe the following properties the two lattices may have and study the relations between them:

A. Every partition of the set M which is determining on L_1 is also determining on L_2 and vice versa.

A1. Every principal determining partition on L_1 is a principal determining partition on L_2 and vice versa.

B. If a set $X \subset M$ forms a convex sublattice in L_1, then X forms a convex sublattice in L_2 and vice versa.

C. Every lattice operation of L_1 is mutually distributive with every lattice operation of L_2.

D. There exist lattices A, B (defined on the sets M_1, M_2) and a mapping $\varphi : M \to M_1 \times M_2$ such that φ is an isomorphism of L_1 onto $A \times B$ and at the same time it is an isomorphism of L_2 onto $\tilde{A} \times B$ ($\tilde{A}$ means the dual of A).[12]

E. There exist two elements $t, t' \in L_1, t'$ being a complement of t in L_1 such that for any $x, y \in M$

$$x \cup y = (x, t, y), \quad x \cap y = (x, t', y).$$

F. The (unoriented) graphs of the lattices L_1, L_2 are isomorphic.

[12]The formulation of this property is not precisely correct in the original paper, which M. KOLIBIAR makes clear in his paper [Kol2].

After a detailed investigation of the behaviour of the described entities the authors prove the following relations between the properties **A** to **F**:

Theorem 4.4. **C** $\Rightarrow$ **B** *and* **D** $\Rightarrow$ **A** *are valid for any lattice.*

Theorem 4.5. **D** $\Rightarrow$ **F** *is valid for finite lattices.*

Theorem 4.6. *The properties* **A, B, C, D** *are equivalent for distributive lattices.*

Theorem 4.7. *The properties* **A, A1, B, C, D, E** *are equivalent for distributive lattices with 0 and 1.*

Theorem 4.8. *All the properties are equivalent for finite distributive lattices.*

Some of the stated results were proved in earlier papers of other mathematicians: the implication **D** $\Rightarrow$ **C** for distributive lattices and **D** $\Rightarrow$ **F** for the case that the factors of the direct product are self-dual in [Kis] and the implication **A** $\Rightarrow$ **D** for distributive lattices may be deduced from [Arn]. The ternary operation $(a, b, c) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ was studied by A. A. GRAU [Gra] in Boolean algebras, and by S. A. KISS [B-K, Kis] in distributive lattices (for more details concerning works dealing with this ternary operation see the analysis of [Kol4]). V. KOŘÍNEK and V. VILHELM draw J. JAKUBÍK's and M. KOLIBIAR's attention to the paper of B. H. ARNOLD [Arn] which described a third, binary, operation on distributive lattices. J. JAKUBÍK and M. KOLIBIAR thus also compare the properties of B. H. ARNOLD's operation with the ternary operation and show the connection of their results to B. H. ARNOLD's. M. KOLIBIAR returned to investigating this ternary operation in his next papers [Kol3, Kol4].

In the course of proving the equivalence of properties **A, D** J. JAKUBÍK and M. KOLIBIAR found lattices A, B from **D** by constructing them:

$A = \{x \in L_1 : x \cap c = x \vee c,\ x \cup c = x \wedge c\}$,
$B = \{x \in L_1 : x \cap c = x \wedge c,\ x \cup c = x \vee c\}$, where $c \in L_1$ is a fixed point.

The equivalence of properties **A** and **D** (for distributive lattices) plays an important role in the final part of the paper: solving G. BIRKHOFF's **Problem 8** [LT–48], p. 20:

> Find a necessary and sufficient condition on a lattice L, in order that every lattice M whose (unoriented) graph is isomorphic with the graph of L be lattice-isomorphic with L.

Graphs of finite lattices are isomorphic iff there exists an isomorphism preserving *neighbouring elements* (J. JAKUBÍK and M. KOLIBIAR call $x, y \in L$ neighbours, or neighbouring elements, and write "x s y" iff x covers y, or y covers x). J. JAKUBÍK and M. KOLIBIAR use the notation $L_1 \stackrel{g}{\sim} L_2$ for L_1, L_2 being graphically isomorphic. When investigating the notion of graphical isomorphism the authors use the concepts of prime quotient, perspective quotients, projective quotients. However, they call perspective quotients *transposes* in accordance with G. BIRKHOFF [LT–48]. They also introduce the term elementary pair: (x, y) is an *elementary pair* iff x, y are neighbouring elements. Transposes of elementary pairs are called *prime transposes.*

If two lattices are graphically isomorphic, then projective prime quotients are carried to projective prime quotients under this isomorphism. By combining this fact with Theorem 10 from [LT–48], p. 77 (the congruence relations on a modular lattice of finite length correspond one-to-one to the sets of classes of projective prime quotients, which they annul) the authors obtain the results that graphical isomorphism of two finite discrete lattices L_1, L_2 is equivalent to the property **A**, and thus **D**, which means that there exist lattices A, B such that $L_1 \cong A \times B$ and $L_2 \cong \tilde{A} \times B$. The answer to G. BIRKHOFF's **Problem 8** is then obvious:

Theorem 4.9. *Let L be a finite distributive lattice. Let M be a distributive lattice whose (unoriented) graph is isomorphic to the graph of L. L is also lattice-isomorphic to M iff every direct factor of L is self-dual.*

4.3.4 *On lattices whose graphs are isomorphic* [Jak2] (1954)

This paper solves the same problem as the previous one [J–K], i.e. **Problem 8** from [LT–48], the result is, however, more general: it provides the answer for the class of modular lattices. Although this answer is the same, the nature of work is different. While the first one [J–K] presents a range of relations between properties of a pair of lattices and the solution of **Problem 8** comes as a consequence, in this article J. JAKUBÍK clearly states the aim to deal with the following questions:

1. find lattices whose (unoriented) graphs are isomorphic with the graph of a given finite lattice,

2. find a necessary and sufficient condition for a finite lattice L such, that every lattice L' whose graph is isomorphic with

the graph of the lattice L, be isomorphic with the lattice L (= G. BIRKHOFF's **Problem 8**).

The author solves the questions for the case of modular lattices and also provides a generalization of a part of the results to discrete modular lattices.

As in [J–K] J. JAKUBÍK defines the notions of neighbouring elements (relation "s"), elementary pairs, prime transposes and graphical isomorphism of lattices. When introducing prime transposes he, untraditionally (instead of a classical version, going back to R. DEDEKIND, defining them as prime quotients in the form $[x \wedge y, x], [y, x \vee y]$), prefers a definition in terms of neighbouring elements:

Definition 4.1. Let L be a modular lattice. $a, b, c, d \in L$. If a s b s c s d s a, then elementary pairs $(a, b), (c, d)$ are called transposes.

The method of proof from [J–K] used properties of distributive lattices which cannot be extended to modular ones. The author therefore provides another proof which is not only more general, but also more straightforward.

Let L, L' be finite modular lattices such that $L \overset{g}{\sim} L'$. For $a, b, c, \dots \in L$ we denote their images (under the graphical isomorphism) in L' by $a', b', c', \dots$ It is obvious that if (a, b) and (c, d) are projective elementary pairs, then so are (a', b') and (c', d'). The set of all elementary pairs of a lattice can be divided into classes T_i where each class contains mutually projective pairs. We define the following congruence relation on L:

$a \equiv b$ iff 1. $a = b$, or 2. every elementary pair $(u, v) \in [a \wedge b, a \vee b]$ belongs to some class T_i.

If we define on L': $a' \equiv b'$ iff $a \equiv b$, we obtain a congruence relation on L' and imposed partitions P on L and P' on L'. It holds that every determining partition on L can be constructed in this way.

Now the author introduces the notion of elementary pairs preserving or reversing the order.

Definition 4.2. Let $L \overset{g}{\sim} L'$, $a, b \in L$. We say that an elementary pair (a, b), where $a < b$ *preserves (reverses) the order* of elements with respect to the graphical isomorphism iff $a' < b'$ ($a' > b'$).

It holds that mutually projective elementary pairs either all preserve, or all reverse the order. We denote by M_1 (M_2) the set of all classes of mutually projective elementary pairs in L which preserve (reverse) the order and by P_1 (P_2) a determining partition on L which is defined by $T_i \in M_1$ ($T_i \in M_2$). J. JAKUBÍK proves that P_1 and P_2 are permutable partitions. If the lattice of the classes of the partition P_1 (P_2) is denoted A_1 (A_2), we obtain $L \cong A_1 \times A_2$, and in a similar way $L' \cong \tilde{A}_1 \times A_2$, where $\tilde{A}_1$ is means the dual of A_1.

Thus we receive the answers for the stated questions:

Theorem 4.10. *Let L, L' be finite modular lattices, $L \overset{g}{\sim} L'$ iff there exist lattices A_1, A_2 such that $L \cong A_1 \times A_2$ and $L' \cong \tilde{A}_1 \times A_2$.*

Theorem 4.11. *Let L be a finite modular lattice. Let a lattice L' be graphically isomorphic to the graph of L. Then L' is also lattice isomorphic to L iff every direct factor of L is self-dual.*

J. JAKUBÍK also looks at the validity of the previous theorem in the case of an infinite lattice. For a discrete lattice (i. e. a lattice in which every chain with 0 and 1 is finite) L it remains true if we make a presumption that it is possible to decompose it into direct product of indecomposable factors. However, it is not generally valid that to every discrete finite lattice there exists its direct product of factors each of which is indecomposable. J. JAKUBÍK gives the following example of such a modular discrete lattice: L is a set of all functions defined on the interval $[0, 1]$ which satisfy 1. the image of all elements is 0 except a finite number of elements, 2. the images are only integers (L is partially ordered in the usual way: $f \leq g$, for $f, g \in L$ iff for each $x \in [0, 1] : f(x) \leq g(x)$).

At the end of the paper J. JAKUBÍK indicates three unsolved problems he is going to deal with in next papers ([Jak3], [Jak9]).

4.3.5 *On the graphical isomorphism of semimodular lattices* [Jak3] (1954)

This paper continues to investigate the problem of graphical isomorphism for even a more general group of lattices than the previous two. J. JAKUBÍK solves the following two questions which he posed at the end of [Jak2]:

> **Q1**. Let L, L' be finite semimodular lattices which are graphically isomorphic. Does it imply (as in the case of modular lattices, see [Jak2]) the existence of such lattices A, B that $L \cong A \times B$ and $L' \cong \tilde{A} \times B$?
>
> **Q2**. Let L be a finite modular lattice, let L' be a lattice graphically isomorphic to L. Is L' also modular?

As in his previous paper [Jak2] J. JAKUBÍK uses the notions of neighbouring elements, elementary pairs, prime transposes and elementary pairs preserving/reversing the order with respect to the given graphical isomorphism (for details see the analysis of the papers [J–K] and [Jak2].) We say that a chain (an interval) preserves/reverses the order with respect to the graphical isomorphism iff every elementary pair of the chain (the interval) preserves/reverses the order. We say that a sublattice L_0 of L preserves/reverses the order iff for any $a, b \in L, a < b$ the interval $[a, b]$ preserves/reverses the order.

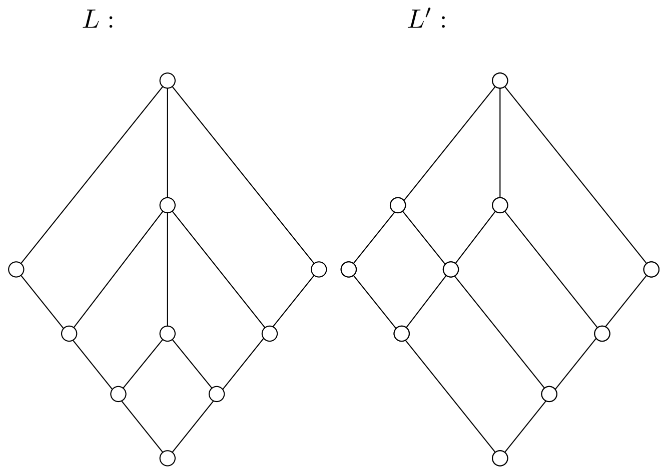

Figure 4.1: Graphically isomorphic semimodular lattices demonstrating the negative answer to **Q1**

J. JAKUBÍK proves that the answer to Question **Q1** is negative. He presents an example of two lattices L, L' from Figure 4.1 which are graphically isomorphic, but L is indecomposable. This negative answer arises from the fact that every semimodular lattice which is not modular contains a sublattice isomorphic with the lattice in Figure 4.2. J. JAKUBÍK calls any lattice isomorphic to this lattice a *lattice of type C*. We can therefore ask whether we may get a positive answer to **Q1** if we place some more assumptions on the behaviour of the sublattices of type C. By studying properties of chains and intervals preserving and reversing the order, and by applying the same method of imposing determining partitions on L as in [Jak2] J. JAKUBÍK proves the following theorem:

Theorem 4.12. *Let L, L' be finite semimodular lattices which are graphically isomorphic. Let all sublattices of type C in L and L' preserve the* *order with respect to the given graphical isomorphism. Then there exist lattices A, B such that*

$$L \cong A \times B,\ L' \cong \tilde{A} \times B$$

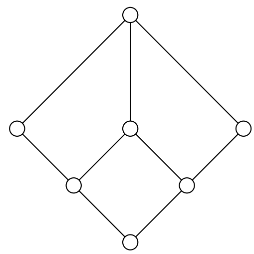

Figure 4.2: A lattice of type C

By investigating the covering conditions in L' J. Jakubík arrives at the following answer to **Q2**:

Theorem 4.13. *Let L be a modular lattice. Let L' be a lattice graphically isomorphic to L. Then L' is also modular.*

J. Jakubík also remarks that the analogous theorem for semimodular lattices is not valid since a lattice of type C is graphically isomorphic to its dual, however, the dual is not semimodular.

4.3.6 *A system of congruence relations on lattices* [Jak4] (1954)

In number theory we know the so called Chinese remainder theorem which gives necessary and sufficient conditions for the system of congruence relations

$$x \equiv a_i (\mathrm{mod}\ m_i), i = 1, \dots, n$$

to be solvable. An analogous problem for the congruence relations on distributive lattices was put forward by V. K. Balachandran:[13]

> **B1** Let L be a distributive lattice with a minimal element, let A, B be ideals in L, and $u, v \in L$. Find a necessary and sufficient condition for elements u, v so that the system of congruence relations
>
> $$x \equiv u (\mathrm{mod}\ A), \quad x \equiv v (\mathrm{mod}\ B) \qquad (4.1)$$

[13] Balachandran, V. K., *The Chinese remainder theorem for distributive lattices*, J. Indian Math. Soc. (N. S.) 13 (1949), 76–80.

is solvable.

The most important results of V. K. BALACHANDRAN are:

Theorem 4.14. *The system* (4.1) *is solvable iff* $u \equiv v(mod\ A \vee B)$.

Theorem 4.15. *If x is a solution of (4.1) and $x \equiv y(mod\ A \wedge B)$, then y is also its solution; for every two elements x, y which are solutions holds $x \equiv y(mod\ A \wedge B)$.*

J. JAKUBÍK proves that V. K. BALACHANDRAN's results are still valid even if we leave out the presumption of **B1** that L has a minimal element. He further generalizes the problem by replacing a congruence relation belonging to an ideal A by a *minimal congruence* $R(A)$, which means the minimal of all congruences which annul all elements of $A \subset L$. (As it was pointed out in the introduction to the analysis J. JAKUBÍK works with the notion of determining partitions, not congruences in the original paper.)

The first section of the paper deals with some important properties of the defined minimal congruence and in the second section the system of congruence relations is investigated for the case of distributive lattices. Let A, B be convex sublattices of L and let us have the following system of two congruence relations:

$$x \equiv u(\text{mod } R(A)), \quad x \equiv v(\text{mod } R(B)). \tag{4.2}$$

It is obvious that a necessary condition for the existence of the solution of (4.2) is the following:

$$u \equiv v(\text{mod } R(A) \vee R(B)), \tag{4.3}$$

however this condition is not a sufficient one. The question to investigate is therefore: *which other condition, apart from* (4.3) *needs to be satisfied so that* (4.3) *implies the existence of a solution of the system* (4.2)*?* In distributive lattices J. JAKUBÍK studies this question in relation to the following statements (the original notations is used for the statements):

Ba. For any $a, b \in L$ and any convex sublattices $A, B \subset L$ the relation (4.3) implies the existence of a solution of (4.2).

Ba(u, v). For given elements $u, v \in L$ and arbitrary convex sublattices $A, B \subset L$ the relation (4.3) implies the existence of a solution of (4.2).

Ba(A, B). For given elements $u, v \in L$ and given convex sublattices $A, B \subset L$ the relation (4.3) implies the existence of a solution of (4.2).

First, the author uses properties of permutable congruences to restate previous statements in equivalent forms:

> **Ba.** Let A, B be any convex sublattices of L. Then the congruences $R(A), R(B)$ are permutable.
>
> **Ba(A, B)**. Let A, B be given convex sublattices of L. Then the congruences $R(A), R(B)$ are permutable.

Then J. JAKUBÍK proves the following theorems:

Theorem 4.16. *A lattice L satisfies the condition* **Ba** *iff L is relatively complemented.*

Theorem 4.17. *A lattice L satisfies the condition* **Ba(u, v)** *iff the lattice*
$[u \wedge v, u \vee v]$ is complemented.

A necessary and sufficient condition for a lattice to satisfy condition **Ba(A, B)** is more complicated, and the author distinguishes two cases - if A and B are disjoint, or not:

Theorem 4.18. *Let A, B be two nondisjoint convex sublattices of L. Then L satisfies the condition* **Ba(A, B)** *iff the sublattice (A, B) generated by A and B is the direct product of A and B.*

Theorem 4.19. *If A, B are disjoint convex sublattices, then a necessary and sufficient condition for L to satisfy* **Ba(A, B)** *is the following: if $A_1'(B_1')$ is an interval projective to an interval $A_1 \subset A(B_1 \subset B)$ and $A_1' \cap B_1' \neq 0$, then the sublattice (A_1', B_1') generated by A_1' and B_1' is the direct product of A_1' and B_1'.*

The third section of the paper deals with modular lattices. The author finds sufficient conditions for the existence of a solution of the system (4.2). First he constructs the minimal congruence $R(C)$ on a modular lattice L whose lattice of classes, denoted by L', is distributive. Some ideals $A', B' \subset L'$ belong to ideals $A, B \subset L$. Thus, we can obtain some properties of elements L from properties of L':

Theorem 4.20. *Let A, B be ideals of a modular lattice. Let*

$$u \equiv v(mod\, R(A \vee B)).$$

Then there exists an element x satisfying

$$x \equiv u(mod\, R(A) \vee R(C)), \quad x \equiv v(mod\, R(B) \vee R(C)).$$

Theorem 4.21. *Let A, B be ideals of a modular lattice. Let*

$$u \equiv v(mod\, R(A \vee B)).$$

Then there exists an element z satisfying

$$z \equiv u(mod\, R(A) \vee (R(B) \vee R(C))), \quad z \equiv v(mod\, R(B) \vee (R(A) \vee R(C))).$$

In the last theorem of this paper J. JAKUBÍK generalizes a theorem which G. BIRKHOFF proved for modular lattices of a finite length in [LT–48], p. 77 (Theorem 10). The generalized theorem is not directly related to the study of the given system of congruences, however, it results from the investigation of congruence relations on lattices presented in the paper. His theorem is the following:

Theorem 4.22. *Let L be a discrete modular lattice. The congruences on L correspond one-to-one to the sets of classes of projective prime quotients which they annul. Hence they form a Boolean algebra.*

J. JAKUBÍK continues in developing his results about permutable congruences and the solvability of the system from this paper later in [Jak16].

4.3.7 *Congruence relations and weak projectivity in lattices* [Jak6] (1955)

The notion of weak projectivity was first introduced by R. D. DILWORTH[14] although a similar idea was used by M. FUNAYAMA [Fun].

Let $i_0 = i, i_1, \ldots, i_{n-1}, i_n = i'$ be intervals in a lattice L. In GRÄTZER's terminology [Grä2] we say that i' is weakly projective into i iff for each $i_{k-1}, (k = 1, \ldots, n)$ the interval i_k is contained in an interval i'_k which is transposed to the interval i_{k-1}. J. JAKUBÍK says in this case that i is weakly projective with i' and we shall write $i \succeq_w i'$. The relation $\succeq_w$ is a quasi–ordering on the set of all intervals of L.

Let L be a discrete lattice and let L be the congruence lattice of L. Let $\mathcal{P}$ be the set of all prime intervals of L quasi-ordered by the relation $\succeq_w$. For $p \in \mathcal{P}$ we will denote by $\bar{p}$ the set of all $p' \in \mathcal{P}$ satisfying both $p \succeq_w p'$ and $p' \succeq_w p$ (we shall call such prime intervals equivalent). Thus we obtain a partition of $\mathcal{P}$ into disjoint classes of mutually equivalent prime quotients. We will denote the set of these classes by X. Let $\bar{p} \geq \bar{q}$

[14]DILWORTH, R. P., *The structure of relatively complemented lattices*, Annals Math. 51 (1950), 348–359.

iff $p \succeq_w q$. Let Y be the set of all functions defined on X having the values 1 or 2 satisfying

$$\bar{p} \leq \bar{q} \Rightarrow f(\bar{p}) \leq f(\bar{q}).$$

For $f_1, f_2 \in Y$ we put $f_1 \leq f_2$ iff $f(\bar{p}) \leq f_2(\bar{q})$ for all $\bar{p} \in X$. J. JAKUBÍK proves the following theorem:

Theorem 4.23. *Let L be a discrete lattice and let* Con$L$ *be its congruence lattice. The partially ordered sets $ConL$ and Y are dually isomorphic.*

This theorem was proved by M. FUNAYAMA [Fun] for L being a finite lattice and its generalization for a lattice satisfying one chain condition was put forward as **Problem 67** in [LT–48], p. 144. J. JAKUBÍK shows on an example that his theorem is not generally valid for lattices satisfying the descending chain condition. Further generalization of Theorem 4.23[15] was provided in [G–S2] in which the authors G. GRÄTZER and E. T. SCHMIDT also continue in the development of the following J. JAKUBÍK's results from this paper and provide full answers to the problems.

J. JAKUBÍK used the notion of weak projectivity while investigating other two problems of [LT–48]:

> **Problem 72** (p. 153): Find necessary and sufficient conditions on a lattice so that its congruence lattice forms a Boolean algebra.
>
> **Problem 73** (p. 161) Find necessary and sufficient conditions in order that the correspondence between the congruence relations and neutral ideals of a lattice be one–to–one.

J. JAKUBÍK solved **Problem 72** for discrete and distributive lattices:

Theorem 4.24. *Let L be a discrete lattice, let* Con$L$ *be its congruence lattice. Then* Con$L$ *is a Boolean algebra iff the relation of weak projectivity of prime intervals in L is symmetric.*

Theorem 4.25. *Let L be a distributive lattice, let $ConL$ be its congruence lattice.* Con$L$ *is a Boolean algebra iff L is a discrete lattice.*

J. JAKUBÍK replaces the condition of **Problem 73** by the following one:

[15]The theorem is valid for semi–discrete lattices, i. e. lattices in which there exists a finite maximal chain between all comparable pairs of elements.

> **(A)** Every non–trivial congruence R on L includes some ideal $I(R)$ as one of its classes and $I(R)$ determines the congruence R uniquely.

The solution of **Problem 73** for distributive lattices is given as follows:

Theorem 4.26. *Let L is a distributive lattice. The condition* **(A)** *is satisfied in L iff to each prime interval p of L there exists an ideal $I(p)$ such that 1. the prime interval p is weakly projective with every prime interval of $I(p)$, 2. in $I(p)$ there exists a prime interval p_1 weakly projective to p.*

J. JAKUBÍK also combines the two problems and finds a lattice satisfying both conditions:

Theorem 4.27. *Let L be a discrete lattice with* 0*. L satisfies* **(A)** *and at the same time* Con$L$ *is a Boolean algebra iff any prime interval p of L 1. is weakly projective with at least one prime interval $[0, x_0]$, 2. if p is weakly projective with some interval $p_0 = [0, y_0]$ then p and p_0 are equivalent.*

Theorem 4.28. *Let L be a discrete lattice without* 0*. L satisfies* **(A)** *and at the same time* Con$L$ *is a Boolean algebra iff L is simple.*

Problem 73 attracted also M. KOLIBIAR's attention (see the analysis of [Kol6]).

4.3.8 *Direct decomposition of the unity in modular lattices* [Jak7] (1955)

Let L be a lattice with 0 and 1. The notion of direct decomposition of 1 was introduced by A. G. KUROŠ in [Kur1, Kur2] for complete lattices, however, it is possible to extend it to lattices of general type. Let

$$\text{(I) } 1 = \bigvee a_\alpha \ (\alpha \in M), \quad \text{(II) } 1 = \bigvee b_\beta \ (\beta \in N)$$

be direct decompositions of the element 1. We can consider the following problem **(P)**:

> **(P)** What is a necessary and sufficient condition for the existence of a common refinement of the decompositions (I) and (II)?

A. G. KUROŠ proved the following theorem **(K)** [Kur1, Kur2]:

Theorem 4.29. **(K)** *Let L be a completely modular lattice. The decompositions* (I) *and* (II) *have a common refinement iff for all $\alpha \in M, \beta \in N$ holds $1\bar{\varphi}_\alpha \Theta_\beta \varphi_\alpha = 0$.*[16]

The method of A. G. KUROŠ's proof lies in exploiting properties of $\varphi_\alpha, \bar{\varphi}_\alpha$, $\Theta_\beta, \bar{\Theta}_\beta$. J. JAKUBÍK generalizes the mentioned theorem, however, he uses a different approach to obtain his results. He reduces the problem to the existence of a common refinement of two direct decompositions of a certain sublattice of L. The introductory part of the paper is, therefore, devoted to an exposition of direct and subdirect decompositions of algebras.

J. JAKUBÍK makes use of the following condition **(B)** which can replace the complete modularity of the lattice L in the theorem **(K)**:

> **(B)** There exists a sublattice $L_1 \subset L$ such that 1. L_1 is a complete sublattice of L, 2. L_1 is completely modular, 3. L_1 contains all elements a_α, b_β.

The summary of J. JAKUBÍK's results is as follows:

Theorem 4.30. *Let a lattice L be complete and modular. Then*

1. A common refinement of the decompositions (I) *and* (II) *exists iff the complete sublattice of L generated by the set of all elements a_α, b_β is distributive.*

2. The theorem **(K)** *holds without supposing the condition* **(B)**.

3. The condition **(B)** *follows from the existence of a common refinement of decompositions* (I) *and* (II).

4. Let A be the set of all elements of L which have a complement. There exists a direct decomposition $1 = \bigvee C_i$ which is a refinement of all direct decompositions of 1 iff the complete sublattice $L_1 \subset L$ generated by the set A is completely distributive.

The work of KUROŠ on direct decomposition influenced also other mathematicians' papers[17] about which J. JAKUBÍK rises a question whether some of their results can be generalized in a way similar to his paper.

The basic result of this paper was also published (without a proof) by M. BENADO.[18]

[16] φ_α, and Θ_β denote projections associated with the decompositions (I) and (II), $\bar{\varphi}_\alpha$ denotes the projection complementary to φ_α.

[17] GRAEV, M. I., *Izomorfizmy prjamych razloženij v dedekindovych strukturach*, Izv. Ak. Nauk SSSR 11 (1947), 33–47 and LIVŠIC, A. CH., *Prjamyje razloženija vpolne dedekindovych struktur*, Mat. Sb. 28 (70) (1951), 481–503.

[18] BENADO, M., *Über eine Frage aus der Theorie der Oreschen Normalitätsbeziehungen*, Comun. Acad. R. P. Romine 5 (1955), 1241–1243.

4.3.9 *Direct decomposition of completely distributive complete lattices* [Jak8] (1955)

Although a direct decomposition into irreducible factors does not generally exist for every lattice, in this note J. JAKUBÍK shows that it does for completely distributive complete lattices:

Theorem 4.31. *Every completely distributive complete lattice is a direct product of directly indecomposable factors.*

In the proof J. JAKUBÍK uses his results of [Jak7]. Further the author presents a weak form of the previous theorem for complete lattices:

Theorem 4.32. *Let L be a complete lattice, $\{L_i\}, i \in M$ the set of all indecomposable factors of L. Then $L \cong \prod L_i$ iff the centre of L is completely distributive complete lattice.*

J. JAKUBÍK also points out it may happen that a lattice which does not have a greatest and least element is not possible to decompose into indecomposable factors although every interval of this lattice has a direct decomposition of indecomposable factors.

4.3.10 *On metric lattices* [Jak9] (1955)

At the end of [Jak2] J. JAKUBÍK stated an unsolved problem concerning an extension of the notion of graphical isomorphism to infinite modular lattices. He suggested to use a topological equivalence instead of graphical isomorphism and posed questions analogous to the problems in [Jak2].

Definition 4.3. A *metric lattice* is defined as a lattice L with a norm. A *norm* on a lattice is a real function $v(x)$ on L satisfying:

$$\begin{aligned} v(x) + v(y) &= v(x \wedge y) + v(x \vee y) \\ x > y &\Rightarrow v(x) > v(y). \end{aligned}$$

If we introduce a distance in L by assigning to each pair $x, y \in L$ the real non-negative number $d(x, y)$:

$$d(x, y) = v(x \vee y) - v(x \wedge y), \tag{4.4}$$

we obtain a metric space in its usual sense. Therefore we use the term metric, or normed, lattices. We will denote the set of all elements of a lattice L (a metric space M) by $|L|$ ($|M|$) and the metric space on L defined by (4.4) will be denoted by $M(L(v))$.

Associating metric spaces with lattices was commenced by V. GLIVENKO [Gli1, Gli2] and a number of mathematicians continued. V. GLIVENKO [Gli1] and M. F. SMILEY and W. R. TRANSUE [S–T] formulated conditions which a given metric space M_1 must satisfy so that there exists a metric lattice $L(v)$ such that:

$$|L| = |M_1|, \; M(L(v)) = M_1. \tag{4.5}$$

They considered lattices with the least element and a non-negative norm. L. M. KELLY [Kel] extended their results to any normed lattices (thus solving **Problem 66** in G. BIRKHOFF's [LT–48]). He also attempted to reconstruct a lattice $L(v)$ satisfying the equations (4.5), however, his construction did not determine all such lattices. L. M. KELLY therefore posed the following question [Kel]:

> Let M_1 be a metric space and $L(v)$ a metric lattice satisfying (4.5). Find all other lattices satisfying (4.5).

For investigating this question J. JAKUBÍK used the results of M. KOLIBIAR from [Kol2] and proved the following two theorems:

Theorem 4.33. *Let M_1 be a metric space and let $L(v)$ be a metric lattice satisfying (4.5). A lattice L' also satisfies (4.5) iff $|L'| = |L|$ and there exist lattices A, B such that*

$$L \cong A \times B, \; L' \cong \tilde{A} \times B,$$

where $\tilde{A}$ is the dual of A and the mapping of $|L|$ on the set $|A \times B| = |\tilde{A} \times B|$ is in both isomorphisms the same.

Theorem 4.34. *Let L be able to decompose into a direct product of indecomposable factors $L \cong \prod L_i, (i \in N)$, let $L(v)$ be a metric lattice. Let $L'(v')$ be a lattice satisfying:*

$$|L'| = |L|, \; M(L'(v')) = M(L(v)).$$

Then L' is isomorphic to L iff every indecomposable direct factor $L_i (i \in N)$ is selfdual.

4.3.11 *On the Jordan–Dedekind chain condition* [Jak10] (1955)

The Jordan–Dedekind Chain Condition (which will be denoted by **(JD)** evolved from ideas originally used to prove the Jordan–Hölder Theorem for groups ([LT–67], p. 164). In connection with lattices it was first studied by R. DEDEKIND in [Ded]. Basing his investigation on R. DEDEKIND, G. BIRKHOFF showed in [LT–40], p. 40 that in a lattice in which all bounded chains are finite each of the covering conditions implies **(JD)**, which means that **(JD)** holds in any finite semimodular lattice. G. SZÁSZ[19] (following R. CROISOT's paper[20] on semimodular lattices of infinite length) showed that the condition of all bounded chains of the lattice being finite can be replaced by a weaker one: there exists at least one finite maximal chain between a and b. G. SZÁSZ[21] introduced a more general version of **(JD)** by defining the length of an infinite chain:

Definition 4.4. If a chain $C(a, b)$ is infinite, its length is the cardinal number of the set $C(a, b)$.

Compared to the original condition, the group of lattices satisfying the general Jordan–Dedekind condition in the sense of G. SZÁSZ (we shall denote it by **(JD 2)**), i. e. if chains $C_1 and C_2$ are maximal chains with the same endpoints, then their cardinal numbers are the same, is surprisingly more limited. G. SZÁSZ stated (in the mentioned paper) the following theorem:[22]

Theorem 4.35. *There exists a distributive lattice which does not satisfy the condition* **(JD 2)**.

In this paper J. JAKUBÍK provides further generalization of this theorem:

Theorem 4.36. *Let α be a cardinal number, $\alpha \geq c$.*[23] *There exists a complete and completely distributive lattice L_α with the least element f_0 and the greatest element f_1 which has the following property: For any cardinal number β which $c \leq \beta \leq \alpha$, there exists, in L_α, a maximal chain $C_\beta(f_0, f_1)$ of the length β.*

[19] SZÁSZ, G., *On the structure of semi–modular lattices of infinite length*, Acta Sci. Math. 14 (1951/1952), 239–245.

[20] CROISOT, R., *Contribution à l'étude des treillis semi–modulaires de longueur infinie*, Annales Sci. Ecole Normale Sup. 68 (1951), 203–265.

[21] SZÁSZ, G., *Generalization of a theorem of Birkhoff concerning maximal chains of a certain type of lattices*, Acta Mathematica Academiae Scientarum Hungaricae 16 (1955), 89–91.

[22] J. JAKUBÍK drew the author's attention to the fact that the original formulation of the example in the proof of this theorem was not correct, G. SZÁSZ therefore provided a correction to the proof: *Correction to my paper "Generalization of a theorem of Birkhoff ... "*, Acta Mathematica Academiae Scientarum Hungaticae 16 (1955), 270.

[23] c denotes the power of the continuum.

J. JAKUBÍK proves the theorem by constructing such a lattice (using the axiom of choice): Let M be a well–ordered set the cardinal number of which is α. We denote by $L^0(M)$ the lattice of all real functions f defined on M such that for every $i \in M, f(i) \in [0,1]$ partially ordered by the usual ordering. J. JAKUBÍK shows that $L^0(M)$ is a complete and completely distributive lattice satisfying the required property.

4.3.12 *On axioms of multilattice theory* [Jak11] (1956)

M. BENADO introduced an interesting generalization of the notion lattice in [Ben1] which he called a multilattice. His definition of a multilattice in terms of two binary operations is the following:

Definition 4.5. A non–void set $\mathcal{M}$ is a multilattice iff to any ordered pair $a, b \in \mathcal{M}$ (any $A, B \subset \mathcal{M}$) exist two subsets of $\mathcal{M}$, which can be void, denoted by $a \vee b,\ a \wedge b$ ($A \vee B,\ A \wedge B$), and the operations $\vee, \wedge$ satisfy axioms (M1)–(M6):

(M1) $a \vee b = b \vee a$, and dually.

(M2) If $M \in (a \vee b) \vee c$, then there exists $M' \in a \vee (b \vee c)$ such that $M \vee M' = M$, and dually.

(M3) If $a \vee b \neq \emptyset$, then $a \wedge (a \vee b) = \{a\}$, and dually.

(M4) $a \vee a \neq \emptyset$, and dually.

(M5) If $a \vee c = b \vee c$, then $a = b$, and dually.

(M6) If $M, M' \in a \vee b, M^* \in M \vee M', M \neq M'$, then $M \neq M^* \neq M'$, and dually.

In this paper J. JAKUBÍK solves two problems which M. BENADO stated in [Ben2], p. 324:

B1. Is the axiom (M6) independent of the axioms (M1) to (M5)?

B2. Is there an associative multilattice which is not a lattice?

J. JAKUBÍK shows that the answer to **B1** is positive by constructing an example of a partially ordered set $\mathcal{P}$ satisfying (M1)–(M5), but not (M6). The set $\mathcal{P}$ is in Figure 4.3 ($\mathcal{P} = \{p, q, u, v\}$, for each $x, y \in \mathcal{P}$, for which $x \leq y$ we set $x \vee y = y \vee x = y, x \wedge y = y \wedge x = x$, and $p \vee q = q \vee p = \{u, v\}, p \wedge q = q \wedge p = \emptyset$).

The answer to **B2** is also positive and can be demonstrated on a simple example of the multilattice $\mathcal{M}_\infty = \{a, b\}$ in which $a \vee a = a \wedge a =$

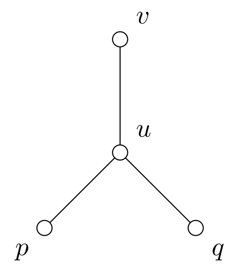

Figure 4.3: A partially ordered set $\mathcal{P}$

a, $b \vee b = b \wedge b = b$, $a \vee b = b \vee a = a \wedge b = b \wedge a = \emptyset$. J. JAKUBÍK shows even more than the answer to **B2**:

Theorem 4.37. *There exists an associative multilattice which is not a lattice. If a multilattice $\mathcal{M}$ is associative and if for any $a, b \in \mathcal{M}$ the sets $a \vee b, a \wedge b$ are non–void, then $\mathcal{M}$ is a lattice.*

At the end of the paper J. JAKUBÍK points out that all his results remain valid even if we replace (M3) by a stronger axiom:

(M3') If $\mathcal{M} \in a \vee b$, then $\mathcal{M} \wedge a = a$, and dually.

4.3.13 *Graphical isomorphism of multilattices* [Jak12] (1956)

In this paper J. JAKUBÍK comes back to the problem of graphical isomorphism (**Problem 8** in [LT–48], see [J–K], [Jak2] and [Jak3]) this time for the case of multilattices (for a definition of a multilattice see Definition 4.5, or Definition 4.18). He investigates whether the following result of his:

Theorem 4.38. *A necessary and sufficient condition for two discrete modular lattices L, L' to be graphically isomorphic is: there exist lattices A, B such that $L \cong A \times B$ (i_1) and $L' \cong A \times \tilde{B}$ (i_2), and the elements $x \in L, x' \in L'$ corresponding to each other under the graphical isomorphism are mapped on the same pair of elements $(a, b), a \in A, b \in B$ in the isomorphisms (i_1) and (i_2).*

remains valid if we replace "lattices" by "multilattices". It will be important if multilattices are directed sets:

Definition 4.6. A multilattice M is called *directed* iff

$$x, y \in M \Rightarrow x \wedge y \neq \emptyset \neq x \vee y.$$

J. JAKUBÍK investigates the cases of distributive and modular multilattices:

Definition 4.7. A multilattice M is *distributive* iff for any $a, b, x \in M$, $a \leq x \leq b$ implies that x has at most one relative complement in $[a, b]$.

Definition 4.8. A multilattice M is *modular* iff the condition (δ) and its dual are valid in M:

(δ) if $[d, a], [d, b]$ are prime intervals and $m \in a \vee b$, then $[a, m], [b, m]$ are prime intervals.

Theorem 4.39. *If a distributive multilattice is not directed, Theorem 4.38 (i. e. its modification created by replacing all words "lattice" with "multilattice" and "modular" with "discrete") is not valid in general.*

Theorem 4.40. *Theorem 4.38 (i. e. its modification obtained by replacing all words "lattice" with "multilattice") is not valid for modular multilattices (even if they are directed).*

The proofs are demonstrated by examples, see Figure 4.4 for distributive multilattices and Figure 4.5 for modular multilattices.

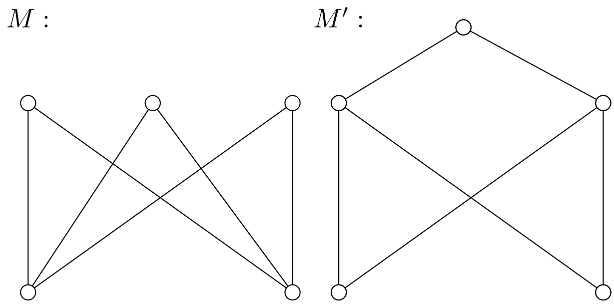

Figure 4.4: Graphically isomorphic distributive multilattices, M is directly irreducible, M, M' are not isomorphic or dually isomorphic.

From an investigation of directed distributive multilattices J. JAKUBÍK derives the following theorems:

Theorem 4.41. *Let M, M' be directed discrete distributive multilattices. A necessary and sufficient condition for the existence of a graphical isomorphism $M \overset{g}{\sim} M'$ is the existence of such multilattices A, B that $M \cong A \times B, M' \cong A \times \tilde{B}$, and elements $x \in M, x' \in M'$ corresponding to each other under the graphical isomorphism are mapped on the same pair $(a, b), a \in A, b \in B$ in both isomorphisms.*

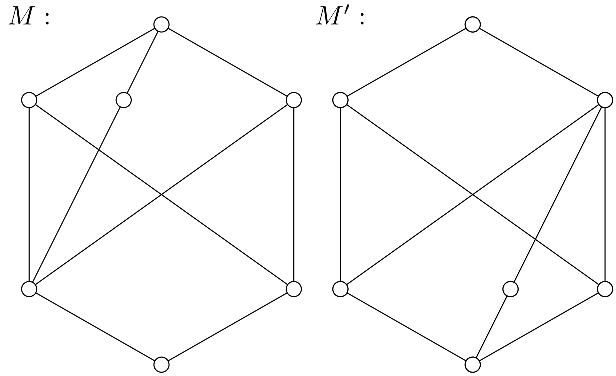

Figure 4.5: Graphically isomorphic modular multilattices, M is directly irreducible, M, M' are not isomorphic or dually isomorphic.

Theorem 4.42. *Let M be a finite directed distributive multilattice. Let M' be a finite distributive multilattice which is graphically isomorphic to M. Then the multilattices M and M' are isomorphic iff every indecomposable direct factor of M is self–dual.*

The method of the proof is similar to the one in [Jak2]. J. Jakubík studies the behaviour of elementary pairs preserving/reversing the order, defines congruences on M and finds multilattices A, B. We shall outline the method: let $x \in M$, we will denote by $\bar{x}^1(\bar{x}^2)$ the set of all elements $y \in M$ for which there exist $z \in x \vee y$ and maximal chains C_1 and C_2 (their least element is x and y respectively, the greatest is z) such that every prime interval contained in them preserves (reverses) the order; J. Jakubík shows that if this condition holds for an element $z_0 \in x \vee y$, then it holds for any $z \in x \vee y$ and any maximal chains between x, z, or y, z, thus if $y \in \bar{x}^1$ $(y \in \bar{x}^2)$, we can write $y \equiv x(\mathrm{mod}\ R_1)$ $(y \equiv x(\mathrm{mod}\ R_2))$ and R_1, R_2 are congruences on L. The author shows that these congruences are permutable and proves that if we take a fixed point $x_0 \in M$ and denote $\bar{x}_0^1 = A$ and $\bar{x}_0^2 = B$, we will obtain an isomorphism of M and $A \times B$ ($x \in L$ is mapped onto $(a, b) \in A \times B$, where $a \in \bar{x}_0^1 \cap \bar{x}^2, b \in \bar{x}_0^2 \cap \bar{x}^1$).

M. Benado developed his concept of multilattices further in later papers, in which he frequently cited J. Jakubík's results from [Jak11] and [Jak12].

4.3.14 *A note on the Jordan–Dedekind chain condition in Boolean algebras* [Jak13] (1957)

In this paper J. JAKUBÍK continues in investigation of the validity of the general Jordan–Dedekind condition **(JD 2)** introduced by G. SZÁSZ (see the analysis of [Jak10]). As in his previous paper he wants to find some "borders" concerning the validity of the **(JD 2)** this time in infinite Boolean algebras. However, the construction from the proof of his Theorem 4.36 about the existence of a complete and completely distributive lattice which does not satisfy **(JD 2)** cannot be applied in the case of Boolean algebras. He, therefore, makes use of the representation theorem proved by A. TARSKI [Tar1]:

Theorem 4.43. *A complete Boolean algebra L is isomorphic to the partially ordered system of all subsets of a set M iff L is completely distributive.*

By A. TARSKI's theorem we can view a complete and completely distributive Boolean algebra L as a system of all subsets of an infinite set M with the usual partial ordering by the set inclusion. Let us write M in the form $M = M_1 \cup M_2$, where $M_1 \cap M_2 = \emptyset$ and M_1 is a countable set. Let L_1 be a partially ordered system of all subsets of M_1. Then L_1 is a convex sublattice of L. By constructing two orderings of M_1 J. JAKUBÍK finds two maximal chains of different length ($\aleph_0$ and c), and thus proves:

Theorem 4.44. *Let L be an infinite complete and completely distributive Boolean algebra. Then it does not satisfy the* **(JD 2)**.

J. JAKUBÍK points out that it is easy to find an example of a countable Boolean algebra which satisfies **(JD 2)**, e. g. the algebra of all finite subsets and its complements of a countable set. L. RIEGER observed that there exist uncountable complete Boolean algebras satisfying **(JD 2)** and even a stronger condition: every two maximal chains are isomorphic (considering their ordering).

4.3.15 *The centre of an infinitely distributive lattice* [Jak14] (1957)

In this paper J. JAKUBÍK investigates the question whether the centre C of a lattice L forms a complete sublattice of L. (Of course this problems is mainly interesting in the case when L itself is not a complete lattice.) As a result the author obtains a theorem concerning a direct decomposition of a special group of lattices.

J. JAKUBÍK presents examples proving the following theorems:

Theorem 4.45. *In general, the centre of a distributive lattice is not a complete sublattice of the lattice.*

Theorem 4.46. *In general, the centre of a complete lattice is not a complete sublattice of the lattice.*

Further J. JAKUBÍK finds a class of lattices whose centre is a complete sublattice:

Theorem 4.47. *The centre of an infinitely distributive complete lattice is a complete sublattice of this lattice.*

M. KOLIBIAR noticed that it is possible to generalize the previous theorem in the following way: if L is a relatively complete infinitely distributive lattice, we can consider instead of its centre a set $C^* \subset L$ such that $x \in C^*$ iff $x \in [a, b] \subset L$ implies the existence of a relative complement of x in the interval $[a, b]$. A proof of this generalized theorem could be performed using the results of M. KOLIBIAR's paper [Kol4].

From the last stated theorem J. JAKUBÍK derives the following result concerning lattice direct decomposition saying that in case of an infinitely distributive complete lattice it is possible to "separate" its irreducible factors from other direct factors:

Theorem 4.48. *If L is an infinitely distributive complete lattice, then there exist lattices A, B such that*

$$L \cong A \times B,$$

and 1. it is possible to decompose A into a direct product of factors each of which is directly indecomposable, 2. if B includes more than one element, it is directly decomposable and each of its direct factors having more than one element is directly decomposable, 3. the decomposition $L \cong A \times B$ having the properties 1. and 2. is unique.

4.3.16 *A note on the endomorphisms of lattices* [Jak15] (1958)

In this note J. JAKUBÍK deals with questions connected with **Problem 93** of [LT–48], p. 209:

> Is the lattice of all join-endomorphisms of an arbitrary lattice semimodular?

Definition 4.9. A *join-endomorphism* of a lattice L is defined as a mapping f of L into itself satisfying:

$$x, y \in L \Rightarrow f(x) \vee f(y) = f(x \vee y).$$

We denote by E a set of all join-endomorphisms of L with the usual partial ordering: for $f, g \in L, f \leq g$ iff $f(x) \leq g(x)$ for all $x \in L$.

J. JAKUBÍK proves the following two theorems:

Theorem 4.49. *If L is a complete lattice, then E is also a complete lattice.*

Theorem 4.50. *There exists a finite lattice L such that the lattice E is not semimodular. There even exists a lattice L such that the set E is not a lattice at all.*

The first part of Theorem 4.50 is proved by the following nice counterexample:

> $L = \{0, 1, 2, 3, 4\}, 0 < 2 < 4 < 1, 0 < 3 < 1$, the elements of L forming the well-known non-modular pentagon. He defines join-endomorphisms $f_i, i = 0, \dots, 4$ of L:
>
> $f_i(0) = 0, f_i(1) = f_i(2) = f_i(4) = 1, f_i(3) = i,$ for $i = 0, \dots, 4.$
>
> By Theorem 4.49 we get that the corresponding E is a lattice, however, it is not semimodular since its diagram is again a pentagon.

The second part of Theorem 4.50 disproves G. BIRKHOFF's statement from Example 4, §4, Ch. XIII of [LT–48], p. 208:

> The join-endomorphisms of any lattice form an l-semigroup.[24]

J. JAKUBÍK describes the following example of a lattice the set E of which is not a lattice:

> $L = \cup A_i (i = 1, \dots, 4)$, where $A_1 = \{1\}, A_2 = \{x_1, x_2, x_3, \dots\}, A_3 = \{y_1, y_2, y_3, \dots\}, A_4 = \{z_1, z_2, z_3, \dots\}$ are disjoint sets. Lattice operations are defined on L as follows (see Figure 4.6):

[24] G. BIRKHOFF defines an l-semigroup as an associative multiplicative lattice with the unity.

- $p \wedge 1 = p,\ p \vee 1 = 1$ for every $p \in L$
- if m, n are integers, denote $u = \min(m,n), v = \max(m,n)$ and for $p_m, p_n \in A_i (i = 2,3,4) : p_m \wedge p_n = p_u, p_m \vee p_n = p_v$
- $y_n \wedge z_m = x_u, y_n \vee z_m = 1$
- $x_n \wedge p_m = x_u, x_n \vee p_m = p_v$, p is any of the symbols y, z

The set L with $\wedge$ and $\vee$ forms a lattice and we define two mappings of L onto L: f_1 is the identity and for $n = 1, 2, \ldots$: $f_2(1) = 1, f_2(x_n) = x_n, f_2(y_n) = z_n, f_2(z_n) = y_n$. Both f_1 and f_2 are join-endomorphism of L, however, there does not exist an element $f_1 \wedge f_2$, and thus the set E is not a lattice, so neither an l-semigroup.

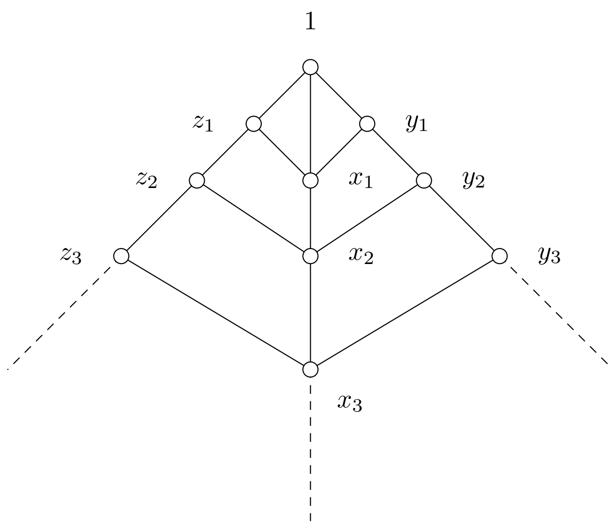

Figure 4.6: A lattice whose join–endomorphisms do not form a lattice.

The solution of **Problem 93** was published independently of J. JAKUBÍK by G. GRÄTZER and E. T. SCHMIDT [G–S3], also some other results of the two papers overlap.

4.3.17 *On permutable congruences on lattices* [Jak16] (1958)

In this paper J. JAKUBÍK reacts to the result stated by H. A. THURSTON[25] investigating congruences on distributive lattices which can be represented by a ring of finite sets. We shall call such lattices *of type (f)*. H. A. THURSTON presented the following proposition:

Proposition 4.51. *If L is a lattice of type (f), then any two congruences on L are permutable.*

This proposition was even cited further,[26] however, as J. JAKUBÍK shows at the next easy example it is not true:

> Let $L = \{x, y, z\}, x < y < z$. It is a finite distributive lattice, which means that L is of type (f). However the partitions $R_1 = \{\{x\}, \{y, z\}\}$ and $R_2 = \{\{x, y\}, \{z\}\}$ are not permutable and so neither are congruences induced by these partitions.

J. JAKUBÍK investigated permutable congruences on lattices in his paper
[Jak4] where he proved the following:

Theorem 4.52. *Let L be a distributive lattice. Any two congruences on L are permutable iff L is relatively complemented.*

Theorem 4.53. *Let L be any lattice. If L is relatively complemented, then any two congruences on L are permutable.*

J. JAKUBÍK continues in developing this investigation to find a condition which is necessary and sufficient for any two congruences on an arbitrary lattice to be permutable. He uses the following notation: if R is a congruence on a lattice L, we will denote by $\bar{L}$ the quotient lattice L/R; if $x \in L$, we will denote the class of R containing x by $\bar{x}$. J. JAKUBÍK shows that a necessary condition for any two congruences R_1 and R_2 on L to be permutable is the following condition **C1** which can be considered a weak form of relative complementarity:

> **C1** If $u, v, x \in L, u < x < v, u \equiv x (\text{mod } R_1), x \equiv v (\text{mod } R_2)$, then in the lattice L/R, where $R = R_1 \cap R_2$, there exists a relative complement of $\bar{x}$ in the interval $[\bar{u}, \bar{v}]$.

[25] THURSTON, H. A., *Congruences on a distributive lattice*, Proc. Edinb. Math. Soc. Ser. 2, 10, Part II (1954), 76–77.

[26] DWINGER, P., *Some theorems on universal algebras I*, Indagationes mathem. 19 (1957), 182–189.

We denote by **C2** the statement which is formed from **C1** by exchanging the indices 1 and 2 and by **C** the statement that **C1** and **C2** are both valid. J. JAKUBÍK proves the following theorem:

Theorem 4.54. *The condition* **C** *is necessary and sufficient for any two congruences R_1 and R_2 on a lattice L to be permutable.*

Let L be a lattice. We will denote for $a, b \in L$ by $R(a,b)$ the intersection of all congruences R_i in which $a \equiv b(\text{mod } R_i)$. If $u, v, x \in L, u < x < v$, we will denote $R(u,v,x) = R(u,x) \cap R(x,v)$. By investigating the following condition:

> **C3** If $u, v, x \in L, u < x < v$ and $R = R(u,v,x)$, then in the lattice L/R there exists a relative complement of $\bar{x}$ in the interval $[\bar{u}, \bar{v}]$.

J. JAKUBÍK proves the theorem:

Theorem 4.55. *The condition* **C3** *is necessary and sufficient for any two congruences on L to be permutable.*

The previous theorem is also re–formulated with the use of weak projectivity introduced in [Jak6].

As this paper generalizes some results of [Jak4] J. JAKUBÍK shows also implications of this generalization to the solvability of the system of congruences

$$x \equiv u(\text{mod } R_1), \quad x \equiv v(\text{mod } R_2). \tag{4.6}$$

A condition which is necessary, however, not sufficient for the existence of a solution of (4.6) is the following:

$$u \equiv v(\text{mod}(R_1 \vee R_2)). \tag{4.7}$$

As in [Jak4] J. JAKUBÍK searches for a condition which would make (4.7) also sufficient and comes to the theorems:

Theorem 4.56. *The condition* **C3** *is necessary and sufficient for the following statement: for any $u, v \in L$ and any congruences R_1, R_2 on L the condition (4.7) implies the existence of a solution of the system (4.6).*

Theorem 4.57. *The condition* **C** *is necessary and sufficient for the following statement: let R_1, R_2 be congruences on L; for any $u, v \in L$ the condition (4.7) implies the existence of a solution of the system (4.6).*

4.3.18 *On chains in Boolean algebras* [Jak17] (1958)

This paper continues the investigation of the Jordan–Dedekind chain condition from the papers [Jak10] and [Jak13]. The aim is to provide a theorem analogous to Theorem 4.36 of [Jak10] about Boolean algebras. J. Jakubík proves the following theorem by constructing a lattice with the given property (using the axiom of choice):

Theorem 4.58. *Let α be a cardinal number, $\alpha \geq \alpha_0$.*[27] *There exists a Boolean algebra B_α with the least element f_0 and the greatest element f_1 satisfying the following property: for any cardinal number $\beta : \alpha_0 \leq \beta \leq \alpha$, there exists in B_α a maximal chain $C_\beta(f_0, f_1)$ whose cardinal number is β.*

We shall present an outline of J. Jakubík's construction: Let L_0 be a Boolean algebra of all subsets of a countable set S, let J be an ideal in L_0 consisting of all finite subsets of S and let L_1 be the quotient Boolean algebra L_0/J. Let M be a set of cardinal number α. We will denote by $L(M)$ the set of all functions defined on M whose functional values belong to L_1 (we consider $L(M)$ with the usual partial ordering: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in M$, f_0 denotes the least element, f_1 the greatest). J. Jakubík shows that $L(M) = B_\alpha$. He writes M in the form $M = M_1 \cup M_2; M_1 \cap M_2 = \emptyset, card(M_1) = \beta, card(M_2) = \alpha$. We will construct $L(M_1), L(M_2)$ analogously to $L(M)$, denote $L(M_1) = A, L(M_2) = B$ and 0_A $(0_B), 1_A$ (1_B) the least and the greatest element of A (B). Then there exists in A a maximal chain $C_1(0_A, 1_A)$ with the cardinal number β and in B a maximal chain $C_2(0_B, 1_B)$ with the cardinal number α_0. But then in $L(M)$ there exists the chain $C_\beta(f_0, f_1)$ whose cardinal number is β.

J. Jakubík also presents several lemmas describing types of maximal chains in a given Boolean algebra:

Lemma 4.59. *Let L be a Boolean algebra containing at least two elements, but no atom. Let $C = C(0,1)$ be a maximal chain in L. Then C is in itself dense.*

Lemma 4.60. *Let L be a complete Boolean algebra, $C(0,1)$ a maximal chain in L, M the set of all atoms of L and M_1 the set of all prime intervals of $C(0,1)$. Then*

$$card(M_1) = card(M).$$

[27] α_0 is a cardinal number of an arbitrary fixed maximal chain C in a lattice with 0 and 1 which is dense in itself, i. e. $x, y \in C, x < y \Rightarrow \exists\, z \in C : x < z < y$.

To show that the preconditions of Lemma 4.60 cannot be replaced by weaker ones J. Jakubík gives an example of a complete distributive lattice with an atom in which a maximal chain does not contain a prime interval, and an example of a Boolean algebra which is not complete, contains an atom, but its maximal chain does not contain a prime interval.

The problem of the Jordan–Dedekind chain condition was studied by other mathematicians in the 1950's as well. G. Grätzer and E. T. Schmidt
[G–S1] showed a different approach. Instead of looking for lattices in which the condition **(JD 2)** generalized by G. Szász (see the analysis of [Jak10]) is valid, they searched for other definitions of the length and maximality of an infinite chain in such a way that in distributive lattices the Jordan–Dedekind condition holds. They define the length of a chain as the power of the set of its different cuts, where by a cut they mean a subdivision of the chain into two non–void convex subchains. They call a chain C strongly maximal iff

(a) C is no proper subchain of any other one with the same endpoints

(b) for every homomorphic image of C, (a) is valid.

The authors prove that in a distributive lattice all strongly maximal chains between fixed endpoints have the same length.

4.3.19 *The Jordan–Dedekind chain condition in direct product of partially ordered sets* [Jak19] (1963)

This paper is devoted to an investigation of the Jordan–Dedekind condition **(JD 2)** (see the analysis of [Jak10]) in partially ordered sets generally, however, it continues in developing ideas of [Jak10, Jak13, Jak17]. The author was inspired by a task of G. Birkhoff ([LT–48], p. 11, ex. 6):

> Prove (or disprove) that the cardinal product of any two partly ordered sets of finite length which satisfy the Jordan–Dedekind chain condition also satisfies it.

In the following we shall consider a partially ordered set S, and we shall denote by $\mathcal{C}(a,b)$ the system of all maximal chains in $[a,b]$, $a,b \in S$, $a \leq b$. Let A, B be a non–void partially ordered sets such that $S = A \times B$,

$$s_i = (a_i, b_i),\ i = 1,2;\ a_i \in A,\ b_i \in B,\ s_1 < s_2.$$

J. JAKUBÍK provides this answer to G. BIRKHOFF's exercise (with a short prove by means of induction):

Theorem 4.61. *If for each* $C_1 \in \mathcal{C}(a_1, a_2)$ *and each* $C_2 \in \mathcal{C}(b_1, b_2)$ *card* $C_1 = n_1$ *and card* $C_2 = n_2$*, where* n_1, n_2 *are natural numbers, then card* $C = n_1 + n_2 - 1$ *for each* $C \in \mathcal{C}(s_1, s_2)$*.*

Naturally, the author carries on with a generalization of the problem to infinite sets. It is easy to see that if partially ordered sets A, B satisfying **(JD 2)** are discrete, then the condition is also satisfied by $S = A \times B$. The same result is valid for a finite number of direct factors of S. However, if the number of direct factors is infinite, it was shown in [Jak13] that the analogous theorem does not hold. J. JAKUBÍK therefore looks for other conditions and he comes to the definition of k–complete partially ordered sets (which need not be lattices):

Definition 4.10. A partially ordered set S will be called *k–complete* iff $u, v \in S, u < v, C \in \mathcal{C}(u, v)$ imply that the chain C is a complete lattice.

Theorem 4.62. *Let* A, B *be k–complete posets which satisfy the condition* **(JD 2)**. *Then* $S = A \times B$ *also satisfies* **(JD 2)**.

If A or B does not satisfy **(JD 2)**, then neither does the direct product $A \times B$. The "simplest" example of a poset which fulfills the condition is a well–ordered set. Let A satisfy **(JD 2)**, then J. JAKUBÍK asks another question: which other properties of A will be preserved in $A \times B$, where B is an arbitrary well–ordered set. His results are the content of the following theorems in which B is a well–ordered set with the least element b_1 and the greatest element b_2 and card $B = m$.

Theorem 4.63. *If* $[a_1, a_2] \subset A,\ C \in \mathcal{C}(a_1, a_2)$*, card* $C = n \leq m$ *and* C *is not a complete lattice, then for each power* n' $(n \leq n' \leq m)$ *there exists a chain* $C' \in \mathcal{C}(s_1, s_2)$ *such that card* $C' = n'$*.*

Theorem 4.64. *Let* A *be a poset which satisfies the condition* **(JD 2)**. *Then the following conditions are equivalent:*

(i) *there exists a well ordered set* B *such that the direct product* $A \times B$ *does not satisfy* **(JD 2)**,

(ii) A *is not k–complete.*

Theorem 4.65. *Let*

$$C_1 \in \mathcal{C}(a_1, a_2),\ C_2 \in \mathcal{C}(b_1, b_2),$$

$$c_1, c_2 \in C_2,\ C_3 = [b_1, c_1] \cap C_2,\ C_4 = [b_1, c_2] \cap C_2,$$

where C_1 is not a complete lattice and card $C_1 = n$, $n \leq$ card $C_3 <$ card C_4. Then the direct product $A \times B$ does not satisfy the condition **(JD 2)**.

4.4 The analysis of Milan Kolibiar's works

4.4.1 Introduction to the analysis

This section analyzes M. KOLIBIAR's papers investigating lattice theoretical problems published by 1963. The works are discussed chronologically as they were published, main results are presented also in relation to previous studies of other mathematicians. As M. KOLIBIAR develops some notions in more papers, we can recognize several areas of interest of his from this period. One of the concepts he found important to investigate was the relation "between" in lattices. His first paper on betweenness was [Kol2] in which he came to various properties of this relation, not knowing, however, that some of them had been published by other mathematicians. Nevertheless, he soon learnt about other papers dealing with the same problem, which led him to generalize some of the existing results, namely finding conditions that an abstract set with the relation between must satisfy so that it would be a lattice [Kol7]. M. KOLIBIAR also shows a connection of the betweenness with a ternary operation [Kol3] which proves to be another strong tool for describing lattices. In [Kol4] he defines an arbitrary lattice with 0 and 1 by means of this ternary operation, thus again generalizing existing results. The paper presenting two systems of two postulates defining modular lattices [Kol5] belongs to his most cited ones. M. BENADO's concept of multilattice found response in M. KOLIBIAR's work too, he investigated mainly its metric properties [Kol8, Kol10]. Naturally, O. BORŮVKA's notion of determining partition plays an important role in M. KOLIBIAR's method of solving problems as well, it has the central role in the papers [Kol1, Kol6]. In the paper [Kol9] he extends the mention "translation" in lattices which was introduced by G. SZÁSZ.

As far as the notation of the analysis is concerned we keep the same principles as in J. JAKUBÍK's case, especially keeping the original "R" for a congruence/determining partition.

Apart from the analyzed papers, M. KOLIBIAR published some other works in the analyzed period which deal mainly with related topics, e. g. chains in posets, congruences or direct products.

4.4.2 *On some properties of a pair of lattices* [J–K] (1954)

The analysis is included in the section dealing with J. JAKUBÍK's papers.

4.4.3 *A note on a representation of a lattice by the partition of a set* [Kol1] (1954)

This paper of M. KOLIBIAR presents an example of O. BORŮVKA's influence, combining the theory of partitions and lattice theory. The author solves the problem of a representation of a distributive lattice in terms of partitions of a set. In case of a non–distributive lattice he puts forward a representation with respect to only one lattice operation.

Let L be a lattice, $a \in L$. For $x, y \in L$ we define $x \equiv y(\text{mod } R^a)$ iff $a \vee x = a \vee y$. Let $\mathcal{P}$ be the lattice of all partitions on L, let $X \subset \mathcal{P}$ be the set of all partitions induced by $R^a, a \in L$. In this note M. KOLIBIAR shows that the set X is $\vee$–isomorphic to L. If L is distributive, then X is a sublattice of $\mathcal{P}$ isomorphic to L.

4.4.4 *On the relation "between" in lattices* [Kol2] (1955)

M. KOLIBIAR learnt about the notion of "betweenness" from M. S. GEL'FAND[28] (M. KOLIBIAR did not see the paper itself, only the review in the *Referativnyj žurnal*) who used the relation between in a lattice L in the same sense as it had been introduced by E. PITCHER, M. F. SMILEY [P-S] who generalized the notion of V. GLIVENKO [Gli1, Gli2]:

Definition 4.11. An element $x \in L$ is between elements $a, b \in L$ iff

$$(a \wedge x) \vee (b \wedge x) = x = (a \vee x) \wedge (b \vee x).$$

We shall call this notion *G–betweenness*. M. S. GEL'FAND proved that in a modular lattice L the set of all elements which are G–between a and b forms a sublattice of L with the least element $a \wedge b$ and the greatest element $a \vee b$. We shall denote such sublattice by $G(a, b)$. M. KOLIBIAR introduced a different notion of betweenness:

Definition 4.12. An element x of a lattice L is between elements $a, b \in L$ iff

$$x \in [a \wedge b, a \vee b]$$

.

[28] GEL'FAND, M. S., *Otrezki v dedekindovoj strukture*, Uč. zap. Mosk. gos. ped. instituta 71 (1953), 199–204.

We shall call M. Kolibiar's notion *K–betweenness* and denote the set of all elements which are K–between a and b by $K(a,b)$.

In this paper the author shows the relationship of the G–betweenness and K–betweenness first, and then returns to some properties of a pair of lattices studied in [J–K], generalizes them and relates them to the notion of betweenness.

M. Kolibiar proves various properties of G–betweenness in modular and distributive lattices, however, as he did not know about the existence of the papers on this topic, he repeats some previous results of other mathematicians. He shows how it is possible to define one notion of betweenness by means of the other one:

Proposition 4.66. $x \in G(a,b)$ *iff* $K(a,x) \cap K(b,x) = \{x\}$.

Proposition 4.67. $K(a,b)$ *is the least convex sublattice of* L *containing* a,b*; convex sublattices can be defined with the notion of G–betweenness: a set* A *is a convex sublattice of* L *iff for each* $a,b \in A : G(a,b) \subset A$.

$G(a,b)$ is obviously a part of $K(a,b)$. M. Kolibiar states two conditions when these two sets equal (the condition of the first theorem was proved earlier[29]):

Proposition 4.68. *Let* L *be a lattice,* $a,b \in L$. $G(a,b) = K(a,b)$ *for every two elements* $a,b \in L$ *iff* L *is distributive.*

Proposition 4.69. *Let* L *be a lattice,* $a,b \in L, A = [a \wedge b, a], B = [a \wedge b, b]$. $G(a,b) = K(a,b)$ *iff* $[a \wedge b, a \vee b] \cong A \times B$*, where the image of* a *is* $(a, a \wedge b)$ *and the image of* b *is* $(a \wedge b, b)$.

Let L_1, L_2 be lattices defined on the same set M. In an earlier paper M. Kolibiar and J. Jakubík investigated relationship between various properties of two lattices (see the analysis of [J–K]). In this paper M. Kolibiar studies again the property **B** and **D**, and also introduces new properties **G** and **H**:

> **B.** If a set $X \subset M$ forms a convex sublattice in L_1, then X forms a convex sublattice in L_2, and vice versa.
>
> **D.** There exist lattices A, B (defined on the sets M_1, M_2) and a one–to–one mapping $\varphi : M \to M_1 \times M_2$ such that $\varphi : L_1 \to A \times B$ and $\varphi : L_2 \to \tilde{A} \times B$ are isomorphisms. ($\tilde{A}$ is the dual of A).

[29] Duthie, W. D., *Segments of ordered sets*, Transactions of the American Mathematical Society 51 (1942), 1–14.

G. If x is G–between $a, b \in L_1$, then it is G–between $a, b \in L_2$, and vice versa.

H. If x is K–between $a, b \in L_1$, then it is K–between $a, b \in L_2$, and vice versa.

It was proved in [J–K] that if L_1, L_2 are distributive, then the properties **B** and **D** are equivalent. M. KOLIBIAR now shows that these two properties are equivalent for arbitrary lattices and are also equivalent to the properties **G** and **H**. At the end of the paper he remarks that each of the properties **B, D, G, H** implies that L_1 and L_2 are graphically isomorphic (which is the property investigated in papers [J–K], [Jak2] and [Jak3]).

M. KOLIBIAR continues with investigating the properties of betweenness in his next papers [Kol3] and mainly [Kol7].

4.4.5 *A ternary operation in lattices* [Kol3] (1956)

In this paper M. KOLIBIAR further develops some results of his previous papers, namely the ternary operation from [J–K] and GEL'FAND's relation between (see [Kol2]).

M. KOLIBIAR defines the following property:

Definition 4.13. Let L be a lattice. We say that an element $t \in L$ has *the property (c)* iff t is a neutral element and t has a relative complement in each interval $[a, b]$ $(a, b \in L)$ containing t.

The author proves that an element $t \in L$ has a property (c) iff there exist a lattice A having the greatest element I and a lattice B having the least element O such that $L \cong A \times B$, and the image of t under the isomorphism is (I, O). The set C of all elements $t \in L$ which have the property (c) forms a sublattice of L. If L has 0 and 1, then C is the centre of L. Let L_1, L_2, A, B be lattices such that $|L_1| = |L_2|$ and $L_1 = A \times B,\ L_2 = \tilde{A} \times B$, then $t \in L_1$ has the property (c) in L_1 iff t has the property (c) is L_2, and vice versa (which means that if L_1 and L_2 have 0 and 1, they have the same centre).

As in [J–K] M. KOLIBIAR works with the ternary operation on L (for details concerning the previous investigation of this operation see the analysis of [Kol4]):

Definition 4.14. Let L be a lattice. If for $a, b, c \in L$:

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a), \qquad (4.8)$$

we define a ternary operation for a, b, c:

$$(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a). \tag{4.9}$$

We will denote by $T(L)$ the set of all elements $[a, b, c]$ $(a, b, c, \in L)$ satisfying (4.8).

Let $t \in L$ be a neutral element. Then the operation (a, t, b) is defined for all $a, b \in L$ and the set $|L|$ with the operation $a \circ b = (a, t, b)$ forms a semilattice. If t has the property (c), then the semilattice is a lattice which is denoted by L_t.

M. KOLIBIAR generalizes results concerning the ternary operation from [J–K] and he also studies relations between the following properties **A, B, C, D** of two lattices L_1, L_2 which are defined on the same set M:

> **A.** $T(L_1) = T(L_2)$ and the operation defined by (4.9) gives the same values in both lattices.
>
> **B.** If $X \subset M$ forms a convex subset in L_1, then it forms a convex subset in L_2, and vice versa.
>
> **C.** There exists an element $t \in L_1$ having the property (c) such that $L_2 = L_t$.
>
> **D.** There exist lattices A, B and a mapping $\varphi : M \to |A| \times |B|$ such that $\varphi : L_1 \to A \times B$ and $\varphi : L_1 \to \tilde{A} \times B$ are isomorphisms.

While investigating the properties the author proves:

Proposition 4.70. *Let L_1, L_2 be lattices with 0 and 1. Then the properties A, B, C, D are equivalent. We can express the operations $\wedge$ and $\vee$ of $L_2 (= L_t)$ by means of the ternary operation in the following way:*

$$a \wedge b = (a, t, b), \; a \vee b = (a, t', b),$$

where t' is the complement of t.

M. KOLIBIAR shows the relationship of the ternary operation and GEL'FAND's relation "between" (this theorem is analogous to the theorem stated for distributive lattices in [B-K]):

Theorem 4.71. *Let $a, b, x \in L$. Then $x \in G(a, b)$ iff $[a, x, b] \in T(L)$ and $(a, x, b) = x$.*

4.4.6 *Characterization of a lattice in terms of a ternary operation* [Kol4] (1956)

This paper is based on results stated in [Kol3] and it generalizes the investigation of S. A. KISS and G. BIRKHOFF [B-K] in which they showed that a distributive lattice with 0 and 1 can be defined in terms of the ternary operation (a, b, c):

$$(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a). \quad (4.10)$$

The symmetric and self–dual ternary operation (4.10), which "plays a unique role in distributive lattices" [LT–67], p. 35, was introduced by A. A. GRAU in his Ph.D. Thesis (*Ternary operations and Boolean algebra*, Univ. of Michigan, 1944), a part of which is the content of his paper [Gra]. He studied it in Boolean algebras. Before him, ternary operations had been discussed in groupoids (A. R. RICHARDSON[30]) and groups (R. BAER and J. CERTAIN[31]); in Boolean algebras an operation different from A. A. GRAU's had been studied by A. L. WHITEMAN.[32] S. A. KISS investigated operations in the Boolean algebra B^n and distributive lattices [Kis] and together with G. BIRKHOFF [B-K] showed the role of the ternary operation (4.10) in distributive lattices with reference to the group of symmetries which it admits. They also presented a definition of a distributive lattice with 0 and 1 in terms of this ternary operation using 5 variables and 5 identities. G. BIRKHOFF included this definition into **Problem 64** in [LT–48], p. 138: "show that some identity can be dispensed with by a suitable permutation of another." This problem was solved by R. CROISOT in [Cro] where he defined a distributive lattice with 0 and 1 in terms of the ternary operation (4.10), 5 variables and 3 independent identities. He also reduced A. A. GRAU's system of postulates [Gra] for Boolean algebras (using the ternary operation (4.10), the operation of complement, 5 variables and 5 identities) to 2 postulates. M. SHOLANDER [Sho] showed that the system of 3 identities defining a distributive lattice with 0 and 1 by means of the ternary operation can be reduced to 2 postulates.

M. KOLIBIAR proves that it is possible to define an arbitrary, not only distributive, lattice L with 0 and 1 in terms of this ternary operation, although the operation is not defined for all triples a, b, c. He

[30] RICHARDSON, A. R., *Algebra of s dimensions*, Proc. London Math. Soc. 47 (1940), 38–59.

[31] CERTAIN, J., *The ternary operation* $(abc) = ab^{(-1)}c$ *of a group*, Bulletin of the American Mathematical Society 49 (1943), 869–877.

[32] WHITEMAN, A. L., *Postulates for Boolean algebra in terms of ternary rejection*, Bulletin of the American Mathematical Society 43 (1937), 293–298.

denotes by $T(L)$ the set of all triples $[a,b,c]$ $(a,b,c \in L)$ for which

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

M. KOLIBIAR's result is described in the following theorem:

Theorem 4.72. *Let M be a set, $O, I \in M$. Let $T \subset M \times M \times M$ have the following properties:*

(a) if $a,b,c \in M, [a,b,c] \in T$, then $[b,c,a],[c,b,a] \in T$;

(b) $[a,b,a] \in T$ for all $a,b \in M$;

(c) $[a,O,b] \in T, [a,I,b] \in T$ for all $a,b \in M$.

Let an element $(a,b,c) \in M$ correspond to each triple $[a,b,c] \in T$ such that

(d1) $(O,a,I) = a$ for every $a \in M$;

(d2) $(a,b,a) = a$ for all $a,b \in M$;

(d3) if $[a,b,c] \in T$, then $(a,b,c) = (b,c,a)$;

(d4) if $[a,b,c],[a,b,d],[(d,b,a),b,c] \in T$, then $[(a,b,c),b,(a,b,d)] \in T$ and

$$((a,b,c),b,(a,b,d)) = ((b,d,a),b,c).$$

Then the set M with the operations

$$a \wedge b = (a,O,b), \ a \vee b = (a,I,b)$$

is a lattice with the greatest element I and the least element O and in which for $[a,b,c] \in T$ the following inequality hold:

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a,b,c) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a). \quad (4.11)$$

M. KOLIBIAR shows an example of L for which $\leq$ in (4.11) is replaced by $<$. He also remarks that the same lattice can be defined by different ternary operations and raises the question whether two different ternary operations defining the same lattice have the same value for each triple (x,y,z) for which they are both defined. The solution is given by T. KATRIŇÁK in [Katr].

4.4.7 *On axioms of modular lattices* [Kol5] (1956)

Several mathematicians worked on finding systems of axioms which would define distributive lattices by means of two binary operations: G. D. BIRKHOFF and G. BIRKHOFF[33] as first developed (following ideas

[33] BIRKHOFF, G. D., BIRKHOFF, G., *Distributive postulates for systems like Boolean algebras*, Transactions of the American Mathematical Society 60 (1946), 3–11.

of M. NEWMAN[34] and M. H. STONE[35]) a set of seven postulates (using three variables) for a distributive lattice with the greatest element. G. BIRKHOFF then stated the question of whether this set is independent as **Problem 65** of [LT–48], p. 139. It was proved by R. CROISOT [Cro] that the postulates are independent and he also showed that it is possible to modify them to be reduced to five. M. SHOLANDER [Sho] described a general distributive lattice by two identities in three variables.

In this note M. KOLIBIAR presents two sets of postulates, however, not for distributive but for modular lattices. The first theorem deals with modular lattices with the greatest element:

Theorem 4.73. *Let L be a set with two binary operations $\wedge, \vee$ having the following properties:*

P1 *For any $a, b, c, d \in L$:*

$$[(a \wedge b) \wedge c] \vee (a \wedge d) = [(d \wedge a) \vee (c \wedge b)] \wedge a. \tag{4.12}$$

P2 *There exist an element $J \in L$ such that any $a \in L$:*

$$a \wedge J = a, \; a \vee J = J.$$

Then L is a modular lattice with the greatest element J and the axioms **P1** *and* **P2** *are independent.*

The identity (4.12) is equivalent to the condition of modularity:

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z.$$

The author remarks that the axiom **P2** cannot be replaced by a weaker one: the existence of elements $I, J \in L$ such that for any $a \in L : a \vee I = I, a \wedge J = a$.

The other theorem of the paper sets the postulates for arbitrary modular lattices:

Theorem 4.74. *Let L be a set with two binary operations $\wedge, \vee$ having the following properties:*

P1 *For any $a, b, c, d \in L$:*

$$[(a \wedge b) \wedge c] \vee (a \wedge d) = [(d \wedge a) \vee (c \wedge b)] \wedge a.$$

[34]NEWMAN, M. H. A., *A Characterization of Boolean lattices and rings*, J. Lond. Math. Soc. 16 (1941), 256–272.

[35]STONE, M. H., *Postulates for Boolean algebras ...*, Amer. J. Math 57 (1935), 703–732.

P2' $[(a \vee (b \wedge b)] \wedge b = b$ *for any* $a, b \in L$.

Then L *is a modular lattice and the axioms* **P1** *and* **P2'** *are independent.*

Further investigation of M. KOLIBIAR's axioms from this paper is done by B. RIEČAN in [Rieč].

As far as the postulates for defining modular lattices is concerned R. PADMANABHAN[36] generalized M. KOLIBIAR's result by providing a uniform method for finding a set of two identities defining not only modular, but any equational class of lattices (which can be defined by a finite number of identities) and R. MCKENZIE[37] proved that the number of identities cannot be reduced to one in case of modular lattices.

4.4.8 *On congruences on distributive lattices* [Kol6] (1956)

This note deals with some properties of congruences which were inspired by the theorem of G. JA. AREŠKIN:[38]

Theorem 4.75. *Let* L *be a distributive lattice with 0. Every congruence on* L *is uniquely determined by its kernel iff* L *is relatively complemented.*

This result is closely connected with G. BIRKHOFF's **Problem 73** ([LT–48], p. 161): Find necessary and sufficient conditions in order that the correspondence between the congruence relations and neutral ideals of a lattice be one–to–one.[39]

M. KOLIBIAR generalizes Theorem 4.75 for distributive lattices without a minimal element and considers also congruences being determined by an arbitrary class, not just an ideal. He comes to several descriptions of relatively complemented distributive lattices. If A is a convex sublattice of a lattice L the author works with the congruence $R_{min}(A)$ which is the least congruence on L that annuls all elements of A and with the congruence $R_{max}(A)$ which is the greatest congruence on L that annuls the elements of A. A convex sublattice A is called characteristic iff $R_{min}(A) = R_{max}(A)$, i. e. there exists only one congruence

[36] PADMANABHAN, R., *Two identities for lattices*, Proceedings of the American Mathematical Society 20 (1969), 409–412.

[37] MCKENZIE, R., *Equational bases for lattice theories* Math. Scand. 27 (1970), 24–38.

[38] AREŠKIN, G. JA., *Ob otnošenijach kongruencii v distributivnych strukturach s nulevym elementom*, Doklady Akad. Nauk SSSR 90 (1953), 485–486.

[39] Compare with the analysis of [Jak6] where J. JAKUBÍK solves this problem by means of weak projectivity of prime intervals.

annulling all elements of A. M. KOLIBIAR also generalizes some results which were proved by J. JAKUBÍK in [Jak4].

The main results of the paper are the following theorems:

Theorem 4.76. *Let* Con$L$ *be a lattice of all congruences on a lattice L. Then* Con$L$ *consists of all minimal congruences which are determined by principal ideals and dual principal ideals of L.*

Theorem 4.77. *Let I be an ideal of a lattice L. Then $R_{min}(I) = R_{max}(I)$ iff the factor lattice $L/R_{min}(I)$ is weakly complemented.*

Theorem 4.78. *Let L be a distributive lattice. Every congruence on L is uniquely determined by a convex sublattice iff L is relatively complemented.*

Before G. J. AREŠKIN Theorem 4.75 had been proved by J. HASHIMOTO.[40] Later G. GRÄTZER and E. T. SCHMIDT [G–S2] proved a theorem which is a generalization of J. HASHIMOTO's and M. KOLIBIAR's results, on which basis L. A. SKORNJAKOV[41] calls one of the theorems the Kolibiar–Hashimoto–Grätzer–Schmidt Theorem:

Theorem 4.79. *Let L be a distributive lattice. Then the following properties are equivalent:*

(i) L is a relatively complemented lattice;

(ii) every ideal of L is the kernel of some congruence on L;

(iii) every dual ideal of L is the kernel of some congruence on L;

(iv) every convex sublattice of of L is a characteristic class of some congruence;

(v) any two congruences on L are permutable.

4.4.9 *Characterization of lattices in terms of the relation "between"* [Kol7] (1958)

In this paper M. KOLIBIAR returns to the investigation of the relation "between" in lattices started in [Kol2], this time with the knowledge of other mathematicians' results concerning this topic. He defines lattice betweenness in the sense of E. PITCHER and M. F. SMILEY [P-S] who generalized metric betweenness in metric lattices introduced by V. GLIVENKO [Gli1, Gli2]:

[40] HASHIMOTO, J., *Ideal theory for lattices*, Math. Japonicae 2 (1952), 149–186.
[41] SKORNJAKOV, L. A., *Elementy teorii struktur*, Nauka, Moskva 1970.

Definition 4.15. Let L be a lattice, $a, b, c \in L$. We say that b is *between the elements* a and c, and write abc, iff

$$(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c).$$

M. KOLIBIAR investigates which properties an abstract system where the relation abc is defined must satisfy in order to be a lattice as well. He generalizes results of L. M. KELLY [Kel] and mainly F. M. SMILEY and W. R. TRANSUE [S–T] who studied abstract systems in which the relation of betweenness abc is defined and found conditions which are necessary and sufficient for this system to be a modular lattice with 0, or an arbitrary lattice with 0. M. KOLIBIAR continues in exploring such systems, and comes to conditions that are necessary and sufficient for a system to be a general lattice. He calls a set K in which the ternary operation abc is given an m–system. In accordance with L. M. KELLY he uses V. GLIVENKO's term [Gli1] of a system being almost ordered:

Definition 4.16. We say that an m–system K is *almost ordered* iff there exists $o \in K$ such that for each two elements $a, b \in K$ there exists a pair of elements $h, l \in B(a, b) = \{x \in K | axb\}$ (high and low) satisfying:

1. olp and oph for each $p \in B(a, b)$,
2. oqa and oqb imply oql; while oaq and obq imply ohq.

M. KOLIBIAR also makes use of the term segment as developed by M. SHOLANDER:[42]

Definition 4.17. Let S be a set of elements $a, b, c, \ldots$ such that to each pair of elements $a, b \in S$ there corresponds a unique subset of S. We will denote this subset by (a, b) and call *the segment* from a to b iff it has the properties (S) and (T):

(S) to each set of three elements a, b, c there corresponds an element d such that $(a, b) \cap (b, c) = (b, d)$.

(T) $(a, b) \subset (a, c) \Rightarrow (a, b) \cap (b, c) = \{b\}$.

where M. SHOLANDER defines the notions of tree, median, between and distributive lattice with 0 and 1 in terms of this concept. M. KOLIBIAR defines a segment (a, b) as a pair of elements of an m–System K for which the set $B(a, b) = \{x | x \in K : axb\}$ equals to its closure.[43] He

[42] SHOLANDER, M., *Trees, lattices, order, and betweenness*, Transactions of the American Mathematical Society 3 (1952), 369–381.

[43] A subset $A \subseteq K$ is closed with respect to the relation between iff for all $a, b \in A$ holds $B(a, b) \subseteq A$; L. M. KELLY [Kel] calls such a set completely convex following L. M. BLUMENTHAL's terminology from BLUMENTHAL, L. M., *Distance Geometries*, University of Missouri Studies, XIII, vol. 2 (1938).

studies the properties of segments in lattices, viewed as m–systems, and then states conditions concerning segments which an m–system must satisfy so that we can define a lattice with 0 and 1 on it.

Apart from others M. KOLIBIAR proves in this paper the following:

Theorem 4.80. *Let K be an m–system. We can define a lattice on K such that the relation abc is a relation of lattice betweenness iff the conditions* (A), (B), (C) *and* (F) *are satisfied:*

(A) *each three elements $a, b, c \in K$ are contained in some segment of K,*

(B) $[a \wedge b, a \vee b] \cap [b \wedge c, b \vee c] \cap [c \wedge a, c \vee a] \neq \emptyset$ *for any $a, b, c \in K$,*

(C) *axb iff $[a \wedge x, a \vee x] \cap [b \wedge x, b \vee x] = \{x\}$,*

(F) *the segments in K can be ordered in the specified way.*

Theorem 4.81. *Let $o, u \in K$ be such that $(o, u) = K$. Then we can define a lattice on K such that the relation abc is a relation of lattice betweenness iff the conditions* (A), (B) *and* (C) *are satisfied. The conditions* (A), (B) *and* (C) *are independent.*

4.4.10 *On metric multilattices I* [Kol8] (1959)

The author investigates multilattices introduced by M. BENADO [Ben2] from the point of view of metric properties and the relation "between". His definition of a metric multilattice corresponds to a normed multilattice by M. BENADO.

Definition 4.18. Let P be a partially ordered set, $a, b \in P$, we denote by $a \vee b$ the set of all such elements $u \in P$ that satisfy: 1. $u \geq a, u \geq b$; 2. if $t \in P, t \geq a, t \geq b, u \geq t$, then $t = u$. If $p \in P, p \geq a, p \geq b$, we define $(a \vee b)_p = \{u \in P : u \leq p, u \in a \vee b\}$. Dually we define $a \wedge b$ and $(a \wedge b)_p$. We call a partially ordered set P *a multilattice* iff

1. for all $a, b, t \in P, t \geq a, t \geq b$: $(a \vee b)_p \neq \emptyset$;
2. for all $a, b, t \in P, t \leq a, t \leq b$: $(a \wedge b)_p \neq \emptyset$.

Definition 4.19. Let M be a multilattice. We say that M is *a normed multilattice* iff there is a real function $v[x], x \in M$, called a *valuation*, satisfying:

1. if $a, b \in M, d \in a \vee b, h \in a \wedge b$, then

$$v[a] + v[b] = v[d] + v[h],$$

2. $a < b \Rightarrow v[a] < v[b]$.

In [Ben2] M. BENADO proved the following theorem:

Theorem 4.82. *Let M be a directed*[44] *normed multilattice. If we define in M a metric in the following way:*

$$\rho(a, b) = v[h] - v[d],$$

where h (d) is an arbitrary element from $a \vee b$ ($a \wedge b$), then M is a metric space.

M. KOLIBIAR says that metric multilattices M, M' are *m-equivalent* iff there exist a one–to–one mapping of M onto M' preserving the relation between. He proves the following:

Theorem 4.83. *Let M, M' be directed distributive*[45] *multilattices. They are m–equivalent iff there exist multilattices A_1, A_2 such that*

$$M \cong A_1 \times A_2,\ M' \cong A_1 \times \tilde{A}_2.$$

Theorem 4.84. *Let (M, ρ) be a metric multilattice and M' a multilattice defined on $|M|$. Let the multilattices M, M' be directed and distributive. (M, ρ) is a metric multilattice iff there exist multilattices A_1, A_2 such that $M \cong A_1 \times A_2$, $M' \cong A_1 \times \tilde{A}_2$, and an element $a \in M$ is carried to the same pair (a_1, a_2) by both these isomorphisms.*

The investigation of betweenness in multilattices can lead also to results concerning graphical isomorphism (for definition see [J–K, Jak12]) of multilattices. M. KOLIBIAR thus arrives at a theorem of J. JAKUBÍK's from [Jak12]:

Theorem 4.85. *Two directed distributive multilattices of finite length are graphically isomorphic iff there exist multilattices A_1, A_2 such that*

$$M \cong A_1 \times A_2,\ M' \cong A_1 \times \tilde{A}_2.$$

In the final notes of this paper M. KOLIBIAR compares the investigated concept of betweenness in multilattices to the case of lattices, and remarks that this notion can be used as a generalization of the notion of graphical isomorphism for infinite modular lattices. This problem was put forward by J. JAKUBÍK in [Jak2] and the same author suggested and solved the generalization with the use of topological equivalence (see the analysis [Jak9]).

[44] See Definition 4.6 for the concept of directed multilattice.
[45] See Definition 4.7 for the concept of distributive multilattice.

4.4.11 *A note on lattice translations* [Kol9] (1961)

The term "translation" was first used in connection with lattices by G. SZÁSZ who adopted it from A. H. CLIFFORD's right- and left–translation of semigroups.[46] G. SZÁSZ studied it first in semilattices[47] and then used the same definition of translation for lattices.[48]

Definition 4.20. A *lattice translation* is defined as a mapping λ of a lattice L into itself satisfying the following condition for all $x, y \in L$:

$$\lambda(x \vee y) = \lambda(x) \vee y.$$

G. SZÁSZ proved that λ is a closure operation and that it is a $\vee$–endomorphism and that $\lambda(L)$ is a dual ideal of L.

In this note M. KOLIBIAR concentrates on connections between a lattice translation and a certain $\vee$–congruence, and on solving the following two questions:

(1) What is a necessary and sufficient condition so that a closure operation ϕ on L is a translation.

(2) Let D be a dual ideal of a lattice L. What are conditions for the existence of a translation λ such that $\lambda(L) = D$?

The answer to (1) was given[49] in the following form: a necessary and sufficient condition is that $\phi(x) \vee y = \phi(x) \vee \phi(y)$ for each $x < y,\ x, y \in L$. M. KOLIBIAR finds a necessary and sufficient condition concerning fixed elements of ϕ:

Theorem 4.86. *A closure operation λ on a lattice L is a translation iff for all $x, y \in L$ holds: $\lambda(x) = x$ and $x \leq y$ implies $\lambda(y) = y$.*

The question (2) is answered by M. KOLIBIAR in the following way:

Theorem 4.87. *Let D be a dual of a lattice L. There exist a translation of L such that $\lambda(L) = D$ iff the intersection of D and any dual principal ideal is a principal ideal.*

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[46] CLIFFORD, A. H., *Extensions of semigroups*, Transactions of the American Mathematical Society 68 (1950), 165–173.

[47] SZÁSZ, G., *Die Translationen der Halbverbände*, Acta Mathematica Academiae Scientarum Hungaticae 17 (1956), 165–169.

[48] SZÁSZ, G., *Translationen der Verbände*, Acta fac. rer. nat. Univ. Comenianae, Mathematica 5 (1961), 449–453.

[49] SZÁSZ, G., SZENDREI, J., *Über die Translationen der Halbverbände*, Acta Sci. Math. 18 (1957), 44–47.

4.4.12 *On metric multilattices II* [Kol10] (1963)

In this note the author comes back to the notion of metric multilattices, however, this time he investigates metric spaces in which it is possible to define a partial ordering in such a way that we obtain a metric multilattice. When solving a similar problem for lattices [Kol7] he used a characterization by means of convex subsets. The method of this paper is different as it applies the concept of *lines* and results of M. ALTWEGG[50] concerning the relation ζ defined in a partially ordered set S as follows:

$$\text{for } x, y, z \in S: \ \zeta(x, y, z) \text{ iff } x \leq y \leq z \text{ or } z \leq y \leq x.$$

M. KOLIBIAR gives the same definition of multilattice as in [Kol8] and he also uses the term m–system from [Kol7] meaning a set with the ternary relation abc.

Definition 4.21. Let a set K be an m–system, $L \subset K$ is called *a line* iff for all $a, b, c, d \in L$ the relations $(\alpha), (\beta), (\gamma)$ are satisfied, and for all $x, y, z \in L$ at least one of the relations xyz, yxz, xzy holds;

(α) $abc \Rightarrow cba$;

(β) abc and $acb \Leftrightarrow \ b = c$;

(γ) abc and $acd \Rightarrow bcd$.

We shall write $l(a, b, \dots)$ iff $a, b, \dots$ lie on one line, if not we shall write $\bar{l}(a, b, \dots)$. If $l(a, b, c)$ and abc hold, we shall write $l^*(a, b, c)$.

Definition 4.22. A system $(M, \leq, \rho)$ will be called *a metric multilattice* iff $(M, \leq)$ is a multilattice, ρ is a metric in M, and the following conditions are satisfied:

M1. $a \leq b \leq c \Rightarrow abc$,

M2. if $c \in a \vee b$ or $c \in a \wedge b$, then acb.

M. KOLIBIAR proved the following theorems (using properties of the relation ζ):

Theorem 4.88. *Let (M, ρ) be a metric space. We can define a partial ordering $\leq$ in M such that $(M, \leq, \rho)$ is a metric lattice iff there exists a system $\mathcal{S}$ of lines in (M, ρ) satisfying:*

(a) *at least one line from $\mathcal{S}$ goes through every point of M;*

(b) *$l^*(a, b, c), l(b, d)$ and not $l^*(a, b, d)$ imply $l^*(c, b, d)$;*

[50] ALTWEGG, M., *Zur Axiomatik der teilweise geordneten Mengen*, Comment. Math. Helv. 24 (1950), 149–155.

(c) *let* $a_0, a_1, \ldots, a_n = a_0, a_{n+1} = a_1$ *be such points in* M *that* $l(a_{i-1}, a_i)$ *and* $\bar{l}(a_{i-1}, a_i, a_{i+1})$ *hold for* $1 \leq i \leq n$*, then* n *is an even number;*

(d) *if* $a_1, a_2, b \in M$ *and* $l(a_i, b)$ $(1 \leq i \leq l)$*, then there exists a point* $m \in M$ *such that* $a_1 m a_2, a_i m b, l(a_i, m)$ *for* $1 \leq i \leq 2$ *holds.*

Theorem 4.89. *Let* (M, ρ) *be a metric space. There exists such a partial ordering* $\leq$ *in* M *that the system* $(M, \leq, \rho)$ *is a directed metric multilattice iff there exists a system* $\mathcal{S}$ *of lines in* (M, ρ) *such that the conditions* (b), (d) *and* (a'), (c') *are satisfied:*

(a') *to each pair of elements* $a_1, a_2 \in M$ *there exist elements* $d, h \in M$ *such that* $l^*(d, a_i, h)$ $(i = 1, 2)$*;*

(c') *if each two points out of three points lie on a line, then all three points lie on a line of* $\mathcal{S}$*.*

Theorem 4.90. *Let* M *be an* m*–system with the properties* $(\alpha), (\beta), (\gamma)$*. There exists such a partial ordering* $\leq$ *in* M *that the system* $(M, \leq)$ *is a directed multilattice in which the conditions M1 and M2 hold iff there exists a system* $\mathcal{S}$ *of lines in* (M, ρ) *such that the conditions* (a'), (b), (c') *and* (d) *are satisfied.*

4.5 Other papers of the period

Apart from J. JAKUBÍK and M. KOLIBIAR we find three more papers dealing with lattice theory by 1963. They were written by mathematicians beginning their research career at that time. BELOSLAV RIEČAN published two papers: one [Rieč] inspired by M. KOLIBIAR's paper concerning postulates for modular lattices [Kol5] and the other one [R-R] written with ZDENA RIEČANOVÁ which continues in M. BENADO's investigation on metric multilattices [Ben2, Ben4]. The last analyzed paper of the period [Katr] was written by TIBOR KATRIŇÁK who gives an answer to M. KOLIBIAR's question from [Kol3].

BELOSLAV RIEČAN graduated from the Faculty of Sciences of Comenius University in 1958 and then became an assistant at the Slovak Technical University in Bratislava. Despite the two mentioned papers touching lattice theory, he has been interested in measure theory since the very beginning of his research. The focus of his scientific activities lies mainly in the areas of measure and integral theory on ordered structures, probability theory and the theory of fuzzy sets.[51]

[51] More information about BELOSLAV RIEČAN can be found in *K životnému jubileu profesora Beloslava Riečana*, Pokroky matematiky, fyziky a astronomie 41 (1996),

TIBOR KATRIŇÁK graduated from the Faculty of Sciences of Comenius University in 1960 and then he started his academic career there. He was recommended to study problems concerning lattice theory by M. KOLIBIAR, also Stone lattices which became his topic for the thesis. In 1965 he received his Ph.D. degree being supervised (as the first one) by M. KOLIBIAR. T. KATRIŇÁK's primary research interests have become lattices and semilattices with pseudocomplementation and he is a world recognized authority in the fields of lattice theory and universal algebra. He is considered to be a successful follower of M. KOLIBIAR not only for the field of their research, but also for his positive teaching activities.[52]

4.5.1 *On axioms of modular lattices* [Rieč] (1957)

In this note B. RIEČAN[53] continues in the investigation of M. KOLIBIAR on the axioms of modular lattices from [Kol5] in which he raised the question whether it is possible to simplify the axioms from Theorem 4.74, suggesting the following two postulates:

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= [(c \wedge a) \vee b] \wedge a,\\ (a \vee b) \wedge b &= b.\end{aligned}$$

B. RIEČAN shows that these identities lead to the theorem:

Theorem 4.91. *Let L be a set with two binary operation $\wedge, \vee$ having the following properties:*

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= [(c \wedge a) \vee b] \wedge a,\\ (a \vee b) \wedge b &= b,\\ (a \vee b) \vee c &= a \vee (b \vee c).\end{aligned}$$

Then L is a modular lattice.

The other theorem of the paper is another modification of M. KOLIBIAR's Theorem 4.74 which is in a way its simplification:

333–335.

[52]More details about T. KATRIŇÁK can be found in HAVIAR, T., ZLATOŠ, P., *Jubilee: The Sixtieth Birthday of Professor Katriňák*, Acta Univ. M. Belii, Math. 5 (1997), 91–98.

[53]The author's first name is wrongly given as "Jan" in this paper, however, "Beloslav" is correct.

Theorem 4.92. *Let L be a set with two binary operation $\wedge, \vee$ having the following properties:*

P1 $(a \wedge b) \vee (a \wedge c) = [(c \wedge a) \vee b] \wedge a$,

P2 $[a \vee (b \vee c)] \wedge c = c$.

Then L is a modular lattice and the identities **P1** *and* **P2** *are independent.*

4.5.2 *A note on metric multilattices* [R-R] (1960)

The authors continue in the investigation of metric multilattices started by M. BENADO in [Ben2] (for defining a multilattice and their types, see the analyses of [Jak11, Jak12] and [Kol8]), and particularly they answer a question stated in his other work on multilattices [Ben4]. M. BENADO introduced three types of valuation functions on a multilattice:

Definition 4.23. Let M be a directed multilattice, $a, b \in M$ and let $v(x)$ be a real function defined on M. If there exist elements $d \in a \vee b$, $m \in a \wedge b$ such that

$$v(a) + v(b) = v(d) + v(m), \tag{4.13}$$

then we shall call $v(x)$ a valuation function of the *first type*. If the equation (4.13) holds for all $d \in a \vee b$ ($m \in a \wedge b$) and for some $m \in a \wedge b$ ($m \in a \vee b$), then we shall call $v(x)$ an *upper (a lower) valuation of type two*. If the equation (4.13) holds for all $d \in a \vee b$ and all $m \in a \wedge b$, then we shall call $v(x)$ a valuation of the *third type*.

In [Ben2] M. BENADO proved that all directed multilattices with a positive valuation of type three are modular. In [Ben4] he generalized this result for directed multilattices with a valuation of the second type:

Theorem 4.93. *Let M be a directed multilattice. Let $v(x)$ be a real function defined on M satisfying the following properties:*

V1. for any pair of elements $a, b \in M$ there exists $d_0 \in a \vee b$ such that

$$v(d_0) \leq v(d) \textit{ for all } d \in a \vee b,$$

V2. for all d_0 satisfying the property V1 and all $m \in a \wedge b$ holds

$$v(a) + v(b) = v(d_0) + v(m),$$

V3. if $a, b \in M, a < b$, then $v(a) < v(b)$.

Then M is a metric space, in which the metric is defined as $\rho(a, b) = v(d_0) - v(m)$ where $D_0 \in a \vee b$ has the property V1 and $m \in a \wedge b$.

Together with this theorem M. BENADO stated the question whether a multilattice satisfying the properties V1–V3 must be modular. B. RIEČAN and Z. RIEČANOVÁ give the negative answer to his question in this paper. They present a multilattice which satisfies the conditions of the previous theorem, but it is not modular. Then the authors prove a modified theorem where the modularity is kept:[54]

Theorem 4.94. *Let M be a multilattice. Let $v(x)$ be a real function defined on M satisfying the following properties:*

M1. for any pair of elements $a, b \in M$ there exists $d_0 \in a \vee b$ such that

$$v(a) + v(b) = v(d_0) + v(m),$$

for all $m \in a \wedge b$,

M2. let $a, b, b', u, t \in M$ such that $u \geq a \geq m, u \geq b \geq b' \geq m, (a \vee b')_u = u, (a \wedge b)_m = m$, let $d_0 \in a \vee b, d'_0 \in a \vee b'$ have the property M1, then $v(d_0) = v(d'_0)$,

M3. if $a, b \in M, a < b$, then $v(a) < v(b)$.

Then M is a modular multilattice.

The second part of the paper is devoted to a generalization of the previous theorem to partially ordered sets.

4.5.3 *On a question concerning characterization of a lattice in term of a ternary operation* [Katr] (1961)

T. KATRIŇÁK continues in investigating the ternary operation defining a lattice with 0 and 1 which was introduced by M. KOLIBIAR in [Kol3] (Theorem 4.72). Specifically, he looks for an answer to KOLIBIAR's question whether different ternary operations defining the same lattice give the same elements for each triple. First he shows the following behaviour of the operations:

Theorem 4.95. *Let M be a lattice with 0 and 1. Let o_1, o_2 be ternary operations defined in $T_1, T_2 (T_1, T_2 \subset M \times M \times M)$ respectively, having the properties* (a), (b), (c), (d1), (d2), (d3) *and* (d4) *of Theorem 4.72 and defining M. Let $(x, y, z) \in T_1, (x, y, z) \in T_2$ and $x \neq y \neq z \neq x$. If at least two elements of the set $\{x, y, z\}$ are comparable, then the triple (x, y, z) corresponds to the same value for both operations o_1 and o_2.*

[54] The properties M1 and M3 are equivalent to the statement that $v(x)$ is a valuation of the second type, the authors also show that a multilattice with a valuation satisfying the properties M1–M3 is not a valuation of the third type.

However, there exists a lattice and ternary operations which do not have the same values as is proved by T. KATRIŇÁK on an example:

Theorem 4.96. *Let L be a lattice in Figure 4.7. Then there exist such ternary operations o_1 and o_2 defining L that $(a, b, c) = 7$ by the operation o_1 and $(a, b, c) = 8$ by the operation o_2.*

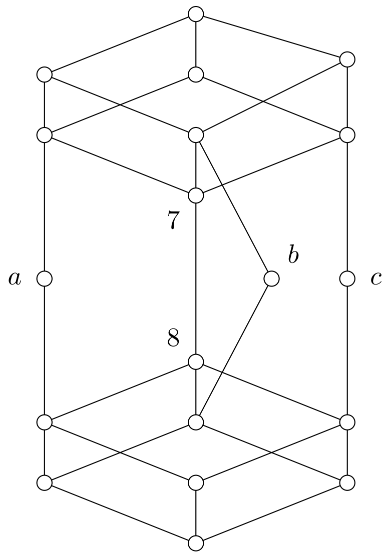

Figure 4.7: Two ternary operations defined on this lattice need not have the same value for all triples.

Chapter 5

Conclusion

The preceding chapters described the activity within Czechoslovak mathematics devoted and related to lattice theory until 1963. If we look into the way lattice theory became a part of research in Czechoslovak mathematics, we identify two different situations when comparing Czech and Slovak mathematics. While in Czech mathematics lattice theory appeared in the course of the implementation of modern algebra at the end of 1930's and in the development of individual algebraists' own research, the introduction of lattice theory into Slovak mathematics is closely tied with the beginnings of mathematical research such as, i. e. with the real establishing of research centers after WWII.

Before summarizing the main achievements of Czech and Slovak mathematicians in the field of lattice theory until 1963, we shall briefly describe the position of lattice theory in Czechoslovak mathematics in comparison with the position of this theory in international mathematics, and we shall also recall the main sources influencing Czechoslovak lattice theoretical research.

5.1 The position of lattice theory in Czechoslovakia and in the world

The reaction of Czech mathematicians to lattice theory at the end of 1930's appeared at the time when the theory was gaining the world's status, however, immediate further development of research was limited by the period of WWII, which influenced the whole European mathematical community. While e. g. in the U.S.A. mathematicians were attracted to and intensively continued developing this new, "fashionable", field from the end of 1930's, in this country, it was only after WWII when lattice theory became a popular area for a wider mathematical community, especially beginning scientists.

Despite the increasing number of mathematicians working in lattice theory in the post–war years , we can see a shift in the opinions on this field compared to the end of 1930's. The optimism surrounding its creation was reduced even from the side of its "originators": G. BIRKHOFF's description of lattice theory of 1938 as a "vigorous and promising younger brother of group theory" ([Bir3], p. 793) is turned down by O. ORE's quite a pessimistic remark in the 1950's: "I think lattice theory is played out" [Rot]. Nevertheless, further development brought lattice theory to a stable place within mathematics though the early hopes that lattices will play central role as universal algebras have not been fulfilled. As R. P. DILWORTH [Dil] expressed: "The emphasis and areas of research in lattice theory had changed since the 1930's, [...] lattice theory provided a useful framework for many topics and developed into a full-fledged member of the algebraic family with an extensive body of knowledge and a collection of exciting problems all of its own, [...]."

The position of lattice theory in Czechoslovak mathematics was set forward by the approach of the leading algebraists O. BORŮVKA and V. KOŘÍNEK from the very beginning: they both recognized it as a theory in its own right, regarded it as worth investigation and presented it to their students. In 1950's O. BORŮVKA often described lattice theory as a modern and intensively developing field with a significant position within contemporary mathematics. V. KOŘÍNEK expressed an opinion that lattice theory "will set the nature of algebraic research in the following years" [Koř5]. The coming generations could therefore profit from having their teachers supporting lattice theoretic research. Even though not all mathematicians making their starts in this theory carried on in this research, the number of papers reveals a continuing interest in this field and played its role in the incorporation of lattice theory into Czech and Slovak mathematics.

A specific position of lattice theory in Slovak mathematics has been mentioned. This subject became one of the first algebraic areas of research and the development of initial activities in this field can be considered an illustration of the process of establishing mathematical research as such. The first step in this process was to provide a new generation of students with good scientific foundations, motivation and inspiration for further research, which was achieved by the work of university professors in Bratislava. The next step in realizing the aim was to show the ability to solve the first rate mathematical problems and at the same time to continue in encouraging students in pursuing academic careers. This was accomplished within the field of lattice theory by the personalities of J. Jakubík and M. Kolibiar who greatly contributed to the successful fast growth of Slovak mathematics.

An important event for Czechoslovak algebraic research was the Conference on Ordered Sets in Brno in 1963, at which the mathematicians had an opportunity to compare the level of their investigation on ordered sets, including lattice theory, with international research. The outcome of this encounter was evaluated by M. Novotný [Nov8] in the following way:

> The scientific benefit of the conference is indisputable. Personal encounters and exchange of opinions of researchers in the same fields from various countries were enabled. It was shown that we work intensively in the area of ordered sets in this country and that the results bear comparison even with a strict international criteria.

Since 1960's lattice theory has become an integral part of mathematical research of Czech and Slovak mathematics with specified areas of interests in which numerous outstanding results have been obtained (let us mention e. g. L. Beran, J. Tůma, P. Pudlák, J. Ježek, T. Katriňák).

5.2 The main influences

The initial influences in introducing lattices to Czech mathematics came especially from O. Ore who is referred to by both O. Borůvka and V. Kořínek. O. Borůvka also cites G. Birkhoff's paper [Bir3] and he must have also known the work of German mathematicians writing on lattice theory. V. Kořínek took the lattice concepts and terminology from G. Köthe [Köt, H–K].

The mentioned sources of O. Ore, G. Birkhoff and G. Köthe suggest that O. Borůvka and V. Kořínek were fully aware of the developing lattice theory at the end of 1930's. However, the war prevented the Czech algebraists to get acquainted with the first edition of G. Birkhoff's monograph *Lattice Theory* promptly and to respond to it. The first edition therefore had no direct impact on Czech mathematics at the time of its publishing, contrary to its second edition which significantly influenced both Czech and Slovak mathematical research.

The second edition of *Lattice theory* [LT–48] and its Russian translation became key literature and inspiration for a great number of mathematicians engaged in the research on posets, lattices, and other algebraic structures. As O. Borůvka points out ([UB], p. 208): "What for the algebraists of the thirties meant Van den Waerden's book *Modern Algebra*, was for the mathematicians after World War II meant by Birkhoff's monograph *Lattice Theory*."

Apart from G. Birkhoff's book, other influences depended on the research area of individual mathematicians. L. Rieger often followed investigations of M. H. Stone, J. M. M. McKinsey, A. Tarski, and R. Sikorski, and naturally drew upon classical works dealing with mathematical logic (K. Gödel, A. Heyting, A. Mostowski, W. Sierpinski). The other Prague mathematicians cited mainly V. Kořínek and O. Ore. Brno mathematicians were often inspired by a variety of sources, depending on their particular interest, even though, many of them were affected by the work of their teacher O. Borůvka.

J. Jakubík's and M. Kolibiar's initial research also bears features of O. Borůvka's algebraic influence, although they soon found specific areas of lattice theory they became engaged in. J. Jakubík often reacted to contemporary works of various mathematicians. In the case of S. Szász and M. Benado, we can recognize mutual responses to each other's papers. In his papers on graphical isomorphism and direct product of lattices, J. Jakubík cites S. A. Kiss and J. Hashimoto several times. J. Jakubík's and M. Kolibiar's papers on metric lattices and M. Kolibiar's works treating the relation "between" in lattices follow the ideas of E. Pitcher, M. F. Smiley, W. R. Transue and L. M. Kelly. When studying the ternary operation in lattices, M. Kolibiar builds mainly upon the works of R. Croisot, G. Birkhoff and S. A. Kiss.

5.3 Achievements of Czech and Slovak mathematicians

5.3.1 The role of O. Borůvka and V. Kořínek

Both O. Borůvka and V. Kořínek are merited for introducing lattice theory into Czech mathematics and for playing a substantial role in drawing the attention of young mathematicians to this field.

As far as their own scientific research is concerned, we can follow two different approaches to the existing lattice theory in the work of

O. BORŮVKA and V. KOŘÍNEK. O. BORŮVKA recognized it as a theory which comprises his theory of decompositions as one of its realizations, thus he only comments on this relationship in appropriate places, since his main investigation within partition theory deals with the properties that cannot be characterized by lattice theory only. V. KOŘÍNEK, on the other hand, concentrated on problems within lattice theory itself, namely questions concerning the Jordan–Hölder–Schreier–Zassenhaus theorem in lattices, which was the subject close to his previous investigation on group theory.

O. BORŮVKA was the first Czech mathematician to use the term "lattice" in his work [Bor1] in 1939, while V. KOŘÍNEK was the first Czech mathematician to write a paper ([Koř1], 1941) dealing wholly with a lattice theoretic problem. Both O. BORŮVKA and V. KOŘÍNEK developed ideas from the mentioned first papers further. O. BORŮVKA's theory of partitions, upon which he built the theory of groupoids and groups, met with favorable acknowledgement in the world mathematics, and his monograph [Bor7a], completing this research, was published by the renowned VEB Deutcher Verlag der Wissenschaften, in German in 1960 and in English in 1975. V. KOŘÍNEK's results on the Schreier–Zassenhaus refinement in lattices were followed not only by his pupils, but also by Romanian mathematicians, and his paper [Koř1] is cited by G. BIRKHOFF in [LT–48], p. 89 and also by A. G. KUROŠ in *Teorija grupp* [Kur2], p. 448.

After WWII several scientists of a new generation started their academic careers in the area of lattice theory, which is to be attributed to the influence of O. BORŮVKA and V. KOŘÍNEK. There existed three centers of studies connected to lattice theory: Prague, Brno and Bratislava (together with Košice after J. JAKUBÍK moved there) in the period after WWII. Each centre was characteristic for its special features.

V. KOŘÍNEK and mainly O. BORŮVKA are credited for their helpful cooperation with Slovak mathematics. T. KATRIŇÁK describes the significance of O. BORŮVKA's role in the following way:

> I am convinced that M. Kolibiar and all his Slovak contemporaries benefited mightily from Professor Borůvka's activities in Bratislava in the late forties and in the fifties. They got a mathematical push that lasted a decade. This is a good example of how Slovak mathematics profited from the collaboration with Czech mathematical community.[1]

[1] KATRIŇÁK, T., *Milan Kolibiar (1922–1994)*, Math. Slovaca 46 (1996), 297–304, p. 298.

5.3.2 Lattice theoretical research in Prague

V. Kořínek's results on the Jordan–Hölder–Schreier–Zassenhaus theorem were extended in the work of his students L. Janoš, Č. Vitner,V. Vilhelm, and V. Havel. Some of their articles are cited by G. Grätzer [Grä2]: V. Havel's paper [Hav2] is referred to when discussing a direct join representation of elements in a lattice (p. 286), and G. Grätzer also mentions V. Vilhelm's paper [Vil2] and O. Hájek's [Háj3, Háj]. From the mentioned Czech mathematicians, only V. Havel continued in work concerning lattice theoretical questions later, the others, apart from some exceptional papers, conducted research mainly in other fields of mathematics. They, however, help to arise and keep interest in lattice theory among Czech mathematicians.

Another Prague mathematician L. Rieger became interested in Boolean algebras after WWII, making contributions towards solving some G. Birkhoff's problems of [LT–48] as well as towards algebraic representation of mathematical logic. His results are mentioned on several places in [LT–67]: the paper [Rie4] on p. 227, [Rie6] on pp. 257 and 260, [Rie8] on p. 252. G. Birkhoff also refers to his investigation of cyclically ordered sets (p. 301) in his book. L. Rieger's results are also included in books on mathematical logic (e. g. R. Sikorski's *Boolean Algebras* [Sik]). G. Grätzer refers to L. Rieger's papers [Rie4] when stating properties of the set of all prime ideals of a lattice ([Grä2], p. 86) and in the chapter about topological representation of lattice ([Grä2], p. 131), the paper [Rie8] is also cited.

5.3.3 Lattice theoretical research in Brno

The situation in Brno differed from the one in Prague and in Bratislava. Neither lattice theory nor Boolean algebras became a primary aim of research, however, many results showed their connection to or application of lattice theory. K. Koutský developed an overall theory of topological lattices, M. Mikulík investigated metric lattices from the impulse of functional analysis, F. Šik was engaged in the investigation of ordered groups and l–groups, and his results became an integral part of this field. The personality of M. Novotný has to be stressed on this place particularly, not only for his first–rate mathematical achievements, but also for his pedagogical and organizational merits. He was a founder of Brno seminar on ordered sets and general algebraic structures, and he greatly participated in organizing regular summer schools on the theory of ordered sets in which lattice theory constituted one of the research areas from the very beginning.

5.3.4 Lattice theoretical research in Slovakia

J. Jakubík and M. Kolibiar belong to the first generation of Slovak mathematicians, graduating and starting their academic careers in the difficult time after WWII. Despite the pioneering conditions they soon succeeded in solving some topical problems of lattice theory and they highly contributed to the fast development of Slovak algebra after the war.

J. Jakubík's results concerned the relationship between graphical and lattice isomorphisms, direct decomposition of lattices, congruence relations, and the Jordan–Dedekind chain condition in lattices and Boolean algebras. M. Kolibiar investigated the relation between and a ternary operation in lattices and developed a set of postulates characterizing modular lattices.

Although the analyzed period presents only a proportion of J. Jakubík and M. Kolibiar work, we can find a number of references to their papers of this time in [LT–67] and [Grä2]. J. Jakubík's results from [Jak16, Jak5] are mentioned as exercises by G. Birkhoff, [LT–67], pp. 163 and 164. G. Grätzer ([Grä2], p. 72) refers to [J–K, Jak2, Jak3] while discussing lattice theory from a graph–theoretic point of view; he also mentions (p. 162) J. Jakubík's considerations on cardinalities of maximal chains in Boolean algebras from [Jak13, Jak17] and the paper [Jak6] is mentioned for J. Jakubík's investigation of weak projectivity of prime intervals (p. 208) and when referring to papers discussing lattices whose congruences form a Boolean lattice (p. 209). M. Kolibiar's and B. Riečan's sets of postulates for modular lattices from [Kol5, Rieč] are referred to by both G. Birkhoff (who calls them "remarkable" [LT–67], p. 36) and G. Grätzer [Grä2], p. 70. It might be also interesting to notice that O. Borůvka [Bor7a] refers to the results from [J–K, Kol1, Kol6].

J. Jakubík and M. Kolibiar also succeeded in introducing lattice theoretic problems to their students. They were both leading algebraic seminars, in Košice and Bratislava respectively, for many years and in the middle of 1960's they started supervising theses. Some of their students themselves became recognized authorities in the field of lattice theory (e. g. T. Katriňák), some others were attracted by "competing" areas of mathematics, however some of them started to work on border subjects (e. g. B. Riečan), thus profiting from the lattice theoretic knowledge.

5.3.5 Birkhoff's problems

In the period until 1963 several of 111 problems stated by G. BIRKHOFF in [LT–48] were solved by Czech and Slovak mathematicians. We shall now present them as they are listed also in the Russian translation of [LT–67][2] in the Appendix "Problemy Birkgofa" by V. N. SALIJ:

- Problem 8 solved by J. JAKUBÍK in [Jak2],
- Problem 33 solved by J. JAKUBÍK in [Jak5],
- Problem 67 solved by J. JAKUBÍK in [Jak6],
- Problem 74 solved by M. KATĚTOV in [Katě] and by L. RIEGER in [Rie8],
- Problem 76 solved by M. KATĚTOV in [Katě],
- Problems 78, 79 and 80 solved by L. RIEGER in [Rie6],
- Problem 93 solved by J. JAKUBÍK in [Jak15],
- Problem 99 solved by J. JAKUBÍK in [Jak18].

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