III. Generalized linear differential equations

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III. *Generalized linear differential equations*

1. The generalized linear differential equation and its basic properties

We assume that $\boldsymbol{A}: [0, 1] \to L(R_n)$ is an $n \times n$-matrix valued function such that $\operatorname{var}_0^1 \boldsymbol{A} < \infty$ and $\boldsymbol{g} \in BV_n[0, 1] = BV_n$.

The generalized linear differential equation will be denoted by the symbol

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x} + \mathrm{d}\boldsymbol{g} \tag{1,1}$$

which is interpreted by the following definition of a solution.

1.1. Definition. Let $[a, b] \subset [0, 1]$, $a < b$; a function $\boldsymbol{x}: [a, b] \to R_n$ is said to be a solution of the *generalized linear differential equation* (1,1) on the interval $[a, b]$ if for any $t, t_0 \in [a, b]$ the equality

$$\boldsymbol{x}(t) = \boldsymbol{x}(t_0) + \int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) + \boldsymbol{g}(t) - \boldsymbol{g}(t_0) \tag{1,2}$$

is satisfied.

In the original papers of J. Kurzweil (cf. [1], [2]) on generalized differential equations and in other papers in this field the notation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \mathrm{D}[\boldsymbol{A}(t)\,\boldsymbol{x} + \boldsymbol{g}(t)]$$

was used for the generalized linear differential equation.

It is evident that the generalized linear differential equation can be given on an arbitrary interval $[a, b] \subset R$ instead of $[0, 1]$.

If $\boldsymbol{x}_0 \in R_n$ and $t_0 \in [a, b] \subset [0, 1]$ are fixed and $\boldsymbol{x}: [a, b] \to R_n$ is a solution of (1,1) on $[a, b]$ such that $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, then $\boldsymbol{x}$ is called the solution of the initial value (Cauchy) problem

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x} + \mathrm{d}\boldsymbol{g}\,, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1,3}$$

on $[a, b]$.

1.2. Remark. If $\boldsymbol{B}$: $[0,1] \to L(R_n)$ is an $n \times n$-matrix valued function, continuous on $[0,1]$ with respect to the norm of a matrix given in I.1.1 and $\boldsymbol{h}$: $[0,1] \to R_n$ is continuous on $[0,1]$, then the initial value problem for the linear ordinary differential equation

$$(1,4) \qquad \boldsymbol{x}' = \boldsymbol{B}(t)\,\boldsymbol{x} + \boldsymbol{h}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

is equivalent to the integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{B}(s)\,\boldsymbol{x}(s)\,\mathrm{d}s + \int_{t_0}^{t} \boldsymbol{h}(s)\,\mathrm{d}s, \qquad t \in [0,1].$$

If we denote $\boldsymbol{A}(t) = \int_0^t \boldsymbol{B}(r)\,\mathrm{d}r$, $\boldsymbol{g}(t) = \int_0^t \boldsymbol{h}(r)\,\mathrm{d}r$ for $t \in [0,1]$, then this equation can be rewritten into the equivalent Stieltjes form

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) + \boldsymbol{g}(t) - \boldsymbol{g}(t_0), \qquad t \in [0,1].$$

The functions $\boldsymbol{A}$: $[0,1] \to L(R_n)$, $\boldsymbol{g}$: $[0,1] \to R_n$ are absolutely continuous and therefore also of bounded variation. In this way the initial value problem (1,4) has become the initial value problem of the form (1,3) with $\boldsymbol{A}$, $\boldsymbol{g}$ defined above and both problems are equivalent. Essentially the same reasoning yields the equivalence of the problem (1,4) to an equivalent Stieltjes integral equation when $\boldsymbol{B}$: $[0,1] \to L(R_n)$, $\boldsymbol{h}$: $[0,1] \to R_n$ are assumed to be Lebesgue integrable and if we look for Carathéodory solutions of (1,4).

1.3. Theorem. *Assume that* $\boldsymbol{A}$: $[0,1] \to L(R_n)$ *is of bounded variation on* $[0,1]$, $\boldsymbol{g} \in BV_n$. *Let* $\boldsymbol{x}$: $[a,b] \to R_n$ *be a solution of the generalized linear differential equation* (1,1) *on the interval* $[a,b] \subset [0,1]$. *Then* $\boldsymbol{x}$ *is of bounded variation on* $[a,b]$.

Proof. By the definition 1.1 of a solution of (1,1) the integral $\int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s)$ exists for every $t, t_0 \in [a,b]$. Hence by I.4.12 the limit $\lim_{t \to t_0+} \int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s)$ exists for $t_0 \in [a,b)$ and $\lim_{t \to t_0-} \int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s)$ exists for $t_0 \in (a,b]$. Hence by (1,2) the solution $\boldsymbol{x}(t)$ of (1,1) possesses onesided limits at every point $t_0 \in [a,b]$ and for every point $t_0 \in [a,b]$ there exists $\delta > 0$ and a constant M such that $|\boldsymbol{x}(t)| \le M$ for $t \in (t_0 - \delta, t_0 + \delta) \cap [a,b]$. By the Heine-Borel Covering Theorem there exists a finite system of intervals of the type $(t_0 - \delta, t_0 + \delta)$ covering the compact interval $[a,b]$. Hence there exists a constant K such that $|\boldsymbol{x}(t)| \le K$ for every $t \in [a,b]$. If now $a = t_0 < t_1 < \ldots < t_k = b$ is an arbitrary subdivision of $[a,b]$, we have by I.4.27

$$|\boldsymbol{x}(t_i) - \boldsymbol{x}(t_{i-1})| \le \left| \int_{t_{i-1}}^{t_i} \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) \right| + |\boldsymbol{g}(t_i) - \boldsymbol{g}(t_{i-1})|$$

$$\le K \operatorname{var}_{t_{i-1}}^{t_i} \boldsymbol{A} + |\boldsymbol{g}(t_i) - \boldsymbol{g}(t_{i-1})|$$

for every $i = 1, \ldots, k$. Hence

$$\sum_{i=1}^{k} |\boldsymbol{x}(t_i) - \boldsymbol{x}(t_{i-1})| \leq K \operatorname{var}_a^b \boldsymbol{A} + \operatorname{var}_a^b \boldsymbol{g}$$

and $\operatorname{var}_a^b \boldsymbol{x} < \infty$ since the subdivision was arbitrary.

Throughout this chapter we use the notations $\Delta^+ \boldsymbol{f}(t) = \boldsymbol{f}(t+) - \boldsymbol{f}(t)$, $\Delta^- \boldsymbol{f}(t) = \boldsymbol{f}(t) - \boldsymbol{f}(t-)$ for any function possessing the onesided limits $\boldsymbol{f}(t+) = \lim_{r \to t+} \boldsymbol{f}(r)$, $\boldsymbol{f}(t-) = \lim_{r \to t-} \boldsymbol{f}(r)$. This applies evidently also to matrix valued functions.

Since by definition the initial value problem (1,3) is equivalent to the Volterra-Stieltjes integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \mathrm{d}[\boldsymbol{A}(s)]\, \boldsymbol{x}(s) + \boldsymbol{g}(t) - \boldsymbol{g}(t_0), \qquad t \in [0,1], \tag{1,5}$$

the following theorem is a direct corollary of II.3.12.

1.4. Theorem. *Assume that* $\boldsymbol{A}: [0,1] \to L(R_n)$ *satisfies* $\operatorname{var}_0^1 \boldsymbol{A} < \infty$. *If* $t_0 \in [0,1)$, *then the initial value problem* (1,3) *possesses for any* $\boldsymbol{g} \in BV_n$, $\boldsymbol{x}_0 \in R_n$ *a unique solution* $\boldsymbol{x}(t)$ *defined on* $[t_0, 1]$ *if and only if the matrix* $\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)$ *is regular for any* $t \in (t_0, 1]$. *If* $t_0 \in (0,1]$, *then the initial value problem* (1,3) *possesses for any* $\boldsymbol{g} \in BV_n$, $\boldsymbol{x}_0 \in R_n$ *a unique solution* $\boldsymbol{x}(t)$ *defined on* $[0, t_0]$ *if and only if the matrix* $\boldsymbol{I} + \Delta^+ \boldsymbol{A}(t)$ *is regular for any* $t \in [0, t_0)$. *If* $t_0 \in [0,1]$, *then the problem* (1,3) *has for any* $\boldsymbol{g} \in BV_n$, $\boldsymbol{x}_0 \in R_n$ *a unique solution* $\boldsymbol{x}(t)$ *defined on* $[0,1]$ *if and only if* $\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)$ *is regular for any* $t \in (t_0, 1]$ *and* $\boldsymbol{I} + \Delta^+ \boldsymbol{A}(t)$ *is regular for any* $t \in [0, t_0)$.

1.5. Remark. Let us mention that by 1.3 the solutions of the problem (1,3) whose existence and uniqueness is stated in Theorem 1.4 are of bounded variation on their intervals of definition. Further, if in the last part of the theorem we have $t_0 = 0$, then the regularity of $\boldsymbol{I} + \Delta^+ \boldsymbol{A}(0)$ is not required. Similarly for $t_0 = 1$ and for the regularity of $\boldsymbol{I} - \Delta^- \boldsymbol{A}(1)$.

Let us mention also that Theorem 1.4 gives the fundamental existence and unicity result for BV_n-solutions of the initial value problem (1,3).

Let us note that if $\boldsymbol{A}: [0,1] \to L(R_n)$ is of bounded variation in $[0,1]$, then there is a finite set of points t in $[0,1]$ such that the matrix $\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)$ is singular and similarly for the matrix $\boldsymbol{I} + \Delta^+ \boldsymbol{A}(t)$. In fact, since $\operatorname{var}_0^1 \boldsymbol{A} < \infty$ the series $\sum_{t \in (a,b]} \Delta^- \boldsymbol{A}(t)$ converges. Hence there is a finite set of points $t \in [0,1]$ such that $|\Delta^- \boldsymbol{A}(t)| \geq \frac{1}{2}$. For all the remaining points in $[0,1]$ we have $|\Delta^- \boldsymbol{A}(t)| < \frac{1}{2}$, and consequently $[\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)]^{-1} = \sum_{k=0}^{\infty} (\Delta^- \boldsymbol{A}(t))^k$ exists since the series on the right-hand side converges at these points. For the matrix $\boldsymbol{I} + \Delta^+ \boldsymbol{A}(t)$ this fact can be shown analogously.

1.6. Proposition. *Assume that* $\mathbf{A}\colon [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$, $\mathbf{g} \in BV_n$. *Let* $\mathbf{x}$ *be a solution of the equation* (1,1) *on some interval* $[a,b] \subset [0,1]$, $a < b$. *Then all the onesided limits* $\mathbf{x}(a+)$, $\mathbf{x}(t+)$, $\mathbf{x}(t-)$, $\mathbf{x}(b-)$, $t \in (a,b)$ *exist and*

$$\mathbf{x}(t+) = [\mathbf{I} + \Delta^+\mathbf{A}(t)]\,\mathbf{x}(t) + \Delta^+\mathbf{g}(t) \qquad \text{for all} \quad t \in [a,b), \tag{1,6}$$
$$\mathbf{x}(t-) = [\mathbf{I} - \Delta^-\mathbf{A}(t)]\,\mathbf{x}(t) - \Delta^-\mathbf{g}(t) \qquad \text{for all} \quad t \in (a,b]$$

holds.

Proof. Let $t \in [a,b)$. By the definition of the solution $\mathbf{x}\colon [a,b] \to R_n$ we have

$$\mathbf{x}(t+\delta) = \mathbf{x}(t) + \int_t^{t+\delta} \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s) + \mathbf{g}(t+\delta) - \mathbf{g}(t)$$

for any $\delta > 0$. For $\delta \to 0+$ we obtain by I.4.13 the equality

$$\begin{aligned}\mathbf{x}(t+) &= \mathbf{x}(t) + (\mathbf{A}(t+) - \mathbf{A}(t))\,\mathbf{x}(t) + \mathbf{g}(t+) - \mathbf{g}(t)\\ &= \mathbf{x}(t) + \Delta^+\mathbf{A}(t)\,\mathbf{x}(t) + \Delta^+\mathbf{g}(t)\end{aligned}$$

where the limit on the right-hand side evidently exists. The second equality in (1,6) can be proved similarly.

1.7. Theorem. *Assume that* $\mathbf{A}\colon [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$, $t_0 \in [0,1]$ *and that* $\mathbf{I} + \Delta^+\mathbf{A}(t)$ *is a regular matrix for all* $t \in [0,t_0)$ *and* $\mathbf{I} - \Delta^-\mathbf{A}(t)$ *is a regular matrix for all* $t \in (t_0, 1]$. *Then there exists a constant* C *such that for any solution* $\mathbf{x}(t)$ *of the initial value problem* (1,3) *with* $\mathbf{g} \in BV_n$ *we have*

$$|\mathbf{x}(t)| \le C(|\mathbf{x}_0| + \operatorname{var}_{t_0}^1 \mathbf{g}) \exp(C \operatorname{var}_{t_0}^t \mathbf{A}) \qquad \text{for} \quad t \in [t_0, 1] \tag{1,7}$$

and

$$|\mathbf{x}(t)| \le C(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g}) \exp(C \operatorname{var}_t^{t_0} \mathbf{A}) \qquad \text{for} \quad t \in [0, t_0]. \tag{1,8}$$

Proof. We consider only the case $t < t_0$ and prove (1,8). The proof of (1,7) can be given in an analogous way. Let us set $\mathbf{B}(t) = \mathbf{A}(t+)$ for $t \in [0,t_0)$ and $\mathbf{B}(t_0) = \mathbf{A}(t_0)$. Hence $\mathbf{B}(t) - \mathbf{A}(t) = \Delta^+\mathbf{A}(t)$ for $t \in [0,t_0)$, $\mathbf{B}(t_0) - \mathbf{A}(t_0) = \mathbf{0}$, i.e. $\mathbf{B}(t) - \mathbf{A}(t) = \mathbf{0}$ for all $t \in [0,t_0]$ except for an at most countable set of points in $[0,t_0)$ and evidently $\operatorname{var}_0^{t_0}(\mathbf{B} - \mathbf{A}) < \infty$. Hence for every $\mathbf{x} \in BV_n$ and $t \in [0,t_0)$ we have by I.4.23

$$\int_t^{t_0} \mathrm{d}[\mathbf{B}(s) - \mathbf{A}(s)]\,\mathbf{x}(s) = -\Delta^+\mathbf{A}(t)\,\mathbf{x}(t)$$

and by the definition we obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_t^{t_0} \mathrm{d}[\mathbf{B}(s)]\,\mathbf{x}(s) - \Delta^+\mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{g}(t) - \mathbf{g}(t_0), \qquad t \in [0,t_0)$$

i.e.

$$\mathbf{x}(t) = [\mathbf{I} + \Delta^+\mathbf{A}(t)]^{-1}\left(\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) + \int_t^{t_0} \mathrm{d}[\mathbf{B}(s)]\,\mathbf{x}(s)\right), \qquad t \in [0,t_0). \tag{1,9}$$

Let us mention that for all $t \in [0, t_0)$ we have

$$(1{,}10) \qquad |[\mathbf{I} + \Delta^+ \mathbf{A}(t)]^{-1}| \le C, \qquad C = \text{const.}$$

This inequality can be proved using the equality $[\mathbf{I} + \Delta^+ \mathbf{A}(t)]^{-1} = \sum_{i=0}^{\infty} (-1)^i (\Delta^+ \mathbf{A}(t))^i$ which holds whenever $|\Delta^+ \mathbf{A}(t)| < 1$. Hence

$$|[\mathbf{I} + \Delta^+ \mathbf{A}(t)]^{-1}| \le \sum_{i=0}^{\infty} |\Delta^+ \mathbf{A}(t)|^i = \frac{1}{1 - |\Delta^+ \mathbf{A}(t)|} < 2$$

provided $|\Delta^+ \mathbf{A}(t)| < \frac{1}{2}$, i.e. for all $t \in [0, t_0)$ except for a finite set of points in $[0, t_0)$. The estimate (1,10) is in this manner obvious. Using (1,10) we obtain by (1,9) the inequality

$$|\mathbf{x}(t)| \le C\left(|\mathbf{x}_0| + |\mathbf{g}(t) - \mathbf{g}(t_0)| + \left|\int_{t_0}^{t} \mathrm{d}[\mathbf{B}(s)]\, \mathbf{x}(s)\right|\right)$$

$t \in [0, t_0]$. This inequality together with I.4.27 yields

$$(1{,}11) \qquad |\mathbf{x}(t)| \le C\left(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g} + \int_t^{t_0} |\mathbf{x}(s)|\, \mathrm{d} \operatorname{var}_0^s \mathbf{B}\right)$$

$$= C(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g}) + C \int_t^{t_0} |\mathbf{x}(s)|\, \mathrm{d}\mathbf{h}(s)$$

where $\mathbf{h}(s) = \operatorname{var}_0^s \mathbf{B}$ is defined on $[0, t_0]$ and is evidently continuous from the right-hand side on $[0, t_0)$ since $\mathbf{B}$ has this property by definition. Using I.4.30 for the inequality (1,11) we obtain

$$|\mathbf{x}(t)| \le C(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g}) \exp(C(h(t_0) - h(t)))$$
$$\le C(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g}) \exp(C(\operatorname{var}_0^{t_0} \mathbf{B} - \operatorname{var}_0^t \mathbf{B}))$$
$$= C(|\mathbf{x}_0| + \operatorname{var}_0^{t_0} \mathbf{g}) \exp(C \operatorname{var}_t^{t_0} \mathbf{B})$$

and this implies (1,8) since $\operatorname{var}_t^{t_0} \mathbf{B} \le \operatorname{var}_t^{t_0} \mathbf{A}$.

Remark. A slight modification in the proof leads to a refinement of the estimates (1,7), (1,8). It can be proved that

$$|\mathbf{x}(t)| \le C(|\mathbf{x}_0| + \operatorname{var}_{t_0}^t \mathbf{g}) \exp(C \operatorname{var}_{t_0}^t \mathbf{A}) \qquad \text{for} \quad t \in [t_0, 1]$$

and

$$|\mathbf{x}(t)| \le C(|\mathbf{x}_0| + \operatorname{var}_t^{t_0} \mathbf{g}) \exp(C \operatorname{var}_t^{t_0} \mathbf{A}) \qquad \text{for} \quad t \in [0, t_0]$$

holds.

1.8. Corollary. *Let* $\mathbf{A}: [0, 1] \to L(R_n)$ *fulfil the assumptions given in* 1.7 *for some* $t_0 \in [0, 1]$, $\mathbf{g}, \tilde{\mathbf{g}} \in BV_n$, $\mathbf{x}_0, \tilde{\mathbf{x}}_0 \in R_n$. *Then if* $\mathbf{x} \in BV_n$ *is a solution of* (1,3) *and* $\tilde{\mathbf{x}} \in BV_n$ *is a solution of*

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\, \mathbf{x} + \mathrm{d}\tilde{\mathbf{g}}, \qquad \mathbf{x}(t_0) = \tilde{\mathbf{x}}_0,$$

we have

$$(1,12)\quad |\mathbf{x}(t) - \tilde{\mathbf{x}}(t)| \le C(|\mathbf{x}_0 - \tilde{\mathbf{x}}_0| + \operatorname{var}_0^{t_0}(\mathbf{g} - \tilde{\mathbf{g}})) \exp(C \operatorname{var}_t^{t_0} \mathbf{A}) \qquad \text{for} \quad t \in [0, t_0]$$

$$|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)| \le C(|\mathbf{x}_0 - \tilde{\mathbf{x}}_0| + \operatorname{var}_{t_0}^1(\mathbf{g} - \tilde{\mathbf{g}})) \exp(C \operatorname{var}_{t_0}^t \mathbf{A}) \qquad \text{for} \quad t \in [t_0, 1],$$

where $C \ge 1$ is a constant. Hence

$$(1,13)\qquad |\mathbf{x}(t) - \tilde{\mathbf{x}}(t)| \le K(|\mathbf{x}_0 - \tilde{\mathbf{x}}_0| + \operatorname{var}_0^1(\mathbf{g} - \tilde{\mathbf{g}}))$$

for all $t \in [0, 1]$ where $K = C \exp(C \operatorname{var}_0^1 \mathbf{A})$.

1.9. Remark. The inequality (1,13) yields evidently $\mathbf{x}(t) = \tilde{\mathbf{x}}(t)$ for all $t \in [0, 1]$ whenever $\mathbf{x}_0 = \tilde{\mathbf{x}}_0$ and $\operatorname{var}_0^1(\mathbf{g} - \tilde{\mathbf{g}}) = 0$. In this way the unicity of solutions of the initial value problem (1,3) is confirmed.

1.10. Theorem. *Assume that $t_0 \in [0, 1]$ is fixed. Let $\mathbf{A}$: $[0, 1] \to L(R_n)$ be such that $\operatorname{var}_0^1 \mathbf{A} < \infty$, $\mathbf{I} - \Delta^- \mathbf{A}(t)$ is a regular matrix for $t \in (t_0, 1]$ and $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ is a regular matrix for $t \in [0, t_0)$. Then the set of all solutions $\mathbf{x}$: $[0, 1] \to R_n$ of the homogeneous generalized differential equation*

$$(1,14)\qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x}$$

with the initial value given at the point $t_0 \in [0, 1]$ is an n-dimensional subspace in BV_n.

Proof. The linearity of the set of solutions is evident from the linearity of the integral. Let us set $\mathbf{e}^{(k)} = (0, \dots, 0, 1, 0, \dots, 0)^* \in R_n$, $k = 1, \dots, n$ (the value 1 is in the k-th coordinate of $\mathbf{e}^{(k)} \in R_n$) and let $\boldsymbol{\varphi}^{(k)}$: $[0, 1] \to R_n$ be the unique solution of (1,14) such that $\boldsymbol{\varphi}^{(k)}(t_0) = \mathbf{e}^{(k)}$, $k = 1, \dots, n$ (they exist by 1.4). The unicity result from 1.4 yields that $\sum_{k=1}^{n} c_k \boldsymbol{\varphi}^{(k)}(t) = \mathbf{0}$, $c_k \in R$ if and only if $c_k = 0$, $k = 1, \dots, n$. If $\mathbf{x}$: $[0, 1] \to R_n$ is an arbitrary solution of (1,14), then clearly

$$\mathbf{x}(t) = \sum_{k=1}^{n} \mathbf{x}_k(t_0)\, \boldsymbol{\varphi}^{(k)}(t)$$

for all $t \in [0, 1]$, i.e. $\mathbf{x}$ is a linear combination of the linearly independent solutions $\boldsymbol{\varphi}^{(k)}$, $k = 1, \dots, n$ and this is our result.

1.11. Example. We give an example of a generalized linear differential equation which demonstrates the role of the assumptions concerning the regularity of the matrices $\mathbf{I} + \Delta^+ \mathbf{A}(t)$, $\mathbf{I} - \Delta^- \mathbf{A}(t)$ in 1.4. Let us set

$$\mathbf{A}(t) = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix}$$

for $0 \le t < \frac{1}{2}$, $\frac{1}{2} \le t \le 1$ respectively; for this 2×2-matrix $\boldsymbol{A}: [0,1] \to L(R_2)$ we have evidently $\Delta^+ \boldsymbol{A}(t) = \boldsymbol{0}$ for all $t \in [0,1)$, $\Delta^- \boldsymbol{A}(t) = \boldsymbol{0}$ for all $t \in (0,1]$, $t \neq \frac{1}{2}$ and

$$\Delta^- \boldsymbol{A}(\tfrac{1}{2}) = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix}.$$

Hence

$$\boldsymbol{I} - \Delta^- \boldsymbol{A}(\tfrac{1}{2}) = \begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix}$$

is not regular. We consider the initial value problem

$$\text{(1,15)} \qquad \mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

where $\boldsymbol{x}_0 = (c_1, c_2)^* \in R_2$. For a solution $\boldsymbol{x}(t)$ of this problem we have

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{x}_0 = (c_1, c_2)^* \qquad \text{if} \quad t \in [0, \tfrac{1}{2}).$$

Further, by 1.6 we obtain $\boldsymbol{x}(\frac{1}{2}-) = [\boldsymbol{I} - \Delta^- \boldsymbol{A}(\frac{1}{2})]\,\boldsymbol{x}(\frac{1}{2})$, i.e. $(c_1, c_2)^* = [\boldsymbol{I} - \Delta^- \boldsymbol{A}(\frac{1}{2})]\,\boldsymbol{x}(\frac{1}{2}) = (x_1(\frac{1}{2}), 0)^*$. This equality is contradictory for $c_2 \neq 0$. Hence the above problem (1,15) cannot have a solution on $[0, \frac{1}{2}]$ when $\boldsymbol{x}_0 = (c_1, c_2)^* \in R_2$ with $c_2 \neq 0$.

Let us now assume that $\boldsymbol{x}_0 = (c_1, 0)^* \in R_2$. Then we have for $t \ge \frac{1}{2}$

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{x}(\tfrac{1}{2}) + \int_{1/2}^t \mathrm{d}[\boldsymbol{A}(s)]\,\boldsymbol{x}(s) = \boldsymbol{x}(\tfrac{1}{2}).$$

By 1.6 necessarily

$$[\boldsymbol{I} - \Delta^- \boldsymbol{A}(\tfrac{1}{2})]\,\boldsymbol{x}(\tfrac{1}{2}) = \begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix} \boldsymbol{x}(\tfrac{1}{2}) = \boldsymbol{x}(\tfrac{1}{2}-) = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}.$$

Hence $\boldsymbol{x}(\frac{1}{2}) = (c_1, d)^*$, where $d \in R$ is arbitrary, satisfies this relation. It is easy to show that any vector valued function $\boldsymbol{x}: [0,1] \to R_2$ defined by $\boldsymbol{x}(t) = (c_1, 0)^*$ for $0 \le t < \frac{1}{2}$, $\boldsymbol{x}(t) = (c_1, d)^*$ for $\frac{1}{2} \le t \le 1$, satisfies our equation.

Summarizing these facts we have the following. If $\boldsymbol{x}(0) = (c_1, c_2)^*$ and $c_2 \neq 0$, then a solution of (1,15) does not exist on the whole interval $[0,1]$. If $\boldsymbol{x}(0) = (c_1, 0)^*$, then the equation (1,15) has solutions on the whole interval $[0,1]$ but the uniqueness is violated.

If we consider the initial value problem $\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x}$, $\boldsymbol{x}(\frac{1}{2}) = (c_1, c_2)^*$ for the given matrix $\boldsymbol{A}(t)$, then it is easy to show that this problem possesses the unique solution $\boldsymbol{x}(t) = (c_1, 0)^*$ if $t \in [0, \frac{1}{2})$, $\boldsymbol{x}(t) = (c_1, c_2)^*$ if $t \in [\frac{1}{2}, 1]$. Hence the singularity of the matrix $\boldsymbol{I} - \Delta^- \boldsymbol{A}(t)$ for $t = \frac{1}{2}$ is irrelevant for the existence and uniqueness of solutions to the initial value problem mentioned above.

2. Variation of constants formula. The fundamental matrix

In this section we continue the consideration of the initial value problem

$$\text{(2,1)} \qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{g}\,, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

with $\mathbf{A}\colon [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$, $\mathbf{g} \in BV_n[0,1] = BV_n$, $t_0 \in [0,1]$, $\mathbf{x}_0 \in R_n$.

2.1. Proposition. *Assume that* $\mathbf{A}\colon [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$, $t_0 \in [0,1]$ *is fixed, the matrix* $\mathbf{I} - \Delta^- \mathbf{A}(t)$ *is regular for all* $t \in (t_0, 1]$ *and the matrix* $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ *is regular for all* $t \in [0, t_0)$.

Then the matrix equation

$$\text{(2,2)} \qquad \mathbf{X}(t) = \tilde{\mathbf{X}} + \int_{t_1}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r)$$

has for every $\tilde{\mathbf{X}} \in L(R_n)$ *a unique solution* $\mathbf{X}(t) \in L(R_n)$ *on* $[t_1, 1]$ *provided* $t_0 \le t_1$ *and on* $[0, t_1]$ *provided* $t_1 \le t_0$.

Proof. Let us denote by $\mathbf{B}_k$ the k-th column of a matrix $\mathbf{B} \in L(R_n)$. For the k-th column of the matrix equation (2,2) we have

$$\text{(2,3)} \qquad \mathbf{X}_k(t) = \tilde{\mathbf{X}}_k + \int_{t_1}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}_k(r)\,, \qquad k = 1, \ldots, n\,.$$

If $t_0 \le t_1$, then for every $t \in (t_1, 1]$ the matrix $\mathbf{I} - \Delta^- \mathbf{A}(t)$ is regular. Hence by 1.4 the equation (2,3) for $\mathbf{X}_k(t)$ has a unique solution on $[t_1, 1]$ for every $k = 1, \ldots, n$ and this implies the existence and unicity of an $n \times n$-matrix $\mathbf{X}(t)\colon [t_1, 1] \to L(R_n)$ satisfying (2,2). The case when $t_1 \le t_0$ can be treated similarly.

2.2. Theorem. *If the assumptions of* 2.1 *are satisfied, then there exists a unique* $n \times n$*-matrix valued function* $\mathbf{U}(t,s)$ *defined for* $t_0 \le s \le t \le 1$ *and* $0 \le t \le s \le t_0$ *such that*

$$\text{(2,4)} \qquad \mathbf{U}(t,s) = \mathbf{I} + \int_{s}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r,s)\,.$$

Proof. If e.g. $t_0 \le s \le 1$ and s is fixed, then the matrix equation

$$\text{(2,5)} \qquad \mathbf{X}(t) = \mathbf{I} + \int_{s}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r)$$

has by 2.1 a uniquely determined solution $\mathbf{X}\colon [s,1] \to L(R_n)$. If we denote this solution by $\mathbf{U}(t,s)$, then $\mathbf{U}(t,s)$ is uniquely determined for $t_0 \le s \le t \le 1$ and satisfies (2,4).

Similarly if $0 \le s \le t_0$, s being fixed, the matrix equation (2,5) has by 2.1 a unique solution $\mathbf{X}\colon [0,s] \to L(R_n)$ which will be denoted by $\mathbf{U}(t,s)$, and $\mathbf{U}(t,s)$ evidently satisfies (2,4) for $0 \le t \le s \le t_0$.

2.3. Lemma. *Suppose that the assumptions of* 2.1 *are fulfilled. Then there exists a constant $M > 0$ such that $|\mathbf{U}(t, s)| \le M$ for all t, s such that $0 \le t \le s \le t_0$ or $t_0 \le s \le t \le 1$. Moreover we have*

$$(2,6) \qquad |\mathbf{U}(t_2, s) - \mathbf{U}(t_1, s)| \le M \operatorname{var}_{t_1}^{t_2} \mathbf{A}$$

for all $0 \le t_1 \le t_2 \le s$ if $s \le t_0$ and all $s \le t_1 \le t_2 \le 1$ if $t_0 \le s$. Consequently $\operatorname{var}_0^s \mathbf{U}(., s) \le M \operatorname{var}_0^s \mathbf{A}$, $\operatorname{var}_s^1 \mathbf{U}(., s) \le M \operatorname{var}_s^1 \mathbf{A}$ *if* $0 \le s \le t_0$, $t_0 \le s \le 1$ *respectively.*

Proof. Since $\mathbf{U}(t, s)$ satisfies (2,4) in its domain of definition, the k-th column ($k = 1, \ldots, n$) of $\mathbf{U}(t, s)$ denoted by $\mathbf{U}_k(t, s)$ satisfies the equation

$$\mathbf{U}_k(t, s) = \mathbf{e}^{(k)} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}_k(r, s)$$

for every $t \in [0, s]$ when $s \le t_0$ ($\mathbf{e}^{(k)}$ means the k-th column of the identity matrix $\mathbf{I} \in L(R_n)$, i.e. $\mathbf{U}_k(t, s)$ is a solution of the problem $\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{g}$, $\mathbf{x}(s) = \mathbf{e}^{(k)}$). Hence by 1.7 we have

$$|\mathbf{U}_k(t, s)| \le C|\mathbf{e}^{(k)}| \exp(C \operatorname{var}_t^s \mathbf{A}) \le C \exp(C \operatorname{var}_0^1 \mathbf{A}), \qquad k = 1, \ldots, n$$

for every $0 \le t \le s \le t_0$ where $C \ge 1$ is a constant and evidently also

$$|\mathbf{U}(t, s)| \le \sum_{k=1}^{n} |\mathbf{U}_k(t, s)| \le nC \exp(C \operatorname{var}_0^1 \mathbf{A}) = M .$$

If $t_0 \le s$, then 1.7 yields the same result for $s \le t \le 1$ and the boundedness of $\mathbf{U}(t, s)$ is proved.

Assume that $0 \le t_1 \le t_2 \le s \le t_0$. Then we have by I.4.16

$$|\mathbf{U}(t_2, s) - \mathbf{U}(t_1, s)| = \left|\int_s^{t_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s) - \int_s^{t_1} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s)\right|$$

$$= \left|\int_{t_1}^{t_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s)\right| \le M \operatorname{var}_{t_1}^{t_2} \mathbf{A} .$$

A similar inequality holds if $t_0 \le s \le t_1 \le t_2 \le 1$ and (2,6) is proved.

2.4. Theorem. *Suppose that the assumptions of* 2.1 *are fullfilled and $t_1 \in [0, 1]$. Then the unique solution of the homogeneous initial value problem*

$$(2,7) \qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x}\,, \qquad \mathbf{x}(t_1) = \tilde{\mathbf{x}}$$

defined on $[t_1, 1]$ if $t_0 \le t_1$ and on $[0, t_1]$ if $t_1 \le t_0$ is given by the relation

$$(2,8) \qquad \mathbf{x}(t) = \mathbf{U}(t, t_1)\, \tilde{\mathbf{x}}$$

on the intervals of definition, where $\mathbf{U}$ is the $n \times n$-matrix from 2.2 *satisfying* (2,4).

Proof. Under the given assumptions the existence and uniqueness of a solution of (2,7) is quaranteed by 1.4. Let us assume that $t_0 \leq t_1$. Since by 2.2 $\mathbf{U}(t, t_1)$ is uniquely defined for $t_1 \leq t \leq 1$, by (2,8) a function $\mathbf{x}$: $[t_1, 1] \to R_n$ is given. By 2.3 we have $\text{var}_{t_1}^1 \mathbf{U}(., t_1) < \infty$ and consequently $\text{var}_{t_1}^1 \mathbf{x} = \text{var}_{t_1}^1 \mathbf{U}(., t_1)\tilde{\mathbf{x}} < \infty$. For $\mathbf{x}$: $[t_1, 1] \to R_n$ given by (2,8) the integral $\int_{t_1}^t \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s)$ evidently exists (see I.4.19) for every $t \in [t_1, 1]$ and by (2,4) we have

$$\int_{t_1}^t \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s) = \int_{t_1}^t \mathrm{d}[\mathbf{A}(s)]\,\mathbf{U}(s, t_1)\,\tilde{\mathbf{x}} = (\mathbf{U}(t, t_1) - \mathbf{I})\,\tilde{\mathbf{x}} = \mathbf{x}(t) - \tilde{\mathbf{x}},$$

i.e. $\mathbf{x}(t) = \mathbf{U}(t, t_1)\,\tilde{\mathbf{x}}$ is a solution of (2,7) on $[t_1, 1]$. The proof of this result for the case $t_1 \leq t_0$ is similar.

2.5. Corollary. *If the assumptions of* 2.1 *are satisfied and* $\mathbf{U}(t, s)$ *is the* $n \times n$*-matrix determined by* (2,4) *for* $t_0 \leq s \leq t \leq 1$ *and* $0 \leq t \leq s \leq t_0$, *then*

$$\mathbf{U}(t, s) = \mathbf{U}(t, r)\,\mathbf{U}(r, s) \tag{2,9}$$

if $t_0 \leq s \leq r \leq t \leq 1$ *or* $0 \leq t \leq r \leq s \leq t_0$ *and*

$$\mathbf{U}(t, t) = \mathbf{I} \tag{2,10}$$

for every $t \in [0, 1]$.

Proof. Let e.g. $0 \leq t \leq r \leq s \leq t_0$, then by (2,4) we obtain

$$\begin{aligned}\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(\varrho)]\,\mathbf{U}(\varrho, s) &= \mathbf{I} + \int_s^r \mathrm{d}[\mathbf{A}(\varrho)]\,\mathbf{U}(\varrho, s) + \int_r^t \mathrm{d}[\mathbf{A}(\varrho)]\,\mathbf{U}(\varrho, s)\\ &= \mathbf{U}(r, s) + \int_r^t \mathrm{d}[\mathbf{A}(\varrho)]\,\mathbf{U}(\varrho, s)\end{aligned}$$

for every $0 \leq t \leq r$. Hence $\mathbf{U}(t, s)$ satisfies the matrix equation

$$\mathbf{X}(t) = \mathbf{U}(r, s) + \int_r^t \mathrm{d}[\mathbf{A}(\varrho)]\,\mathbf{X}(\varrho)$$

for $0 \leq t \leq r$ and by 2.4 this solution can be expressed in the form $\mathbf{U}(t, r)\,\mathbf{U}(r, s)$, i.e. (2,9) is satisfied. If $t_0 \leq s \leq r \leq t \leq 1$, then (2,9) can be proved analogously. The relation (2,10) obviously follows from (2,4).

2.6. Lemma. *If the assumptions of* 2.1 *are satisfied, then for* $\mathbf{U}(t, s)$ *given by* 2.2 we have

$$|\mathbf{U}(t, s_2) - \mathbf{U}(t, s_1)| \leq M^2 \text{var}_{s_1}^{s_2} \mathbf{A} \tag{2,11}$$

for any s_1, s_2 *such that* $t_0 \leq s_1 \leq s_2 \leq t \leq 1$ *or* $0 \leq t \leq s_1 \leq s_2 \leq t_0$ *where* M *is the bound of* $\mathbf{U}(t, s)$ (see 2.3). *Hence* $\text{var}_{t_0}^t \mathbf{U}(t, .) \leq M^2 \text{var}_{t_0}^t \mathbf{A}$ *if* $t_0 \leq t$ *and* $\text{var}_t^{t_0} \mathbf{U}(t, .) \leq M^2 \text{var}_t^{t_0} \mathbf{A}$ *if* $t \leq t_0$.

Proof. Let us consider the case when $t_0 \leq s_1 \leq s_2 \leq t$. By (2,4) we have

$$\mathbf{U}(t, s_2) - \mathbf{U}(t, s_1) = \int_{s_2}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_2) - \int_{s_1}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1)$$

$$= \int_{s_2}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_2) - \int_{s_2}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1) - \int_{s_1}^{s_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1),$$

i.e. the difference $\mathbf{U}(t, s_2) - \mathbf{U}(t, s_1)$ satisfies the matrix equation

$$\mathbf{X}(t) = - \int_{s_1}^{s_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1) + \int_{s_2}^{t} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{X}(r)$$

for $s_2 \leq t \leq 1$. Hence by 2.4 we obtain

$$\mathbf{U}(t, s_2) - \mathbf{U}(t, s_1) = \mathbf{U}(t, s_2)\left(- \int_{s_1}^{s_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1)\right)$$

and by 2.3 and I.4.16 it is

$$|\mathbf{U}(t, s_2) - \mathbf{U}(t, s_1)| \leq M \left| \int_{s_1}^{s_2} \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s_1)\right| \leq M^2 \operatorname{var}_{s_1}^{s_2} \mathbf{A}.$$

The proof for the case $0 \leq t \leq s_1 \leq s_2 \leq t_0$ can be given similarly and (2,11) is valid.

2.7. Lemma. *Suppose that the assumptions of* 2.1 *are satisfied. Let us define*

$$(2,12) \qquad \begin{aligned} \tilde{\mathbf{U}}(t, s) &= \mathbf{U}(t, s) && \text{for} \quad t_0 \leq s \leq t \leq 1, \\ \tilde{\mathbf{U}}(t, s) &= \mathbf{U}(t, t) = \mathbf{I} && \text{for} \quad t_0 \leq t \leq s \leq 1, \end{aligned}$$

and

$$(2,13) \qquad \begin{aligned} \tilde{\mathbf{U}}(t, s) &= \mathbf{U}(t, s) && \text{for} \quad 0 \leq t \leq s \leq t_0, \\ \tilde{\mathbf{U}}(t, s) &= \mathbf{U}(t, t) = \mathbf{I} && \text{for} \quad 0 \leq s \leq t \leq t_0, \end{aligned}$$

where $\mathbf{U}(t, s) \in L(R_n)$ *is given by* 2.2.

Then for the twodimensional variations of $\tilde{\mathbf{U}}$ *on the squares* $[t_0, 1] \times [t_0, 1]$ *and* $[0, t_0] \times [0, t_0]$ *on which* $\tilde{\mathbf{U}}$ *is defined we have* $\mathrm{v}_{[t_0,1] \times [t_0,1]}(\tilde{\mathbf{U}}) < \infty$ *and* $\mathrm{v}_{[0,t_0] \times [0,t_0]}(\tilde{\mathbf{U}}) < \infty$.

Proof. Assume that $t_0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ is an arbitrary subdivision of the interval $[t_0, 1]$ and $J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]$, $i, j = 1, \ldots, k$ the corresponding net-type subdivision of $[t_0, 1] \times [t_0, 1]$. We consider the sum (see I.6.2, I.6.3)

$$\sum_{i,j=1}^{k} |m_{\tilde{U}}(J_{ij})| = \sum_{i=1}^{k} \left(\sum_{j=1}^{i-1} |m_{\tilde{U}}(J_{ij})| + |m_{\tilde{U}}(J_{ii})| + \sum_{j=i+1}^{k} |m_{\tilde{U}}(J_{ij})| \right).$$

where we use the convention that $\sum_{j=1}^{0} |m_{\tilde{U}}(J_{ij})| = 0$ and $\sum_{j=k+1}^{k} |m_{\tilde{U}}(J_{ij})| = 0$. By (2,12) we have $m_{\tilde{U}}(J_{ij}) = m_U(J_{ij})$ if $j \le i - 1$,

$$\begin{aligned} m_{\tilde{U}}(J_{ii}) &= \tilde{\boldsymbol{U}}(\alpha_i, \alpha_i) - \tilde{\boldsymbol{U}}(\alpha_i, \alpha_{i-1}) - \tilde{\boldsymbol{U}}(\alpha_{i-1}, \alpha_i) + \tilde{\boldsymbol{U}}(\alpha_{i-1}, \alpha_{i-1}) \\ &= \tilde{\boldsymbol{U}}(\alpha_i, \alpha_i) - \tilde{\boldsymbol{U}}(\alpha_i, \alpha_{i-1}) = \boldsymbol{U}(\alpha_i, \alpha_i) - \boldsymbol{U}(\alpha_i, \alpha_{i-1}) \end{aligned}$$

and $m_{\tilde{U}}(J_{ij}) = \boldsymbol{0}$ if $i + 1 \le j$. Hence

$$(2,14) \qquad \sum_{i,j=1}^{k} |m_{\tilde{U}}(J_{ij})| = \sum_{i=1}^{k} \sum_{j=1}^{i-1} |m_U(J_{ij})| + \sum_{i=1}^{k} |\boldsymbol{U}(\alpha_i, \alpha_i) - \boldsymbol{U}(\alpha_i, \alpha_{i-1})| .$$

If $j \le i - 1$, then $\alpha_{j-1} < \alpha_j \le \alpha_{i-1} < \alpha_i$ and by 2.5

$$\begin{aligned} m_U(J_{ij}) &= \boldsymbol{U}(\alpha_i, \alpha_j) - \boldsymbol{U}(\alpha_i, \alpha_{j-1}) - \boldsymbol{U}(\alpha_{i-1}, \alpha_j) + \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1}) \\ &= \boldsymbol{U}(\alpha_i, \alpha_{i-1})\, \boldsymbol{U}(\alpha_{i-1}, \alpha_j) - \boldsymbol{U}(\alpha_{i-1}, \alpha_j) - \boldsymbol{U}(\alpha_i, \alpha_{i-1})\, \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1}) + \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1}) \\ &= [\boldsymbol{U}(\alpha_i, \alpha_{i-1}) - \boldsymbol{I}]\, \boldsymbol{U}(\alpha_{i-1}, \alpha_j) - [\boldsymbol{U}(\alpha_i, \alpha_{i-1}) - \boldsymbol{I}]\, \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1}) \\ &= [\boldsymbol{U}(\alpha_i, \alpha_{i-1}) - \boldsymbol{I}]\, [\boldsymbol{U}(\alpha_{i-1}, \alpha_j) - \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1})] \\ &= [\boldsymbol{U}(\alpha_i, \alpha_{i-1}) - \boldsymbol{U}(\alpha_{i-1}, \alpha_{i-1})]\, [\boldsymbol{U}(\alpha_{i-1}, \alpha_j) - \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1})] . \end{aligned}$$

Hence by 2.3 and 2.6 we obtain

$$\begin{aligned} |m_U(J_{ij})| &= |\boldsymbol{U}(\alpha_i, \alpha_{i-1}) - \boldsymbol{U}(\alpha_{i-1}, \alpha_{i-1})|\, |\boldsymbol{U}(\alpha_{i-1}, \alpha_j) - \boldsymbol{U}(\alpha_{i-1}, \alpha_{j-1})| \\ &\le M(\operatorname{var}_{\alpha_{i-1}}^{\alpha_i} \boldsymbol{A})\, M^2 \operatorname{var}_{\alpha_{j-1}}^{\alpha_j} \boldsymbol{A} = M^3 \operatorname{var}_{\alpha_{i-1}}^{\alpha_i} \boldsymbol{A} \operatorname{var}_{\alpha_{j-1}}^{\alpha_j} \boldsymbol{A} \end{aligned}$$

and

$$\sum_{i=1}^{k} \sum_{j=1}^{i-1} |m_U(J_{ij})| \le M^3 \sum_{i=1}^{k} \operatorname{var}_{\alpha_{i-1}}^{\alpha_i} \boldsymbol{A} \sum_{j=1}^{i-1} \operatorname{var}_{\alpha_{j-1}}^{\alpha_j} \boldsymbol{A} \le M^3 (\operatorname{var}_{t_0}^{1} \boldsymbol{A})^2 .$$

Further, by (2,11) from 2.6 we have

$$\sum_{i=1}^{k} |\boldsymbol{U}(\alpha_i, \alpha_i) - \boldsymbol{U}(\alpha_i, \alpha_{i-1})| \le \sum_{i=1}^{k} M^2 \operatorname{var}_{\alpha_{i-1}}^{\alpha_i} \boldsymbol{A} = M^2 \operatorname{var}_{t_0}^{1} \boldsymbol{A} .$$

Hence by (2,14) we have

$$\sum_{i,j=1}^{k} |m_{\tilde{U}}(J_{ij})| \le M^3 (\operatorname{var}_{t_0}^{1} \boldsymbol{A})^2 + M^2 \operatorname{var}_{t_0}^{1} \boldsymbol{A}$$

and since the net-type subdivision was chosen arbitrarily, we have by the definition also

$$\mathrm{v}_{[t_0,1] \times [t_0,1]}(\tilde{\boldsymbol{U}}) \le M^3 (\operatorname{var}_{t_0}^{0} \boldsymbol{A})^2 + M^2 \operatorname{var}_{t_0}^{1} \boldsymbol{A} < \infty .$$

The finiteness of $\mathrm{v}_{[0,t_0] \times [0,t_0]}(\tilde{\boldsymbol{U}})$ can be proved similarly.

2.8. Theorem (variation-of-constants formula). *Let* $\mathbf{A}\colon [0,1] \to L(R_n)$ *satisfy the assumptions given in* 2.1 *where* $t_0 \in [0,1]$ *is fixed. Then for every* $\mathbf{x}_0 \in R_n$, $\mathbf{g} \in BV_n$ *the unique solution of the initial value problem* (2,1) *can be expressed in the form*

$$\mathbf{x}(t) = \mathbf{U}(t, t_0)\,\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) - \int_{t_0}^{t} \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) \tag{2,15}$$

where $\mathbf{U}$ *is the uniquely determined matrix satisfying* (2,4) *from* 2.2.

Proof. We verify by computation that $\mathbf{x}\colon [0,1] \to R_n$ from (2,15) is really a solution of (2,1). Let us assume that $t < t_0$. Then

$$\begin{aligned}
\int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{x}(r) &= \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r,t_0)\,\mathbf{x}_0 + \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,(\mathbf{g}(r) - \mathbf{g}(t_0)) \\
&\quad - \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)] \int_{t_0}^{r} \mathrm{d}_s[\mathbf{U}(r,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) \\
&= (\mathbf{U}(t,t_0) - \mathbf{I})\,\mathbf{x}_0 + \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,(\mathbf{g}(r) - \mathbf{g}(t_0)) - \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)] \int_{t_0}^{r} \mathrm{d}_s[\mathbf{U}(r,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0))
\end{aligned} \tag{2,16}$$

since $\mathbf{U}$ satisfies 2.4. Let us now consider the last term from the right-hand side in (2,16). We have

$$\int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)] \int_{t_0}^{r} \mathrm{d}_s[\mathbf{U}(r,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) = \int_{t}^{t_0} \mathrm{d}[\mathbf{A}(r)] \int_{t}^{t_0} \mathrm{d}_s[\tilde{\mathbf{U}}(r,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0))$$

where $\tilde{\mathbf{U}}$ is defined in 2.7 and satisfies by 2.7, 2.3 and 2.6 the assumptions of I.6.20 on the square $[t, t_0] \times [t, t_0]$. Hence we interchange by I.6.20 the order of integration and obtain by the definition of $\mathbf{U}$

$$\begin{aligned}
\int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)] \int_{t_0}^{r} \mathrm{d}_s[\mathbf{U}(r,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) &= \int_{t}^{t_0} \mathrm{d}_s\left[\int_{t}^{t_0} \mathrm{d}[\mathbf{A}(r)]\,\tilde{\mathbf{U}}(r,s)\right](\mathbf{g}(s) - \mathbf{g}(t_0)) \\
&= \int_{t}^{t_0} \mathrm{d}_s\left[\int_{t}^{s} \mathrm{d}[\mathbf{A}(r)]\,\tilde{\mathbf{U}}(r,s) + \int_{s}^{t_0} \mathrm{d}[\mathbf{A}(r)]\,\tilde{\mathbf{U}}(r,s)\right](\mathbf{g}(s) - \mathbf{g}(t_0)) \\
&= \int_{t_0}^{t} \mathrm{d}_s\left[\int_{s}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r,s) + \int_{t_0}^{s} \mathrm{d}[\mathbf{A}(r)]\right](\mathbf{g}(s) - \mathbf{g}(t_0)) \\
&= \int_{t_0}^{t} \mathrm{d}_s[\mathbf{U}(t,s) - \mathbf{I} + \mathbf{A}(s) - \mathbf{A}(t_0)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) \\
&= \int_{t_0}^{t} \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) + \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)).
\end{aligned}$$

Using this expression we obtain by (2,16)

$$\int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,\mathbf{x}(r) = \mathbf{U}(t,t_0)\,\mathbf{x}_0 - \mathbf{x}_0 + \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(r)]\,(\mathbf{g}(r) - \mathbf{g}(t_0))$$

$$- \int_{t_0}^{t} \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) - \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0))$$

$$= \mathbf{U}(t,t_0)\,\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) - \int_{t_0}^{t} \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)) - (\mathbf{g}(t) - \mathbf{g}(t_0)) - \mathbf{x}_0$$

$$= \mathbf{x}(t) - \mathbf{x}_0 - (\mathbf{g}(t) - \mathbf{g}(t_0)).$$

Hence $\mathbf{x}(t)$ is a solution of (2,1) for $t \le t_0$. For the case $t_0 \le t$ the proof can be given analogously. Using 1.4 the solutions of (2,1) are uniquely determined and this completes the proof.

2.9. Remark. Let us mention that the operator $\mathbf{x} \in BV_n \to \int_{t_0}^{t} \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s)$ appearing in the definition of the generalized linear differential equation (2,1) can be written in the Fredholm-Stieltjes form $\int_0^1 \mathrm{d}_s\,[\mathbf{K}(t,s)]\,\mathbf{x}(s)$ where $\mathbf{K}\colon [0,1] \times [0,1] \to L(R_n)$ is defined as follows: if $t_0 \le t \le 1$, then

$$\begin{aligned} \mathbf{K}(t,s) &= \mathbf{A}(t_0) && \text{for } 0 \le s \le t_0\,,\\ \mathbf{K}(t,s) &= \mathbf{A}(s) && \text{for } t_0 \le s \le t\,,\\ \mathbf{K}(t,s) &= \mathbf{A}(t) && \text{for } t \le s \le 1\,, \end{aligned}$$

and if $0 \le t \le t_0$, then

$$\begin{aligned} \mathbf{K}(t,s) &= -\mathbf{A}(t) && \text{for } 0 \le s \le t\,,\\ \mathbf{K}(t,s) &= -\mathbf{A}(s) && \text{for } t \le s \le t_0\,,\\ \mathbf{K}(t,s) &= -\mathbf{A}(t_0) && \text{for } t_0 \le s \le 1\,. \end{aligned}$$

If this fact is used and II.2.5 is taken into account, then the solution of the equation (2,1) can be given by the resolvent formula (II.2.16) in the form

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t,s)]\,(\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0))\,, \tag{2,17}$$

for $t \in [0,1]$ since (2,1) has a solution uniquely defined for every $\mathbf{x}_0 \in R_n$, $\mathbf{g} \in BV_n$. The resolvent kernel $\boldsymbol{\Gamma}\colon [0,1] \times [0,1] \to L(R_n)$ satisfies

$$\boldsymbol{\Gamma}(t,s) = \mathbf{K}(t,s) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t,r)]\,\boldsymbol{\Gamma}(r,t)\,.$$

If we set $\mathbf{U}(t,s) = \mathbf{I} + \boldsymbol{\Gamma}(t,s) - \boldsymbol{\Gamma}(t,t)$, then the variation-of-constants formula (2,15) can be derived from (2,17).

In the following we consider the initial value problem (2,1) with the assumptions on $\mathbf{A}$: $[0, 1] \to L(R_n)$ strengthened.

2.10. Theorem. *Assume that the matrix* $\mathbf{A}: [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$ *is such that* $\mathbf{I} - \Delta^- \mathbf{A}(t)$ *is regular for all* $t \in (0, 1]$ *and* $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ *is regular for all* $t \in [0, 1)$.

Then there exists a unique $n \times n$*-matrix valued function* $\mathbf{U}: [0,1] \times [0,1] \to L(R_n)$ *such that*

$$\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\, \mathbf{U}(r, s) \tag{2,18}$$

for all $t, s \in [0, 1]$.

The matrix $\mathbf{U}(t, s)$ *determined by* (2,18) *has the following properties.*

(i) $\mathbf{U}(t, t) = \mathbf{I}$ *for all* $t \in [0, 1]$.

(ii) *There exists a constant* $M > 0$ *such that* $|\mathbf{U}(t, s)| \leq M$ *for all* $t, s \in [0, 1]$, $\operatorname{var}_0^1 \mathbf{U}(t, .) \leq M$, $\operatorname{var}_0^1 \mathbf{U}(., s) \leq M$ *for all* $t, s \in [0, 1]$.

(iii) *For any* $r, s, t \in [0, 1]$ *the relation*

$$\mathbf{U}(t, s) = \mathbf{U}(t, r)\, \mathbf{U}(r, s) \tag{2,19}$$

holds.

(iv)
$$\begin{aligned}
\mathbf{U}(t+, s) &= [\mathbf{I} + \Delta^+ \mathbf{A}(t)]\, \mathbf{U}(t, s) && \text{for } t \in [0, 1),\ s \in [0, 1],\\
\mathbf{U}(t-, s) &= [\mathbf{I} - \Delta^- \mathbf{A}(t)]\, \mathbf{U}(t, s) && \text{for } t \in (0, 1],\ s \in [0, 1],\\
\mathbf{U}(t, s+) &= \mathbf{U}(t, s)\, [\mathbf{I} + \Delta^+ \mathbf{A}(s)]^{-1} && \text{for } t \in [0, 1],\ s \in [0, 1),\\
\mathbf{U}(t, s-) &= \mathbf{U}(t, s)\, [\mathbf{I} - \Delta^- \mathbf{A}(s)]^{-1} && \text{for } t \in [0, 1],\ s \in (0, 1].
\end{aligned}$$

(v) *The matrix* $\mathbf{U}(t, s)$ *is regular for any* $t, s \in [0, 1]$.

(vi) *The matrices* $\mathbf{U}(t, s)$ *and* $\mathbf{U}(s, t)$ *are mutually reciprocal, i.e.* $[\mathbf{U}(t, s)]^{-1} = \mathbf{U}(s, t)$ *for every* $t, s \in [0, 1]$.

(vii) *The twodimensional variation of* $\mathbf{U}$ *is finite on* $[0, 1] \times [0, 1]$, *i.e.* $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{U}) < \infty$.

Proof. By 2.1 for every fixed $s \in [0, 1]$ the matrix equation

$$\mathbf{X}(t) = \tilde{\mathbf{X}} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\, \mathbf{X}(r), \qquad \tilde{\mathbf{X}} \in L(R_n)$$

has a unique solution $\mathbf{X}: [0, 1] \to L(R_n)$, which is defined on the whole interval $[0, 1]$. Hence the existence of $\mathbf{U}(t, s)$ satisfying (2,18) is quaranteed.

(i) is obvious from (2,18). (ii) follows immediately from 2.3 and 2.6. For (iii) we have

$$\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{U}(\varrho, s) = \mathbf{I} + \int_s^r \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{U}(\varrho, s) + \int_r^t \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{U}(\varrho, s)$$

$$= \mathbf{U}(r, s) + \int_r^t \mathrm{d}[\mathbf{A}(\varrho)]\, \mathbf{U}(\varrho, s),$$

i.e. $\boldsymbol{U}(t,s)$ satisfies the matrix equation

$$\boldsymbol{X}(t) = \boldsymbol{U}(r,s) + \int_r^t \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{X}(r).$$

Hence by 2.4 we obtain $\boldsymbol{U}(t,s) = \boldsymbol{U}(t,r)\,\boldsymbol{U}(r,s)$ for every $r,s,t \in [0,1]$, and (2,19) is satisfied.

The first two relations in (iv) are simple consequences of 1.6. To prove the third relation in (iv) let us mention that for any $t \in [0,1]$, $s \in [0,1)$ and sufficiently small $\delta > 0$ we have by definition

$$\begin{aligned}\boldsymbol{U}(t, s+\delta) - \boldsymbol{U}(t,s) &= \int_{s+\delta}^t \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r, s+\delta) - \int_s^t \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r,s)\\ &= \int_{s+\delta}^t \mathrm{d}[\boldsymbol{A}(r)]\,(\boldsymbol{U}(r,s+\delta) - \boldsymbol{U}(r,s)) - \int_s^{s+\delta} \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r,s),\end{aligned}$$

i.e. the difference $\boldsymbol{U}(t, s+\delta) - \boldsymbol{U}(t,s)$ satisfies the matrix equation

$$\boldsymbol{X}(t) = -\int_s^{s+\delta} \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r,s) + \int_{s+\delta}^t \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{X}(r)$$

and consequently by 2.4 it is

$$\boldsymbol{U}(t, s+\delta) - \boldsymbol{U}(t,s) = \boldsymbol{U}(t, s+\delta)\left(-\int_s^{s+\delta} \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r,s)\right).$$

For $\delta \to 0+$ this equality yields

$$\boldsymbol{U}(t, s+) - \boldsymbol{U}(t,s) = -\boldsymbol{U}(t,s+)\,\Delta^+\boldsymbol{A}(s)\,\boldsymbol{U}(s,s) = -\boldsymbol{U}(t,s+)\,\Delta^+\boldsymbol{A}(s).$$

Hence $\boldsymbol{U}(t,s) = \boldsymbol{U}(t,s+)\,[\boldsymbol{I} + \Delta^+\boldsymbol{A}(s)]$ for any $t \in [0,1]$, $s \in [0,1)$ and the assumption of the regularity of the matrix $\boldsymbol{I} + \Delta^+\boldsymbol{A}(s)$ gives the existence of the inverse $[\boldsymbol{I} + \Delta^+\boldsymbol{A}(s)]^{-1}$ and also the third equality from (iv). The fourth equality in (iv) can be proved analogously.

By (iii) we have $\boldsymbol{U}(t,s)\,\boldsymbol{U}(s,t) = \boldsymbol{I}$ and $\boldsymbol{U}(s,t)\,\boldsymbol{U}(t,s) = \boldsymbol{I}$ for every $t,s \in [0,1]$. Hence $\boldsymbol{U}(t,s) = \boldsymbol{U}(s,t)^{-1}$ and $\boldsymbol{U}(s,t) = \boldsymbol{U}(t,s)^{-1}$ and (vi) is proved. From (vi) the statement (v) follows immediately. (In this connection we note that a direct proof of (v) can be given without using (iii), see Schwabik [1].)

Finally by (iii) we have $\boldsymbol{U}(t,s) = \boldsymbol{U}(t,0)\,\boldsymbol{U}(0,s)$ for every $(t,s) \in [0,1] \times [0,1]$. By (ii) it is $\operatorname{var}_0^1 \boldsymbol{U}(.,0) < \infty$ and $\operatorname{var}_0^1 \boldsymbol{U}(0,.) < \infty$. Hence by I.6.4 we have $v_{[0,1]\times[0,1]}(\boldsymbol{U}) < \infty$ and (vii) is also proved.

2.11. Corollary. *If* $\boldsymbol{A}\colon [0,1] \to L(R_n)$, $\operatorname{var}_0^1 \boldsymbol{A} < \infty$, *satisfies the assumptions given in* 2.10, *then*

(2,20) $$\boldsymbol{U}(t,s) = \boldsymbol{X}(t)\,\boldsymbol{X}^{-1}(s) \qquad \textit{for every} \quad s,t \in [0,1]$$

where $\mathbf{X}: [0,1] \to L(R_n)$ *satisfies the matrix equation*

$$\mathbf{X}(t) = \mathbf{I} + \int_0^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r), \qquad t \in [0,1]. \tag{2,21}$$

Proof. Since the matrix equation (2,21) has a unique solution, it is easy to compare it with (2,18) and state that $\mathbf{X}(t) = \mathbf{U}(t,0)$. By (iii) from 2.10 we have $\mathbf{U}(t,s) = \mathbf{U}(t,0)\,\mathbf{U}(0,s)$ and by (vi) from 2.10 it follows $\mathbf{U}(0,s) = [\mathbf{U}(s,0)]^{-1} = \mathbf{X}^{-1}(s)$. Hence (2,20) hold.

2.12. Remark. If the matrix $\mathbf{A}: [0,1] \to L(R_n)$ satisfies the assumptions of 2.10, then evidently the assumptions of 1.4, 2.1–2.8 are satisfied for every $t_0 \in [0,1]$. Hence by 1.4 the initial value problem (2,1) has for every $t_0 \in [0,1]$, $\mathbf{x}_0 \in R_n$, $\mathbf{g} \in BV_n$ a unique solution $\mathbf{x}: [0,1] \to R_n$ defined on the whole interval $[0,1]$.

The variation-of-constants formula 2.8 leads to the following.

2.13. Theorem (variation-of constants formula). *Let us assume that* $\mathbf{A}: [0,1] \to L(R_n)$ *satisfies the conditions given in* 2.10. *Then for any* $t_0 \in [0,1]$, $\mathbf{x}_0 \in R_n$, $\mathbf{g} \in BV_n$ *the solution of the nonhomogeneous initial value problem* (2,1) *is given by the expression*

$$\mathbf{x}(t) = \mathbf{U}(t,t_0)\,\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) - \int_{t_0}^t \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0)), \qquad t \in [0,1]$$

where $\mathbf{U}(t,s): [0,1] \times [0,1] \to L(R_n)$ *is the matrix whose existence was stated in* 2.10.

The proof follows immediately from 2.8.

2.14. Corollary. *If* $\mathbf{A}: [0,1] \to L(R_n)$ *satisfies the assumptions from* 2.10, *then the above variation-of-constants formula can be written in the form*

$$\mathbf{x}(t) = \mathbf{g}(t) - \mathbf{g}(t_0) + \mathbf{X}(t)\left\{\mathbf{X}^{-1}(t_0)\,\mathbf{x}_0 - \int_{t_0}^t \mathrm{d}_s[\mathbf{X}^{-1}(s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0))\right\} \tag{2,22}$$

for $t \in [0,1]$ *where* $\mathbf{X}: [0,1] \to L(R_n)$ *is the uniquely determined solution of the matrix equation* (2,21).

The proof follows immediately from 2.13 and from the product decomposition (2,20) given in 2.11.

2.15. Proposition. *If* $\mathbf{A}: [0,1] \to L(R_n)$ *satisfies the assumptions given in* 2.10 *and* $\mathbf{X}: [0,1] \to L(R_n)$ *is the unique solution of the matrix equation* (2,21), *then*

$$\mathbf{X}^{-1}(s) = \mathbf{I} + \mathbf{A}(0) - \mathbf{X}^{-1}(s)\,\mathbf{A}(s) + \int_0^s \mathrm{d}[\mathbf{X}^{-1}(r)]\,\mathbf{A}(r) \tag{2,23}$$

for every $s \in [0,1]$.

Proof. For $\boldsymbol{X}$: $[0, 1] \to L(R_n)$ we have by (2,21)

$$\boldsymbol{X}(s) - \boldsymbol{I} = \int_0^s \mathrm{d}[\boldsymbol{A}(r)]\, \boldsymbol{X}(r) = \int_0^s \mathrm{d}[\boldsymbol{A}(r)]\,(\boldsymbol{X}(r) - \boldsymbol{I}) + \boldsymbol{A}(s) - \boldsymbol{A}(0)$$

for every $s \in [0, 1]$. Using the variation-of-constants formula (2,22) in the matrix form we get

$$\boldsymbol{X}(s) - \boldsymbol{I} = \boldsymbol{A}(s) - \boldsymbol{A}(0) - \boldsymbol{X}(s) \int_0^s \mathrm{d}[\boldsymbol{X}^{-1}(r)]\,(\boldsymbol{A}(r) - \boldsymbol{A}(0))$$

$$= \boldsymbol{A}(s) - \boldsymbol{A}(0) - \boldsymbol{X}(s) \int_0^s \mathrm{d}[\boldsymbol{X}^{-1}(r)]\,\boldsymbol{A}(r) + \boldsymbol{X}(s)\,[\boldsymbol{X}^{-1}(s) - \boldsymbol{X}^{-1}(0)]\,\boldsymbol{A}(0)$$

$$= \boldsymbol{A}(s) - \boldsymbol{X}(s)\,\boldsymbol{A}(0) - \boldsymbol{X}(s) \int_0^s \mathrm{d}[\boldsymbol{X}^{-1}(r)]\,\boldsymbol{A}(r)\,.$$

Multiplying this relation from the left by $\boldsymbol{X}^{-1}(s)$ we obtain for every $s \in [0, 1]$

$$\boldsymbol{I} - \boldsymbol{X}^{-1}(s) = -\boldsymbol{A}(0) + \boldsymbol{X}^{-1}(s)\,\boldsymbol{A}(s) - \int_0^s \mathrm{d}[\boldsymbol{X}^{-1}(r)]\,\boldsymbol{A}(r)$$

and (2,23) is satisfied.

2.16. Definition. The matrix $\boldsymbol{U}(t, s)$: $[0, 1] \times [0, 1] \to L(R_n)$ given by 2.10 is called the *fundamental matrix* (or *transition matrix*) for the homogeneous generalized linear differential equation $\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x}$.

2.17. Remark. If $\boldsymbol{B}$: $[0, 1] \to L(R_n)$ is an $n \times n$-matrix, continuous on $[0, 1]$ and $\dot{\boldsymbol{x}} = \boldsymbol{B}(t)\,\boldsymbol{x}$ is the corresponding ordinary linear differential system, then in the theory of ordinary differential equations the transition matrix $\boldsymbol{\Phi}(t, t_0)$ is defined as a solution of the matrix differential equation

$$\boldsymbol{X}' = \boldsymbol{B}(t)\,\boldsymbol{X}$$

satisfying the condition $\boldsymbol{X}(t_0) = \boldsymbol{I} \in L(R_n)$. Hence for $\boldsymbol{\Phi}$ we have

$$\boldsymbol{\Phi}(t, t_0) = \boldsymbol{I} + \int_{t_0}^t \boldsymbol{B}(\tau)\,\boldsymbol{\Phi}(\tau, t_0)\,\mathrm{d}\tau\,,$$

i.e. $\boldsymbol{\Phi}$ satisfies the generalized matrix differential equation

$$\boldsymbol{\Phi}(t, t_0) = \boldsymbol{I} + \int_{t_0}^t \mathrm{d}[\boldsymbol{A}(\tau)]\,\boldsymbol{\Phi}(\tau, t_0)$$

where $\boldsymbol{A}(t) = \int_0^t \boldsymbol{B}(\tau)\,\mathrm{d}\tau$ (see also 1.3). The variation-of-constant formula for the generalized linear differential equation

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}[\boldsymbol{A}]\,\boldsymbol{x} + \mathrm{d}\boldsymbol{g}\,, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

where $\mathbf{g}(t) = \int_{t_0}^t \mathbf{h}(s)\,\mathrm{d}s$, which corresponds by 1.3 to the ordinary linear system

$$\mathbf{x}' = \mathbf{B}(t)\,\mathbf{x} + \mathbf{h}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has the form

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\,\mathbf{x}_0 + \mathbf{g}(t) - \mathbf{g}(t_0) - \int_{t_0}^t \mathrm{d}_s[\boldsymbol{\Phi}(t, s)]\,(\mathbf{g}(s) - \mathbf{g}(t_0))$$

$$= \boldsymbol{\Phi}(t, t_0)\,\mathbf{x}_0 + \int_{t_0}^t \mathbf{h}(s)\,\mathrm{d}s + \int_{t_0}^t \boldsymbol{\Phi}(t, s)\,\mathrm{d}\left(\int_{t_0}^s \mathbf{h}(\sigma)\,\mathrm{d}\sigma\right) - \boldsymbol{\Phi}(t, t)\int_{t_0}^t \mathbf{h}(s)\,\mathrm{d}s$$

$$= \boldsymbol{\Phi}(t, t_0)\,\mathbf{x}_0 + \int_{t_0}^t \boldsymbol{\Phi}(t, s)\,\mathbf{h}(s)\,\mathrm{d}s\,.$$

This is the usual form of the variation-of-constants formula for ordinary linear differential equations.

2.18. Definition. The $n \times n$-matrix $\mathbf{U}(t, s)$ defined for $t, s \in [0, 1]$ is called *harmonic* if $\operatorname{var}_0^1 \mathbf{U}(t, .) < \infty$ for every $t \in [0, 1]$, $\operatorname{var}_0^1 \mathbf{U}(., s) < \infty$ for every $s \in [0, 1]$.

(2,19) $\quad \mathbf{U}(t, s) = \mathbf{U}(t, r)\,\mathbf{U}(r, s) \quad$ for any three points $r, s, t \in [0, 1]$,

(2,24) $\quad \mathbf{U}(t, t) = \mathbf{I} \quad$ for any $t \in [0, 1]$.

For the concept of harmonic matrices see e.g. Hildebrandt [2], Mac Nerney [1], Wall [1].

As was shown in 2.10 for $\mathbf{A}$: $[0, 1] \to L(R_n)$, $\operatorname{var}_0^1 \mathbf{A} < \infty$ with the matrices $\mathbf{I} - \Delta^- \mathbf{A}(t)$, $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ regular for $t \in (0, 1]$, $t \in [0, 1)$ respectively, the corresponding fundamental matrix $\mathbf{U}(t, s)$ is harmonic (see (i), (ii) and (iii) in 2.10). In other words, to any $n \times n$-matrix valued function $\mathbf{A}$: $[0, 1] \to L(R_n)$ with the above mentioned properties through the relation

$$\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r, s), \qquad t, s \in [0, 1]$$

a uniquely determined harmonic matrix $\mathbf{U}(t, s)$ corresponds. In the opposite direction the following holds.

2.19. Theorem. *If the $n \times n$-matrix $\mathbf{U}(t, s)$: $[0, 1] \times [0, 1] \to L(R_n)$ is harmonic, then there exists $\mathbf{A}$: $[0, 1] \to L(R_n)$ such that $\operatorname{var}_0^1 \mathbf{A} < \infty$, the matrices $\mathbf{I} - \Delta^- \mathbf{A}(t)$, $\mathbf{I} + \Delta^+ \mathbf{A}(t)$ are regular for all $t \in (0, 1]$, $t \in [0, 1)$, respectively and $\mathbf{U}$ satisfies the relation*

(2,25) $$\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r, s), \qquad t, s \in [0, 1],$$

i.e. $\mathbf{U}(t, s)$ is the fundamental matrix for the homogeneous generalized linear differential equation with the matrix $\mathbf{A}$ (see 2.16).

Proof. Let us set

$$\boldsymbol{A}(t,\tau)=\int_0^t \mathrm{d}_r[\boldsymbol{U}(r,\tau)]\,\boldsymbol{U}(\tau,r)$$

for $t,\tau\in[0,1]$. This integral exists for every t,τ by I.4.19. For every $t,\tau\in[0,1]$ we have by (2,19) and (2,24)

$$\boldsymbol{A}(t,\tau)=\int_0^t \mathrm{d}_r[\boldsymbol{U}(r,\tau)\,\boldsymbol{U}(\tau,0)]\,\boldsymbol{U}(0,\tau)\,\boldsymbol{U}(\tau,r)=\int_0^t \mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,r)=\boldsymbol{A}(t,0).$$

Hence the matrix $\boldsymbol{A}(t,\tau)$ is independent of τ and we denote $\boldsymbol{A}(t)=\boldsymbol{A}(t,\tau)=\boldsymbol{A}(t,0)$ for $t\in[0,1]$. Evidently $\operatorname{var}_0^1\boldsymbol{A}<\infty$ by I.4.27. Further we have by the definition of $\boldsymbol{A}$, by the substitution theorem I.4.25 and by (2,19), (2,24)

$$\begin{aligned}\int_s^t \mathrm{d}[\boldsymbol{A}(r)]\,\boldsymbol{U}(r,s)&=\int_s^t \mathrm{d}_r\left[\int_0^r \mathrm{d}_\varrho[\boldsymbol{U}(\varrho,0)]\,\boldsymbol{U}(0,\varrho)\right]\boldsymbol{U}(r,s)\\&=\int_s^t \mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,r)\,\boldsymbol{U}(r,s)=\int_s^t \mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,s)\\&=(\boldsymbol{U}(t,0)-\boldsymbol{U}(s,0))\,\boldsymbol{U}(0,s)=\boldsymbol{U}(t,s)-\boldsymbol{I},\end{aligned}$$

i.e. $\boldsymbol{U}(t,s)$ satisfies (2,25) for every $t,s\in[0,1]$. Finally we show that $\boldsymbol{A}\colon[0,1]\to L(R_n)$ satisfies the regularity conditions for $\boldsymbol{I}-\Delta^-\boldsymbol{A}(t)$, $\boldsymbol{I}+\Delta^+\boldsymbol{A}(t)$. By definition we have for $t\in(0,1]$

$$\begin{aligned}\Delta^-\boldsymbol{A}(t)&=\boldsymbol{A}(t)-\lim_{\delta\to0+}\boldsymbol{A}(t-\delta)\\&=\int_0^t \mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,r)-\lim_{\delta\to0+}\int_0^{t-\delta}\mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,r)\\&=\lim_{\delta\to0+}\int_{t-\delta}^t \mathrm{d}_r[\boldsymbol{U}(r,0)]\,\boldsymbol{U}(0,r)=\lim_{\delta\to0+}(\boldsymbol{U}(t,0)-\boldsymbol{U}(t-\delta,0))\,\boldsymbol{U}(0,t)\\&=\boldsymbol{U}(t,0)\,\boldsymbol{U}(0,t)-\lim_{\delta\to0+}\boldsymbol{U}(t-\delta,0)\,\boldsymbol{U}(0,t)=\boldsymbol{I}-\lim_{\delta\to0+}\boldsymbol{U}(t-\delta,t),\end{aligned}$$

where I.4.13 was used. Hence

$$\boldsymbol{I}-\Delta^-\boldsymbol{A}(t)=\lim_{\delta\to0+}\boldsymbol{U}(t-\delta,t)=\boldsymbol{U}(t-,t)\tag{2,26}$$

for every $t\in(0,1]$. Since $\boldsymbol{U}$ is assumed to be harmonic, we have $\boldsymbol{U}(t-\delta,t)\,\boldsymbol{U}(t,t-\delta)=\boldsymbol{I}$ for any sufficiently small $\delta>0$. $\boldsymbol{U}(t,s)$ is of bounded variation in each variable, the limits $\lim_{\delta\to0+}\boldsymbol{U}(t-\delta,t)=\boldsymbol{U}(t-,t)$ and $\lim_{\delta\to0+}\boldsymbol{U}(t,t-\delta)=\boldsymbol{U}(t,t-)$ exist. Hence

$$\boldsymbol{U}(t-,t)\,\boldsymbol{U}(t,t-)=\lim_{\delta\to0+}\boldsymbol{U}(t-\delta,t)\,\boldsymbol{U}(t,t-\delta)=\boldsymbol{I}$$

and the matrix $\boldsymbol{U}(t-,t)$ is evidently regular since it has an inverse $[\boldsymbol{U}(t-,t)]^{-1}=\boldsymbol{U}(t,t-)$. This yields by (2,26) the regularity of $\boldsymbol{I}-\Delta^-\boldsymbol{A}(t)$ for every $t\in(0,1]$. The regularity of $\boldsymbol{I}+\Delta^+\boldsymbol{A}(t)$ for every $t\in[0,1)$ can be proved analogously.

3. Generalized linear differential equations on the whole real axis

In this section let us assume that $\mathbf{A}\colon R \to L(R_n)$ is an $n \times n$-matrix defined on the whole real axis R and is of locally bounded variation in R, i.e. $\operatorname{var}_a^b \mathbf{A} < \infty$ for every compact interval $[a, b] \subset R$. We consider the generalized linear differential equation

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{g} \tag{3,1}$$

where $\mathbf{g}\colon R \to R_n$ is of locally bounded variation in R.

The basic existence and uniqueness result follows from 1.4.

3.1. Theorem. *Assume that $\mathbf{A}\colon R \to L(R_n)$ is of locally bounded variation in R and $\mathbf{I} - \Delta^-\mathbf{A}(t)$, $\mathbf{I} + \Delta^+\mathbf{A}(t)$ are regular matrices for all $t \in R$. Then for any $t_0 \in R$, $\mathbf{x}_0 \in R_n$ and $\mathbf{g}\colon R \to R_n$ of locally bounded variation in R there is a unique solution $\mathbf{x}\colon R \to R_n$ of the equation (3,1) with $\mathbf{x}(t_0) = \mathbf{x}_0$ and this solution is of locally bounded variation in R.*

Proof. This theorem follows immediately from 1.4 and 1.7 since evidently the assumptions of 1.4 are satisfied on every compact interval $[a, b] \subset R$.

In this way our preceding arguments on generalized linear differential equations are applicable to the case of equations on the whole real axis R. Especially the fundamental matrix $\mathbf{U}(t, s)$ determined uniquely by the equation

$$\mathbf{U}(t, s) = \mathbf{I} + \int_s^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{U}(r, s)$$

is defined for all $t, s \in R$, has the properties (i), (iii), (iv), (v), (vi) from 2.10 and is of locally bounded variation in R in each variable separately (see (ii) in 2.10). Moreover, the twodimensional variation of $\mathbf{U}$ on every compact interval $I = [a, b] \times [c, d] \subset R_2$ is finite.

Now we prove a result which is analogous to the Floquet theory for linear systems of ordinary differential equations.

3.2. Theorem. *Assume that $\mathbf{A}\colon R \to L(R_n)$ is of locally bounded variation in R such that $\mathbf{I} - \Delta^-\mathbf{A}(t)$, $\mathbf{I} + \Delta^+\mathbf{A}(t)$ are regular matrices for every $t \in R$. Moreover let*

$$\mathbf{A}(t + \omega) - \mathbf{A}(t) = \mathbf{C} \qquad \text{for every} \quad t \in R$$

where $\omega > 0$ and $\mathbf{C} \in L(R_n)$ is a constant $n \times n$-matrix. If $\mathbf{X}\colon R \to L(R_n)$ is the solution of the matrix equation

$$\mathbf{X}(t) = \mathbf{I} + \int_0^t \mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r), \qquad t \in R$$

(i.e. $\mathbf{X}(t) = \mathbf{U}(t, 0)$) then there exists a regular $n \times n$-matrix $\mathbf{P}\colon R \to L(R_n)$, which is

periodic with the period ω $(\mathbf{P}(t+\omega)=\mathbf{P}(t))$ *and a constant* $n\times n$*-matrix* $\mathbf{Q}\in L(R_n)$ *such that*

$$\mathbf{X}(t)=\mathbf{P}(t)\,\mathrm{e}^{t\mathbf{Q}}$$

is satisfied for every $t\in R$.

Proof. By definition we have

$$\mathbf{X}(t+\omega)=\mathbf{I}+\int_0^{t+\omega}\mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r)=\mathbf{X}(\omega)+\int_\omega^{t+\omega}\mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r)$$

$$=\mathbf{X}(\omega)+\int_0^t\mathrm{d}[\mathbf{A}(r+\omega)]\,\mathbf{X}(r+\omega)=\mathbf{X}(\omega)+\int_0^t\mathrm{d}[\mathbf{A}(r)+\mathbf{C}]\,\mathbf{X}(r+\omega)$$

$$=\mathbf{X}(\omega)+\int_0^t\mathrm{d}[\mathbf{A}(r)]\,\mathbf{X}(r+\omega)$$

for every $t\in R$. Using the variation of constants formula 2.14 in the matrix form we get

$$\mathbf{X}(t+\omega)=\mathbf{X}(t)\,\mathbf{X}(\omega)\qquad\text{for every}\quad t\in R\,.$$

By (v) from 2.10 the matrix $\mathbf{X}(\omega)=\mathbf{U}(\omega,0)$ is regular. Using the standard argument we conclude that there is a constant real $n\times n$-matrix $\mathbf{Q}\in L(R_n)$ ($\mathbf{Q}$ is not unique) such that $\mathbf{X}(\omega)=\mathrm{e}^{\omega\mathbf{Q}}$ (see e.g. Coddington, Levinson [1], III.1.), i.e.

$$\mathbf{X}(t+\omega)=\mathbf{X}(t)\,\mathrm{e}^{\omega\mathbf{Q}}.$$

Let us define $\mathbf{P}(t)=\mathbf{X}(t)\,\mathrm{e}^{-t\mathbf{Q}}$ for every $t\in R$. We have

$$\mathbf{P}(t+\omega)=\mathbf{X}(t+\omega)\,\mathrm{e}^{-(t+\omega)\mathbf{Q}}=\mathbf{X}(t)\,\mathrm{e}^{\omega\mathbf{Q}}\,\mathrm{e}^{-\omega\mathbf{Q}}\,\mathrm{e}^{-t\mathbf{Q}}=\mathbf{X}(t)\,\mathrm{e}^{-t\mathbf{Q}}=\mathbf{P}(t)$$

for all $t\in R$, i.e. $\mathbf{P}$ is periodic with the period ω. The regularity of $\mathbf{P}(t)$ is obvious by the regularity of $\mathbf{X}(t)$ and $\mathrm{e}^{-t\mathbf{Q}}$. Hence $\mathbf{X}(t)=\mathbf{P}(t)\,\mathrm{e}^{t\mathbf{Q}}$ and the result is proved.

Remark. This theorem is a basis for more detailed considerations concerning the linear system (3,1) with $\mathbf{A}\colon R\to L(R_n)$ satisfying the "periodicity" condition $\mathbf{A}(t+\omega)-\mathbf{A}(t)=\text{const}$. Some special results are contained in Hnilica [1].

4. Formally adjoint equation

Let $\mathbf{B}\colon[0,1]\to L(R_n)$, $\operatorname{var}_0^1\mathbf{B}<\infty$ and $\mathbf{g}\in BV_n$. Let us consider the generalized linear differential equation for a row n-vector valued function $\mathbf{y}^*$

$$\mathrm{d}\mathbf{y}^*=-\mathbf{y}^*\,\mathrm{d}[\mathbf{B}]+\mathrm{d}\mathbf{g}^*\quad\text{on }[0,1]\,,\tag{4,1}$$

which is equivalent to the integral equation

$$\mathbf{y}^*(s)=\mathbf{y}^*(s_0)-\int_{s_0}^{s}\mathbf{y}^*(t)\,\mathrm{d}[\mathbf{B}(t)]+\mathbf{g}^*(s)-\mathbf{g}^*(s_0)\,,\qquad s,s_0\in[0,1]\,.$$

Obviously, $\mathbf{y}^*\colon [0,1] \to R_n$ is a solution to (4,1) on $[a,b] \subset [0,1]$ if and only if $\mathbf{y}$ verifies the equation

$$\mathbf{y}(s) = \mathbf{y}(s_0) - \int_{s_0}^{s} \mathrm{d}[\mathbf{B}^*(t)]\,\mathbf{y}(t) + \mathbf{g}(s) - \mathbf{g}(s_0) \tag{4,2}$$

for every $s, s_0 \in [a,b]$. Thus taking into account that $\mathbf{I} - \Delta^-(-\mathbf{B}^*)(s) = [\mathbf{I} + \Delta^-\mathbf{B}(s)]^*$ on $(0,1]$, $\mathbf{I} + \Delta^+(-\mathbf{B}^*)(s) = [\mathbf{I} - \Delta^+\mathbf{B}(s)]^*$ on $[0,1)$ we may easily obtain the basic results for the equation (4,1) as consequences of the corresponding theorems from the foregoing sections.

Given $\mathbf{y}_0^* \in R_n^*$, the equation (4,1) possesses a unique solution $\mathbf{y}^*$ on $[0,1]$ such that $\mathbf{y}^*(1) = \mathbf{y}_0^*$ or $\mathbf{y}^*(0) = \mathbf{y}_0^*$ if and only if

$$\det[\mathbf{I} - \Delta^+\mathbf{B}(s)] \neq 0 \quad \text{on } [0,1) \tag{4,3}$$

or

$$\det[\mathbf{I} + \Delta^-\mathbf{B}(s)] \neq 0 \quad \text{on } (0,1], \tag{4,4}$$

respectively (cf. 1.4).

If (4,3) holds, then by 2.2 there exists a unique $n \times n$-matrix valued function $\mathbf{W}(t,s)$ defined for $t, s \in [0,1]$ such that $s \geq t$ and fulfilling for all such t, s the relation

$$\mathbf{W}(t,s) = \mathbf{I} - \int_s^t \mathrm{d}[\mathbf{B}^*(r)]\,\mathbf{W}(r,s).$$

Furthermore, given $t, s \in [0,1]$, $\operatorname{var}_0^s \mathbf{W}(.,s) + \operatorname{var}_t^1 \mathbf{W}(t,.) < \infty$, $\mathbf{W}(t+,s) = [\mathbf{I} - \Delta^+\mathbf{B}(t)]^*\,\mathbf{W}(t,s)$ if $t < s$ and $\mathbf{W}(t-,s) = [\mathbf{I} + \Delta^-\mathbf{B}(t)]^*\,\mathbf{W}(t,s)$ if $t \leq s$ (cf. 2.10). It follows that the function $\mathbf{V}(t,s) = \mathbf{W}^*(s,t)$ for $t \geq s$ is a unique $n \times n$-matrix valued function which fulfils for $t, s \in [0,1]$, $t \geq s$ the relation

$$\mathbf{V}(t,s) = \mathbf{I} + \int_s^t \mathbf{V}(t,r)\,\mathrm{d}[\mathbf{B}(r)]. \tag{4,5}$$

Moreover, given $t, s \in [0,1]$

$$\operatorname{var}_0^t \mathbf{V}(t,.) + \operatorname{var}_s^1 \mathbf{V}(.,s) < \infty$$

and

$$\mathbf{V}(t,s+) = \mathbf{V}(t,s)\,[\mathbf{I} - \Delta^+\mathbf{B}(s)] \quad \text{if } t > s, \tag{4,6}$$

$$\mathbf{V}(t,s-) = \mathbf{V}(t,s)\,[\mathbf{I} + \Delta^-\mathbf{B}(s)] \quad \text{if } t \geq s. \tag{4,7}$$

If $\mathbf{y}_0^* \in R_n^*$ is given, the unique solution $\mathbf{y}^*$ of (4,1) on $[0,1]$ with $\mathbf{y}^*(1) = \mathbf{y}_0^*$ is given on $[0,1]$ by

$$\mathbf{y}^*(s) = \mathbf{y}_0^*\,\mathbf{V}(1,s) + \mathbf{g}^*(s) - \mathbf{g}^*(1) + \int_s^1 (\mathbf{g}^*(t) - \mathbf{g}^*(1))\,\mathrm{d}_t[\mathbf{V}(t,s)] \tag{4,8}$$

(cf. 2.8).

If (4,4) holds, then the fundamental matrix $\mathbf{V}(t,s)$ for (4,1) is defined and fulfils (4,5) for $t \le s$, (4,6) holds for $t \le s$ and (4,7) holds for $t < s$. Furthermore, $\operatorname{var}_0^s \mathbf{V}(.,s) + \operatorname{var}_t^1 \mathbf{V}(t,.) < \infty$ for all $t,s \in [0,1]$ and given $\mathbf{y}_0^* \in R_n^*$, the unique solution $\mathbf{y}^*$ of (4,1) on $[0,1]$ with $\mathbf{y}^*(0) = \mathbf{y}_0^*$ is given on $[0,1]$ by

$$(4,9) \qquad \mathbf{y}^*(s) = \mathbf{y}_0^* \mathbf{V}(0,s) + \mathbf{g}^*(s) - \mathbf{g}^*(0) - \int_0^s (\mathbf{g}^*(t) - \mathbf{g}^*(0))\, \mathrm{d}_t[\mathbf{V}(t,s)]\,.$$

If both (4,3) and (4,4) hold, then there exists $M < \infty$ such that given $t,s \in [0,1]$

$$|\mathbf{V}(t,s)| + \operatorname{var}_0^1 \mathbf{V}(t,.) + \operatorname{var}_0^1 \mathbf{V}(.,s) + \mathrm{v}_{[0,1]\times[0,1]}(\mathbf{V}) \le M < \infty\,.$$

Moreover, in this case, given $t,s,r \in [0,1]$,

$$(4,10) \qquad \mathbf{V}(t,r)\,\mathbf{V}(r,s) = \mathbf{V}(t,s) \quad \text{and} \quad \mathbf{V}(t,t) = \mathbf{I}$$

(cf. 2.10).

The equation (4,1) is said to be *formally adjoint* to (1,1) if

$$(4,11) \qquad \mathbf{B}(t+) - \mathbf{A}(t+) = \mathbf{B}(t-) - \mathbf{A}(t-) = \mathbf{B}(0) - \mathbf{A}(0) \quad \text{on } [0,1]\,.$$

(According to the convention introduced in I.3 we have

$$\mathbf{B}(0-) - \mathbf{A}(0-) = \mathbf{B}(0) - \mathbf{A}(0) = \mathbf{B}(1+) - \mathbf{A}(1+) = \mathbf{B}(1) - \mathbf{A}(1)\,.)$$

The condition (4,11) ensures that

$$(4,12) \qquad \int_0^1 \mathbf{y}^*(t)\, \mathrm{d}[\mathbf{B}(t) - \mathbf{A}(t)]\, \mathbf{x}(t) = 0 \qquad \text{for all} \quad \mathbf{x}, \mathbf{y} \in BV_n$$

(cf. I.4.23). (4,11) holds e.g. if $\mathbf{B}(t) \equiv \mathbf{A}(t)$ on $[0,1]$ or

$$(4,13) \qquad \begin{aligned} \mathbf{B}(t) &= \mathbf{A}_*(t) = \mathbf{A}(t-) + \Delta^+ \mathbf{A}(t) \qquad \text{on } (0,1)\,, \\ \mathbf{B}(0) &= \mathbf{A}_*(0) = \mathbf{A}(0)\,, \qquad \mathbf{B}(1) = \mathbf{A}_*(1) = \mathbf{A}(1)\,. \end{aligned}$$

Without any loss of generality we may assume that $\mathbf{A}(0) = \mathbf{B}(0)$.

4.1. Theorem. *Let the $n \times n$-matrix valued functions $\mathbf{A}$, $\mathbf{B}$ be of bounded variation on $[0,1]$ and such that (4,11) with $\mathbf{A}(0) = \mathbf{B}(0)$ holds.*

(i) *If*

$$(4,14) \qquad \det(\mathbf{I} - \Delta^- \mathbf{A}(t)) \det(\mathbf{I} - \Delta^+ \mathbf{B}(t)) \det(\mathbf{I} + \Delta^+ \mathbf{A}(t)) \neq 0 \qquad \text{on } [0,1]$$

or

$$(4,15) \qquad \det(\mathbf{I} - \Delta^- \mathbf{A}(t)) \det(\mathbf{I} - \Delta^+ \mathbf{B}(t)) \det(\mathbf{I} + \Delta^- \mathbf{B}(t)) \neq 0 \qquad \text{on } [0,1]\,,$$

then the fundamental matrices $\mathbf{U}(t,s)$ *to* (1,1) *and* $\mathbf{V}(t,s)$ *to* (4,1) *fulfil the relation*

$$(4,16)\quad \begin{aligned} \mathbf{V}(t,s) = {} & \mathbf{U}(t,s) + \mathbf{V}(t,s)[\mathbf{A}(s)-\mathbf{B}(s)] - [\mathbf{A}(t)-\mathbf{B}(t)]\,\mathbf{U}(t,s) \\ & + \mathbf{V}(t,s)\,\Delta^+\mathbf{B}(s)\,\Delta^+\mathbf{A}(s) - \Delta^-\mathbf{B}(t)\,\Delta^-\mathbf{A}(t)\,\mathbf{U}(t,s) \\ & + \sum_{s<\tau<t} \mathbf{V}(t,\tau)[\Delta^+\mathbf{B}(\tau)\,\Delta^+\mathbf{A}(\tau) - \Delta^-\mathbf{B}(\tau)\,\Delta^-\mathbf{A}(\tau)]\,\mathbf{U}(\tau,s) \quad \text{if } t>s, \\ \mathbf{V}(t,t) = {} & \mathbf{U}(t,t) = \mathbf{I}. \end{aligned}$$

(ii) *If*

$$(4,17)\quad \det(\mathbf{I}+\Delta^+\mathbf{A}(t))\det(\mathbf{I}+\Delta^-\mathbf{B}(t))\det(\mathbf{I}-\Delta^+\mathbf{B}(t)) \neq 0 \quad \text{on } [0,1]$$

or

$$(4,18)\quad \det(\mathbf{I}+\Delta^+\mathbf{A}(t))\det(\mathbf{I}+\Delta^-\mathbf{B}(t))\det(\mathbf{I}-\Delta^-\mathbf{A}(t)) \neq 0 \quad \text{on } [0,1],$$

then

$$(4,19)\quad \begin{aligned} \mathbf{V}(t,s) = {} & \mathbf{U}(t,s) + \mathbf{V}(t,s)[\mathbf{A}(s)-\mathbf{B}(s)] - [\mathbf{A}(t)-\mathbf{B}(t)]\,\mathbf{U}(t,s) \\ & + \mathbf{V}(t,s)\,\Delta^-\mathbf{B}(s)\,\Delta^-\mathbf{A}(s) - \Delta^+\mathbf{B}(t)\,\Delta^+\mathbf{A}(t)\,\mathbf{U}(t,s) \\ & + \sum_{t<\tau<s} \mathbf{V}(t,\tau)[\Delta^-\mathbf{B}(\tau)\,\Delta^-\mathbf{A}(\tau) - \Delta^+\mathbf{B}(\tau)\,\Delta^+\mathbf{A}(\tau)]\,\mathbf{U}(\tau,s) \quad \text{if } t<s, \\ \mathbf{V}(t,t) = {} & \mathbf{U}(t,t) = \mathbf{I}. \end{aligned}$$

(*In* (4,14)–(4,19) $\Delta^-\mathbf{A}(0) = \Delta^-\mathbf{B}(0) = \mathbf{0}$ *and* $\Delta^+\mathbf{A}(1) = \Delta^+\mathbf{B}(1) = \mathbf{0}$.)

Proof. Let e.g. (4,14) hold. Then $\mathbf{U}(t,s)$ is defined for all $t,s\in[0,1]$ and $\mathbf{V}(t,s)$ is defined for $t\geq s$. Let $t,s\in[0,1]$, $t>s$ be given and let us consider the expression

$$\mathbf{W} = \int_s^t \mathrm{d}_\tau[\mathbf{V}(t,\tau)]\,\mathbf{U}(\tau,t) + \int_s^t \mathbf{V}(t,\tau)\,\mathrm{d}_\tau[\mathbf{U}(\tau,t)].$$

Inserting into $\mathbf{W}$ from (2,4) and (4,5) and making use of the subsitution theorem I.4.25 we easily obtain

$$\mathbf{W} = \int_s^t \mathbf{V}(t,\tau)\,\mathrm{d}[\mathbf{A}(\tau)-\mathbf{B}(\tau)]\,\mathbf{U}(\tau,t)$$

and according to (4,11) and I.4.23

$$\begin{aligned} \mathbf{W} &= \mathbf{V}(t,s)[\Delta^+\mathbf{A}(s)-\Delta^+\mathbf{B}(s)]\,\mathbf{U}(s,t) + [\Delta^-\mathbf{A}(t)-\Delta^-\mathbf{B}(t)] \\ &= -\mathbf{V}(t,s)[\mathbf{A}(s)-\mathbf{B}(s)]\,\mathbf{U}(s,t) + [\mathbf{A}(t)-\mathbf{B}(t)] \end{aligned}$$

because the components of $\mathbf{A}(t)-\mathbf{B}(t)$ are evidently break functions on $[0,1]$. On the other hand, the integration-by-parts theorem I.4.33 yields

$$\begin{aligned} \mathbf{W} = \mathbf{I} - \mathbf{V}(t,s)\,\mathbf{U}(s,t) - \Delta_2^+\mathbf{V}(t,s)\,\Delta_1^+\mathbf{U}(s,t) + \Delta_2^-\mathbf{V}(t,t)\,\Delta_1^-\mathbf{U}(t,t) \\ + \sum_{s<\tau<t} [\Delta_2^-\mathbf{V}(t,\tau)\,\Delta_1^-\mathbf{U}(\tau,t) - \Delta_2^+\mathbf{V}(t,\tau)\,\Delta_1^+\mathbf{U}(\tau,t)], \end{aligned}$$

where $\Delta_1^+ \mathbf{Z}(t,s) = \mathbf{Z}(t+,s) - \mathbf{Z}(t,s)$, $\Delta_2^+ \mathbf{Z}(t,s) = \mathbf{Z}(t,s+) - \mathbf{Z}(t,s)$, $\Delta_1^- \mathbf{Z}(t,s) = \mathbf{Z}(t,s) - \mathbf{Z}(t-,s)$ and $\Delta_2^- \mathbf{Z}(t,s) = \mathbf{Z}(t,s) - \mathbf{Z}(t,s-)$ for $\mathbf{Z} = \mathbf{U}$ and $\mathbf{Z} = \mathbf{V}$. Taking into account the relations (4,6), (4,7), (4,10) and 2.10 we obtain immediately (4,16).

The remaining cases can be treated similarly. If (4,17) or (4,18) holds, then instead of the expression $\mathbf{W}$ we should handle the expression

$$\int_s^t \mathrm{d}_\tau[\mathbf{V}(s,\tau)]\,\mathbf{U}(\tau,s) + \int_s^t \mathbf{V}(s,\tau)\,\mathrm{d}_\tau[\mathbf{U}(\tau,s)]\,.$$

4.2. Theorem (Lagrange identity). *Let* $\mathbf{A}$: $[0,1] \to L(R_n)$ *and* $\mathbf{B}$: $[0,1] \to L(R_n)$ *be of bounded variation on* $[0,1]$ *and let* (4,11) *hold. Then for any* $\mathbf{x} \in BV_n$ *left-continuous on* $(0,1]$ *and right-continuous at* 0 *and any* $\mathbf{y} \in BV_n$ *right-continuous on* $[0,1)$ *and left-continuous at* 1

$$(4,20)\qquad \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}\left[\mathbf{x}(t) - \int_0^t \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s)\right] + \int_0^1 \mathrm{d}\left[\mathbf{y}^*(s) - \int_s^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{B}(t)]\right]\mathbf{x}(s)$$
$$= \mathbf{y}^*(1)\,\mathbf{x}(1) - \mathbf{y}^*(0)\,\mathbf{x}(0)\,.$$

Proof. Applying the substitution theorem I.4.25 the left-hand side of (4,20) reduces to

$$\int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{x}(t)] + \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\,\mathbf{x}(t) + \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{B}(t) - \mathbf{A}(t)]\,\mathbf{x}(t)\,.$$

The integration-by-parts formula I.4.33 yields

$$\int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{x}(t)] + \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\,\mathbf{x}(t) = \mathbf{y}^*(1)\,\mathbf{x}(1) - \mathbf{y}^*(0)\,\mathbf{x}(0)$$

whence by (4,11) and (4,12) our assertion follows.

4.3. Remark. The relations (4,16) and (4,19) are considerably simplified if

$$(4,21)\qquad \Delta^+\mathbf{B}(t)\,\Delta^+\mathbf{A}(t) = \Delta^-\mathbf{B}(t)\,\Delta^-\mathbf{A}(t) \quad \text{on } [0,1]\,.$$

This together with (4,11) and $\mathbf{A}(0) = \mathbf{B}(0)$ is true e.g. if

(i) $\mathbf{B} = \mathbf{A}$ and $(\Delta^+\mathbf{A}(t))^2 = (\Delta^-\mathbf{A}(t))^2$ on $[0,1]$, or
(ii) $\mathbf{B} = \mathbf{A}_*$ (cf. (4,13)), $(\Delta^+\mathbf{A}(0))^2 = (\Delta^-\mathbf{A}(1))^2 = \mathbf{0}$ and $\Delta^+\mathbf{A}(t)\,\Delta^-\mathbf{A}(t) = \Delta^-\mathbf{A}(t)\,\Delta^+\mathbf{A}(t)$ on $(0,1)$.

5. Two-point boundary value problem

Let $\mathbf{M}$ and $\mathbf{N}$ be $m \times n$-matrices and $\mathbf{r} \in R_m$. The problem of determining a solution $\mathbf{x}$: $[0,1] \to R_n$ to

$$(5,1)\qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{f}$$

on $[0,1]$, which fulfils in addition the relation

$$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) = \mathbf{r}\,, \tag{5,2}$$

is called the *two-point boundary value problem.*

5.1. Assumptions. *Throughout the section,* $\mathbf{A}, \mathbf{B}$ *are* $n \times n$*-matrix valued functions of bounded variation on* $[0,1]$. *Moreover we suppose that* (4,11) *with* $\mathbf{A}(0) = \mathbf{B}(0)$, (4,21) *and at least one of the conditions* (4,14), (4,15), (4,17), (4,18) *are satisfied.* (*In particular, the assumptions of* 4.1 *are fulfilled.*) $\mathbf{M}$ *and* $\mathbf{N}$ *are* $m \times n$*-matrices,* $\mathbf{f} \in BV_n$ *and* $\mathbf{r} \in R_m$, $m \geq 1$.

Making use of the variation-of-constants formula (2,15) we may reduce the boundary value problem (5,1), (5,2) to a linear nonhomogeneous algebraic equation.

5.2. Lemma. *If* (4,14) *or* (4,15) *holds, then* $\mathbf{x}\colon [0,1] \to R_n$ *is a solution of the problem* (5,1), (5,2) *if and only if*

$$\mathbf{x}(t) = \mathbf{U}(t,0)\,\mathbf{c} + \mathbf{f}(t) - \mathbf{f}(0) - \int_0^t \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) \qquad \textit{on } [0,1]\,, \tag{5,3}$$

where $\mathbf{c} \in R_n$ *is a solution to the algebraic equation*

$$[\mathbf{M} + \mathbf{N}\,\mathbf{V}(1,0)]\,\mathbf{c} = \mathbf{r} + \mathbf{N}\left\{\mathbf{V}(1,0)\,\mathbf{f}(0) - \mathbf{f}(1) + \int_0^1 \mathrm{d}_s[\mathbf{V}(1,s)]\,\mathbf{f}(s)\right\}.$$

If (4,17) *or* (4,18) *holds, then* $\mathbf{x}\colon [0,1] \to R_n$ *is a solution to* (5,1), (5,2) *if and only if*

$$\mathbf{x}(t) = \mathbf{U}(t,1)\,\mathbf{c} + \mathbf{f}(t) - \mathbf{f}(1) + \int_t^1 \mathrm{d}_s[\mathbf{U}(t,s)]\,(\mathbf{f}(s) - \mathbf{f}(1)) \qquad \textit{on } [0,1]\,,$$

where

$$[\mathbf{M}\,\mathbf{V}(0,1) + \mathbf{N}]\,\mathbf{c} = \mathbf{r} + \mathbf{M}\left\{-\mathbf{f}(0) + \mathbf{V}(0,1)\,\mathbf{f}(1) - \int_0^1 \mathrm{d}_s[\mathbf{V}(0,s)\,\mathbf{f}(s)\right\}.$$

Proof. Let (4,14) or (4,15) hold. Then by 2,15 $\mathbf{x}\colon [0,1] \to R_n$ is a solution of the given problem if and only if it is given by (5,3), where $\mathbf{c} \in R_n$ fulfils the equation

$$[\mathbf{M} + \mathbf{N}\,\mathbf{U}(1,0)]\,\mathbf{c} = \mathbf{r} + \mathbf{N}\left\{\mathbf{U}(1,0)\,\mathbf{f}(0) - \mathbf{f}(1) + \int_0^1 \mathrm{d}_s[\mathbf{U}(1,s)]\,\mathbf{f}(s)\right\}.$$

By (4,16) and (4,21)

$$\mathbf{V}(1,s) = \mathbf{U}(1,s) + \mathbf{V}(1,s)\,(\mathbf{A}(s) - \mathbf{B}(s)) + \mathbf{V}(1,s)\,\Delta^+\mathbf{B}(s)\,\Delta^+\mathbf{A}(s) \tag{5,4}$$

and thus

$$\mathbf{V}(1,s+) - \mathbf{U}(1,s+) = \mathbf{V}(1,s-) - \mathbf{U}(1,s-)$$

for any $s \in [0,1]$. (In particular $\mathbf{V}(1,0) = \mathbf{U}(1,0)$, $\mathbf{V}(1,1) = \mathbf{U}(1,1)$.) This implies by I.4.23

$$\int_0^1 d_s[\mathbf{U}(1,s)]\,\mathbf{v}(s) = \int_0^1 d_s[\mathbf{V}(1,s)]\,\mathbf{v}(s) \qquad \text{for any} \quad \mathbf{v} \in BV_n$$

wherefrom our assertion follows.

The cases (4,17) and (4,18) could be treated analogously. ($\mathbf{V}(0,s) = \mathbf{U}(0,s) + \mathbf{V}(0,s)\,(\mathbf{A}(s) - \mathbf{B}(s)) + \mathbf{V}(0,s)\,\Delta^-\mathbf{B}(s)\,\Delta^-\mathbf{A}(s)$ on $[0,1]$.)

5.3. Remark. Consequently, in the cases (4,14) or (4,15) the problem (5,1), (5,2) has a solution if and only if

$$\lambda^*[\mathbf{M} + \mathbf{N}\,\mathbf{V}(1,0)] = \mathbf{0} \tag{5,5}$$

implies

$$\lambda^*\mathbf{N}\,\mathbf{V}(1,1)\,\mathbf{f}(1) - \lambda^*\mathbf{N}\,\mathbf{V}(1,0)\,\mathbf{f}(0) - \int_0^1 d_s[\lambda^*\mathbf{N}\,\mathbf{V}(1,s)]\,\mathbf{f}(s) = \lambda^*\mathbf{r}\,. \tag{5,6}$$

Let us denote $\mathbf{y}_\lambda^*(s) = \lambda^*\mathbf{N}\,\mathbf{V}(1,s)$ for $s \in [0,1]$ and $\lambda \in R_m$. Then (5,6) becomes

$$\mathbf{y}_\lambda^*(1)\,\mathbf{f}(1) - \mathbf{y}_\lambda^*(0)\,\mathbf{f}(0) - \int_0^1 d[\mathbf{y}_\lambda^*(s)]\,\mathbf{f}(s) = \lambda^*\mathbf{r}\,.$$

By (4,8) for any $\lambda^* \in R_m^*$ and $s, s_0 \in [0,1]$

$$\mathbf{y}_\lambda^*(s) = \mathbf{y}_\lambda^*(s_0) + \int_s^{s_0} \mathbf{y}_\lambda^*(t)\,d[\mathbf{B}(t)]\,.$$

Moreover, if $\lambda^* \in R_m^*$ verifies (5,5), then $\mathbf{y}_\lambda^*(0) = \lambda^*\mathbf{N}\,\mathbf{V}(1,0) = -\lambda^*\mathbf{M}$ and $\mathbf{y}_\lambda^*(1) = \lambda^*\mathbf{N}$. Analogously, if (4,17) or (4,18) holds, the problem (5,1), (5,2) possesses a solution if and only if $\lambda^*[\mathbf{M}\,\mathbf{V}(0,1) + \mathbf{N}] = \mathbf{0}$ implies

$$\mathbf{y}_\lambda^*(1)\,\mathbf{f}(1) - \mathbf{y}_\lambda^*(0)\,\mathbf{f}(0) - \int_0^1 d[\mathbf{y}_\lambda^*(s)]\,\mathbf{f}(s) = \lambda^*\mathbf{r}\,,$$

where $\mathbf{y}_\lambda^*(s) = -\lambda^*\mathbf{M}\,\mathbf{V}(0,s)$ on $[0,1]$.

5.4. Lemma. *Let $\mathbf{g} \in BV_n$ and $\mathbf{p}, \mathbf{q} \in R_n$. If (4,14) or (4,15) holds, then $\mathbf{y}^*\colon [0,1] \to R_n^*$ is a solution to the generalized differential equation*

$$d\mathbf{y}^* = -\mathbf{y}^*\,d[\mathbf{B}] + d\mathbf{g}^* \qquad \text{on } [0,1] \tag{5,7}$$

and together with $\lambda^* \in R_m^*$ verifies the relations

$$\mathbf{y}^*(0) + \lambda^*\mathbf{M} = \mathbf{p}^*\,, \qquad \mathbf{y}^*(1) - \lambda^*\mathbf{N} = \mathbf{q}^* \tag{5,8}$$

if and only if

$$\mathbf{y}^*(s) = (\lambda^*\mathbf{N} + \mathbf{q}^*)\,\mathbf{V}(1,s) + \mathbf{g}^*(s) - \mathbf{g}^*(1) + \int_s^1 (\mathbf{g}^*(t) - \mathbf{g}^*(1))\,d_t[\mathbf{V}(t,s)] \tag{5,9}$$

on $[0,1]$ *and*

$$\lambda^*[\mathbf{M} + \mathbf{N}\,\mathbf{U}(1,0)]$$

$$= \mathbf{p}^* - \mathbf{q}^*\,\mathbf{U}(1,0) - \mathbf{g}^*(0) - \mathbf{g}^*(1)\,\mathbf{U}(1,0) + \int_0^1 \mathbf{g}^*(t)\,\mathrm{d}_t[\mathbf{U}(t,0)]\,.$$

(*By* (4,16) $\mathbf{V}(t,0) - \mathbf{U}(t,0) = (\mathbf{A}(t) - \mathbf{B}(t))\,\mathbf{U}(t,0) + \Delta^-\mathbf{B}(t)\,\Delta^-\mathbf{A}(t)\,\mathbf{U}(t,0)$.)

If (4,17) *or* (4,18) *holds, then* $\mathbf{y}^*$: $[0,1] \to R_n^*$ *and* $\lambda^* \in R_m^*$ *verify the system* (5,7), (5,8) *if and only if*

$$(5,10)\quad \mathbf{y}^*(s) = (\mathbf{p}^* - \lambda^*\mathbf{M})\,\mathbf{V}(0,s) + \mathbf{g}^*(s) - \mathbf{g}^*(0) - \int_0^s (\mathbf{g}^*(t) - \mathbf{g}^*(0))\,\mathrm{d}_t[\mathbf{V}(t,s)] \quad \text{on } [0,1]$$

and

$$\lambda^*[\mathbf{M}\,\mathbf{U}(0,1) + \mathbf{N}]$$

$$= \mathbf{p}^*\,\mathbf{U}(0,1) - \mathbf{q}^* + \mathbf{g}^*(1) - \mathbf{g}^*(0)\,\mathbf{U}(0,1) - \int_0^1 \mathbf{g}^*(t)\,\mathrm{d}_t[\mathbf{U}(t,1)]\,.$$

($\mathbf{V}(t,1) - \mathbf{U}(t,1) = (\mathbf{A}(t) - \mathbf{B}(t))\,\mathbf{U}(t,1) + \mathbf{V}(t,1)\,\Delta^+\mathbf{B}(t)\,\Delta^+\mathbf{A}(t)$ *by* (4,19).)

Proof. In virtue of our assumption (4,21) the fundamental matrices $\mathbf{U}(t,s)$ and $\mathbf{V}(t,s)$ fulfil the relation (5,4). Inserting (4,8) or (4,9) into (5,8) we complete the proof.

5.5. Theorem. *Under the assumptions* 5.1 *the given problem* (5,1), (5,2) *possesses a solution if and only if*

$$(5,11)\qquad \mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{y}^*(t)]\,\mathbf{f}(t) = \lambda^*\mathbf{r}$$

for any solution $(\mathbf{y}^*, \lambda^*)$ *of the homogeneous system*

$$(5,12)\qquad \mathrm{d}\mathbf{y}^* = -\mathbf{y}^*\,\mathrm{d}[\mathbf{B}] \quad \text{on } [0,1]\,,$$

$$(5,13)\qquad \mathbf{y}^*(0) + \lambda^*\mathbf{M} = \mathbf{0}\,, \qquad \mathbf{y}^*(1) - \lambda^*\mathbf{N} = \mathbf{0}\,.$$

Proof follows immediately from 5.2 (cf. also 5.3).

5.6. Theorem. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{N}$ *fulfil* 5.1. *Then given* $\mathbf{g} \in BV_n$ *and* $\mathbf{p}, \mathbf{q} \in R_n$ *the system* (5,7), (5,8) *possesses a solution if and only if*

$$\mathbf{g}^*(1)\,\mathbf{x}(1) - \mathbf{g}^*(0)\,\mathbf{x}(0) - \int_0^1 \mathbf{g}^*(s)\,\mathrm{d}[\mathbf{x}(s)] = \mathbf{q}^*\,\mathbf{x}(1) - \mathbf{p}^*\,\mathbf{x}(0)$$

for any solution $\mathbf{x}$ *of the homogeneous equation*

$$(5,14)\qquad \mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} \quad \text{on } [0,1]$$

which fulfils also

$$(5,15)\qquad \mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) = \mathbf{0}\,.$$

Proof. If (4,14) or (4,15) holds, then by 5.4 the system (5,7), (5,8) possesses a solution if and only if

$$[\boldsymbol{M} + \boldsymbol{N}\,\boldsymbol{U}(1,0)]\,\boldsymbol{c} = \boldsymbol{0} \tag{5,16}$$

implies

$$\boldsymbol{q}^*\,\boldsymbol{x}_c(1) - \boldsymbol{p}^*\,\boldsymbol{x}_c(0) = \boldsymbol{g}^*(1)\,\boldsymbol{x}_c(1) - \boldsymbol{g}^*(0)\,\boldsymbol{x}_c(0) - \int_0^1 \boldsymbol{g}^*(s)\,\mathrm{d}[\boldsymbol{x}_c(s)]\,,$$

where $\boldsymbol{x}_c(t) = \boldsymbol{U}(t,0)\,\boldsymbol{c}$ for $t \in [0,1]$ and $\boldsymbol{c} \in R_n$. By 5.2 $\boldsymbol{x}$: $[0,1] \to R_n$ is a solution to (5,14), (5,15) if and only if $\boldsymbol{x}(t) = \boldsymbol{U}(t,0)\,\boldsymbol{c}$ on $[0,1]$ where $\boldsymbol{c} \in R_n$ verifies (5,16). Now, our assertion follows readily.

5.7. Definition. The system (5,12), (5,13) of equations for $\boldsymbol{y}^*$: $[0,1] \to R_n^*$ and $\lambda^* \in R_m^*$ is called the *adjoint boundary value problem* to the problem (5,1), (5,2) (or (5,14), (5,15)).

5.8. Definition. The homogeneous problem (5,14), (5,15) (or (5,12), (5,13)) has *exactly k linearly independent solutions* if it has at least k linearly independent solutions on $[0,1]$, while any set of its solutions which contains at least $k+1$ elements is linearly dependent on $[0,1]$.

Another interesting question is the *index of the boundary value problem*, i.e. the relationship between the number of linearly independent solutions to the homogeneous problem (5,14), (5,15) and its adjoint.

5.9. Remark. Without any loss of generality we may assume rank $[\boldsymbol{M}, \boldsymbol{N}] = m$. In fact, if rank $[\boldsymbol{M}, \boldsymbol{N}] = m_1 < m$, then there exists a regular $m \times n$-matrix $\boldsymbol{\Theta}$ such that

$$\boldsymbol{\Theta}[\boldsymbol{M}, \boldsymbol{N}] = \begin{bmatrix} \boldsymbol{M}_1, & \boldsymbol{N}_1 \\ \boldsymbol{0}, & \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{M}_1, \boldsymbol{N}_1 \in L(R_n, R_{m_1})$ are such that rank $[\boldsymbol{M}_1, \boldsymbol{N}_1] = m_1$. Let $\boldsymbol{r} \in R_m$, $\boldsymbol{\Theta r} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}$, $\boldsymbol{r}_1 \in R_{m_1}$ and $\boldsymbol{r}_2 \in R_{m-m_1}$. Then either $\boldsymbol{r}_2 \neq 0$ and the equation for $\boldsymbol{d} \in R_{2n}$

$$[\boldsymbol{M}, \boldsymbol{N}]\,\boldsymbol{d} = \boldsymbol{r} \tag{5,17}$$

possesses no solution or $\boldsymbol{r}_2 = \boldsymbol{0}$ and (5,17) is equivalent to $[\boldsymbol{M}_1, \boldsymbol{N}_1]\,\boldsymbol{d} = \boldsymbol{r}_1$.

5.10. Theorem. *Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{N}$ *fulfil* 5.1 *and* rank $[\boldsymbol{M}, \boldsymbol{N}] = m$. *Then both the homogeneous problem* (5,14), (5,15) *and its adjoint* (5,12), (5,13) *possesses at most a finite number of linearly independent solutions on* $[0,1]$. *Let* (5,14), (5,15) *possess exactly k linearly independent solutions on* $[0,1]$ *and let* (5,12), (5,13) *possess exactly* k^* *linearly independent solutions on* $[0,1]$. *Then* $k^* - k = m - n$.

Proof. Let us assume e.g. (4,14). By 5.2 the system (5,14), (5,15) possesses exactly $k = n - \operatorname{rank}[\boldsymbol{M} + \boldsymbol{N}\,\boldsymbol{U}(1,0)]$ linearly independent solutions on $[0,1]$. (If $\boldsymbol{c}_j \in R_n$ are linearly independent solutions to (5,16), then since $\boldsymbol{U}(0,0) = \boldsymbol{I}$, the functions $\boldsymbol{x}_j(t) = \boldsymbol{U}(t,0)\,\boldsymbol{c}_j$ are linearly independent solutions on $[0,1]$ of the system (5,14), (5,15).)

On the other hand, the equation (5,5) has exactly $m - \operatorname{rank}[\boldsymbol{M} + \boldsymbol{N}\,\boldsymbol{U}(1,0)] = h$ linearly independent solutions. Let Λ denote an arbitrary $h \times n$-matrix whose rows $\boldsymbol{\lambda}_1^*, \boldsymbol{\lambda}_2^*, \ldots, \boldsymbol{\lambda}_h^*$ are linearly independent solutions of (5,5). Let us assume that the functions $\boldsymbol{y}_j^*(s) = \boldsymbol{\lambda}_j^*\boldsymbol{N}\,\boldsymbol{V}(1,s)$ are linearly dependent on $[0,1]$, i.e. there is $\boldsymbol{\alpha} \in R_h$, $\boldsymbol{\alpha} \neq \boldsymbol{0}$ such that $\boldsymbol{\alpha}^*\Lambda\boldsymbol{N}\,\boldsymbol{V}(1,s) \equiv \boldsymbol{0}$ on $[0,1]$. In particular, $\boldsymbol{0} = \boldsymbol{\alpha}^*\Lambda\boldsymbol{N}\,\boldsymbol{V}(1,1) = \boldsymbol{\alpha}^*\Lambda\boldsymbol{N}$ and $\boldsymbol{0} = \boldsymbol{\alpha}^*\Lambda\boldsymbol{N}\,\boldsymbol{V}(1,0) = -\boldsymbol{\alpha}^*\Lambda\boldsymbol{M}$. Since (5,17), $\boldsymbol{\alpha}^*\Lambda = \boldsymbol{0}$ and by the definition of Λ it is $\boldsymbol{\alpha} = \boldsymbol{0}$. This being a contradiction, $k^* = m - \operatorname{rank}[\boldsymbol{M} + \boldsymbol{N}\,\boldsymbol{U}(1,0)]$ and $k^* - k = m - n$.

5.11. Definition. Given $m \times n$-matrices $\boldsymbol{M}$, $\boldsymbol{N}$ with $\operatorname{rank}[\boldsymbol{M}, \boldsymbol{N}] = m$, any $(2n - m) \times n$-matrices $\boldsymbol{M}^c$, $\boldsymbol{N}^c$ such that

(5,18) $$\det\begin{bmatrix} \boldsymbol{M}, & \boldsymbol{N} \\ \boldsymbol{M}^c, & \boldsymbol{N}^c \end{bmatrix} \neq 0$$

are called the *complementary matrices to* $[\boldsymbol{M}, \boldsymbol{N}]$.

5.12. Proposition. *Let* $\boldsymbol{M}, \boldsymbol{N} \in L(R_n, R_m)$, $\operatorname{rank}[\boldsymbol{M}, \boldsymbol{N}] = m$ *and let* $\boldsymbol{M}^c, \boldsymbol{N}^c \in L(R_n, R_{2n-m})$ *be arbitrary matrices complementary to* $[\boldsymbol{M}, \boldsymbol{N}]$. *Then there exist uniquely determined matrices* $\boldsymbol{P}, \boldsymbol{Q} \in L(R_{2n-m}, R_n)$ *and* $\boldsymbol{P}^c, \boldsymbol{Q}^c \in L(R_m, R_n)$ *such that*

(5,19) $$\det\begin{bmatrix} \boldsymbol{P}^c, & \boldsymbol{Q} \\ \boldsymbol{Q}^c, & \boldsymbol{Q} \end{bmatrix} \neq 0$$

and $\boldsymbol{y}_1^*\boldsymbol{x}_1 - \boldsymbol{y}_0^*\boldsymbol{x}_0 = (\boldsymbol{y}_0^*\boldsymbol{P}^c + \boldsymbol{y}_1^*\boldsymbol{Q}^c)(\boldsymbol{M}\boldsymbol{x}_0 + \boldsymbol{N}\boldsymbol{x}_1) + (\boldsymbol{y}_0^*\boldsymbol{P} + \boldsymbol{y}_1^*\boldsymbol{Q})(\boldsymbol{M}^c\boldsymbol{x}_0 + \boldsymbol{N}^c\boldsymbol{x}_1)$ *for all* $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}_0, \boldsymbol{y}_1 \in R_n$.

Proof. Let $\boldsymbol{P}, \boldsymbol{Q} \in L(R_{2n-m}, R_n)$ and $\boldsymbol{P}^c, \boldsymbol{Q}^c \in L(R_m, R_n)$ be such that

(5,20) $$\begin{bmatrix} \boldsymbol{M}, & \boldsymbol{N} \\ \boldsymbol{M}^c, & \boldsymbol{N}^c \end{bmatrix}^{-1} = \begin{bmatrix} -\boldsymbol{P}^c, & -\boldsymbol{P} \\ \boldsymbol{Q}^c, & \boldsymbol{Q} \end{bmatrix}.$$

Then

(5,21) $$-\boldsymbol{P}^c\boldsymbol{M} - \boldsymbol{P}\boldsymbol{M}^c = \boldsymbol{I}_n, \qquad -\boldsymbol{P}^c\boldsymbol{N} - \boldsymbol{P}\boldsymbol{N}^c = \boldsymbol{0},$$
$$\boldsymbol{Q}^c\boldsymbol{M} + \boldsymbol{Q}\boldsymbol{M}^c = \boldsymbol{0}, \qquad \boldsymbol{Q}^c\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{N}^c = \boldsymbol{I}_n$$

and

(5,22) $$\begin{bmatrix} \boldsymbol{P}^c, & \boldsymbol{P} \\ \boldsymbol{Q}^c, & \boldsymbol{Q} \end{bmatrix}\begin{bmatrix} \boldsymbol{M}, & \boldsymbol{N} \\ \boldsymbol{M}^c, & \boldsymbol{N}^c \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I}_n, & \boldsymbol{0} \\ \boldsymbol{0}, & \boldsymbol{I}_n \end{bmatrix}.$$

Thus, given $\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}_0, \mathbf{y}_1 \in R_n$,

$$\begin{aligned}
\mathbf{y}_1^*\mathbf{x}_1 - \mathbf{y}_0^*\mathbf{x}_0 &= (\mathbf{y}_0^*, \mathbf{y}_1^*)\begin{bmatrix} -\mathbf{I}_n, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}_n \end{bmatrix}\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix} \\
&= (\mathbf{y}_0^*, \mathbf{y}_1^*)\begin{bmatrix} \mathbf{P}^c, & \mathbf{P} \\ \mathbf{Q}^c, & \mathbf{Q} \end{bmatrix}\begin{bmatrix} \mathbf{M}, & \mathbf{N} \\ \mathbf{M}^c, & \mathbf{N}^c \end{bmatrix}\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix} \\
&= (\mathbf{y}_0^*\mathbf{P}^c + \mathbf{y}_1^*\mathbf{Q}^c)(\mathbf{M}\mathbf{x}_0 + \mathbf{N}\mathbf{x}_1) + (\mathbf{y}_0^*\mathbf{P} + \mathbf{y}_1^*\mathbf{Q})(\mathbf{M}^c\mathbf{x}_0 + \mathbf{N}^c\mathbf{x}_1).
\end{aligned}$$

5.13. Remark. It follows from (5,20) that according to 5.12 the matrices $\mathbf{P}, \mathbf{Q} \in L(R_{2n-m}, R_n)$ and $\mathbf{P}^c, \mathbf{Q}^c \in L(R_m, R_n)$ associated to $\mathbf{M}, \mathbf{N}, \mathbf{M}^c, \mathbf{N}^c$ fulfil besides (5,21), (5,22) also

$$\begin{bmatrix} -\mathbf{M}, & \mathbf{N} \\ -\mathbf{M}^c, & \mathbf{N}^c \end{bmatrix}\begin{bmatrix} \mathbf{P}^c, & \mathbf{P} \\ \mathbf{Q}^c, & \mathbf{Q} \end{bmatrix} = \mathbf{I}_{2n},$$

i.e.

(5,23) $$-\mathbf{M}\mathbf{P}^c + \mathbf{N}\mathbf{Q}^c = \mathbf{I}_m, \qquad -\mathbf{M}\mathbf{P} + \mathbf{N}\mathbf{Q} = \mathbf{0},$$

(5,24) $$-\mathbf{M}^c\mathbf{P}^c + \mathbf{N}^c\mathbf{Q}^c = \mathbf{0}, \qquad -\mathbf{M}^c\mathbf{P} + \mathbf{N}^c\mathbf{Q} = \mathbf{I}_{2n-m}.$$

The following assertion is evident.

5.14. Proposition. *Let $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$,* rank $[\mathbf{M}, \mathbf{N}] = m$ *and let $\mathbf{P}, \mathbf{Q} \in L(R_{2n-m}, R_n)$ and $\mathbf{P}^c, \mathbf{Q}^c \in L(R_m, R_n)$ be such that (5,19) and (5,23) hold. Then $\mathbf{P}_1, \mathbf{Q}_1 \in L(R_{2n-m}, R_n)$ and $\mathbf{P}_1^c, \mathbf{Q}_1^c \in L(R_m, R_n)$ fulfil also (5,19) and (5,23) if and only if there exist a regular matrix $\mathbf{E} \in L(R_{2n-m})$ and $\mathbf{F} \in L(R_m, R_{2n-m})$ such that*

(5,25) $$\mathbf{P}_1 = \mathbf{P}\mathbf{E}, \qquad \mathbf{Q}_1 = \mathbf{Q}\mathbf{E}$$

and

(5,26) $$\mathbf{P}_1^c = \mathbf{P}^c + \mathbf{P}\mathbf{F}, \qquad \mathbf{Q}_1^c = \mathbf{Q}^c + \mathbf{Q}\mathbf{F}.$$

5.15. Definition. Let $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$ and let $\mathbf{P}, \mathbf{Q} \in L(R_{2n-m}, R_n)$ and $\mathbf{P}^c, \mathbf{Q}^c \in L(R_m, R_n)$ be such that (5,19) and (5,23) hold. Then the matrices $\mathbf{P}, \mathbf{Q}$ are called *adjoint matrices associated to* $[\mathbf{M}, \mathbf{N}]$ and the matrices $\mathbf{P}^c, \mathbf{Q}^c$ are called *complementary adjoint matrices associated to* $[\mathbf{M}, \mathbf{N}]$.

5.16. Remark. If $\mathbf{M}, \mathbf{N} \in L(R_m, R_n)$, rank $[\mathbf{M}, \mathbf{N}] = m$ and if $\mathbf{P}, \mathbf{Q} \in L(R_{2n-m}, R_n)$ are arbitrary adjoint matrices associated to $\mathbf{M}, \mathbf{N}$, then

(5,27) $$\operatorname{rank}\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = 2n - m$$

and the rows of the $m \times 2n$-matrix $[-\mathbf{M}, \mathbf{N}]$ form a basis in the space of all solutions $\mathbf{d}^* \in R_{2n}^*$ to the equation

(5,28) $$\mathbf{d}^*\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \mathbf{0}.$$

5.17. Remark. Let $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$ and $\operatorname{rank}[\mathbf{M}, \mathbf{N}] = m$. Let $\mathbf{P}, \mathbf{Q}$ and $\mathbf{P}^c, \mathbf{Q}^c$ be respectively adjoint and complementary adjoint matrices to $[\mathbf{M}, \mathbf{N}]$. If $\mathbf{y}^*: [0, 1] \to R_n^*$ and $\boldsymbol{\lambda}^* \in R_m^*$ fulfil (5,13), then

$$\mathbf{y}^*(0)\,\mathbf{P} + \mathbf{y}^*(1)\,\mathbf{Q} = \mathbf{0} \tag{5,29}$$

and

$$\mathbf{y}^*(0)\,\mathbf{P}^c + \mathbf{y}^*(1)\,\mathbf{Q}^c = \boldsymbol{\lambda}^*. \tag{5,30}$$

On the other hand, if $\mathbf{y}^*: [0, 1] \to R_n^*$ fulfils (5,29), then there exists $\boldsymbol{\lambda}^* \in R_m^*$ such that (5,13) and consequently also (5,30) hold (cf. 5.16).

5.18. Corollary. *Let the assumptions* 5,1 *be fulfilled. Then the boundary value problem* (5,1), (5,2) *has a solution if and only if*

$$\mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{y}^*(s)]\,\mathbf{f}(s) = [\mathbf{y}^*(0)\,\mathbf{P}^c + \mathbf{y}^*(1)\,\mathbf{Q}^c]\,\mathbf{r} \tag{5,31}$$

for any solution $\mathbf{y}^*: [0, 1] \to R_n^*$ *of the system* (5,12), (5,29) *where* $\mathbf{P}, \mathbf{Q}$ *and* $\mathbf{P}^c, \mathbf{Q}^c$ *are respectively adjoint and complementary adjoint matrices associated to* $[\mathbf{M}, \mathbf{N}]$.

Proof follows immediately from 5.5 and 5.17.

5.19. Remark. If $\mathbf{P}_1, \mathbf{Q}_1$ and $\mathbf{P}_1^c, \mathbf{Q}_1^c$ are also adjoint and complementary adjoint matrices associated to $[\mathbf{M}, \mathbf{N}]$, then by 5.14 there exist a regular matrix $\mathbf{E} \in L(R_{2n-m})$ and $\mathbf{F} \in L(R_m, R_{2n-m})$ such that for all $\mathbf{y}_0^*, \mathbf{y}_1^* \in R_n^*$ we have $\mathbf{y}_0^*\mathbf{P}_1 + \mathbf{y}_1^*\mathbf{Q}_1 = [\mathbf{y}_0^*\mathbf{P} + \mathbf{y}_1^*\mathbf{Q}]\,\mathbf{E}$ and $\mathbf{y}_0^*\mathbf{P}_1^c + \mathbf{y}_1^*\mathbf{Q}_1^c = \mathbf{y}_0^*\mathbf{P}^c + \mathbf{y}_1^*\mathbf{Q}^c + [\mathbf{y}_0^*\mathbf{P} + \mathbf{y}_1^*\mathbf{Q}]\,\mathbf{F}$. Thus $\mathbf{y}_0^*\mathbf{P} + \mathbf{y}_1^*\mathbf{Q} = \mathbf{0}$ and $\mathbf{y}_0^*\mathbf{P}^c + \mathbf{y}_1^*\mathbf{Q}^c = \boldsymbol{\lambda}^*$ if and only if also $\mathbf{y}_0^*\mathbf{P}_1 + \mathbf{y}_1^*\mathbf{Q}_1 = \mathbf{0}$ and $\mathbf{y}_0^*\mathbf{P}_1^c + \mathbf{y}_1^*\mathbf{Q}_1^c = \boldsymbol{\lambda}^*$. This means that neither the boundary condition (5,29) nor the condition (5,31) depend on the choice of the adjoint and complementary adjoint matrices associated to $[\mathbf{M}, \mathbf{N}]$.

5.20. Remark. The matrix valued functions $\mathbf{A}: [0, 1] \to L(R_n)$ and $\mathbf{B}: [0, 1] \to L(R_n)$ of bounded variation on $[0, 1]$ fulfil 5.1 e.g. if

(i) $\mathbf{A}$ is left-continuous on $(0, 1]$ and right-continuous at 0, $\det[\mathbf{I} + \Delta^+\mathbf{A}(t)] \neq 0$ on $[0, 1]$ and $\mathbf{B} = \mathbf{A}_*$ (cf. (4,13)), or

(ii) $(\Delta^+\mathbf{A}(0))^2 = (\Delta^-\mathbf{A}(1))^2 = \mathbf{0}$, $(\Delta^+\mathbf{A}(t))^2 = (\Delta^-\mathbf{A}(t))^2$ on $(0, 1)$, $\det[\mathbf{I} - (\Delta^+\mathbf{A}(t)]^2 \neq 0$ on $[0, 1]$ and $\mathbf{B} = \mathbf{A}$, or

(iii) $\Delta^+\mathbf{A}(t) = \Delta^-\mathbf{A}(t)$ on $[0, 1]$, $(\Delta^+\mathbf{A}(t))^2 = \mathbf{0}$ on $[0, 1]$ and $\mathbf{B} = \mathbf{A}$.

(In the case (iii)

$$[\mathbf{I} + \Delta^+\mathbf{A}(t)]\,[\mathbf{I} - \Delta^-\mathbf{A}(t)] = \mathbf{I} - (\Delta^+\mathbf{A}(t))^2 = \mathbf{I}\,.)$$

We shall see later that the problems of the type (5,1), (5,2) cover also problems with a more general side condition (cf. V.7.19).

Notes

The theory of generalized differential equations was initiated by J. Kurzweil [1], [2], [4]. It is based on the generalization of the concept of the Perron integral; special results needed in the linear case are given in I.4. Differential equations with discontinuous solutions are considered e.g. in Stallard [2], Ligęza [2].

The paper by Hildebrandt [2] is devoted to linear differentio-Stieltjes integral equations. These equations are essentially generalized linear differential equations in our setting where the Young integral is used for the definition of a solution. Some results for the equations of this type can be found in Atkinson [1], Hönig [1], Schwabik [1], [4], Schwabik, Tvrdý [1], Mac Nerney [1], Wall [1].

Boundary value problems for generalized differential equations were for the first time mentioned in Atkinson [1] (Chapter XI). They appeared also in Halanay, Moro [1] as adjoints to boundary value problems with Stieltjes integral side conditions. A systematic study of such problems was initiated in Vejvoda, Tvrdý [1] and Tvrdý [1], [2]. Further related references are Krall [6], [8], Ligęza [1] and Zimmerberg [1], [2].