11 pounds avoirdupois, 200 grammes are 7 ounces av., 4 litres are 7 pints, which the draper's information led me to calculate. I may state, by the way, that we come within one unit of the truth up to 1000 times the French units of measure (for metres up to 11000) by adding 1 part in 400 to the yards and pounds, subtracting 6 parts in 1000 from the miles, and adding the same to pints, and subtracting 12 parts in 1000 from the acres. The calculation is much easier than for decimals, and the results furnish admirable materials for exercising pupils in approximating to ratios of magnitudes arithmetically.

(vii.) Observe that if $mA = nB+D$, and we do not know the limit of the value of D, we can tell by the mere division $m'n \div m = n' +$ proper fraction, that $m'A$ lies between $n'B$ and $(n'+2)B$, but that we cannot tell whether it lies between $n'B$ and $(n'+1)B$, or between $(n'+1)B$ and $(n'+2)B$, however great m may be. If, then, we want to find, not V, but $m'V$ within the limit E, we must find $m'mA = n''B+D'$, where $D' < B$. This is important in settling the limits of error, or "the number of decimal places required."

(viii.) But the processes of finding multiples, or throwing into a continued fraction, are alike illusory when certainty is required, as the suggested trials shew. Then arises the great problem of higher geometry: to find a series of terms (taken as geometrical magnitudes) continually diminishing, and *connected by a law* such that when a few are known any required number can be found, and such also that their (geometrical) sum continually approaches to the required limit, and may be made to differ from that limit by less than any assigned amount. The *practical* problem is then perfectly solved, but that practical problem gives birth to a *theoretical* problem. Suppose V to be the fixed limit toward which the series S converges, then $V-S$ will be a magnitude (a straight line, see ii.) of continually diminishing size, which can be made less than any assignable magnitude, while at every moment $V-(V-S) = S$. Can we then *neglect* $V-S$, and deal with S as if it were V, not merely for a practical approximation, but for theoretical exactness?

II. "CARNOT'S PRINCIPLE" FOR LIMITS.

10. "*Carnot's Principle.*"—(i.) The only satisfactory answer which I have been able to find to the question just propounded, (and I have paid minute attention to the subject at various times for nearly 40 years,) is contained in *Réflexions sur la Métaphysique du Calcul Infinitésimal par* CARNOT (3rd ed., Paris, 1839, pp. 254), which the name of the writer is enough to recommend to the careful study of all teachers. I wish here to state the principle in connection with the author's name, in that simple geometrical form which is suitable for learners, without any anticipation of the infinitesimal calculus.

(ii.) Let A, B be two homogeneous magnitudes, of which we only know that they are invariable; and let X, Y be two other magnitudes homogeneous with A, B, of which we at first only know that they are constantly changing. And then in addition suppose that we know, by given geometrical or other relations, that through all the changes of X and Y, the following condition subsists: $A-X=B-Y$. What relation between A and B will such a condition allow us to infer?

(iii.) First the given relation makes $A-B=X-Y$; and as A and B are invariable, $A-B$ is invariable, and hence $X-Y$ is invariable. Hence also we cannot have one of the two variables X and Y increasing and the other diminishing, because in that case the difference $X-Y$ would necessarily vary. X and Y must *both* increase, or *both* diminish.

(iv.) If X and Y both increase, $X-Y$ may remain constantly equal to any unknown homogeneous magnitude whatever, as M, and then $A-B=M$, that is, some unknown. In this case, then, the relation $A-X=B-Y$ leads to no result.

(v.) But if X and Y both decrease, and each can become less than any assignable homogeneous magnitude, the difference $X-Y$ must also vary and become less and less, unless $X=Y$ at all times. The condition that the difference should not vary, entails therefore the necessity that $X=Y$, and as a necessary consequence that $A=B$.

(vi.) If, then, $A-X=B-Y$ under these last circumstances (v.), we know with theoretical exactness, without any approximation at all, that $A=B$ and $X=Y$.

(vii.) If, then, our interest consists only in finding the relation between A and B, and we have no sort of interest at all in knowing that between X and Y, we may from the moment that the relation $A-X=B-Y$ has been established, *neglect* the consideration of X and Y, and infer, with perfect exactness, that $A=B$. What we have neglected is *not* a decreasing magnitude, nor anything which affects the relation of A to B, but only something which affects the relation of X to Y, which we do not care about, but which we could at any time revert to if desired.

(viii.) By neglecting variable infinitesimals, then, when seeking relations between invariable finites, we merely simplify the chain of argument, without impairing exactness. We do not neglect them as magnitudes so small that they are of no consequence (on the principle *dē minimīs nōn cūrat lēx*), but merely leave them out of consideration, because we do not happen to want to know anything about them. See the citations in Appendix I.

11. *Examples.*—(i.) It is important that the pupil should appreciate the working of this principle by applying it to the two main cases in elementary geometry; the expression of the ratios of the circumferences and areas of two circles.

(ii.) Within any two circles describe, say, hexagons, Euc. iv. 15, as offering the least geometrical difficulty. These are similar polygons, and the ratio of their perimeters, $P : P'$, is the same as that of the corresponding radii, $R : R'$. By bisecting the arcs subtended by the sides of these hexagons, and so on, we get other similar polygons, for all of which $P : P' :: R : R'$. Now if C, C' be the invariable circumferences of the circles, and D, D' the values of $C-P$, $C'-P'$, which constantly diminish, and may be made less than any assignable, the last proportion may be written $C-D : C'-D' :: R : R'$; whence by Euc. vi. 16,

$$\text{rect.}\ (C, R') - \text{rect.}\ (D, R') = \text{rect.}\ (C', R) - \text{rect.}\ (D', R).$$

The second rectangles on each side may become less than any assignable, and hence, by "Carnot's principle," we have *always*, whether D, D' are large or small,

$$\text{rect.}\ (C, R') = \text{rect.}\ (C', R),\ \text{and rect.}\ (D, R') = \text{rect.}\ (D', R),$$

that is, *both* $C : C' :: R : R'$, *and* $D : D' :: R : R'$.

The last result was not wanted, (although it is useful to draw attention to it when D is large,) and hence, as soon as we had stated the proportion as $C-D : C'-D' :: R : R'$, we might have inferred $C : C' :: R : R'$, which was all of the truth we wanted, although not the whole truth.

(iii.) Apply the reduced process to find the ratio $A : A'$ of the areas of the circles, $Q : Q'$ being that of the areas of the similar polygons, which is the same as that of sq. on R : sq. on R'. E and E' being the varying differences between the invariable areas of the circles and of the variable areas of the polygons, which differences may become less than any assignable areas, the constant proportion $Q : Q' ::$ sq. on R : sq. on R' can be expressed as

$$A-E : A'-E' :: \text{sq. on } R : \text{sq. on } R',$$

and hence by "Carnot's principle" we infer

$$A : A' :: \text{sq. on } R : \text{sq. on } R',$$

and also, if required, $E : E' :: A : A'$. This solves Art. 3. v.

(iv.) It is obvious that the application of this principle in higher algebraical geometry, the differential calculus, &c., is impossible unless we assume that we can deal with incommensurable expressions by the ordinary laws of commutative algebra. I have never seen any attempt to prove the justifiability of this condition, which seems to be taken as an axiom. Yet it is evident that if the limit of a convergent series is incommensurable, we cannot, from conclusions drawn from the (commensurable) sum of any finite, or ever increasing (infinite) number of its terms, conclude an exact relation which depends solely on its limit, until we know what are the laws by which we may calculate with incommensurables. The ordinary algebraical proof that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, by "squaring each side," is absurd if we do not know the *meaning* of *multiplying* $\sqrt{6}$ by $\sqrt{6}$ or of multiplying $\sqrt{2}$ by $\sqrt{3}$. To this question, then, the next Tract is devoted.

III. The Laws of Tensors, or the Algebra of Proportion.

12. *Proportion expressed by Tensors.*—(i.) Euc. v. and vi. are assumed. It is also assumed that no ratio is known till two straight lines have been found having that ratio. And only known ratios are here dealt with.

(ii.) OI, fig. 3, is a straight line continued indefinitely beyond I, on which