Witold Więsław

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TWO ABEL'S THEOREMS

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1 Motivations. Abel's anniversary

The bicentenary anniversary of the birth of Niels Henrik Abel (1802–1829) was over in the foregoing year 2002. To celebrate it, I would like to recall two important Abel's theorems. Since the literature on the subject is rather available, I am going to give here only a sketch of fundamental research concerning these theorems.

Abel's life is described in details in [22].

2 The First Theorem. Algebraic equations

Abel claims in the paper [2] that:

Bekanntlich kann man algebraische Gleichung bis zum vierten Grade allgemein aufzulösen, Gleichungen von höhern Graden aber nur in einzelnen Fällen, und irre ich nicht, so ist die Frage:
Ist es möglich Gleichung von höhern Graden als dem vierten Grade allgemein aufzulösen ?
noch nicht befriedigend beantwortet worden. Der gegenwärtige Aufsatz hat diese Frage zum Gegenstande.

He was not right. Paolo Ruffini [20] gave rather satisfactory proof using, in fact, combinatorial methods (cf. also [4], [5], [10], [17], [18]). Ruffini presented later five modified versions of the proof. His book [20] was written too early, in the sense, that in the last years of XVIII century it was still believed that algebraic equations of any degree can be, perhaps in very complicated ways, solved algebraically or, as we say now, by radicals. Leonhard Euler, Joseph-Louis Lagrange and others had many results dealing with algebraic equations, but without a definite result. For example Euler found the form of roots of an equation of degree five in the case when the equation is solvable by radicals and gave numerous examples, e. g.

$$\alpha = \sqrt[5]{75\cdot\left(5+4\sqrt{10}\right)} + \sqrt[5]{75\cdot\left(5-4\sqrt{10}\right)} +$$
$$+ \sqrt[5]{225\cdot\left(35+11\sqrt{10}\right)} + \sqrt[5]{225\cdot\left(35-11\sqrt{10}\right)}$$

is the root of equation $x^5 - 2\,625x - 61\,500 = 0$. However, Euler was neither able to determine the problem nor to find conditions for the solvability in such cases. Moreover, his activities in many other fields of science fulfiled the rest of his days.

Let S_n be the symmetric group of n elements and $Q\left[x_1, x_2, \ldots, x_n\right]$ the ring of polynomials in n variables with rational coefficients. The main idea of Ruffini's proof lies in the fact, that there exists no polynomial $W \in Q\left[x_1, x_2, \ldots, x_n\right] (n > 4)$ taking three or four values when permuting its variables, i.e. such that the cardinality of the set

$$\left\{W\left(x_{g(1)}, \ldots, x_{g(n)}\right):\ g \in S_n\right\} \tag{1}$$

is three or four. Abel proved that if the set (1) has less than five elements, its cardinality can be neither three nor four. Next he assumes that an equation of degree five is solvable algebraically and finds a general form of any element of degree five over the field $Q\left(x_1, \ldots, x_5\right)$ of rational functions of five variables (I use our contemporary terminology). Finally he applies his theory to the equation $x^5 - R = 0$, concluding that its root must have the form

$$s_1\sqrt[5]{R} = x_1 + \alpha^4 x_2 + \alpha^3 x_3 + \alpha^2 x_4 + \alpha x_5 \tag{2}$$

where s_1 is a given element (for details consult [2]) and α is a fifth root of unity (i.e. $\alpha^5 = 1$). Now permuting variables one can easily see that the left side of (2) can take only five values (the roots of the binomial equation $x^5 - R = 0$) and the left side of (2) can take 120 values.

Now it is well-known that both proofs given by Ruffini and by Abel have gaps, which, however, can be removed. A detailed analysis of Abel's proof was given by Hamilton [14] in the year 1839. The state and methods used in the theory of algebraic equations in XIX century were described in the papers [9] and [10]. Results of Ruffini and Abel can be stated as

The First Theorem (Ruffini [20], Abel [2])

A general algebraic equation of degree five cannot be solved algebraically [i. e. by radicals].

In the next paper on the subject [3] Abel found sufficient conditions for solvability of equations by radicals. He was motivated by Gauss equation ([12], Sectio Septima, DE AEQVATIONIBVS CIRCVLI SECTIONES DEFINIENTIBVS) for the roots of unity of degree p (p being prime), i. e. by the equation $X^p - 1 = 0$. The main result of [3] can be formulated as follows.

Let x be a root of an irreducible algebraic equation $f(X) = 0$ such that:

(i) every root of the equation has a form $g(x)$, g being a rational function of one variable,

(ii) $gh(x) = hg(x)$ holds for every pair of roots $g(x)$ and $h(x)$ of the equation $f(X) = 0$. Then the equation $f(X) = 0$ is solvable by radicals.

For these reasons groups satisfying (ii) are called now Abelian, or commutative. Note here that the notion of *commutativity* was introduced for the first time by Servois ([21], page 101).

Abel formulated and proved in fact ten theorems in [2].

3 Elliptic functions as a method for solving algebraic equations

Charles Hermite [15] solved an arbitrary equation of five degree by applying elliptic functions. Every equation of degree five can be reduced to the form $x^5 - x - a = 0$ using Georg Birch Jerrard method (1835). Cockle and Harley also obtained this result (1858–59). Arthur Cayley repeated the result in the year 1861. This theorem was obtained by analysis of identities for elliptic functions. Similar idea can be used to obtain general formulae for the roots of algebraic equations of arbitrary degree (cf. [8]). Practical application of these formulae is rather doubtful, since hypergeometric series are very complicated. However, the above mentioned results on algebraic equations show that in any case roots of the equation can be effectively calculated from its coefficients in finite or infinite number of steps.

$$f\mathrm{f}\mathrm{f}^{-1}z=\mathrm{f}\mathrm{f}^{-1}fz\ ;$$

or, (15)

$$f\mathrm{f}\mathrm{f}^{-1}z=\mathrm{f}f\mathrm{f}^{-1}z\ ;$$

donc

$$\mathrm{f}f\mathrm{f}^{-1}z=\mathrm{f}\mathrm{f}^{-1}\mathrm{f}z\ ;$$

et, en prenant de part et d'autre la fonction f^{-1},

$$f\mathrm{f}^{-1}z=\mathrm{f}^{-1}fz\ .$$

C'est le premier des théorèmes (16), et le deuxième se démontrerait de la même manière. Quant au troisième on a (1)

$$f^{-1}f\mathrm{f}^{-1}z=\mathrm{f}^{-1}f^{-1}fz\ ;$$

et, d'après le premier des théorèmes (16),

$$f^{-1}\mathrm{f}^{-1}fz=\mathrm{f}^{-1}f^{-1}fz\ ;$$

laquelle devient le troisième théorème (16), en y changeant fz en z.

9. Des théorèmes (n.os 7, 8) on conclut, sans discussion, les formules qui suivent.

Quand f, f, φ, ... étant commutatives entre elles, k, m, n, ... sont des nombres entiers positifs, on a

$$\mathrm{f}^nf^mz=f^m\mathrm{f}^nz\ ; \qquad (17)$$

puis, en désignant $\mathrm{f}.fz$ par φz,

$$\varphi^nz=\mathrm{f}^nf^nz=f^n\mathrm{f}^nz\ ; \qquad (18)$$

enfin, en désignant f^nf^mz par ψz,

$$\psi^kz=\mathrm{f}^{kn}f^{km}z=f^{km}\mathrm{f}^{kn}z\ . \qquad (19)$$

10. Si les fonctions monômes d'une fonction polynôme sont à la fois distributives et commutatives entre elles, tous les ordres de la fonction polynôme seront des fonctions distributives (on le sait déjà d'après le n.° 6) et commutatives, non seulement avec les différens ordres des composantes, mais aussi avec tous les ordres des fonctions distributives qui sont commutatives avec ces dernières.

Soit

A page from [21]

4 The Second Theorem. Dividing Bernoulli's lemniscate

A lemniscate called now Bernoulli's lemniscate was discovered by Jacob Bernoulli in the year 1694 (cf. [7]). There are some equivalent definitions of this curve.

Definition 1. The lemniscate is the locus of a point which moves so that the product of the distances from the two given points in the plain is a constant.

Taking $(-a, 0)$ and $(a, 0)$ for the points it is easily seen that

Definition 2. The lemniscate is given in Cartesian coordinates by the equation

$$\left(x^2 + y^2\right)^2 = a^2 \cdot \left(x^2 - y^2\right).$$

Definition 3. An arc of the lemniscate is determined by a flexible elastic band constrained by its own weight (cf. [6], [7], [11]).

The definition 3 led Jacob Bernoulli [7] to the study of the lemniscate integral, i. e. to

$$\int \frac{\mathrm{d}x}{\sqrt{1 - x^4}} \tag{3}$$

Many authors studied Bernoulli's lemniscate in XVIII century, including Giulio Carlo, Count de Fagnano (1682–1766) and Leonhard Euler (1707–1783). It was conjectured in the XVIII century that the lemniscate integral is not expressible by elementary functions, but no proof was given.

The study of the integral (3) included two important problems:

I. calculate the length of an arc of Bernoulli's lemniscate,

II. decide into how many parts the lemniscate can be divided by a geometric construction with compasses and ruler.

Fagnano obtained relations between different elliptic integrals of the form (3) and proved that Bernoulli's lemniscate can be divided geometrically into 2, 3 and 5 equal parts, and generally, into 2^n, $3 \cdot 2^n$ and $5 \cdot 2^n$ for any natural number n. Euler stated in a paper published in 1756/57 (cf. [6]), that if $N = 2^n \cdot (1 + 2^m)$, then Bernoulli's lemniscate can be divided into N equal parts. It is however not true, since the lemniscate cannot be divided into 33 parts. Ayoub (loc. cit.) doubtes whether Euler actually had a proof. However Euler published numerous papers dealing with Bernoulli's lemniscate, e. g. [11].

Gauss was interested in Bernoulli's lemniscate already in the year 1797 (cf. [13]). He wrote in his *Tagebuch* (21th March 1797): *Lemniscata geometrice in quinque partes dividitur.* He proved not only the possibility of dividing lemniscate geometrically into five parts, but also discovered fundamental properties of elliptic functions connected with the lemniscate (loc. cit.). Unfortunately his results had no influence on the development of mathematics, since *Tagebuch* was found many years after Gauss's death. A small trace of his research in this domain can be found in [12]. Gauss states there ([12], Sectio Septima, page 593):

Ceterum principia theoriae, quam exponere aggredimur, multo latius patent, quam hic extenduntur. Namque non solum ad functiones circulares, sed pari successu ad multas alias functiones transscendendes applicari possunt, e. g. ad eas quae ab integrali $\int \frac{\mathrm{d}x}{\sqrt{1-x^4}}$ *pedent [...].*

In other words, foundations of the theory [of a cicrle division] can be extended [...] and can be applied not only to circular functions, but also to many other transcendental functions, e.g. to these which depend upon the integral [...].

An ultimate result of dividing Bernoulli's lemniscate was obtained by Abel. He knew the result already in 1826 (cf. [1], Correspondence) but published it as a corollary from a general theory of elliptic functions ([1] ed. 1838, vol.I, XII. *Recherches sur les fonctions elliptiques*, 141–252).

Abel's research can be stated as

The Second Theorem (Abel [1], ed. 1839, vol. I, p. 229–230) If $N = 2^n p_1 \dots p_s$, where p_k are pairwise different Fermat primes (i.e. every prime has a form $2^m + 1$), then Bernoulli's lemniscate can be geometrically divided into N equal parts and conversly: if the lemniscate can be geometrically divided into N equal parts, then N is the product of some power of two and different Fermat primes.

Abel proved the first part of the theorem. The converse theorem was discovered later (for details see [19], where some additional references are given). Fagnano knew geometric divisions of the lemniscate into two, three and five equal parts. Gauss rediscovered the division into five parts. Kiepert [16] gave an effective construction for division the lemniscate into seventeen equal parts.

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là, par la formule (226) la valeur de $\varphi\left(\frac{n\omega}{\alpha^2+\beta^2}\right)=\varphi\left(\frac{n\omega}{4\nu+1}\right)$, en extrayant des racines carrées.

39.

Un autre cas, où la valeur de $\varphi\left(\frac{m\omega}{n}\right)$ peut être determinée par des racines carrées est celui où n est une puissance de 2, comme nous l'avons vu No. 13. Donc on connait les fonctions:

$$\varphi\left(\frac{m\omega}{2^n}\right), \varphi\left(\frac{m\omega}{1+2^n}\right),$$

où dans la dernière $1+2^n$ est un nombre premier.

Soient maintenant $1+2^n$, $1+2^{n_1}$, $1+2^{n_2}, \dots 1+2^{n_\mu}$ plusieurs nombres premiers, on connait les fonctions:

$$\varphi\left(\frac{m\omega}{2^n}\right), \varphi\left(\frac{m_1\omega}{1+2^{n_1}}\right), \varphi\left(\frac{m_2\omega}{1+2^{n_2}}\right), \dots \varphi\left(\frac{m_\mu\omega}{1+2^{n_\mu}}\right),$$

et de là la fonction:

$$\varphi\left(\frac{m}{2^n}+\frac{m_1}{1+2^{n_1}}+\frac{m_2}{1+2^{n_2}}+\dots+\frac{m_\mu}{1+2^{n_\mu}}\right)\omega$$

$$=\varphi\left(\frac{m'.\omega}{2^n(1+2^{n_1})(1+2^{n_2})\dots(1+2^{n_\mu})}\right),$$

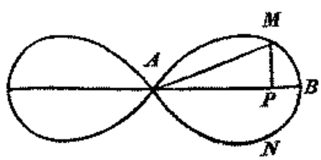
où m' est un nombre entier, qui, à cause des indéterminés $m, m_1, m_2, \dots m_\mu$ peut avoir une valeur quelconque. On peut donc établir le théorème suivant: "La valeur de la fonction $\varphi\left(\frac{m\omega}{n}\right)$ peut être exprimée par *des racines carrées* "toutes les fois que n est un nombre de la forme 2^n ou $1+2^n$, le dernier "nombre étant premier, ou même un produit de plusieurs nombres de ces deux formes."

40.

En appliquant ce qui précède à la lemniscate, on parviendra au théorème énoncé No. 22.

Soit l'arc $AM=\alpha$, la corde $AM=x$ et l'angle $MAP=\theta$, on aura

$$d\alpha=\frac{dx}{\sqrt{(1-x^4)}}.$$



En effet, l'équation polaire de la lemniscate est

$$x=\sqrt{(\cos 2\theta)},$$

A page from [1]

5 Why Gauss's condition for dividing a circle is the same as Abel's condition for a lemniscate?

It is easy to see that in the circle case (applying some elementary Galois theory) the geometric division of the circle into N equal arcs is possible, if and only if, $\phi(N) = 2^n$ for some n, where ϕ is Euler's arithmetical function. It is also a standard number theory exercise that it holds exactly in the case when N is the product of a power of two and different Fermat primes.

In the case of the lemniscate, the situation is much more complicated (cf. e. g. [19]). In this case the result follows from the properties of elliptic functions connected with the lemniscate. But why we obtain the same conditions for an integer N? It seems to be mysterious. Perhaps the circle and the lemniscate have common properties implying the result. Maybe there are other curves with this property.

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Instytut Matematyczny
Uniwersytetu Wrocławskiego
plac Grunwaldzki 2/4
50-384 Wrocław
Poland
e-mail: wieslaw@math.uni.wroc.pl