Chapter 6

Kurzweil and McShane product integrals

The introduction of Lebesgue integration signified a revolution in mathematical analysis: Every Riemann integrable function is also Lebesgue integrable, but the class of functions having Lebesgue integral is considerably larger.

However, there exist functions f which are Newton integrable and

$$(N)\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a),$$

where F is an antiderivative of f, but the Lebesgue integral $(L)\int_a^b f(t)\,\mathrm{d}t$ does not exist. Consider for example the function

$$F(x) = \begin{cases} x^2\sin(1/x^2) & \text{if } x\in(0,1],\\ 0 & \text{if } x=0.\end{cases}$$

This function has a derivative $F'(x) = f(x)$ for every $x\in[0,1]$ and

$$f(x) = \begin{cases} 2x\sin(1/x^2) - (2/x)\cos(1/x^2) & \text{if } x\in(0,1],\\ 0 & \text{if } x=0.\end{cases}$$

The function f is therefore Newton integrable and

$$(N)\int_0^1 f(t)\,\mathrm{d}t = F(1) - F(0).$$

If we denote

$$a_k = \frac{1}{\sqrt{(k+1/2)\pi}},\quad b_k = \frac{1}{\sqrt{k\pi}}$$

for every $k\in\mathbf{N}$, then

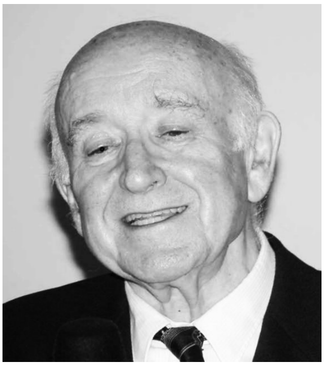
$$\int_{a_k}^{b_k} |f(t)|\,\mathrm{d}t \geq \left|\int_{a_k}^{b_k} f(t)\,\mathrm{d}t\right| = |F(b_k) - F(a_k)| = \frac{1}{(k+1/2)\pi},$$

which implies that

$$\int_0^1 |f(t)|\,\mathrm{d}t \geq \sum_{k=1}^{\infty}\frac{1}{(k+1/2)\pi} = \infty.$$

The Lebesgue integral $(L)\int_0^1 f(t)\,\mathrm{d}t$ therefore does not exist.

Jaroslav Kurzweil (and later independently Ralph Henstock) introduced a new definition of integral which avoids the above mentioned drawback of the Lebesgue integral. The Kurzweil integral (also known as the gauge integral or the Henstock-Kurzweil integral) encompasses the Newton and Lebesgue (and consequently also Riemann) integrals. Another benefit is that the definition of Kurzweil integral is obtained by a gentle modification of Riemann's definition and is considerably simpler than Lebesgue's definition.



Jaroslav Kurzweil[1]

Edward J. McShane[2]

The definition of integral due to E. J. McShane is similar to Kurzweil's definition and in fact represents an equivalent definition of Lebesgue integral.

In this chapter we first summarize the definitions of Kurzweil and McShane integrals; in the second part we turn our attention to product analogies of these integrals. The proofs in this chapter are often omitted and may be found in the original papers (the references are included).

6.1 Kurzweil and McShane integrals

A finite collection of point-interval pairs $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^m$ is called an M-partition of interval $[a, b]$ if

$$a = t_0 < t_1 < \cdots < t_m = b,$$

$$\xi_i \in [a, b], \quad i = 1, \ldots, m.$$

A K-partition is a M-partition which moreover satisfies

$$\xi_i \in [t_{i-1}, t_i], \quad i = 1, \ldots, m.$$

Given a function $\Delta : [a, b] \to (0, \infty)$ (the so-called gauge on $[a, b]$), a partition D is called Δ-fine if

$$[t_{i-1}, t_i] \subset (\xi_i - \Delta(\xi_i), \xi_i + \Delta(\xi_i)), \quad i = 1, \ldots, m.$$

[1] Photo taken by Š. Schwabik

[2] Photo from [McT]

For a given function f on $[a,b]$ and a M-partition D of $[a,b]$ denote

$$S(f,D) = \sum_{i=1}^{m} f(\xi_i)(t_i - t_{i-1}).$$

Definition 6.1.1. Let X be a Banach space. A vector $S_f \in X$ is called the Kurzweil (McShane) integral of function $f : [a,b] \to X$ if for every $\varepsilon > 0$ there is a gauge $\Delta : [a,b] \to (0,\infty)$ such that

$$\|S(f,D) - S_f\| < \varepsilon$$

for every Δ-fine K-partition (M-partition) D of interval $[a,b]$. We define

$$(K)\int_a^b f(t)\,\mathrm{d}t = S_f \quad \text{or} \quad (M)\int_a^b f(t)\,\mathrm{d}t = S_f, \quad \text{respectively.}$$

We state the following theorems without proofs; they can be found (together with more information about the Kurzweil, McShane and Bochner integrals) in the book [SY]; other good sources are [Sch2, RG].

Theorem 6.1.2. Let X be a Banach space. Then every McShane integrable function $f : [a,b] \to X$ is also Kurzweil integrable (but not vice versa) and

$$(K)\int_a^b f(t)\,\mathrm{d}t = (M)\int_a^b f(t)\,\mathrm{d}t.$$

Theorem 6.1.3. Let X be a Banach space. Then every Lebesgue (Bochner) integrable function $f : [a,b] \to X$ is also McShane integrable and

$$(M)\int_a^b f(t)\,\mathrm{d}t = (L)\int_a^b f(t)\,\mathrm{d}t.$$

The converse statement holds if and only if X is a finite-dimensional space.

6.2 Product integrals and their properties

We now proceed to the definitions of Kurzweil and McShane product integrals. The definition of Kurzweil product integral appeared for the first time in the paper [JK]; the authors speak about the Perron product integral and use the notation

$$(PP)\int_a^b (I + A(t)\,\mathrm{d}t).$$

The McShane product integral was studied in [Sch1, SS].

For the sake of simplicity we confine our exposition only to matrix functions $A : [a,b] \to \mathbf{R}^{n\times n}$ instead of working with operator-valued functions $A : [a,b] \to \mathcal{L}(X)$ or even with functions $A : [a,b] \to X$ with values in a Banach algebra X. For an arbitrary M-partition D of $[a,b]$ and a matrix function $A : [a,b] \to \mathbf{R}^{n\times n}$ denote

$$P(A,D) = \prod_{i=m}^{1} (I + A(\xi_i)(t_i - t_{i-1})).$$

Definition 6.2.1. Consider function $A : [a,b] \to \mathbf{R}^{n\times n}$. A matrix $P_A \in \mathbf{R}^{n\times n}$ is called the Kurzweil (McShane) product integral of A if for every $\varepsilon > 0$ there is a gauge $\Delta : [a,b] \to (0,\infty)$ such that

$$\|P(A,D) - P_A\| < \varepsilon$$

for every Δ-fine K-partition (M-partition) D of interval $[a,b]$. We define

$$(K)\prod_a^b (I + A(t)\,\mathrm{d}t) = P_A, \quad \text{or} \quad (M)\prod_a^b (I + A(t)\,\mathrm{d}t) = P_A, \quad \text{respectively.}$$

We also denote

$$KP([a,b],\mathbf{R}^{n\times n}) = \left\{ A : [a,b] \to \mathbf{R}^{n\times n};\ (K)\prod_a^b (I + A(t)\,\mathrm{d}t) \text{ exists} \right\},$$

$$MP([a,b],\mathbf{R}^{n\times n}) = \left\{ A : [a,b] \to \mathbf{R}^{n\times n};\ (M)\prod_a^b (I + A(t)\,\mathrm{d}t) \text{ exists} \right\}.$$

The right product integrals can be introduced using the products

$$P^*(A,D) = \prod_{i=1}^{m} (I + A(\xi_i)(t_i - t_{i-1})),$$

but we limit our discussion to the left integrals.

Example 6.2.2. Assume that the Riemann product integral $(R)\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists, i.e. for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\left\| P(A,D) - \prod_a^b (I + A(t)\,\mathrm{d}t) \right\| < \varepsilon$$

for every partition D of $[a,b]$ which such that $\nu(D) < \delta$. If we put

$$\Delta(x) = \frac{\delta}{2}, \quad x \in [a,b],$$

then every Δ-fine K-partition D of $[a,b]$ satisfies $\nu(D) < \delta$. This means that the Kurzweil product integral of A exists and

$$(K)\prod_a^b (I + A(t)\,\mathrm{d}t) = (R)\prod_a^b (I + A(t)\,\mathrm{d}t).$$

Example 6.2.3. Consider the function

$$f(x) = \begin{cases} -1/x & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

It can be proved (see [SS]) that

$$(M)\prod_0^1 (1 + f(x)\,\mathrm{d}x) = 0.$$

It is worth noting that neither the Riemann integral $(R)\prod_0^1(1 + f(x)\,\mathrm{d}x)$ nor the Lebesgue integral $(L)\prod_0^1(1 + f(x)\,\mathrm{d}x)$ exist; this follows e.g. from Theorem 6.2.10.

Theorem 6.2.4.[1] Consider function $A : [a,b] \to \mathbf{R}^{n\times n}$. Then the following conditions are equivalent:

1) The integral $(K)\prod_a^b(I + A(t)\,\mathrm{d}t)$ exists and is invertible.
2) There exists an invertible matrix P_A such that for every $\varepsilon > 0$ there is a gauge $\Delta : [a,b] \to (0,\infty)$ such that

$$\left\| \prod_{i=m}^{1} e^{A(\xi_i)(t_i - t_{i-1})} - P_A \right\| < \varepsilon$$

whenever $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^m$ is a Δ-fine K-partition of $[a,b]$.

If one of these conditions is fulfilled, then

$$(K)\prod_a^b (I + A(t)\,\mathrm{d}t) = P_A.$$

A similar statement holds also for McShane product integral.

Theorem 6.2.5. Consider function $A : [a,b] \to \mathbf{R}$. The integral $(K)\int_a^b A(t)\,\mathrm{d}t$ exists if and only if the integral $(K)\prod_a^b(1 + A(t)\,\mathrm{d}t)$ exists and is different from zero. In this case the following equality holds:

$$(K)\prod_a^b (1 + A(t)\,\mathrm{d}t) = \exp\left((K)\int_a^b A(t)\,\mathrm{d}t\right).$$

---

[1] [JK], p. 651, and [SS]

A similar statement holds also for McShane product integral.

Proof. Assume that $(K)\int_a^b A(t)\,\mathrm{d}t = S_A$ exists and choose $\varepsilon > 0$. Since the exponential function is continuous at the point S_A, there is a $\delta > 0$ such that

$$\left|e^x - e^{S_A}\right| < \varepsilon, \quad x \in (S_A - \delta, S_A + \delta). \tag{6.2.1}$$

Let $\Delta : [a,b] \to (0,\infty)$ be a gauge such that

$$|S(A,D) - S_A| < \delta$$

for every Δ-fine K-partition of interval $[a,b]$. Each of these partitions satisfies

$$\left|\prod_{i=m}^{1} e^{A(\xi_i)(t_i - t_{i-1})} - e^{S_A}\right| = \left|e^{S(A,D)} - e^{S_A}\right| < \varepsilon$$

and using Theorem 6.2.4 we obtain

$$(K)\prod_a^b (1 + A(t)\,\mathrm{d}t) = \exp\left((K)\int_a^b A(t)\,\mathrm{d}t\right).$$

The reverse implication is proved in a similar way using the equality

$$S(A,D) = \log\left(\prod_{i=m}^{1} e^{A(\xi_i)(t_i - t_{i-1})}\right).$$

□

Remark 6.2.6. The previous theorem no longer holds for matrix functions $A : [a,b] \to \mathbf{R}^{n\times n}$. Jaroslav Kurzweil and Jiří Jarník constructed[1] two functions $A, B : [-1,1] \to \mathbf{R}^{2\times 2}$ such that

$$(K)\int_{-1}^{1} A(t)\,\mathrm{d}t \text{ exists}, \quad (K)\prod_{-1}^{1}(I + A(t)\,\mathrm{d}t) \text{ doesn't exist},$$

$$(K)\int_{-1}^{1} B(t)\,\mathrm{d}t \text{ doesn't exist}, \quad (K)\prod_{-1}^{1}(I + B(t)\,\mathrm{d}t) \text{ exists}.$$

Theorem 6.2.7. Every McShane product integrable function is also Kurzweil product integrable (but not vice versa) and

$$(K)\prod_a^b (I + A(t)\,\mathrm{d}t) = (M)\prod_a^b (I + A(t)\,\mathrm{d}t).$$

[1] [JK], p. 658

Proof. The inclusion $MP \subseteq KP$ follows from the fact that every K-partition is also a M-partition; the equality of the two product integrals is then obvious. We only have to prove that $MP \neq KP$. For an arbitrary function $f : [a,b] \to \mathbf{R}$ denote

$$A_f(t) = I \cdot f(t),$$

where I is the identity matrix of order n (A_f is therefore a matrix-valued function on $[a,b]$). Then evidently

$$P(A,D) = I \cdot P(f,D)$$

for every partition D of $[a,b]$ and A_f is product integrable (in the Kurzweil or McShane sense) if and only if f is product integrable. Theorem 6.1.2 guarantees the existence of a function $f : [a,b] \to \mathbf{R}$ that is Kurzweil integrable, but not McShane integrable; then (according to Theorem 6.2.5) the corresponding function $A_f : [a,b] \to \mathbf{R}^{n\times n}$ is Kurzweil product integrable, but not McShane product integrable. □

Theorem 6.2.8.[1] Consider function $A : [a,b] \to \mathbf{R}^{n\times n}$. Suppose that the integral $(M)\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists and is invertible. Then for every $x \in (a,b)$ the integral

$$Y(x) = (M)\prod_a^x (I + A(t)\,\mathrm{d}t)$$

exists as well and the function Y satisfies

$$Y'(x) = A(x)Y(x)$$

almost everywhere on $[a,b]$.

Remark 6.2.9. In Chapter 3 we have defined the Lebesgue (or Bochner) product integral $(L)\prod_a^b (I + A(t)\,\mathrm{d}t)$; the definition was based on the approximation of A by a sequence of step functions which converge to A in the norm of space $L([a,b],\mathbf{R}^{n\times n})$. The following theorem describes the relationship between McShane and Lebesgue product integrals.

Theorem 6.2.10.[2] Consider function $A : [a,b] \to \mathbf{R}^{n\times n}$. The following conditions are equivalent:

1) A is Lebesgue (Bochner) integrable.
2) The McShane product integral $(M)\prod_a^b (I + A(t)\,\mathrm{d}t)$ exists and is invertible.

If one of these conditions is fulfilled, then

$$(M)\prod_a^b (I + A(t)\,\mathrm{d}t) = (L)\prod_a^b (I + A(t)\,\mathrm{d}t).$$

---

1 [JK], p. 652–656 and [SS]
2 [Sch1], p. 329–334 and [SS]

Remark 6.2.11. We conclude this chapter by comparing the classes of functions which are integrable according to different definitions presented in the previous text.

Let R, L, M and K be the classes of all functions $A : [a,b] \to \mathbf{R}^{n\times n}$ which are integrable in the sense of Riemann, Lebesgue, McShane and Kurzweil, respectively. In a similar way let RP and LP denote the classes of Riemann product integrable and Lebesgue product integrable functions. Instead of working with the classes KP and MP it is more convenient to concentrate on the classes

$$KP^* = \left\{ A : [a,b] \to \mathbf{R}^{n\times n};\ (K)\prod_a^b (I + A(t)\,\mathrm{d}t) \text{ exists and is invertible} \right\},$$

$$MP^* = \left\{ A : [a,b] \to \mathbf{R}^{n\times n};\ (M)\prod_a^b (I + A(t)\,\mathrm{d}t) \text{ exists and is invertible} \right\}.$$

The following diagram shows the inclusions between the above mentioned classes.

$$\begin{array}{ccccccc} R & \subset & L & = & M & \subset & K \\ = & & = & & = & & \neq \\ RP & \subset & LP & = & MP^* & \subset & KP^* \end{array}$$