Similarly, on taking OJ on OP of the length OI, so that

$$\text{rect. } (OI, OJ) = \text{square on } OI,$$

we have $$OJ : OP :: \text{rect. } (OI, OJ) : \text{rect. } (OI, OP),$$

so that, on using p for the tensor of OP,

$$\text{rect. } (OI, OP) = p \,.\, \text{square on } OI,$$

whence $\text{rect. } (OM, OP) = mp \,.\, \text{square on } OI = OM \,.\, OP \,.\, \text{square on } OI$, on using the ordinary notation. The proof is here conducted by proportion only, and quite independently of commensurability, so that the objections to the "algebraical proof"—really, commensurable proof—of Euc. ii. no longer hold.

Referred to cube on OI, it becomes a relation of rectangular solids, having one constant dimension OI.

(iv.) The demonstration of all these relations flows at once from the laws of tensors. But there is no room for negatives or imaginaries in an algebra derived from the geometry of magnitude only. The laws of both are obtained at once from the geometry of direction, as follows.

IV. THE LAWS OF CLINANTS, OR THE ALGEBRA OF SIMILAR TRIANGLES LYING ON THE SAME PLANE.

20. *Data from the Geometry of Direction.*—(i.) The following propositions are borrowed from the geometry of *direction*, as opposed to that of *ratio*, or magnitude only. See fig. 5.

(ii.) AB, without further limitation, always represents the line AB, as respects both magnitude and direction, considered as the trace of the motion of a point along a straight line *from A to B*, A being its *initial* and B its *final* point. When length is considered without direction, write len AB, and read "length of AB."

(iii.) The equation $AB = CD$ implies that AB and CD are the opposite sides of the parallelogram $ABDCA$, the points lying in this order.

(iv.) *Directional Addition* is defined by the equation

$$AB + AC = AB + BD = AD,$$

or the sum of two adjacent sides of a parallelogram measured from their point of intersection is the included diagonal. And since in this case

$$AC + AB = AC + CD = AD,$$

directional addition (which is quite different in its results from quantitative or *rational* addition) is commutative. It is also easily shewn to be associative.

(v.) *Directional Subtraction* is defined by the equation

$$AB - AC = AB + CA = AB + BE = AE = CB,$$

or the directional difference of two adjacent sides of a parallelogram is the transverse (non-included) diagonal, measured from the final extremity of the subtrahend to that of the minuend.

(vi.) IOI', JOJ' (fig. 6 and 8) are diameters of a *standard* (or unit) circle, drawn at right angles. The position of centre O (taken as *origin*) and radius OI, (taken as the *standard* of *length* and of *original* direction,) and the direction of right angle IOJ, (taken as standard of angular rotation,) are arbitrary, but once fixed remain throughout the problem, and determine the plane IOJ on which all points are situate. The above lettering (founded on Sir W. R. Hamilton's) is assumed throughout.

(vii.) M, N, P (fig. 6) being any points *in the plane*, (these words omitted in future,) and M', N', P' the points in which the *straight* (this word omitted in future) lines OM, ON, OP cut the standard circle, then the arc $M'N'$ is measured from M' to N' through *not more than* a semicircle, so that its direction is not ambiguous except for a semicircle such as arc II', which may be either arc IJI' or arc $IJ'I'$. If the length of chord $M'N'$ = length of chord $M''N''$, and direction of arcs $M'N'$, $M''N''$ the same, then arc $M'N'$ = arc $M''N''$; but ch $M'N'$ is not = ch $M''N''$, unless they are coincident, and so of all similar cases.

(viii.) Always, arc $M'N'$+arc $N'P'$ = arc $M'P'$,
arc $M'P'-N'P'$= arc $M'P'$+arc $P'N'$= arc $M'N'$,
and the law of association also holds. All this is similar to, but different from (iv. v.)

(ix.) The directional sum of any number of directed arcs is therefore a directed arc not exceeding a semicircle. If m be an integer, then (see fig. 7, where $m=3$) m.arc IX = arc IV, determines IV unambiguously when IX is known; but there are m different directed arcs $IX_1, IX_2 \dots IX_m$ which satisfy the condition m . arc IX = arc IV when IV is known. And, assuming the power of finding an arc whose *length* bears any given ratio to that of a given arc, if m is an incommensurable tensor, the above equation admits of an infinite number of solutions. This is the source of the ambiguity of equations in all cases.

(x.) The conclusions in vii., viii., ix., hold for any *directed* angle, or $\angle MON$, fig. 6, subtended by the directed arc $M'N'$, and having OM for its initial and ON for its final arm. Under these conditions no directed angle greater than the directional sum of two right angles in the same direction can occur (art. 3. iv.). $\angle IOI$ is called a *null* angle, $\angle IOI'$ a *straight* angle. If $AB = OM$ and $CD = ON$, by $\angle (AB, CD)$ is meant $\angle MON = \angle M'ON'$. When the *amount of rotation* in angles is alone considered, independently of the direction, the angles are said to be *rationally* equal. Shew this thus, amt MON, and read "amount of angle MON."

(xi.) By *directionally* similar triangles are meant similar triangles in which the rationally equal angles are also *directionally equal*, so that their *differences* two and two = $\angle IOI$. By *conjugately* similar triangles are meant similar triangles in which the rationally equal angles are *directionally opposite*, so that their *sums* two and two = $\angle IOI$. Any three separate points determine a triangle, whether they do or do not lie on the same straight line. The notation $ABC \triangle A'B'C'$ (read "ABC sim $A'B'C'$") denotes the directional, and $ABC \bigtriangledown A'B'C'$ (read "ABC con-sim $A'B'C'$") the conjugate similarity of the triangles ABC, $A'B'C'$. Sir W. R. Hamilton (Elements of Quaternions, p. 112, art. 118) uses the terms *directly* and *inversely* similar, and the notations $\triangle ABC \propto A'B'C'$, and $\triangle ABC \propto' A'B'C'$, for the present $ABC \triangle A'B'C'$ and $ABC \bigtriangledown A'B'C'$ respectively.

21. *Directionally similar Triangles expressed by Clinants.*—(i.) This being premised, let A, B, fig. 8, be any points, and determine C, so that $IOA \triangle BOC$, which results from a simple geometrical construction. In this case also $IOB \triangle AOC$.

(ii.) Then OC is found by an *operation* on OB, determined by the point A, which operation will be called a *clinant*, a term introduced by myself in 1855, see Appendix III. It will be marked by the small letter a, corresponding to the large letter A, by which the point is noted. The *tensor* of OA (art. 12. iii.) will henceforth be marked Ta, as explained in art. 26. ii. It is sometimes convenient to use small Greek letters, α, β, γ, δ, ϵ, for clinants. In such cases, when the corresponding capitals are the same in the Latin and Greek alphabets, I find it necessary to distinguish the latter by an apostrophe, which is not otherwise used, thus A', B', Γ, Δ, E', see fig. 33. The result is then written (read $a . OB$ as "a ante OB"), $OC = a . OB$, from $IOA \;\Delta\; BOC$,

and $OC = b . OA$, from $IOB \;\Delta\; AOC$.

22. *Commutative and Associative Multiplication of Clinants.*—(i.) From art. 21. ii., $OA = a . OI$, &c., fig. 8; and expressing c by ab in the first case, and ba in the second, we obtain, as in art. 13, $c = ab = ba$, or *clinants are commutative in multiplication.*

(ii.) In this case

len OI : len OA :: len OB : len OC, so that $Tc = Ta . Tb$,

and $\angle IOC = \angle IOB + \angle BOC = \angle IOB + \angle IOA = \angle IOA + \angle IOB$.

(iii.) Then if A, B, C are any points, the proved association of tensors in multiplication, art. 15. ii., and of directed angles in directional addition, immediately establishes that $a . bc = ab . c$, or that *clinants are associative in multiplication.*

23. *Division of Clinants.*—(i.) When we have given A, C to find B (fig. 8), or B, C to find A, on the condition that $c = ab$, the geometrical operations are of the same kind as before, and will be represented by

$$b = \frac{c}{a}, \quad a = \frac{c}{b};$$

whence $$\frac{c}{a} \cdot a = ba = c, \quad \frac{ab}{a} = \frac{c}{a} = b,$$

which are the laws of division.

(ii.) If then (fig. 9) we have $m = \dfrac{b}{a}$, and $m = \dfrac{d}{c}$, A, B, C, D being different points and M determined as above, we have $\dfrac{b}{a} = \dfrac{d}{c}$, and $IOM \;\Delta\; AOB$, and $IOM \;\Delta\; COD$; whence $AOB \;\Delta\; COD$, so that $\dfrac{b}{a} = \dfrac{d}{c}$ represents the relation of *directional similarity* between these triangles.

(iii.) All the relations found for tensors in art. 16. can now be proved for clinants, and in each case establish relations between the positions of points, or relations of directional similarity between triangles, and hence of directed angles and directed arcs.

24. *Addition, Subtraction, and Distributive Character of Clinants.*—(i.) The relations in art. 20, on putting

$$OA + OB = OC, \quad OC - OB = OA,$$

and properly defining $a+b$, $c-b$, after the model of art. 17, give

$$a+b=c, \quad c-b=a,$$

so that the associative laws of directional addition make clinants associative in addition. Similarly the reduction of directional subtraction to directional addition (art. 20. v.) effects the same for clinants.

(ii.) But the equation $OA = OC-OB = BO+OC = BC$, by art. 20. iv v., shews that $BC = OA = a.OI = (c-b).OI$, and thus gives us power to find the clinant of any finite directed line on a plane.

(iii.) Now if (fig. 9) $PQ=OA$, $PR=OB$, $P'Q'=OC$, $P'R'=OD$, and $\frac{a}{b}=\frac{c}{d}$, so that $BOA \triangle DOC$, then will also $RPQ \triangle R'P'Q'$. But under these circumstances $a=q-p$, $b=r-p$, $c=q'-p'$, $d=r'-p'$. Hence the equation $\frac{q-p}{r-p}=\frac{q'-p'}{r'-p'}$ means $RPQ \triangle R'P'Q'$, and consequently gives a perfect algebraical representation of this geometrical relation, implying an equality of angle and proportionality of length, without any reference to commensurability or limits.

(iv.) Let COD (fig. 10) be any triangle. Alter the lengths of the sides OC, OD by extending or contracting them to OC' and OD' in such a way that *both* C', D' lie on OC, OD, on the C and D side of O respectively. Then if

$$\text{len } OC : \text{len } OC' :: \text{len } OD : \text{len } OD' :: \text{len } OI : \text{len } ON,$$

we have, by art. 21. ii., $c'=Tn.c$ and $d'=Tn.d$.

(v.) Next suppose the whole triangle to be revolved about the point O, into the position $C''OD''$, so that $\angle C'OC'' = \angle D'OD'' = \angle ION$, in which case also $\angle (C'D', C''D'') = \angle C'OC'' = \angle ION$, because $C'D'$ can not have rotated differently from the arm OC' to which it is attached. Consequently $c''=nc$, $d''=nd$, and $C''D''=n.CD=n.(d-c).OI$, as is well shewn in fig. 10, where $C''E=CD$, and $ION \triangle COC'' \triangle DOD'' \triangle EC''D''$. But $C''D''=OD''-OC''=nd.OI-nc.OI$. Consequently

$$n.(d-c) = nd-nc,$$

or *clinants are distributive.*

25. *On the originality of these Conceptions of Tensors and Clinants.*— The slow and painful degrees by which I have at length arrived, after twenty years of thought and detailed work, at the above extremely simple fundamental laws and notation for tensors and clinants, and at the results to be subsequently sketched, may be seen by reference to Appendix III. Since the year 1855, when I first became acquainted with Sir W. R. Hamilton's Quaternions, I have as far as possible made use of his terminology and notation, which however I have been obliged to modify to suit my own objects. Although in some respects *clinants* may be regarded as *complanar* quaternions, and hence the theory of clinants may be brought under the theory of quaternions, the introduction of three dimensions, and all its complications, with its generally non-commutative algebra, was opposed to the object I had in view; and hence I have had to pursue a completely independent course, and in especial my term *vector* (art. 26. v.) has not precisely the same meaning as Sir W. R. Hamilton's, but only a correlative signification. The

two may be distinguished as *clinant* and *quaternion* vectors when needed. Before seeing Sir W. R. Hamilton's Quaternions, I had used *cumbent* and *sistent* for what are here termed *scalar* and *vector*. Some terms and expressions, and most of the algebra, are entirely my own, though the reader must carefully attribute to Sir W. R. Hamilton whatever can be fairly traced to him, as I have had his magnificent labours constantly in mind and at hand. But notwithstanding his views, I believe that I may claim originality for the *conceptions* I have formed of tensors and clinants, as derived from pure geometrical proportion and similar triangles, and for my *demonstrations* of their laws. In particular I cannot recollect having seen elsewhere an approach to my proof of the associative character of tensors. And I know how gladly I should have availed myself of any such help, and how readily I should have acknowledged it. For general work I am of course deeply indebted to Augustus De Morgan and Martin Ohm, and all the usual sources of information on the subject of imaginaries and complex numbers. See also art. 35. and Appendix II.

26. *Subsidiary clinants.*—(i.) C, fig. 11, being any point, the biradial (art. 6. vi.) IOC determines the clinant c. Then the following subsidiary clinants can be readily formed.

(ii.) *Tensors.* With centre O and radius OC describe a circle cutting OI, on the I side of O, in T', then t' is the *tensor* of c, and is written $t' = Tc$. T^2c means $(Tc)^2$, see (xii.), and $= T(c^2)$ or Tc^2.

(iii.) *Versors.* Let the unit circle cut OC, on the C side of O, in U', then u' is called the *versor* of c, and written $u' = Uc$. Observe that $t'u' = c$, or $c = Tc \,.\, Uc$.

(iv.) *Scalars.* Let a perpendicular from C cut OI, on either side of O, in S', then s' is called the *scalar* of c, and written $s' = Sc$. Scalars constitute the real or *possible*, positive and negative, expressions of ordinary algebra, and are always represented by points on II' produced either way.

(v.) *Vectors* and *Jactors* (my own term). From C let fall a perpendicular on JOJ' cutting it in V', and make $V'OW' \; \triangle \; JOI$, so that W' falls on the I' side of O, if V' falls on the J' side of O. Then v' is called the *vector*, and w' (which is always scalar) the *jactor* of c, and they are written $v' = Vc$, $w' = Wc$, in which case $jw' = v'$ or $Vc = jWc$, the letter W being used for *jactor* in preference to J, to shew its relation to V. Observe that $c = s' + v' = Sc + Vc = Sc + j \,.\, Wc$. As Sc, Wc are both scalars, the last is the usual form of *imaginaries*, which c represents when C does not lie on IOI'. By the *reduced jactor* is meant $W_r c = Wc \pm 2r\pi i$, where r is so chosen that $TW_r c$ is not greater than πi. In fig. 11, Tw' being already $< \pi i$, $W_r c = Wc$.

(vi.) *Conjugates.* Continue CS' to K' where $S'K' = CS'$, then k' is called the *conjugate* of c, and is written $k' = Kc$. Observe that $IOC \; \nabla \; IOK'$, and hence conjugates furnish the method of dealing with conjugately similar triangles. Observe that

$$KKc = c, \quad Kc = Sc - Vc, \quad SKc = Sc, \quad VKc = i'.Vc,$$
$$Kc = Tc \div Uc = Tc \,.\, UKc, \quad c \,.\, Kc = T^2c, \quad Uc \,.\, UKc = i,$$
$$c \div Kc = U^2c, \quad 2Sc = c + Kc, \quad 2Vc = c - Kc.$$

(vii.) *Reciprocals.* Find R' so that $COI \triangle IOR'$, then r' is called the *reciprocal* of c, and written $r' = Rc$. Observe $c \,.\, Rc = i$, $URc = UKc$.

(viii.) *Angles, Amplitudes,* and *Cissals* (my own term). The $\angle IOC$ is called the *angle* of the clinant c, and written $\angle IOC = \angle c$. If we take length of $OA' =$ to the *rectified* length of arc IU', and put A' on IOI', to the I side of O, if $\angle c$ is in the same direction as $\angle j$, and to the I' side of O if $\angle c$ is in the same direction as $\angle j'$, then a' is called the amplitude of c, and written $a' = Ac$. Observe that Ac is always scalar, and that TAc never exceeds πi. Also, since $\angle (c_1 c_2 \ldots c_m) = \angle c_1 + \angle c_2 + \ldots + \angle c_m$ never exceeds $\angle i'$, and hence $TA\,(c_1 c_2 \ldots c_m)$ cannot exceed πi, we have $A\,(c_1 c_2 \ldots c_m) = Ac_1 + Ac_2 + \ldots + Ac_m \pm 2r\pi i$, where r is an integer so chosen that the tensor of this sum never exceeds πi. Define $\cos \angle c$, $\sin \angle c$, $\tan \angle c$, $W_S c$, $V_S c$ by the equations $Tc . \cos \angle c = Sc$, $Tc . \sin \angle c = Wc$, $\tan \angle c = Wc \div Sc = W_S c = j' V_S c$, and $\cos Ac$, &c. by the usual scalar series, giving $\cos Ac = \cos \angle c$, &c. The *cissal* of c is a term for Uc in the form $\cos \angle c + j \sin \angle c$, or $\cos Ac + j \sin Ac$, where it is expressed in terms of the $\angle c$, or Ac, and it is so called because of Sir W. R. Hamilton's extremely convenient abbreviation cis $\angle c$ or cis Ac. If for Ac we substitute $A_n c = Ac + 2n\pi i$, the result is called the nth *amplitude* of c, and cis $A_n c =$ cis Ac, by the properties of the well known scalar series for $\cos A_n c$ and $\sin A_n c$.

(ix.) *Logometers* (De Morgan's term). The napierian logarithm of any *tensor* is a *scalar*, and the usual process is supposed to be known, and *assumed* to be executable geometrically by some arrangement like "Peaucellier's cell." It is easily shewn that series in which the sum of *tensors* of the terms form a converging series, converge to a definite *clinant*. It is worth while constructing several terms of such a series as $i + x + x^2 + \ldots$, where $x = \frac{1}{2} j$, and seeing what is meant by its continually approaching to y, where $(i - \frac{1}{2} j)\, y = i$. All the laws of convergent series therefore hold for clinants.

Represent the napierian logarithm of Tc by λTc, and find the points L_1, L_2, L' so that $l_1 = \lambda Tc$, $l_2 = jAc$, and $OL' = OL_1 + OL_2$, then l' is called the *logometer* of c, and written $l' = Lc$. If we take $l_{2n} = jA_n c$, and $OL'_n = OL_1 + OL_{2n}$, l'_n is called the nth logometer of c, and written $l'_n = L_n c$. Hence $Lc = L_0 c$, or the *original* logometer of c. Also, $SL_n c = l = \lambda Tc$.

(x.) *Metrands* (my own term). Using the expression Ex for the well known series $i + x + \dfrac{x^2}{1 . 2} + \dfrac{x^3}{1 . 2 . 3} + \ldots$, when x is any clinant, and having found $E \,.\, Sc$ which will always be a tensor, set off a point M' so that $Tm' = E \,.\, Sc$ and $Am' = Wc$, then m' is called the *metrand* of c, and written $m' = Mc$. This gives $Mc = ESc \,.\,$cis Wc, and hence $ML_n c = ESL_n c \,.\,$cis $WL_n c$. But $SL_n c = SLc = \lambda Tc$, and hence $ESL_n c = E\lambda Tc = Tc$. And cis $WL_n c =$ cis $A_n c =$ cis Ac, so that $ML_n c = Tc \,.\,$cis $Ac = c$. If $Ac = o$, and hence $c = Tc$, then $Lc = \lambda c$, and $MLc = c = E\lambda c = ELc$.

(xi.) If (fig. 12), $c = \dfrac{m - p}{m - n}$, then the construction of the figure compared with fig. 11, shews that $\dfrac{m - t''}{m - n} = t' = Tc$, $\dfrac{m - u''}{m - n} = u' = Uc$,

$\frac{m-s''}{m-n}=s'=Sc$, $\frac{m-v''}{m-n}=v'=Vc$, $\frac{m-w''}{m-n}=w'=Wc$, $\frac{m-k''}{m-n}=k'=Kc$, $\frac{m-r''}{m-n}=r'=Rc$, $\angle NMP=\angle IOC=\angle c$, and hence all these can be obtained without a previous reduction to the form c, but Ac, Lc, Mc require that reduction, or its equivalent.

(xii.) Observe that when any one of these signs T, U, S, V, W, K, R, &c., are employed they refer to *all* letters which follow until either a point (.) or a (+, —) sign intervenes. Thus $Tab=T(ab)$, $UaRb=U(a.Rb)$, and not $Ua.Rb$. But the point may be used thus $U.aRb$, if thought more distinct. Also that $T^n a$, $U^n a$, &c., mean $(Ta)^n$, $(Ua)^n$, &c., as in trigonometry $\cos^n x$ commonly means $(\cos x)^n$. Since $T(Ta)=Ta$, $U(Ua)=Ua$, &c., this is the most convenient notation. Thus also $L^n a=(La)^n$, and LLa must be used for $L(La)$, which is seldom required.

27. *Some Relations of Subsidiary Clinants.*—The hints given in art. 26. shew the relations of these important subsidiary clinants, excluding logometers and metrands, for single clinants. The following gives some of their relations for combined clinants:

$S(a+b)=Sa+Sb$, $V(a+b)=Va+Vb$, $K(a+b)=Ka+Kb$;
$Tab=Ta.Tb$, $T(a\div b)=Ta\div Tb$, or $TaRb=Ta.RTb$;
$Uab=Ua.Ub$, $UaRb=Ua.RUb$;
$Kab=Ka.Kb$, $KaRb=Ka.RKb$; $Rab=Ra.Rb$;
$Sab=Sa.Sb+Va.Vb$, $Vab=(Va.Sb+Sa.Vb)$.

The two last equations are found by putting $ab=(Sa+Va).(Sb+Vb)$, and when divided by $T.ab$ give the trigonometric formulae for $\cos(Aa+Ab)$ and $\sin(Aa+Ab)$, see art. 26. viii., being their most general independent proofs.

$SaRb=(Sa.Sb-Va.Vb).T^2Rb$, $VaRb=(Va.Sb-Sa.Vb).T^2Rb$.

These two last equations are found by putting

$$aRb=(Sa+Va).(Sb-Vb).T^2Rb,$$

and when divided by $T.aRb$ give the usual trigonometric formulae for $\cos(Aa-Ab)$ and $\sin(Aa-Ab)$, being their most general independent proofs.

$$S(Sa.Sb)=Sa.Sb, \qquad V(Sa.Sb)=o;$$
$$S(Va.Vb)=Va.Vb, \qquad V(Va.Vb)=o;$$
$$S(Sa.Vb)=o, \qquad V(Sa.Vb)=Sa.Vb;$$
$$T^2(a+b)=T^2a+T^2b+2T^2b.SaRb=T^2a+T^2b+2T^2a.SbRa,$$

which contain Euc. ii. 12, 13, and i. 47, of which they form independent proofs. On putting for $SaRb$, $SbRa$ the above values, they also contain the whole trigonometric theory of the solution of triangles.

28. *General Exponential Expressions.*—The clinant power I define unambiguously by the equation $a^b=M.bLa$,
throwing all variety of values on the solution of exponential equations (art. 29), so that $\sqrt[m]{Ta}$ has only its one tensor value. Then, m, n, p being integers,

$a^b . a^c = a^{b+c}$ generally;

$(a^b)^c = a^{bc} . M2n\pi cj, \quad (a^c)^b = a^{bc} . M2p\pi bj, \quad a^c . b^c = (ab)^c . M2m\pi cj,$

where m, n, p depend on adjustment of amplitudes (art. 26. viii.);

$L_m Ma = a + 2n\pi j$, where $Wa + 2n\pi = W_r a + 2m\pi$ (see art. 26. v.);

$L_m a + L_n b = L_p ab$, where $A_m a + A_n b = A_p ab$;

$L_m a^b = bLa + 2n\pi j$, where $Wa^b + 2n\pi = W_r a^b + 2m\pi$ (see art. 26. v.);

$M(a+b) = Ma . Mb$,

$M2n\pi j = 1, \qquad Ma = M(a + 2n\pi j),$

$Mj Wa = \text{cis } Wa, \qquad Ma = ESa . \text{cis } Wa, \qquad Ea = Ma.$

29. *Solution of General Exponential Equations.*

(i.) Given $\qquad x^a = b = MaLx,$

then $\qquad x = M(Lb . Ra) . M(2r\pi j . Ra).$

If a be a scalar integer, this gives the usual expressions. The equation $b = MaL_p x$ has the same solutions.

(ii.) Given $\quad b = a^x = MxLa$; then $\quad x = L_r b . RLa.$

But if $\quad b = MxLa . M2p\pi jx$, then $\quad x = L_r b . RL_p a$,

which is Martin Ohm's general solution.

30. *Logometric and Binomial Series.*—If (art. 26. x.) we put

$$Ey = i + x, \quad \text{then} \quad y = L_n(i+x) = x - \tfrac{1}{2}x^2 + \dots$$

in the usual way, n being determined so that the amplitudes should be the same on both sides, and Tx being $< i$. Also, taking

$$(i+a)^b = EbL(i+a),$$

we find in the usual way

$$(i+a)^b = 2m\pi j + i + ba + \frac{b(b-i)}{1.2} . a^2 + \frac{b(b-i)(b-2i)}{1.2.3} . a^3 + \dots,$$

where the value of m has to be adjusted for amplitude, Ta being supposed to be small enough for convergence.

The first series gives

$$L(i+j) = 2m\pi j + j - \frac{1}{2}j^2 + \frac{1}{3}j^3 - \dots$$

$$= 2m\pi j + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right) i + \left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right) j,$$

where both series are convergent, the first $= \lambda\sqrt{2}$, and the second lies between 1 and $1 - \frac{1}{3}$, so that m, which has to be adjusted so that the whole amplitudes must be $< \pi i$, will $= 0$; hence

$$L(i+j) = i . \lambda\sqrt{2} + j\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right).$$

But $\quad A(i+j) = \frac{1}{4}\pi . i, \quad T(i+j) = \sqrt{2} . i$, so that

$$L(i+j) = iL(\sqrt{2} . \text{cis } \tfrac{1}{4}\pi) = i . \lambda\sqrt{2} + \tfrac{1}{4}\pi j,$$

whence, on comparing, we have Leibnitz's well-known series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right).$$

Similarly, from $L\,\frac{1}{2}(i\,.\sqrt{3}+j)$ we find

$$\pi = 2\sqrt{3}\,.\left(1-\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{5}\cdot\frac{1}{3^2}-\frac{1}{7}\cdot\frac{1}{3^3}+\ldots\right).$$

Taking $2\sqrt{3} = 3{\cdot}4641016$, and proceeding only as far as $+\frac{1}{21}\cdot\frac{1}{3^{10}}$ $= 0{\cdot}0000008$, this series gives $\pi = 3{\cdot}1415911$, or five places correct.

31. *General Goniometric Series.*—(i.) When x is any clinant, let

$$Gx = \tfrac{1}{2}\,(Exj + Exj'),$$
$$Zx = \tfrac{1}{2}j'\,(Exj - Exj'),$$
$$Px = Zx\,.\,RGx,$$
$$Qx = Gx\,.\,RZx,$$

so that when $x = Sy$, these expressions become cos Sy, sin Sy, tan Sy, cot Sy, respectively, of which series they are the clinant generalisations.

(ii.) Let $G\breve{}x$, $Z\breve{}x$, $P\breve{}x$, $Q\breve{}x$, (read, "G-invert of x" &c.,) be determined by the equations

$$G\breve{}x = j'\,L\,[x+j\,\sqrt{(i-x^2)}],$$
$$Z\breve{}x = j'\,L\,[\sqrt{(i-x^2)}+jx],$$
$$P\breve{}x = \tfrac{1}{2}j'\,[L\,(i+xj)-L\,(i-xj)],$$
$$Q\breve{}x = \tfrac{1}{2}j\,\,[L\,(x+j)-L\,(x-j)].$$

Then $GG\breve{}x = ZZ\breve{}x = PP\breve{}x = QQ\breve{}x = x$, but the order of the symbols is important. In the scalar case I use cŏs Sy, sĭn Sy, tăn Sy, cŏt Sy for the inverts, in place of $\cos^{-1} Sy$ &c., which are inappropriate whether $\cos^2 x$ mean $(\cos x)^2$ or cos (cos x), because the signs cos, $\cos^{-1}$ would not be commutative.

(iii.) Given $Gx = c$, then $x = 2n\pi + G\breve{}c.$

Given $Zx = c$, then either $x = 2n\pi + Z\breve{}c$, or $(2n+1)\pi - Z\breve{}c.$

Given $Px = c$, then $x = n\pi + P\breve{}c.$

Given $Qx = c$, then $x = n\pi + Q\breve{}c,$

whence all the scalar cases may be deduced, giving forms equivalent to those in Martin Ohm's Versuch eines vollkommen consequenten Systems der Mathematik, vol. 2, *third* edition, 1855, chap. viii.

32. *Completion of the Laws of Clinants.*—(i.) This completes the whole of the fundamental laws of clinants, which are shewn to be those of ordinary algebra, including imaginaries; and as each clinant expression can be perfectly constructed on the principles already given, by the elementary process of forming directionally or conjugately similar triangles, every one of these so-called *imaginary* forms comes to be expressed by a real point on the plane IOJ.

(ii.) The fact that $fx = x^m + ax^{m-1} + \ldots + px + q$, where all the coefficients are clinant, but m is an integer, can be expressed as a product $(x-a_1)\,(x-a_2)\ldots(x-a_m)$, is proved by a process precisely similar to that in Sir W. R. Hamilton's *Elements of Quaternions*, Book II., Chap. ii., section 5, p. 265, which however admits of considerable simplification.

(iii.) The solutions of quadratic, cubic, and biquadratic equations are conducted in the usual way, but with material simplifications, and the roots are constructible by an elementary process which is rendered occasionally troublesome by the practical difficulty of drawing equal angles with sufficient exactness when the arms are greatly extended. But the process is always strictly geometrical, granted the power of dividing angles in any ratio, and of interposing any number of geometrical means. I have myself constructed every case, assuming the coefficients to be given by any points on a plane, with sufficient exactness to verify all the relations between the roots and the coefficients by geometric construction.

33. *Geometrical Construction of Clinant Combinations.*—It would be beyond the purpose of these rough notes to enter upon details, but the following simple and frequently recurring cases should be noted:

(i.) $ax = c$, make $IOX \bigtriangleup AOC$, so that X is the B of fig. 8.

(ii.) $\dfrac{x-m}{x-n} = \dfrac{a-b}{a-c}$, make $MNX \bigtriangleup BCA$, as in fig. 13.

(iii.) $x^2 = ab$ gives $\dfrac{a}{x} = \dfrac{x}{b}$, so that $XOA \bigtriangleup BOX$; and hence, fig. 14, if COD bisects $\angle AOB$, and len $OC =$ len OD is a geometrical mean between len OA and len OB, the points C, D construct the values of x. Either of the lines OC, OD is called the *mean bisector* of OA, OB, or of the biradial AOB. When O, A, B are not collinear, A, C, B, D are concircular, for $ab = x^2 = i' . cd$, whence $\dfrac{a-c}{a-d} \cdot \dfrac{b-d}{b-c} = i'$, and hence $\angle CAD + \angle DBC = \angle i'$.

(iv.) $x^2 = a^2 - b^2 = (a+b)(a-b) = hk$, fig. 15, from A draw $AH = OB$, $AK = BO$, and construct OX', OX'' as mean bisectors of OH, OK, by (iii.)

(v.) $y^2 = a^2 + b^2 = a^2 - b'^2$, if $b' = jb$, fig. 15. Draw OB' of the len OB, making $\angle BOB' = \angle j$, $K'A = AH' = OB'$, and find OY', OY'' as mean bisectors of OH', OK', by (iii.)

(vi.) $ax^2 + bx + c = o$, then $2ax = bi' \pm \surd(b^2 - 4ac)$. Find A', B', by (iii., iv.), so that $a'^2 = 2a . 2c$, $b'^2 = b^2 - a'^2$, then $2ax = bi' \pm b'$, and the two positions of X are constructed by (i.) See an example with various cases in art. 34. viii., figs. 14, 21, 22.

These simple constructions suffice for all the cases considered in the next Tract.

34. *Applications of Clinants.*—(i.) In Sir W. R. Hamilton's *Elements of Quaternions*, Book I., chap. ii., and Book II., chap. ii., will be found a large quantity of geometry suitable for direct treatment by clinants, and this treatment will be found to introduce much simplification. To these I need merely refer. The following will suffice to shew the nature of direct clinant treatment of geometrical problems, and will give some results required in the next Tract.

(ii.) If A, B, C are collinear, $V\dfrac{a-b}{a-c} = o$.

(iii.) (1) If the straight lines AB, CD, EF converge to a common point X, then $V\dfrac{a-x}{a-b} = V\dfrac{c-x}{c-d} = V\dfrac{e-x}{e-f} = o$,

whence $$V\frac{a-b}{e-f}\cdot V\frac{c-a}{c-d}=V\frac{a-b}{c-d}\cdot V\frac{e-a}{e-f},$$
which by forming the auxiliar points B', C', D', E' may be made to take the form
$$\frac{a-b'}{e-f}\cdot\frac{c'-a}{c-d}=\frac{a-d'}{c-d}\cdot\frac{e'-a}{e-f}\ \text{ or }\ \frac{a-b'}{a-d'}=\frac{a-e'}{a-c'},$$
or $B'AD' \triangle E'AC'$, as the condition of convergence.

(2) If three straight lines known as perpendiculars to OA, OB, OC (fig. 16), converge to X, we have
$$S\frac{x}{a}=S\frac{x}{b}=S\frac{x}{c}=i,$$
and since
$$S\frac{x}{c}=S\left(\frac{a}{c}\cdot\frac{x}{a}\right)=S\frac{a}{c}\cdot S\frac{x}{a}+V\frac{a}{c}\cdot V\frac{x}{a},\ \text{by art. 27,}$$
we have $$i=S\frac{a}{c}+V\frac{a}{c}\cdot V\frac{x}{a},$$
and similarly $$i=S\frac{a}{b}+V\frac{a}{b}\cdot V\frac{x}{a},$$
and hence eliminating $V\frac{x}{a}$, we find
$$V\frac{a}{c}\cdot\left(i-S\frac{a}{b}\right)=V\frac{a}{b}\cdot\left(i-S\frac{a}{c}\right),$$
which, by forming the auxiliary points M, N, as in art. 26. xi., may be reduced to $\frac{a-m}{c}\left(i-\frac{n}{b}\right)=\frac{a-n}{b}\left(i-\frac{m}{c}\right)$ or $\frac{a-m}{c-m}=\frac{a-n}{b-n}$,
or $AMC \triangle ANB$, as the condition of convergence.

(iv.) Let $$(abcd)=\frac{(a-b).(c-d)}{(a-d).(c-b)},$$
the letters being carefully written in this order. Then $(abcd)$ will be called the *anral* (*an*-harmonic *r*-atio $+$ *al*) of the four points A, B, C, D anywhere situate on a plane. (See Appendix III. for the principle of this terminology.) It is also convenient to shew the omission of the terms involving any one of the points by the notations
$$(..bcd)=\frac{c-d}{c-b},\ (a..cd)=\frac{c-d}{a-d},\ (ab..d)=\frac{a-b}{a-d},\ \text{and } (abc..)=\frac{a-b}{c-b}.$$
Then $$(abcd)=(badc)=(cdab)=(dcba),$$
$$(abcd).(adcb)=i,\quad (abcd).(acdb).(adbc)=i',$$
and, since $(a-b)(c-d)+(b-c)(a-d)+(c-a)(b-d)=o$,
also $(abcd)+(acbd)=i$, $(abcd)+R.(adbc)=i$,
and $(acdb)=[(abcd)-i].R.(abcd)$,
so that all the 24 possible anrals can be expressed in terms of one.

If $a'=a+m$, $b'=b+m$, $c'=c+m$, $d'=d+m$, $(abcd)=(a'b'c'd')$.

If $V.(abcd)=o$, then $\angle BAD+\angle DCB=\angle i$ or $\angle i'$, and the four points $ABCD$ are *either* on a straight line *or* on a circle. Putting $(abcd)=(a'b'c'd')$, the *first* case only is the foundation of all Chasles's theories of homography and involution, and the first and second cases *combined* form the basis of Möbius's *Kreisverwandtschaft*, or circular-relationship, all the results of which are much more simply written and obtained by means of clinants—as I have found by actual work.

If $(abcd) = i'$, the anral becomes a *harmal* (*harm*-onic-ratio + *al*), and the points lie harmonically on a straight line, or on a circle. For example, in fig. 29, $(edfd') = (a'eaf) = i'$, and $EDFD'$ is a straight line and $A'EAF$ a circle. The troubles experienced by Chasles (Géom. Sup., chap. V.) arising from imaginary points in harmonic ratios, at once disappear, and the investigations are not only simplified but generalised, and, as will be seen, are capable of still further generalisation (art. 44).

(v.) If A, B be any two points on a plane, the *annal* (*an*-gle + *al*) and *tannal* (*tan*-gent + *al*) of the directed *biradial* AIB (written bir AIB), which gives not merely the $\angle AIB$, but the length and direction of both arms IA, IB (compare art. 6. vi.), are written as follows, and express the following functions of a, b respectively:

$$\text{an } AB = \frac{b-i}{a-i}, \quad \text{tal } AB = \frac{a-b}{i-ab},$$

the letter I being always understood between A and B, and the order of the letters being important. For the use of these forms, see art. 39. iv.

If $Sa = Sb = Sc = Sd = o$, and hence $Va = a$, $Vb = b$, $Vc = c$, $Vd = d$, then

$$V_S . \text{an } AB = \frac{V . \text{an } AB}{S . \text{an } AB} = \tan AB = j \tan AIB,$$

and
$$(abcd) = \frac{\sin AIB . \sin CID}{\sin AID . \sin CIB}.$$

The general expressions will be found to include the whole geometrical theory of imaginary angles (as distinguished from the algebraical theory of art. 31). The following properties shew some analogies and solve some previously incomprehensible relations. They should be all constructed geometrically.

Generally
$$\text{tal } AB + \tan BA = o.$$

$$\text{tal } AB = \frac{\text{tal } AE - \text{tal } BE}{i - \text{tal } AE . \text{tal } BE},$$

$$\text{tal } AB + \text{tal } BC + \text{tal } CA + \text{tal } AB . \text{tal } BC . \tan CA = o.$$

And if $a + a' = b + b' = c + c' = o$, and $\text{tal } AB = c$,

then
$$\text{tal } AO = a, \qquad \text{tal } OA = a'.$$

$$\text{tal } AC = b, \qquad \text{tal } BA = c', \qquad \text{tal } BC' = a,$$

$$\text{tal } CA = b', \qquad \text{tal } CB' = a, \qquad \text{tal } AB' = \text{tal } BA'.$$

If $\text{tal } AX = \text{tal } XB$, then IX_1 and IX_2 are the *medials* of bir AIB, where x_1, x_2 are the roots of $x^2 - 2 . (i + ab) . R(a+b) . x + i = o$. In this case $(x_1 a x_2 b) = i'$, or X_1, X_2 are harmonically situate with respect to A, B; see (iii.). And since $x_1 x_2 = i = i^2$, X_1, X_2 are also harmonically situate with respect to I and I'; see art. 44. iii. X_1, X_2 are the points of intersection of the two figures $X_1 A X_2 B$ and $X_1 I X_2 I'$, of which one may be a straight line and the other a circle, or both circles.

Although tal II and tal $I'I'$ are indeterminate, yet when A is neither I nor I', tal $AA = o$, tal $AI = i'$, tal $IA = i$, tal $AI' = i$, tal AI . tal $AI' = i'$.

If $b = Ra$, or $ab = i$, R tal $AB = o$, and tal AB does not exist. In this case, if $Sa = Sb = o$, or A, B lie on OJ, $\angle AIB$ is a right angle. Hence, in the general case, I say that the biradial AIB is *orthal* ($\grave{o}\rho\theta$-$\acute{o}\varsigma + al$), a conception of considerable importance in stigmatic geometry. Bir X_1IX_2, formed by the medials of the bir AIB, is orthal, because $x_1x_2 = i$.

If an $AY =$ an YB, then $(y-i)^2 = (a-i)(b-i)$; and y_1, y_2 being the roots of this equation, IY_1, IY_2 are the *mean bisectors* of the bir AIB (art. 33. iii.). If both A and B lie on OY, one of the means coincides in direction with one of the medials, but the lengths are different. (See art. 44. iii.)

(vi.) If A, B, C be points in a circle of which O is the centre, $T^2a = T^2b = T^2c$, or $a \, . \, Ka = b \, . \, Kb = c \, . \, Kc$, whence $\frac{a-c}{c} = K\frac{c-a}{a}$, $\frac{b-c}{c} = K\frac{c-b}{b}$, and then $K\frac{b}{a} = \frac{a-c}{K(a-c)} \, . \, \frac{K(b-c)}{b-c}$, or $U\frac{a}{b} = U^2\frac{a-c}{b-c}$, that is $\angle AOB = 2 \, . \angle ACB$. This is a general and independent proof of Euc. iii. 20. 21. 22, in their proper statement for directed angles. See also art. 48. x.

If D be a fourth point on the circle, it follows that $U^2\frac{a-c}{b-c} = U^2\frac{a-d}{b-d}$, or $U(acbd) = i$ or i', or $V(acbd) = o$; compare (iv.)

(vii.) To find the points X, Y, Z on the unit circle so that $\angle IOX = 2 \, . \angle OXI$, (the solution of which is evident,) $\angle OYI = 2 \, . \angle IOY$, (which is Euc. iv. 10, with directed angles,) and $\angle IOZ = 3 \, . \angle OZI$, (which is immediately constructible in Euclidean geometry from Euc. iv. 10).

The properties of the unit circle give $Tx = i$, $Kx = Rx$, $Ux = x$, and similarly for y, z. The statement of the three problems in clinants is

$$Ux = U^2\frac{x-i}{x}, \quad U\frac{y-i}{y} = U^2y, \quad \text{and} \quad U^3\frac{z-i}{z} = Uz,$$

or $$U^3x = U^2(x-i), \quad U(y-i) = U^3y, \quad \text{and} \quad U^3(z-i) = U^4z.$$

Since $U^2(x-i) = \frac{x-i}{K(x-i)} = \frac{x-i}{Rx-i} = i' . x$, and similarly for y and z, the first equation gives $x^3 = i' . x$, or $x^2 = i'$, whence $x_1 = j$, $x_2 = j'$, the two triangles being IOJ and IOJ', as is evident. The other two equations, on being squared (which introduces adventitious roots) and reduced, give $y^5 = i'$ and $z^5 = i'$, and as one root of each is i', all the roots can be readily found. On calling them y_1, y_2, y_3, y_4, y_5, as the points Y_1, Y_2, Y_3, Y_4, Y_5 lie in order on the circumference of the unit circle in the direction IJI', we find that y_1, y_3, y_5 and $z_2 = y_2$, $z_3 = y_3$, $z_4 = y_4$ give the required solutions.

(viii.) A, B, (fig. 17. 18. 19. 20.) being any points, find X and Y, so that $IOX \; \Delta \; ABX$, and $IOY \; \nabla \; ABY$. This is selected as giving rise to general simple equations,

$$\frac{x-b}{a-b} = x, \quad \text{and} \quad \frac{y-b}{a-b} = Ky.$$

(1) In no case can $a-b=o$, or A coincide with B.

(2) If $b=o$, or B coincide with O, then $x=ax$, or $a=i$, that is, A also coincides with I, and x is indeterminate; of course $IOX \triangle IOX$ wherever X may be. But $y=aKy$, or $U^2y=a$, whence $Ta=i$, or OA is a radius on the unit circle, and any point Y on the line bisecting the the $\angle IOA$, fig. 17, makes $IOY \nabla ABY$, that is ∇AOY.

(3) Excluding these cases, make $OC=BA$, or $c=a-b$; first let $c=i$, or $BA=OI$; then the first equation gives $x-b=x$, or $b=o$, (which reduces this to case 2). The second equation gives $y-b=Ky$, or $y-Ky=b$; and operating with K on each side, $Ky-y=Kb$, so that, on adding, $o=b+Kb$, or $Sb=o$, that is, B must lie on OJ, and $2Vy=b$, or Y lies anywhere on the line through D (where $OD=\frac{1}{2}OB$) parallel to OI, fig. 18.

(4) Take $c=a-b$ as before, we have $x-b=cx$, or $OXB \triangle IOC$, fig. 19, or $(i-c)x=b$; or, making $d=i-c$, we have $dx=b$, that is, $IOX \triangle DOB$, from either of which X is immediately constructed.

(5) The second equation gives

$$y-b=c\,.\,Ky, \quad \text{whence} \quad Ky-Kb=Kc\,.\,y, \quad \text{and } y=\frac{b+c\,.\,Kb}{i-c\,.\,Kc}.$$

Now if $b+c\,.\,Kb=o$, both b and Kb may $=o$, in which case $y=o$, and we have the evident similarity $AOO \triangle IOO$. But generally $Kb+Kc\,.\,b=o$ gives $cKc=i$, which leads to a mere identity $o\,.\,y=o$, from which nothing can be determined.

Reserving this case, and first putting $b'=Kb$, $c'=Kc$, fig. 20, and then $cb'=d$, $cc'=e$, $b+d=g$, $i-e=h$, we shall have $hy=g$, or $IOY \triangle HOG$, and then $IOY \nabla ABY$.

(6.) For the reserved case, put $y=\eta+\eta'$, $b=\beta+\beta'$, $c=\gamma+\gamma'$, where η, β, γ are scalars, and η', β', γ' are vectors. Since then $Ky=\eta-\eta'$ &c., the two equations $b+c\,.\,Kb=o$, and $y=b+c\,.\,Ky$, give, on equating the scalar and vector portions of each,

$$\beta+\gamma\beta-\gamma'\beta'=o, \qquad \beta'+\gamma'\beta-\gamma\beta'=o,$$
$$\beta+\gamma\eta-\gamma'\eta'=\eta, \qquad \beta'+\gamma'\eta-\gamma\eta'=\eta'.$$

Eliminating η and η' alternately from the two last equations, we obtain their value in the general case 5, from

$$\beta(\gamma+1)-\beta'\gamma'+[(\gamma^2-1)-\gamma'^2]\,\eta=o,$$
$$\beta\gamma'-\beta'(\gamma-1)+[(\gamma'^2-(\gamma^2-1)]\,\eta=o;$$

but applying the two first equations, the parts independent of η, η' each $=o$, and hence $\eta=o$, $\eta'=o$, and $y=o$, which necessitates $b=o$, so that the reserved case only gives, as before, $IOY \triangle AOO$.

(ix.) Let A, B, C be any three points, and D a fourth, so that $OD=CO$. To find a point X so that the mean bisectors of XA, XB shall be equal to OC and OD. This is selected as giving the general quadratic equation

$$(x-a)(x-b)=c^2=d^2,$$
or
$$x^2-(a+b)x=c^2-ab.$$

(1) If OC, OD are the mean bisectors of OA, OB, fig. 14, $c^2-ab=o$, and $x=o$, or $a+b$, that is, X is at either of the two extremities of the diagonal OE, of the parallelogram of which OA, OB are adjacent sides; thus EF is clearly $=OD$.

(2) If $c^2 - ab$ is not $= o$, then, fig. 21, putting $2m = a + b$ and $2p = a - b$, on adding m^2 to each side we have

$$(x-m)^2 = m^2 - ab + c^2 = p^2 + c^2 = n^2 \text{ or } n'^2,$$

where N, N' are constructed as in art. 33. v., and $x = m + n = x'$, or $x = m - n = m + n' = x''$. The mean bisectors of $X'A$, $X'B$ and $X''A$, $X''B$ are $X'E$, $X'E'$ and $X''F$, $X''F'$, which are $= OC$, OD respectively.

(3) If $p^2 + c^2 = o$, or OP, OC are of the same length and at right angles to each other, fig. 22; then $n = n' = o$, and the two positions X', X'' coalesce at M, so that there is only one position which satisfies the conditions. The mean bisectors of MA, MB are MG, MG'.

(x.) To determine the points where a line perpendicular to OA cuts a circle with radius OB.

As in iii. (2) we have in the line $S . xRa = i$, and as in (vi.) for the circle $T^2x = T^2b$. Then, by art. 26. vi., $xRa + K . xRa = 2SxRa = 2i$, and $xKx = T^2b$, whence eliminating Kx, we have

$$x^2 = 2ax - U^2a . T^2b,$$

and $\quad x = a \pm Ua \sqrt{(T^2a - T^2b)}$;

whence $\quad x . Ra = i \pm RTa . \sqrt{(T^2a - T^2b)}$.

Unless then $Ta = < Tb$, $S . xRa$ will not $= i$, and this is therefore the condition of possibility. There are no "imaginary" intersections. No "imagination" can make $i = i \pm k$, where k is not $= o$, for this would lead to the impossibility of Appendix II. A circle and straight line have therefore *no* "imaginary" intersections. This term applies only to a derived case, considered in art. 49. v. The meaning of this distinction is assigned in art. 36. v.

When $Ta = Tb$, $x = a$, and there is only one point of intersection A. When $Ta < Tb$, $x = a \pm j . Ua . \sqrt{(T^2b - T^2a)}$, which gives the two points determined by drawing $X'AX''$ perpendicular to OA, and making len AX' = len AX'' = length of the perpendicular of a right-angled triangle, of which the lengths of base and hypothenuse are the lengths of OA and OB respectively.

V. Stigmatic Geometry, or the Correspondence of Points in a Plane.

35. *No previous complete representation of Algebra by Geometry.*— Some of the results hitherto adduced have been already obtained (although less directly, and always by a more or less implied use of limits) from various geometrical "explanations" of "imaginaries," advanced with some degree of hesitation, often on metaphysical grounds, and (except by Sir W. R. Hamilton) always by means of "complex numbers," or clinants of the form $Sa + jWa$, where Sa, Wa were considered as the limits of convergent "possible" (that is, scalar) series. The class of problems embraced under the theory of Stigmatics have also been attacked with immense acuteness and wide success, in particular instances, but the occurrence of imaginaries have constantly baffled