# PEXIDER'S MATHEMATICAL PUBLICATIONS

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The mathematical works of J. V. Pexider have been published in the period 1898–1909.

Pexider's doctoral dissertation from 1898 bears the title *Theorie variačního počtu dle Weierstrasse* (Theory of the calculus of variations according to Weierstrass). Pexider considers the problem, which he posed himself in the introduction as follows:

*Find the quantities $x$ and $y$ as functions of the variable $t$ in such a way that if the curve given by the equations $x = \varphi(t)$, $y = \psi(t)$ is changed arbitrarily in an arbitrarily small way then the change of the integral*

$$J = \int_{t_0}^{t_1} F(x, y, x', y')\,\mathrm{d}t$$

*is still positive if the value of the integral has to be minimal and still negative if the value of the integral has to be maximal.*

This means that the extrema of the parametrically given functional $J$ have to be determined, i.e. we are looking for a plane curve $x = \varphi(t)$, $y = \psi(t)$, $t \in [t_0, t_1]$, for which the functional $J$ possesses an extremum. This is for a long time a well-known problem of the calculus of variations; in its exact form it was treated especially by Weierstrass. Pexider follows in his dissertation closely and thoroughly Weierstrass's ideas.

In the first part of the work (with the subtitle *Absolute maxima and minima*), necessary conditions for an extremum are given; the second part concerns sufficient conditions, where the central role is played by the Weierstrass criterion presented via the known Weierstrass function $\mathcal{E}$.

From our point of view, Pexider's dissertation is undoubtedly a compilation only; new ideas, different from those of Weierstrass and his students, cannot be found there.

What can be said in this connection is that Pexider is presenting the theme exactly and with understanding. In this sense, the dissertation can be understood as an attempt to present this relatively new approach to the Czech scientific public. However, Pexider's dissertation was never published in printed form.

Pexider's first published paper [P1] contains two examples of calculating the derivative of a function given by a functional equation. From the today's point of view the work is inaccurate in formulations and no essentially new results are given there.

In the paper [P2] Pexider states that if the functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are linearly independent (*no one of them is an algebraically linear function of the remaining ones*) and if the linear combinations

$$u_0 + u_1\varphi_1(x) + u_2\varphi_2(x) + \cdots + u_n\varphi_n(x)$$

and

$$v_0 + v_1\varphi_1(x) + v_2\varphi_2(x) + \cdots + v_n\varphi_n(x)$$

*coincide for infinitely many different values $x_1, x_2, \dots$ numerically less than an arbitrary value $A$*, then

$$u_0 = v_0,\ u_1 = v_1,\ \dots,\ u_n = v_n.$$

It is unclear where the functions are defined. What Pexider is showing can be nowadays, with some exaggeration, presented as the statement: the system of funcions $1, \varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ is linearly independent if and only if it is linearly independent.

The paper [P3] is close to [P1]; the author starts with a functional equation for a certain function $f$, e.g. with the functional equation

$$f(uz) = f(u) + f(z),$$

and an easy-going formal attitude to the "differential" and integration leads him to some formulae for the integral of $f$.

Pexider's works [P4] and [P5] make a whole under the title *A study on functional equations.*

They consist of four parts ([P4] contains parts I and II, parts III and IV are in [P5]):

I. *Determination of functions, satisfying certain conditions,*
II. *Determination of integrals of functions, satisfying certain conditions,*
III. *Abel's theorem on the existence of a function, satisfying a functional equation of a certain form and a parallel theorem,*
IV. *Functional equations for elementary functions.*

In part I Pexider considers the group of functional equations

$$\begin{aligned} f(x+y) &= f(x) + f(y),\\ f(x+y) &= f(x)\cdot f(y),\\ f(xy) &= f(x)\cdot f(y),\\ f(xy) &= f(x) + f(y). \end{aligned}$$

Functions $f$ satisfying these equations have to be determined if it is required that

1. $f$ is a continuous real function,

or that

2. $f$ is a continuous and analytic complex function.

Part II is devoted to representing the integrals of continuous functions, which satisfy one of the equations

$$\begin{aligned} \Phi(x+y) &= f(x) + f(y),\\ \Phi(x+y) &= f(x)\cdot f(y),\\ \Phi(xy) &= f(x)\cdot f(y),\\ \Phi(xy) &= f(x) + f(y). \end{aligned}$$

This part is related to Pexider's paper [P3]. The relations for the integrals are derived by a general approach, which is again not very precise and the presentation is – especially with respect to the assumptions – very short-handed.

In part III, Pexider shortly proves Abel's theorem ([P5], p. 1):

*If the function $\varphi(x, y)$ of two variables $x$ and $y$ is such that $\varphi(z, \varphi(x, y))$ is symmetric in the arguments $x$, $y$ and $z$, then there is a function $f$ that satisfies*

$$f(\varphi(x, y)) = f(x) + f(y).$$

After this the *parallel theorem* ([P5], p. 6) is presented:

*If the function $\varphi(x, y)$ of two variables $x$ and $y$ is such that $\varphi(z, \varphi(x, y))$ is symmetric in the arguments $x$, $y$ and $z$, then there is a function $f$ that satisfies*

$$f(\varphi(x, y)) = f(x) \cdot f(y).$$

Hence the result is that, under the given assumption on the function $\varphi$, the functional equations

$$f(\varphi(x, y)) = f(x) + f(y)$$

and

$$f(\varphi(x, y)) = f(x) \cdot f(y)$$

for the unknown function $f$ always possess a solution.

In part IV, Pexider derives functional equations for elementary functions of both real and complex variables.

Abel's theorem was one of the great topics of analysis in the nineteenth century. Pexider's paper [P6] was devoted exactly to this theme. His knowledge came probably from his studies abroad and undoubtedly also from an extensive study of primary sources. The work [P6] is an extensive treatise of 64 pages accompanied by three pages of references. Pexider published it on his own costs.

The Abelian integral is of the form

$$\int R(u, z)\, \mathrm{d}z,$$

where $R(u, z)$ is a rational function and $u$ in the integral is an algebraic function of the variable $z$ given implicitly by $f(u, z) = 0$, where $f$ is a certain polynomial.

Abel's theorem states, roughly speaking, that the sum of integrals of the given type can be written as the sum of $p$ such integrals, to which some algebraic and logarithmic terms have to be added. The number $p$ depends on the function $f$ only. If we have e.g. $f(u, z) = u^2 - P(z)$ and $P$ is a sixth-order polynomial, then $p = 2$ and

$$\sum_{n=1}^{N} \int_0^{z_n} R(u, z)\, \mathrm{d}z = \int_0^a R(u, z)\, \mathrm{d}z + \int_0^b R(u, z)\, \mathrm{d}z + R_0 + \sum_k c_k \log R_k,$$

where $R_0$ and $R_k$ are rational functions of the variables $z_m$, $a$, $b$, $u_m = u(z_m)$, $u(a)$, $u(b)$ and the bounds $a$ and $b$ in the integrals on the right hand side depend algebraically on $z_m, u_m$.

In the foreword to this publication, Pexider mentioned that in its final part dealing with historical aspects of the problem, he made use of the "thorough treatment *Bericht über die Entwickelung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit* by Brill and Noether from 1894".

Pexider then used his knowledge of sources concerning Abel's Theorem also in the historically oriented paper [P10], where he presented a chronological survey of the work on Abel's Theorem and its various proofs.

In an "enlightening" series of four papers [P9a] published in the journal *Časopis pro pěstování mathematiky a fysiky*, Pexider devoted himself to the *representation of numbers by lengths and vice versa.* This was inspired by Hilbert's lectures on the foundations of geometry from 1902 as well as topics summarized by Hilbert e.g. in the second edition of his book *Grundlagen der Geometrie.*

Geometry is treated axiomatically in Hilbert's style. Pexider shows that *it is possible to fix such a system of axioms, ... and that ... among other new things, based on this system the representation of numbers by lengths and measuring lengths by numbers can be solved satisfactorily in spirit of the traditional requirement, that is the requirement of both-sided unique correspondence between numbers and lengths* ([P9a], p. 16).

In the journal *Časopis pro pěstování mathematiky a fysiky* this essay is somewhat surprising. The theme was extraordinarily actual (together with its philosophical aspects) at this time and it appeared on the pages of the journal in a very recent state. It seems that Pexider was led to this topic, among others, also on the basis of his dissension with Eduard Weyr. The problem of representing a number by a length in geometry and conversely a length by a number was one of his admonitions toward Weyr. Pexider considered necessary to inform the Czech readership on the problem and he made this successfully and in due time. Not all essential mathematical discoveries reached the pages of the journal so quickly.

Pexider's work [P9b] appeared on his own costs and differs from [P9a] in unessential details only.

Pexider's paper [P11] was devoted to the properties of symmetric functions of the variables $x_1, x_2, \dots, x_n$, i.e. of functions $F$ whose values remain unchanged under any permutation of the variables. Besides this simple form of symmetry, referred to by Pexider as "forms of the first sort" (*Formen erster Gattung*), he considers also other ones (e.g. a form of the second sort is generated in such a way that $y_k$, $k = 1, 2, \dots, n$ are functions of the independent arguments $x_{k_1}, x_{k_2}, \dots, x_{k_s}$ and $F$ is a function of $y_1, y_2, \dots, y_n$, which is symmetric in the series of arguments $x_{k_1}, x_{k_2}, \dots, x_{k_s}$ for $k = 1, 2, \dots, n$ and $s \geq 2$).

General and not very complicated reasoning is used by Pexider for special forms of symmetric functions, e.g. for functions of the form

$$F(x_1, x_2, \dots, x_n) = \varphi_1(x_1) + \varphi_2(x_2) + \dots + \varphi_n(x_n),$$

which has to be symmetric. He obtains that

$$F(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} \varphi(x_j) + C,$$

where $\varphi$ is a certain function and $C$ is a constant.

In the work [P12] Pexider returns to an earlier problem concerning functional equations from his older papers [P4] and [P5]. He studies the collection

$$\begin{aligned} f_1(z) + \varphi_1(u) &= \psi_1(z+u), \\ f_2(z) \times \varphi_2(u) &= \psi_2(z+u), \\ f_3(z) \times \varphi_3(u) &= \psi_3(zu), \\ f_4(z) + \varphi_4(u) &= \psi_4(zu) \end{aligned}$$

of functional equations and poses the question as to which continuous functions $f_j, \varphi_j, \psi_j$, $j = 1, 2, 3, 4$ satisfy these equations.

This is a generalization of the case

$$f_j = \varphi_j = \psi_j, \quad j = 1, 2, 3, 4,$$

which was considered by Cauchy.

The above mentioned functional equations are in a certain sense coupled each to the other, therefore let us consider the first one only, i.e. look at the functional equation

$$f(z) + \varphi(u) = \psi(z+u), \tag{1}$$

called in the specialized literature the *Pexider equation.* Let us follow for this case the reasoning of Pexider from [P12].

*Because of the symmetry of the right hand side of the equation (e.g. the symmetry with respect to $z$ and $u$) we have*

$$\varphi(z) = f(z) + C,$$

*where $C$ is a constant, whose value can be determined e.g. by setting $z = 0$. Substituting this for $u$ into the equation (1), we get the relations*

$$f(z) + \varphi(0) = \varphi(z) + f(0) = \psi(z),$$

*which can be used for eliminating two of the functions $f, \varphi, \psi$ in the equation (1); for example, let us eliminate the functions $\varphi$ and $\psi$. Substituting first $f(z) + \varphi(0) = \psi(z)$ into* (1), *we get*

$$f(z) + \varphi(u) = f(z+u) + \varphi(0),$$

*and finally using the equality $f(z) + \varphi(0) = \varphi(z) + f(0)$ for $z = u$, i.e. $f(u) + \varphi(0) - f(0) = \varphi(u)$, we get*

$$f(z) + f(u) + \varphi(0) - f(0) = f(z+u) + \varphi(0),$$

*or in other words*

$$f(z) + f(u) = f(z + u) + f(0).$$

*Introducing a new function $g$ defined by the relation $f(z) = f(0) + g(z)$ into the last equality, we obtain a functional equation for the function $g$ in the well-known Cauchy form*

$$g(z) + g(u) = g(z + u),$$

*and its solution is known to be the linear function $g(z) = az + b$.*

In this way the solution of the complicated looking Pexider equation $f(z) + \varphi(u) = \psi(z + u)$ reduces to the known functional equation for the linear function. The way back from $g$ to the functions $f, \varphi$ and $\psi$ can be found easily by the reader himself.

It is because of this paper [P12] that Pexider's name occurs also in recent mathematics. Indications of similar reasoning (for two unknown functions) can be found also in the paper [P4], which was published in Czech. The work [P12] became widely known and introduced to mathematics equations connected with Pexider's name.

In fact, Pexider was showing that a more generally posed problem does not give essentially new knowledge in comparison with the results obtained by Cauchy.

In the second part of [P12], Pexider returns in a more general way to the problems he considered in his first published article [P1].

This concerns the equation

$$F[f_1(z_1), f_2(z_2), \dots , f_n(z_n)] = 0,$$

where the functions $f_j$ have to be determined, the variables $z_1, z_2, \dots , z_\kappa$ are assumed to be independent and the remaining ones $z_{\kappa+1}, \dots , z_n$ depend on $z_1, z_2, \dots , z_\kappa$. The necessary condition for solvability of the problem (similar to the condition for the validity of the implicit function theorem) is in some cases also sufficient, and this situation is analyzed by Pexider, together with simple examples of evaluation of the derivatives of the function given by the functional equation. He finally discovers the fact that, under some circumstances, from the functional relation for determining functions $f_j$ it is possible to deduce also relations determining the integrals of these functions. The paper [P13] concerns the same topic. This work also represents a return to the older theme from [P3] in a more general and more ordered form.

The paper [P15] is devoted to expressions for the number-theoretic function $\psi(x)$, which determines the number of primes not greater than a positive real number $x$, and for the function

$$\Psi(x) = \psi(x) - \psi(\sqrt{x}).$$

In [P18] Pexider asks for the number of roots of the equation

$$E\left(\frac{n}{x}\right) - E\left(\frac{n}{x+1}\right) = 0,$$

where $E(y)$ is the integer part of the number $y$; he also uses the more traditional notation $[x]$. The number $A(n)$ of the roots of the given equation was expressed by Matyáš Lerch in 1895 using some special functions. Pexider comes to the fact that this number can be given by

$$A(n) = E(n) - E(\sqrt{n}) - E\left(\frac{n}{E(\sqrt{n})+1}\right),$$

and that the roots in consideration are the numbers

$$x = E\left(\frac{n}{\alpha}\right) + 1,\ E\left(\frac{n}{\alpha}\right) + 2,\ \dots,\ E\left(\frac{n}{\alpha-1}\right) - 1,$$

where $\alpha = 2, 3, \dots, E(\sqrt{n}) + 1$ and $\alpha$ fulfills the condition

$$E\left(\frac{n}{\alpha-1}\right) - E\left(\frac{n}{\alpha}\right) > 1.$$

Assume that $n \in \mathbb{N}$ and $r_1, r_2, \dots, r_\kappa, \dots$ are the smallest positive remainders of the powers $1^n, 2^n, \dots, \kappa^n, \dots$ by the integer modulus $m$. Then the remainders $r_1, r_2, \dots, r_m$ of the congruence

$$(\kappa + qm)^n \equiv \kappa^n \pmod{m}$$

$(r_{\kappa+qm} = r_\kappa)$, where $q$ is an integer, form a family of $m$ integers, which are periodically repeated in the infinite series of $n$-th power remainders $r_1, r_2, \dots, r_\kappa, \dots$

In the paper [P19], Pexider examined the relations among the elements of such a period of power remainders, depending on whether the value of the exponent $n$ is odd or even. He studied the case of quadratic ($n = 2$), cubic ($n = 3$) and biquadratic ($n = 4$) remainders and examined their sums.

The papers [P14], [P16] and [P17] are devoted to actuarial mathematics.

Following the time line of Pexider's mathematical articles which appeared in printed form, an indubitable development of their content and form can be observed. As the time passes, he returns to his themes and the exposition gets more exact and comprehensive. For example, the papers [P4] and [P12] are related, [P12] was published 4 years after [P4] and this time made the exposition more transparent and exact. There is also a qualitative difference between the work published in Czech and in foreign languages. The second ones are surely more ordered and exact, maybe because they have been published later. It is maybe worth to mention that till 1903, Pexider was publishing in Czech; after this time, all his work is written in German or French.

The works in which Pexider tried to inform the Czech mathematical community about new or less common things concerning mathematics are interesting and remarkable. His paper [P6] on Abel's Theorem or the paper [P9a] mentioned above in more detail fall into this category. Originality of mathematical results and their exposition was not the aim there, the papers should be considered as informative surveys.

Looking at the mathematical publications of J. Pexider, another circumstance has to be mentioned. Pexider submitted the papers [P1]–[P6] as his habilitation thesis. The referee for the habilitation application was professor Eduard Weyr, who on the basis of his examination of the treatise on March 3, 1902 presented *to the respectable faculty the proposition to refuse the supplication of dr. J. Pexider for permitting his habilitation.*

Eduard Weyr concluded correctly that the presented papers lack a good quality, being mostly compilations. He analyzed the things elaborately and documented his opinion in a five pages long report. The mentioned works are really of the sort as Weyr characterized them. Even from our recent point of view, nothing can be changed on Weyr's condemnation. Nothing is changed even in spite of the fact that in the work [P4], functional equations are dealt with which later introduced Pexider's name into recent mathematics. The mathematical contribution of Pexider was really negligible.

The situation with his supplication for habilitation was for Pexider a challenge for a combat. Against the decision of the faculty of the Prague university, which was approved by the Ministry of Cult and Education, Pexider appealed in June 1902 by insinuating Weyr of incompetence. After this, he attacked Eduard Weyr from an other point in connection with Weyr's book on differential calculus. This particular controversy was long lasting, full of emotion and proceeded also in printed form (see [P7], [P8]). The quarrel was described in detail and assessed in the book [Be].

The dispute over Pexider's habilitation finished in the year 1907. The work of Pexider was again judged by professor Karel Petr, who considered also the original opinion of the late Eduard Weyr. The conclusion of the committee (the members have been Koláček, Strouhal, Sobotka, Petr and Raýman) was very brusk and related also to other things than mathematics:

> ... *the committee cannot conciliate its opinion on scientific and moral principles of a normal university teacher with the way in which dr. Pexider bear himself for long years in relation to mathematics and also to the scientific community.*

Further, the committee proposed

> ... *that the dean of the faculty, when communicating to dr. Pexider ... the settlement of his two applications ... should at once attach a remark saying that an eventual new request of Pexider for habilitation at our faculty ... will be rejected immediately.*

This was, after six years, the end of the story concerning Pexider's habilitation.

## References

[Be] J. Bečvář et al., *Eduard Weyr 1852–1903*, Dějiny matematiky 2, Prometheus, Prague, 1995.