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Hadi Amirnia; Alireza Khastan

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A GENERALIZATION OF THE MEAN-SQUARE DERIVATIVE FOR FUZZY STOCHASTIC PROCESSES AND SOME PROPERTIES

Hadi Amirnia and Alireza Khastan

The purpose of this paper is to generalize and develop a mean-square calculus for fuzzy stochastic processes and study their differentiability and integrability properties. Some results for second-order fuzzy stochastic processes are presented.

Keywords: fuzzy numbers, Hukuhara difference, random variables, second-order fuzzy stochastic processes, mean-square calculus

Classification: 03E72 , 26E50, 28E10

1. INTRODUCTION

Stochastic differential equations (SDEs) are used in numerous applications to model classical problems in control theory, physics, biology, economics and engineering [20]. In such studies, random disturbances are the only source of uncertainty. To handle these situations, methods of stochastic analysis are used. However, in some real world problems, we encounter a second source of uncertainty: vagueness (sometimes called imprecision, fuzziness, ambiguity or softness). This is mostly observed when a state of a considered system is described by linguistic variables. It is known that the fuzzy set theory plays an appropriate role to deal with such type of uncertainty [20]. Many fuzzy stochastic phenomena in nature which directly interest us are expressed mathematically in terms of limiting sums, derivatives, integrals, and differential and integral equations. The theory of fuzzy random variables, fuzzy martingales, various limit theorems, fuzzy differentials and integrals of non-random fuzzy mapping was developed in recent years [7] (more details exist in [15, 16, 18, 19, 22, 24, 25, 26]). For fuzzy-number-valued functions, Dubois–Prade derivative was introduced in [6]. Thereafter, other definitions of fuzzy derivative such as Hukuhara derivative (or Puri–Ralescu derivative) in [24], Goetschel–Voxman derivative in [16], Seikkala derivative in [28] and Friedman-Ming-Kandel derivative in [12] were used. Among the mentioned fuzzy derivatives, Hukuhara and Seikkala derivatives are mostly known [21]. The Hukuhara derivative (H-derivative, for short) is defined based on the Hukuhara difference (H-difference) and Seikkala derivative is defined based on derivatives of the lower and upper levels of fuzzy functions. However, the research

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results have revealed that these derivatives suffer from a number of major limitations among which the most serious is that the diameter of the fuzzy functions under study needs to be necessarily non-decreasing [21]. This shortcoming was solved by the concept of strongly generalized Hukuhara (SGH) derivative in [2]. Also the authors in [13, 14] studied fuzzy and interval differential equations using a new geometric approach.

The mean-square calculus (or for short m.s. calculus) of fuzzy stochastic processes is important for several practical reasons as in the case of real-valued stochastic processes. First of all, its importance lies in the fact that simple, powerful and well-developed methods exist. Secondly, the development of m.s. calculus and its applications in science and engineering follows, broadly the same steps used in considering calculus of non-random fuzzy functions and non-fuzzy random functions. It is thus easier to grasp for engineers and scientists who have a solid background in the fuzzy analysis [7].

This paper is organized as follows. Section 2 is devoted to notations, essential definitions, and foundational theory of the L^p space, $1 \leq p < \infty$, for fuzzy random variables, and the concept of m.s. continuity for fuzzy stochastic processes. The concept of generalized m.s. derivative and its properties are introduced in Section 3. In Section 4, we define the m.s. integral for second-order fuzzy stochastic processes and establish some properties. In Section 5, we illustrate our results by some examples. Conclusions are given in Section 6.

2. PRELIMINARIES

In this section, we give some definitions and useful results, see for example [3, 17], and introduce the necessary notation which will be used throughout the paper.

2.1. Space of fuzzy numbers

Let $\mathcal{P}_c(\mathbb{R})$ denote the family of all nonempty compact convex subsets of $\mathbb{R}$, i. e., bounded closed intervals on the real line. Define the addition and scalar multiplication in $\mathcal{P}_c(\mathbb{R})$ as usual. We denote the Pompeiu–Hausdorff semi-metric by

$$h(A,B) = \sup_{a\in A} \inf_{b\in B} |a-b|,$$

for $A, B \in \mathcal{P}_c(\mathbb{R})$, where $|\ .\ |$ denotes usual Euclidean norm in $\mathbb{R}$. It is clear that $h(A,B) = 0 \iff A \subseteq B$ and $h(A,C) \leq h(A,B) + h(B,C)$, for $A, B, C \in \mathcal{P}_c(\mathbb{R})$. The Pompeiu–Hausdorff metric d is defined by

$$d(A,B) = \max\{h(A,B), h(B,A)\}, \quad A, B \in \mathcal{P}_c(\mathbb{R}), \tag{1}$$

and the norm of an $A \in \mathcal{P}_c(\mathbb{R})$ is defined by

$$\|A\| = d(A, \{0\}) = \sup_{a\in A} |a|.$$

A fuzzy set u in $\mathbb{R}$ is characterized by its membership function $u : \mathbb{R} \longrightarrow [0,1]$ and $u(x)$ for each $x \in \mathbb{R}$ is interpreted as the degree of membership of element x in the fuzzy set u. As the value $u(x)$ expresses a "degree of membership of x in u" or a "degree of satisfying by x a property", one can work with imprecise information.

Denote $E^1 = \{u : \mathbb{R} \longrightarrow [0,1] \mid u \text{ satisfies (a)-(d)}\}$, where

(a) u is normal, i. e., there exists an x_0 such that $u(x_0) = 1$,

(b) u is fuzzy convex, i. e., $u(rx + (1-r)y) \geq \min\{u(x), u(y)\},\ r \in [0,1]$,

(c) u is upper semi-continuous,

(d) $\overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The elements of E^1 are called fuzzy numbers. Let $a \leq b \leq c$ be real numbers. The fuzzy set u is a triangular fuzzy number if

$$u(x) = \begin{cases} 0, & \text{if } x < a, \\ \frac{x-a}{b-a}, & \text{if } a \leq x < b, \\ \frac{c-x}{c-b}, & \text{if } b \leq x < c, \\ 0, & \text{if } c \leq x. \end{cases}$$

Symbolically, we write $u = (a, b, c)$.

Denote $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$ for $0 < r \leq 1$ and $[u]^0 = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$, the r-level set of u. Then, it is well-known that for each $0 \leq r \leq 1$, $[u]^r$ is a bounded closed interval and denote $[u]^r = [u^r_-, u^r_+]$. If $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a classic function, then according to Zadeh's extension principle, one can extend g to $\tilde{g} : E^1 \times E^1 \longrightarrow E^1$ by

$$\tilde{g}(u, v)\big(z\big) = \sup_{z = g(x,y)} \min\{u(x), v(y)\}. \tag{2}$$

It is well known that if g is continuous, then $[\tilde{g}(u, v)]^r = g([u]^r, [v]^r)$, for all $u, v \in E^1, r \in [0,1]$ [23]. Especially, for addition and scalar multiplication in fuzzy number space E^1, we have

$$[u + v]^r = [u]^r + [v]^r, \quad [\lambda u]^r = \lambda [u]^r, \tag{3}$$

where $u, v \in E^1, \lambda \in \mathbb{R}$ and $r \in [0,1]$. Let $A \in \mathcal{P}_c(\mathbb{R})$ and I_A be the characteristic function of A, then $I_A \in E^1$.

The Pompeiu–Hausdorff distance D on E^1 is defined by

$$D(u, v) = \sup_{0 \leq r \leq 1} d([u]^r, [v]^r), \quad u, v \in E^1. \tag{4}$$

The norm of fuzzy number $u \in E^1$ is defined by

$$\|u\| = D(u, \hat{0}), \tag{5}$$

where $\hat{0}$ is the fuzzy number in E^1 which membership function equals 1 at 0 and zero elsewhere.

Theorem 2.1. (Puri and Ralescu [25]) (E^1, D) is a complete metric space.

Also, the following results and concepts are known.

Theorem 2.2. (Anastassiou and Gal [1], Rojas-Medar et al. [27]).

(a) $\hat{0} \in E^1$ is neutral element, i. e., $u + \hat{0} = \hat{0} + u = u$, for all $u \in E^1$.

(b) With respect to $\hat{0}$, none of $E^1 \setminus \mathbb{R}$ has inverse in E^1 (with respect to $+$).

(c) For any $a, b \in \mathbb{R}$ with $ab \geq 0$ and any $u \in E^1$, we have $(a+b)u = au + bu$.

(d) For any $\lambda \in \mathbb{R}$ and any $u, v \in E^1$, we have $\lambda(u+v) = \lambda u + \lambda v$.

(e) For any $\lambda, \mu \in \mathbb{R}$ and any $u \in E^1$, we have $\lambda(\mu u) = (\lambda\mu)u$.

Corollary 2.3. (Anastassiou and Gal [1]). For any $a, b \in \mathbb{R}$ with $ab \geq 0$, and any $u \in E^1$, we have

$$D(au, bu) = |a-b| \|u\|. \tag{6}$$

2.2. The L^p space

The concepts and definitions of this section are adapted from [7]. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, where Ω is the sample space, which is the set of all possible outcomes, $\mathcal{A}$ is the Borel σ-algebra of subsets of Ω, which is a set of events, P is a Lebesgue measure on $(\Omega, \mathcal{A})$, which assigns, to each event in the event space, a probability, which is a number between 0 and 1. A fuzzy random variable (f.r.v., for short) is a Borel measurable function $F : (\Omega, \mathcal{A}) \longrightarrow (E^1, D)$. If F is an f.r.v., then $[F]^r$ is a random compact convex set for every $0 \leq r \leq 1$ [8]. Let $1 \leq p < \infty$. Then, F is called a p-order f.r.v. provided

$$E(\|F\|^p) < \infty,$$

where E denotes the expectation operator that is defined for r.v.'s X by

$$E(X) = \int_\Omega X(\omega)\, \mathrm{d}P(\omega).$$

The family of all p-order f.r.v.'s is denoted by $L^p(E^1)\big(L^p$, for short$\big)$. Any two f.r.v.'s F and G are called equivalent if $P(F \neq G) = 0$ [7] , i. e., if the same output is produced when the same values are input to each f.r.v. (either as input parameters, as values made available during the f.r.v., or all).

We define [7]

$$\rho(F, G) = \Big[E\big(D(F,G)\big)^p\Big]^{\frac{1}{p}}, \qquad F, G \in L^p. \tag{7}$$

It is easy to check that (L^p, ρ) is a metric space [7]. The norm $\|F\|_p$ of an element $F \in L^p$ is defined by

$$\|F\|_p = \rho(F, \hat{0}) = \Big(E(\|F\|^p)\Big)^{\frac{1}{p}}. \tag{8}$$

From [7], for any $F, G, Z, Y \in L^p$ and $\lambda \in \mathbb{R}$, we have

(a) $\rho(F+Z, G+Z) = \rho(F, G)$,

(b) $\rho(\lambda F, \lambda G) = |\lambda| \rho(F, G)$,

(c) $\rho(F+Y, G+Z) \leq \rho(F, G) + \rho(Y, Z)$.

Theorem 2.4. (Feng [7]) Let $\{F_n\}_{n \geq 1}$ be a sequence in L^p. The following conditions are equivalent:

(a) $F \in L^p$ and $F_n \longrightarrow F$, i. e., $\rho(F_n, F) \longrightarrow 0$,

(b) $\{F_n\}_{n\geq 1}$ is a Cauchy sequence in L^p,

(c) $(\|F_n\|_p^p, n \geq 1)$ is uniformly integrable and $D(F_n, F) \longrightarrow 0$, i. e., F converges in probability to some f.r.v. F in metric D.

Corollary 2.5. (Feng [7]) (L^p, ρ) is a complete metric space.

In the rest of the paper, we fix $p = 2$, although many of the results below hold true for each $1 \leq p < \infty$. Let $\{F_n\}_{n\geq 1}$ be a sequence in L^2. We say that F_n converges in mean-square or m.s. converges to F as $n \longrightarrow \infty$ if $F_n \xrightarrow{L_2} F$, and write $F_n \xrightarrow{m.s.} F$ [7].

Definition 2.6. Let T be an interval (finite or an infinite) in $\mathbb{R}$. A mapping $F : T \longrightarrow L^2$ is called a second-order fuzzy stochastic process (f.s.p., for short), i. e., $L^2 = \{F :$ F is f.r.v. with $E(\|F\|^2) < \infty\}$ [7].

If F is an f.r.v., then $[F]^r = [F_-^r, F_+^r]$ (F_r, for short), $r \in [0, 1]$ is a random closed and F_-^r, F_+^r are real-valued random variables [7, 11].

Similar to Definition 2.6, the second-order interval-valued stochastic process can be defined.

A second-order fuzzy stochastic process is a parameterized collection of f.r.v.'s $\{F_t\}_{t\in T}$ defined on a complete probability space $(\Omega, \mathcal{A}, P)$ the values of which are in E^1. Note that, we have a f.r.v.

$$\omega \longrightarrow F_t(\omega), \qquad \omega \in \Omega.$$

On the other hand, fixing $\omega \in \Omega$, we can consider the function

$$t \longrightarrow F_t(\omega), \qquad t \in T,$$

which is called a realization or path of F_t. Thus, we may also regard the process as a function of two variables

$$(t, \omega) \longrightarrow F(t, \omega),$$

from $T \times \Omega$ to E^1, i. e., $F(\cdot, \omega)$ is a fuzzy-valued function for a fixed $\omega \in \Omega$ (this function will be called a trajectory) and $F(t, \cdot)$ is a fuzzy random variable for any fixed $t \in T$.

Definition 2.7. Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$, then we call w the Hukuhara difference (H-difference, for short) of u and v, and denote it by $u \ominus v$.

If $F \ominus G$ exists, its r-level sets are

$$[F \ominus G]^r = [F_-^r - G_-^r, F_+^r - G_+^r],\ r \in [0, 1].$$

From the Definition of L^2 and using Theorem 2.2 and Corollary 2.3, we have the following results.

(a) $\hat{0} \in L^2$ is neutral element with respect to addition, i. e., $F + \hat{0} = \hat{0} + F = F$, for all $F \in L^2$.

(b) None $F \in L^2 \setminus \mathbb{R}$ has inverse in L^2 (with respect to addition).

(c) For any $\lambda, \mu \in \mathbb{R}$ and any $F \in L^2$, we have $\lambda(\mu F) = (\lambda\mu)F$.

(d) For any $\lambda \in \mathbb{R}$ and any $F, G \in L^2$, we have $\lambda(F + G) = \lambda F + \lambda G$.

(e) For any $a, b \in \mathbb{R}$ with $ab \geq 0$ and any $F \in L^2$, we have $(a + b)F = aF + bF$ and $\rho(aF, bF) = |a - b| \|F\|_2$.

Corollary 2.8. For any F, G and $Z \in L^2$, if $F \ominus G$ and $F \ominus Z$ exist, then

$$\rho(F \ominus G, F \ominus Z) = \rho(G, Z). \tag{9}$$

Definition 2.9. If F is continuous at $t \in T$ with respect to the metric ρ, then we say that F is continuous in mean-square or F is m.s. continuous at t. If F is m.s. continuous at every $t \in T$, then we say that F is m.s. continuous on T.

Lemma 2.10. (Bede et al. [4]) The H-difference is a continuous function in both of its arguments.

Lemma 2.11. Let $F(t) : T \longrightarrow L^2$ be a second-order fuzzy stochastic process. Then, if

(a) $\lim_{t \to t_0} [F(t)]^r = F_r = [F_-^r, F_+^r]$, uniformly with respect to $r \in [0, 1]$ for $t_0 \in T$,

(b) F_-^r and F_+^r fulfill the conditions in LU-representations (Theorem 1.1 in [16]), then $\lim_{t \to t_0} F(t) = F$, with $F_r = [F_-^r, F_+^r]$.

Proof. By condition (b), the intervals F_r define a second-order fuzzy random variable, denoted by F. Then, by condition (a), we have

$$\lim_{t \to t_0} \rho(F(t), F) = \lim_{t \to t_0} \Big[E\big(D^2(F(t), F)\big) \Big]^{\frac{1}{2}} = \lim_{t \to t_0} \Big[E\Big(\sup_{0 \leq r \leq 1} d([F(t)]^r, [F]^r) \Big)^2 \Big]^{\frac{1}{2}} = 0,$$

i. e., $\lim_{t \to t_0} F(t) = F$. □

An f.r.v. F is called integrably bounded if $E(\|F\|) < \infty$ and the expected value $E(F)$ is defined as the unique fuzzy number $E(F) = u \in E^1$ which satisfies the property: $[u]^r = [E(F)]^r = E([F]^r) = [E(F_-^r), E(F_+^r)]$ for $r \in [0, 1]$. For further details the reader is referred to [7, 8, 11, 25, 26].

The expectation of f.r.v.'s has several interesting properties which are similar to the ones of real-valued r.v.'s, for example,

$$(a)\ E(F + G) = E(F) + E(G), \tag{10}$$

$$(b)\ E(\lambda F) = \lambda E(F) \text{ for any } \lambda \in \mathbb{R}, \tag{11}$$

$$(c)\ E(u) = u, \text{ for any } u \in E^1, \tag{12}$$

where F and G are f.r.v.'s with $E(\|F\|) < \infty, E(\|G\|) < \infty$ [8]. It is well known that the inequality

$$D(E(F), E(G)) \leq E(D(F, G)),$$

holds for any f.r.v.'s F and G [25].

3. MEAN-SQUARE DIFFERENTIATION

In [2], the concept of strongly generalized differentiability is introduced for fuzzy-number-valued functions, as follows.

Definition 3.1. Let $g : T \longrightarrow E^1$. We say that g is strongly generalized Hukuhara (SGH) differentiable at $t \in T$ with derivative $g'(t) \in E^1$, if for all sufficiently small $h > 0$, the H-differences and the limits exist in at least one of the following cases:

$$
\begin{aligned}
(i)\ &\lim_{h\to 0^+} D\Big(\frac{g(t+h)\ominus g(t)}{h}, g'(t)\Big) = \lim_{h\to 0^+} D\Big(\frac{g(t)\ominus g(t-h)}{h}, g'(t)\Big) = 0,\\
(ii)\ &\lim_{h\to 0^+} D\Big(\frac{g(t)\ominus g(t+h)}{-h}, g'(t)\Big) = \lim_{h\to 0^+} D\Big(\frac{g(t-h)\ominus g(t)}{-h}, g'(t)\Big) = 0,\\
(iii)\ &\lim_{h\to 0^+} D\Big(\frac{g(t+h)\ominus g(t)}{h}, g'(t)\Big) = \lim_{h\to 0^+} D\Big(\frac{g(t-h)\ominus g(t)}{-h}, g'(t)\Big) = 0,\\
(iv)\ &\lim_{h\to 0^+} D\Big(\frac{g(t)\ominus g(t+h)}{-h}, g'(t)\Big) = \lim_{h\to 0^+} D\Big(\frac{g(t)\ominus g(t-h)}{h}, g'(t)\Big) = 0.
\end{aligned}
$$

We say that g is (i)-differentiable at $t \in T$, if the case (i) in Definition 3.1 is satisfied. We have analogous notations for the cases (ii), (iii) and (iv).

The (i)-derivative of $g(t) \in E^1$ in Definition 3.1 coincides with the H-derivative of fuzzy-number-valued function, introduced in [24].

The concept of m.s. differentiability for second-order fuzzy stochastic processes in [7] is introduced as follows.

Definition 3.2. Let $F : T \longrightarrow L^2$. We say that F is m.s. differentiable at $t \in T$ with derivative $F'(t) \in L^2$, if for all sufficiently small $h > 0$, the H-differences and the following limits exist

$$
\lim_{h\to 0^+} \rho\Big(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t)\ominus F(t-h)}{h}, F'(t)\Big) = 0. \tag{13}
$$

Now, similar to SGH derivative of fuzzy functions, we introduce a new generalized concept of differentiability which extends the m.s. differentiability for f.s.p.'s.

Definition 3.3. Let $F : T \longrightarrow L^2$. We say that F is generalized m.s. differentiable at $t \in T$ with derivative $F'(t) \in L^2$, if for all sufficiently small $h > 0$, the H-differences and the limits exist in at least one of the following cases:

$$
\begin{aligned}
(i)\ &\lim_{h\to 0^+} \rho\Big(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t)\ominus F(t-h)}{h}, F'(t)\Big) = 0,\\
(ii)\ &\lim_{h\to 0^+} \rho\Big(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t-h)\ominus F(t)}{-h}, F'(t)\Big) = 0,\\
(iii)\ &\lim_{h\to 0^+} \rho\Big(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t-h)\ominus F(t)}{-h}, F'(t)\Big) = 0,\\
(iv)\ &\lim_{h\to 0^+} \rho\Big(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t)\ominus F(t-h)}{h}, F'(t)\Big) = 0.
\end{aligned}
$$

At the end points of T, we consider the one-side derivatives. We say that F is (i)-m.s. differentiable at $t \in T$, if the case (i) in Definition 3.3 is satisfied. We have analogous notations for the cases (ii), (iii) and (iv).

The following notations for simplicity are used. We say that F satisfies the condition (H_1) at $t \in T$, if $F(t+h) \ominus F(t)$ and $F(t) \ominus F(t-h)$ exists for sufficiently small $h > 0$. Similarly, we say that F satisfies the condition (H_2) at $t \in T$, if $F(t) \ominus F(t+h)$ and $F(t-h) \ominus F(t)$ exists for sufficiently small $h > 0$.

Remark 3.4. The (i)-m.s. differentiability of $F : T \longrightarrow L^2$ coincides with the differentiability in Definition 3.2.

Theorem 3.5. Let $F : T \longrightarrow L^2$. If F is generalized m.s. differentiable at $t \in T$, then the derivative is unique.

Proof. We suppose that F is (i)-m.s. differentiable in the sense of Definition 3.3. The proofs of other cases are similar. Assume that $F'(t)$ and $G'(t) \in L^2$ are two derivatives of F at $t \in T$. So,

$$\lim_{h \to 0^+} \rho(\frac{F(t+h) \ominus F(t)}{h}, F'(t)) = \lim_{h \to 0^+} \rho(\frac{F(t) \ominus F(t-h)}{h}, F'(t)) = 0,$$

and

$$\lim_{h \to 0^+} \rho(\frac{F(t+h) \ominus F(t)}{h}, G'(t)) = \lim_{h \to 0^+} \rho(\frac{F(t) \ominus F(t-h)}{h}, G'(t)) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that for $|h| < \delta$, we have

$$\rho(F(t+h) \ominus F(t), hF'(t)) < |h|\frac{\epsilon}{2},$$

and

$$\rho(F(t+h) \ominus F(t), hG'(t)) < |h|\frac{\epsilon}{2}.$$

Therefore,

$$\rho(hF'(t), hG'(t)) < \rho(hF'(t), F(t+h) \ominus F(t)) + \rho(hG'(t), F(t+h) \ominus F(t)) < |h|\epsilon.$$

Thus

$$\rho(F'(t), G'(t)) < \epsilon,$$

and proof is complete.

Now, we consider the case that F is m.s. differentiable at $t \in T$ in different cases. For example, let F be (i)-m.s. differentiable with $F'(t)$ derivative and (ii)-m.s. differentiable with $G'(t)$ derivative at $t \in T$. Then

$$\lim_{h \to 0^+} \rho(\frac{F(t+h) \ominus F(t)}{h}, F'(t)) = \lim_{h \to 0^+} \rho(\frac{F(t) \ominus F(t-h)}{h}, F'(t)) = 0,$$

and

$$\lim_{h \to 0^+} \rho(\frac{F(t) \ominus F(t+h)}{-h}, G'(t)) = \lim_{h \to 0^+} \rho(\frac{F(t-h) \ominus F(t)}{-h}, G'(t)) = 0,$$

then

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{|-h|}\rho(F(t)\ominus F(t+h),-hG'(t))\\
&=\lim_{h\to 0^+}\frac{1}{h}\rho(F(t)\ominus F(t+h),-hG'(t))\\
&=\lim_{h\to 0^+}\frac{1}{h}\rho(F(t)\ominus F(t+h)+hF'(t),hF'(t)-hG'(t))\\
&=\lim_{h\to 0^+}\frac{1}{h}\rho(F(t)+hF'(t),F(t+h)+hF'(t)-hG'(t))\\
&=0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\rho(\hat{0},hF'(t)-hG'(t))\\
&=\lim_{h\to 0^+}\frac{1}{h}\rho(F(t+h),F(t+h)+hF'(t)-hG'(t))\\
&\le\lim_{h\to 0^+}\frac{1}{h}\rho(F(t+h),F(t)+hF'(t))+\lim_{h\to 0^+}\frac{1}{h}\rho(F(t)+hF'(t),F(t+h)+hF'(t)-hG'(t))\\
&=0.
\end{aligned}
$$

Thus

$$\lim_{h\to 0^+}\frac{1}{h}\rho(\hat{0},hF'(t)-hG'(t))=\rho(\hat{0},F'(t)-G'(t))=0.$$

Therefore, $F'(t)-G'(t)=\hat{0}$. So the derivative is a singleton and $F'(t)=G'(t)$. The proofs of other cases are similarly obtained as the previous cases. □

Theorem 3.6. Let $F:T\longrightarrow L^2$ be a second-order fuzzy stochastic process. If F has a generalized m.s. derivative at $t\in T$ in at least two cases in Definition 3.3, then $F'(t)$ is crisp.

P r o o f. Suppose that F is (iii)-m.s. differentiable on t. Then, the H-differences $F(t+h)\ominus F(t)$ and $F(t-h)\ominus F(t)$ exist for sufficiently small $h>0$. So, we have $F(t+h)=F(t)+u(t,h)$ and $F(t-h)=F(t)+v(t,h)$ for sufficiently small $h>0$. Thus

$$F(t)=F(t+h)+v(t+h,h).$$

Then $F(t+h)=F(t)+u(t,h)=F(t+h)+v(t+h,h)+u(t,h)$, which follows that $v(t+h,h)+u(t,h)=\hat{0}$. Therefore, $v(t+h,h)$ and $u(t,h)$ are crisp for sufficiently small $h>0$. Then, it is easy to see that $F'(t)=\lim_{h\to 0^+}\frac{u(t,h)}{h}$. If F is (iv)-m.s. differentiable, the reasoning is similar. □

Theorem 3.7. Let $F:T\longrightarrow L^2$ be generalized m.s. differentiable at $t\in T$. Then it is m.s. continuous at t.

Proof. For example, let us suppose that F is (iii)-m.s. differentiable at $t \in T$. It follows that for any $\epsilon > 0$, there exists $\delta > 0$, such that for $0 <| h |< \delta$, we have

$$\rho\left(\frac{F(t+h)\ominus F(t)}{h}, F'(t)\right) < \epsilon,$$

and

$$\rho\left(\frac{F(t-h)\ominus F(t)}{-h}, F'(t)\right) < \epsilon.$$

Then $\rho(F(t+h), F(t)+hF'(t)) < \epsilon|h|$, and $\rho(F(t), F(t-h)+hF'(t)) < \epsilon|h|$. Therefore, we have

$$\rho(F(t+h), F(t)) \leq \rho(F(t+h), F(t)+hF'(t))+\rho(F(t)+hF'(t), F(t)) \leq \epsilon|h|+|h|\rho(F'(t), \hat{0}).$$

Passing to the limit as $h \to 0^+$, we obtain

$$\lim_{h\to 0^+} \rho(F(t+h), F(t)) = 0.$$

Similarly, if $\lim_{h\to 0^-} \rho(F(t-h), F(t)) = 0$, then F is continuous at t. The proofs of the other cases can be obtained similarly. □

Theorem 3.8. Let $F : T \longrightarrow L^2$ be generalized m.s. differentiable at $t \in T$. Then $F(t) = \textit{constant}$ for all $t \in T$ if and only if $F'(t) = \hat{0}$ for all $t \in T$.

Proof. Let $F_0 \in L^2$ and suppose that $F : T \longrightarrow L^2, F(t) = F_0$ for all $t \in T$ is a constant function and (iv)-m.s. differentiable, then

$$\rho(\hat{0}, F'(t)) = \lim_{h\to 0^+} \rho(\frac{F_0 \ominus F_0}{-h}, F'(t)) = \lim_{h\to 0^+} \rho(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)) = 0.$$

Thus $F'(t) = \hat{0}$. The proofs of the other cases can be obtained similarly.

Reciprocally, let us consider that $F : T \longrightarrow L^2$ is (iv)-m.s. differentiable and $F'(t) = \hat{0}$, i.e.,

$$\lim_{h\to 0^+} \rho(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)) = \lim_{h\to 0^+} \rho(\frac{F(t)\ominus F(t+h)}{-h}, \hat{0}) = 0,$$

and

$$\lim_{h\to 0^+} \rho(\frac{F(t)\ominus F(t-h)}{h}, F'(t)) = \lim_{h\to 0^+} \rho(\frac{F(t)\ominus F(t-h)}{h}, \hat{0}) = 0.$$

Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for $0 <|h| < \delta$, we have $\rho(F(t), F(t-h)) <|h|\epsilon$.

Let $t, s \in T$. Since $\rho(F(s), F(t)) \leq \rho(F(s), F(t-h))+\rho(F(t-h), F(t))$ and $\rho(F(s), F(t-h)) \leq \rho(F(s), F(t)) + \rho(F(t), F(t-h))$, then for $0 <|h| < \delta$ we have $|\rho(F(s), F(t)) - \rho(F(s), F(t-h))| \leq|h|\epsilon$. Let $h \longrightarrow 0$, the real valued function $t \longrightarrow \rho(F(s), F(t))$, whose derivative is equal to zero, for every $t \in T$, must be constant. For $t = s$, it is identically 0, so $F(t) = F(s)$ for all $t \in T$, which is the desired result. The proofs of other cases are similarly obtained as the previous case. □

Theorem 3.9. Let $F, G : T \longrightarrow L^2$ be generalized m.s. differentiable at $t \in T$ in the same case for (i)-(iv) and $\lambda \in \mathbb{R}$, then

(a) $F + G$ is m.s. differentiable at t and $(F + G)'(t) = F'(t) + G'(t)$.

(b) λF is m.s. differentiable at t and $(\lambda F)'(t) = \lambda F'(t)$.

Proof. Case (a). Suppose that F, G are (iii)-m.s. differentiable. It follows that for all $\epsilon > 0$, there exists $\delta > 0$ such that for $-\delta < -h < 0$, we have $\lim_{h\to 0^+} \rho(\frac{F(t-h)\ominus F(t)}{-h}, F'_-(t)) = 0$, then $\rho(\frac{F(t-h)\ominus F(t)}{-h}, F'_-(t)) < \frac{\epsilon}{2}$. So $\rho(F(t-h) \ominus F(t), |h|F'_-(t)) < \frac{\epsilon|h|}{2}$, then $\rho(F(t-h), F(t)+|h|F'_-(t)) < \frac{\epsilon|h|}{2}$. Similarly, we have $\rho(G(t-h), G(t)+|h|G'_-(t)) < \frac{\epsilon|h|}{2}$. Then

$$\begin{aligned}
&\rho\Big(F(t-h) + G(t-h), F(t) + G(t)+|h|F'_-(t)+|h|G'_-(t)\Big) \\
&\leq \rho\Big(F(t-h), F(t)+|h|F'_-(t)\Big) + \rho\Big(G(t-h), G(t)+|h|G'_-(t)\Big) \\
&\leq \frac{\epsilon|h|}{2} + \frac{\epsilon|h|}{2}.
\end{aligned}$$

Similarly, for $0 < h < \delta$, we have

$$\rho\Big(F(t+h) + G(t+h), F(t) + G(t)+|h|F'_+(t)+|h|G'_+(t)\Big) \leq \epsilon|h|.$$

Now, passing to the limit as $h \to 0^+$, we obtain $(F+G)'(t) = F'(t) + G'(t)$ in the sense (iii)-m.s. differentiability. The proof of the other cases can be obtained similarly.

Case (b). Suppose that F is (ii)-m.s. differentiable, then

$$\lim_{h\to 0^+} \rho\Big(\frac{F(t) \ominus F(t+h)}{-h}, F'(t)\Big) = \lim_{h\to 0^+} \rho\Big(\frac{F(t-h) \ominus F(t)}{-h}, F'(t)\Big) = 0.$$

Therefore, we have

$$\lim_{h\to 0^+} \rho\Big(\frac{\lambda F(t) \ominus \lambda F(t+h)}{-h}, \lambda F'(t)\Big) =|\lambda| \lim_{h\to 0^+} \rho\Big(\frac{F(t) \ominus F(t+h)}{-h}, F'(t)\Big) = 0,$$

and

$$\lim_{h\to 0^+} \rho\Big(\frac{\lambda F(t-h) \ominus \lambda F(t)}{-h}, \lambda F'(t)\Big) =|\lambda| \lim_{h\to 0^+} \rho\Big(\frac{F(t-h) \ominus F(t)}{-h}, F'(t)\Big) = 0.$$

Then $\lambda F(t)$ is (ii)-m.s. differentiable and $(\lambda F)'(t) = \lambda F'(t)$. The proof of the other cases can be obtained similarly. □

Corollary 3.10. Let $F, G : T \longrightarrow L^2$ be generalized m.s. differentiable at $t \in T$ in the same case (i)-(iv) and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 F + \lambda_2 G$ is m.s. differentiable at t and $(\lambda_1 F + \lambda_2 G)'(t) = \lambda_1 F'(t) + \lambda_2 G'(t)$.

Theorem 3.11. Let $F, G : T \longrightarrow L^2$ be generalized m.s. differentiable on T and $(\alpha, \beta) \subseteq T$.

(a) If F is (i)-m.s. differentiable and G is (ii)-m.s. differentiable on an interval (α,β) and the H-difference $F(t)\ominus G(t)$ exists for $t\in(\alpha,\beta)$, then $F\ominus G$ is (i)-m.s. differentiable and

$$(F\ominus G)'(t)=F'(t)+(-1)G'(t),\quad \text{for all } t\in(\alpha,\beta).$$

(b) If F is (ii)-m.s. differentiable and G is (i)-m.s. differentiable on an interval (α,β) and the H-difference $F(t)\ominus G(t)$ exists for $t\in(\alpha,\beta)$, then $F\ominus G$ is (ii)-m.s. differentiable and

$$(F\ominus G)'(t)=F'(t)+(-1)G'(t),\quad \text{for all } t\in(\alpha,\beta).$$

Proof. We present the details for the case (a), the case (b) is analogous.

Case (a). Since F is (i)-m.s. differentiable, it follows that there exist $u_1(t,h)$ and $u_2(t,h)\in L^2$ such that

$$F(t+h)=F(t)+u_1(t,h),\qquad \lim_{h\to 0^+}\rho(F'(t),\frac{u_1(t,h)}{h})=0,$$

and

$$F(t)=F(t-h)+u_2(t,h),\qquad \lim_{h\to 0^+}\rho(F'(t),\frac{u_2(t,h)}{h})=0.$$

Analogously, since G is (ii)-m.s. differentiable, there exist $v_1(t,h)$ and $v_2(t,h)\in L^2$ such that

$$G(t)=G(t+h)+v_1(t,h),\qquad \lim_{h\to 0^+}\rho(G'(t),\frac{v_1(t,h)}{-h})=0,$$

and

$$G(t-h)=G(t)+v_2(t,h),\qquad \lim_{h\to 0^+}\rho(G'(t),\frac{v_2(t,h)}{-h})=0.$$

Then

$$F(t+h)+G(t)=F(t)+G(t+h)+u_1(t,h)+v_1(t,h).$$

Since the H-differences $F(t)\ominus G(t)$ and $F(t+h)\ominus G(t+h)$ exist for sufficiently small $h>0$, we get

$$F(t+h)\ominus G(t+h)=F(t)\ominus G(t)+u_1(t,h)+v_1(t,h).$$

Then

$$(F(t+h)\ominus G(t+h))\ominus(F(t)\ominus G(t))=u_1(t,h)+v_1(t,h).$$

Therefore,

$$\begin{aligned}&\lim_{h\to 0^+}\rho\Big(\frac{(F(t+h)\ominus G(t+h))\ominus(F(t)\ominus G(t))}{h},F'(t)+(-1)G'(t)\Big)\\ &\le \lim_{h\to 0^+}\rho\Big(\frac{(F(t+h)\ominus G(t+h))\ominus(F(t)\ominus G(t))}{h},\frac{u_1(t,h)}{h}+\frac{v_1(t,h)}{h}\Big)\end{aligned}$$

$$+\lim_{h\to 0^+}\rho\Big(\frac{u_1(t,h)}{h}+\frac{v_1(t,h)}{h},F'(t)+(-1)G'(t)\Big)$$
$$\le \lim_{h\to 0^+}\rho(\frac{u_1(t,h)}{h},F'(t))+\lim_{h\to 0^+}\rho(\frac{v_1(t,h)}{h},(-1)G'(t))$$
$$=0.$$

Then the proof for case (a) is complete. The case when F is (ii)-m.s. differentiable and G is (i)-m.s. differentiable is similar to the previous one. □

Corollary 3.12. Let $F:[a,b]\longrightarrow L^2$ be generalized m.s. differentiable on $[a,b]$.

(a) If F is (i)-m.s. differentiable on $[a,b]$, then $F(b)\ominus F(t)$ is (ii)-m.s. differentiable for $t\in[a,b]$ and
$$(F(b)\ominus F(t))'=(-1)F'(t).$$

(b) If F is (ii)-m.s. differentiable on $[a,b]$, then $F(a)\ominus F(t)$ is (i)-m.s. differentiable for $t\in[a,b]$
$$(F(a)\ominus F(t))'=(-1)F'(t).$$

The following results extend Theorems 3.9 and 3.11 for different types of differentiability.

Theorem 3.13. Let $F,G:T\longrightarrow L^2$be generalized m.s. differentiable at $t\in T$.

(a) If F is (i)-m.s. differentiable, G is (ii)-m.s. differentiable at t and $F+G$ satisfies (H_1) at t, then $F+G$ is (i)-m.s. differentiable at t and
$$(F+G)'(t)=F'(t)\ominus(-1)G'(t).$$

(b) If F is (i)-m.s. differentiable, G is (ii)-m.s. differentiable at t and $F+G$ satisfies (H_2) at t, then $F+G$ is (ii)-m.s. differentiable at t and
$$(F+G)'(t)=G'(t)\ominus(-1)F'(t).$$

P r o o f. Case (a). Since F is (i)-m.s. differentiable at t, the H-differences $F(t+h)\ominus F(t)$ and $F(t)\ominus F(t-h)$ exist for sufficiently small $h>0$, i. e., there exist $u_1(t,h),u_2(t,h)\in L^2$ such that
$$F(t+h)=F(t)+u_1(t,h),\qquad F(t)=F(t-h)+u_2(t,h).$$
Analogously, since G is (ii)-m.s. differentiable at t, there exist $v_1(t,h)$ and $v_2(t,h)\in L^2$ such that for sufficiently small $h>0$, we have
$$G(t)=G(t+h)+v_1(t,h),\qquad G(t-h)=G(t)+v_2(t,h).$$
Since $F+G$ satisfies (H_1) at t, then the H-differences $(F(t+h)+G(t+h))\ominus(F(t)+G(t))$ and $(F(t)+G(t))\ominus(F(t-h)+G(t-h))$ exist for sufficiently small $h>0$, i. e., there exist $w_1(t,h)$ and $w_2(t,h)\in L^2$ such that
$$F(t+h)+G(t+h)=F(t)+G(t)+w_1(t,h),\quad F(t)+G(t)=F(t-h)+G(t-h)+w_2(t,h).$$

Then

$$(F(t+h)\ominus F(t))\ominus(G(t)\ominus G(t+h)) = w_1(t,h),\ (F(t)\ominus F(t-h))\ominus(G(t-h)\ominus G(t)) = w_2(t,h).$$

Multiplying with $\frac{1}{h}$ and passing to limit as $h \to 0^+$, we get that $F+G$ is (i)-differentiable at t and

$$(F+G)'(t) = F'(t) \ominus (-1)G'(t).$$

The case (b) is similar. □

Theorem 3.14. Let $F, G : T \longrightarrow L^2$ be generalized m.s. differentiable functions.

(a) If F, G are (i)-m.s. differentiable functions on T and $F \ominus G$ exists on T and $F \ominus G$ satisfies (H_1) at every $t \in T$, then $F \ominus G$ is (i)-m.s. differentiable and

$$(F \ominus G)'(t) = F'(t) \ominus G'(t), \qquad \text{for all t} \in T.$$

(b) If F, G are (i)-m.s. differentiable functions on T and $F \ominus G$ exists on T and $F \ominus G$ satisfies (H_2) at every $t \in T$, then $F \ominus G$ is (ii)-m.s differentiable and

$$(F \ominus G)'(t) = -(G'(t) \ominus F'(t)), \qquad \text{for all t} \in T.$$

(c) If F, G are (ii)-m.s. differentiable functions on T and $F \ominus G$ exists on T and $F \ominus G$ satisfies (H_1) at every $t \in T$, then $F \ominus G$ is (i)-m.s. differentiable and

$$(F \ominus G)'(t) = -(G'(t) \ominus F'(t)), \qquad \text{for all t} \in T.$$

(d) If F, G are (ii)-m.s. differentiable functions on T and $F \ominus G$ exists on T and $F \ominus G$ satisfies (H_2) at every $t \in T$, then $F \ominus G$ is (ii)-m.s. differentiable and

$$(F \ominus G)'(t) = F'(t) \ominus G'(t), \qquad \text{for all t} \in T.$$

Proof. We present the details for the case (a), the other cases are analogous.

Case(a). Since F, G are (i)-m.s. differentiable at $t \in T$, the H-differences $F(t+h)\ominus F(t)$, $F(t) \ominus F(t-h)$, $G(t+h) \ominus G(t)$ and $G(t) \ominus G(t-h)$ exist for sufficiently small $h > 0$, i. e., there exist $u_1(t,h)$, $u_2(t,h)$, $v_1(t,h)$ and $v_2(t,h)$ in L^2 such that

$$F(t+h) = F(t) + u_1(t,h), \qquad F(t) = F(t-h) + u_2(t,h),$$

and

$$G(t+h) = G(t) + v_1(t,h), \qquad G(t) = G(t-h) + v_2(t,h).$$

Since $F \ominus G$ exists and satisfies (H_1) at t, then the H-differences $(F(t+h) \ominus G(t+h)) \ominus (F(t) \ominus G(t))$ and $(F(t) \ominus G(t)) \ominus (F(t-h) \ominus G(t-h))$ exist for sufficiently small $h > 0$, i. e., there exist $w_1(t,h)$ and $w_2(t,h) \in L^2$ such that

$$F(t+h) \ominus F(t) = G(t+h) \ominus G(t) + w_1(t,h),\ F(t) \ominus F(t-h) = G(t) \ominus G(t-h) + w_2(t,h).$$

Multiplying with $\frac{1}{h}$ and passing to limit as $h \to 0^+$, we get $F'(t) = G'(t) + (F(t) \ominus (-1)G(t))'$, and conclusion in case (a) is obtained. □

Theorem 3.15. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $G : \mathbb{R} \longrightarrow L^2$ be two differentiable functions (where G is (i)-m.s. differentiable or (ii)-m.s. differentiable).

(a) If $f(t)f'(t) > 0$ and G is (i)-m.s. differentiable, then fG is (i)-m.s. differentiable and

$$(fG)'(t) = f'(t)G(t) + f(t)G'(t).$$

(b) If $f(t)f'(t) < 0$ and G is (ii)-m.s. differentiable, then fG is (ii)-m.s. differentiable and

$$(fG)'(t) = f'(t)G(t) + f(t)G'(t).$$

(c) If $f(t)f'(t) > 0$ and G is (ii)-m.s. differentiable and fG satisfies (H_1) at t, then fG is differentiable and

$$(fG)'(t) = f'(t)G(t) \ominus (-f(t))G'(t).$$

(d) If $f(t)f'(t) > 0$ and G is (ii)-m.s. differentiable and fG satisfies (H_2) at t, then fG is differentiable and we have

$$(fG)'(t) = f(t)G'(t) \ominus (-f'(t))G(t).$$

(e) If $f(t)f'(t) < 0$ and G is (i)-m.s. differentiable and fG satisfies (H_1) at t, then fG is differentiable and

$$(fG)'(t) = f(t)G'(t) \ominus (-f'(t))G(t).$$

(f) If $f(t)f'(t) < 0$ and G is (i)-m.s. differentiable and fG satisfies (H_2) at t, then fG is differentiable and

$$(fG)'(t) = f'(t)G(t) \ominus (-f(t))G'(t).$$

Proof. We present the details for the cases (b), (c) and (d), the other cases are analogous.

Case (b). Since f is continuous, for sufficiently small $h > 0$, $f(t)$, $f(t-h)$ and $f(t+h)$ have the same sign. Since G is (ii)-differentiable at t, the H-difference $G(t) \ominus G(t+h)$ exists for sufficiently small $h > 0$, i. e., there exists $u_1(t,h) \in L^2$ such that

$$G(t) = G(t+h) + u_1(t,h).$$

Also,

$$f(t) = f(t+h) + v_1(t,h),$$

where $v_1(t,h) = f(t) - f(t+h)$ has the same sign as $f(t)$ and $f(t+h)$ for sufficiently small $h > 0$. By Theorem 2.2 (c) and (d), we get

$$f(t)G(t) = f(t+h)G(t+h) + f(t,h)u_1(t+h) + v_1(t,h)G(t+h) + v_1(t,h)u_1(t,h).$$

So, the H-difference $f(t)G(t) \ominus f(t+h)G(t+h)$ exists and we have

$$f(t)G(t) \ominus f(t+h)G(t+h) = f(t+h)u_1(t,h) + v_1(t,h)G(t+h) + v_1(t,h)u_1(t,h).$$

Multiplying with $\frac{1}{-h}$ and passing to limit as $h \to 0^+$, we get

$$\lim_{h\to 0^+} \frac{f(t)G(t) \ominus f(t+h)G(t+h)}{-h} = \lim_{h\to 0^+} f(t+h)\frac{u_1(t,h)}{-h} + G(t+h)\frac{v_1(t,h)}{-h} + \frac{v_1(t,h)}{-h}u_1(t,h).$$

It can be easily observed that the product of an element of L^2 by a crisp number is continuous, by Definition 3.3, we get

$$\lim_{h\to 0^+} \frac{f(t)G(t) \ominus f(t+h)G(t+h)}{-h} = f(t)G'(t) + f'(t)G(t) + f'(t) \lim_{h\to 0^+} u_1(t,h).$$

Since G is continuous, by Lemma 2.10, the last term is $\hat{0}$ and

$$\lim_{h\to 0^+} \frac{f(t)G(t) \ominus f(t+h)G(t+h)}{-h} = f(t)G'(t) + f'(t)G(t).$$

Similarly, we get

$$\lim_{h\to 0^+} \frac{f(t-h)G(t-h) \ominus f(t)G(t)}{-h} = f(t)G'(t) + f'(t)G(t),$$

and finally the required conclusion is obtained.

Case (c). Similar to the previous case, we have

$$G(t) = G(t+h) + u_2(t,h),$$

and

$$f(t+h) = f(t) + v_2(t,h),$$

where $u_2(t,h) = G(t) \ominus G(t+h)$ and $v_2(t,h) = f(t+h) - f(t)$ have the same sign for sufficiently small $h > 0$. Therefore, we have

$$f(t)G(t) + v_2(t,h)G(t) = f(t+h)G(t+h) + f(t+h)u_2(t+h).$$

Since $f(t+h)G(t+h) \ominus f(t)G(t)$ exists, it follows that the H-difference $v_2(t,h)G(t) \ominus f(t+h)u_2(t,h)$ exists and

$$f(t+h)G(t+h) \ominus f(t)G(t) = v_2(t,h)G(t) \ominus f(t+h)u_2(t,h).$$

Multiplying with $\frac{1}{h}$ and passing to limit as $h \to 0^+$, by Lemma 2.10, we get

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(t+h)G(t+h) \ominus f(t)G(t)}{h} &= \lim_{h\to 0^+} \frac{v_2(t,h)}{h}G(t+h) \ominus f(t+h)\frac{u_2(t,h)}{h} \\ &= f'(t)G(t) \ominus f(t)G'(t).\end{aligned}$$

Analogously, we get

$$\lim_{h\to 0^+} \frac{f(t)G(t) \ominus f(t-h)G(t-h)}{h} = f'(t)G(t) \ominus -f(t)G'(t).$$

Therefore, the conclusion in the case (c) is obtained.

Case (d). Similar to case (c), we have

$$f(t)G(t) + v_2(t,h)G(t) = f(t+h)G(t+h) + f(t+h)u_2(t+h).$$

By (H_2), the H-difference $f(t)G(t) \ominus f(t+h)G(t+h)$ exists and so the H-difference $f(t+h)u_2(t,h) \ominus v_2(t,h)G(t)$ exists and we have

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(t)G(t) \ominus f(t+h)G(t+h)}{h} &= \lim_{h\to 0^+} f(t+h)\frac{u_2(t+h)}{-h} \ominus \frac{v_2(t+h)}{-h}G(t)\\ &= f'(t)G(t) \ominus -f(t)G'(t).\end{aligned}$$

Similarly,

$$\lim_{h\to 0^+} \frac{f(t-h)G(t-h) \ominus f(t)G(t)}{h} = f'(t)G(t) \ominus -f(t)G'(t).$$

and this equation leads to the required conclusion. □

Lemma 3.16. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable and G_0 be constant from L^2.

(a) If $f(t)f'(t) > 0$, then fG_0 satisfies (H_1) at $t \in \mathbb{R}$.

(b) If $f(t)f'(t) < 0$, then fG_0 satisfies (H_2) at $t \in \mathbb{R}$.

P r o o f. Case(a). Since f is continuous, for sufficiently small $h > 0$, $f(t)$, $f(t+h)$ and $f(t-h)$ have the same sign as $f(t+h) - f(t)$, then $f(t+h)G_0 = ((f(t+h) - f(t)) + f(t))G_0 = (f(t+h) - f(t))G_0 + f(t)G_0$.

Therefore, the H-difference $f(t+h)G_0 \ominus f(t)G_0$ exists. The proof of case (b) is similarly obtained. □

Corollary 3.17. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable and G_0 be constant in L^2.

(a) If $f(t)f'(t) > 0$, then fG_0 is (i)-m.s. differentiable and we have

$$(fG_0)'(t) = f'(t)G_0.$$

(b) If $f(t)f'(t) < 0$, then fG_0 is (ii)-m.s. differentiable and we have

$$(fG_0)'(t) = f'(t)G_0.$$

P r o o f. It is an immediate consequence of Theorems 3.8, 3.15 and Corollary 3.16. □

Similar to strongly generalized Hukuhara derivative of interval-valued functions in [5, 29], the generalized m.s. differentiability for a second-order interval-valued stochastic process can be defined, as follows.

Definition 3.18. Let $X(t)$ be a second-order interval-valued stochastic process. We say that X is generalized m.s. differentiable at $t \in T$ with derivative $X'(t)$, if for all sufficiently small $h > 0$, the H-differences and the limits exist in at least one of the following cases:

$$(i) \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t+h)\ominus X(t)}{h}, X'(t)\big)\Big) = \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t)\ominus X(t-h)}{h}, X'(t)\big)\Big),$$
$$(ii) \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t)\ominus X(t+h)}{-h}, X'(t)\big)\Big) = \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t-h)\ominus X(t)}{-h}, X'(t)\big)\Big),$$
$$(iii) \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t+h)\ominus X(t)}{h}, X'(t)\big)\Big) = \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t-h)\ominus X(t)}{-h}, X'(t)\big)\Big),$$
$$(iv) \lim_{h\to 0^+} \Big(D^2\big(\frac{X(t)\ominus X(t+h)}{-h}, X'(t)\big)\Big) = \lim_{h\to 0^+} E\Big(D^2\big(\frac{X(t)\ominus X(t-h)}{h}, X'(t)\big)\Big).$$

At the end points of T, we consider the one-side derivatives. We say that F is (i)-m.s. differentiable at $t \in T$, if the case (i) in Definition 3.18 is satisfied. We have analogous notations for the cases (ii), (iii) and (iv). We have the following results for second-order f.s.p.s

Theorem 3.19. Let $F : T \longrightarrow L^2$ be a second-order f.s.p. and denote $F_r(t) = [F(t)]^r = [F_-^r, F_+^r]$, for all $r \in [0,1]$.

(a) If $F(t)$ is (i)-m.s. differentiable at t uniformly for all $r \in [0,1]$, then $F_r'(t) = [F'(t)]^r$, for all $r \in [0,1]$.

(b) If $F(t)$ is (ii)-m.s. differentiable at t uniformly for all $r \in [0,1]$, then $F_r'(t) = [F'(t)]^r$, for all $r \in [0,1]$.

Proof. We present the details for the case (b), the case (a) is analogous.
Case (b). If $F(t)$ is (ii)-m.s. differentiable at $t \in T$, then given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
\rho\Big(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\Big) &= \int_\Omega D^2\Big(\frac{F(t)\ominus F(t+h)}{-h}, F'(t)\Big)\mathrm{d}P(\omega)\\
&= \int_\Omega \Big(\sup_{0\le r\le 1} \mathrm{d}\big([\frac{F(t)\ominus F(t+h)}{-h}]^r, [F'(t)]^r\big)\Big)^2 \mathrm{d}P(\omega)\\
&= \int_\Omega \Big(\sup_{0\le r\le 1} \mathrm{d}\big(\frac{[F(t)]^r\ominus [F(t+h)]^r}{-h}, [F'(t)]^r\big)\Big)^2 \mathrm{d}P(\omega)\\
&= \int_\Omega \Big(\sup_{0\le r\le 1} \mathrm{d}\big(\frac{F_r(t)\ominus F_r(t+h)}{-h}, [F'(t)]^r\big)\Big)^2 \mathrm{d}P(\omega)\\
&< \epsilon.
\end{aligned}$$

Similarly, we obtain

$$\int_\Omega \Big(\sup_{0\le r\le 1} \mathrm{d}\big(\frac{F_r(t-h)\ominus F_r(t)}{-h}, [F'(t)]^r\big)\Big)^2 \mathrm{d}P(\omega) < \epsilon,$$

for all $h < \delta$. Therefore, $F_r(t)$ is (ii)-m.s. differentiable at t uniformly for all $r \in [0,1]$ and $F_r'(t) = [F'(t)]^r$ for all $r \in [0,1]$. □

Theorem 3.20. Let $F : T \longrightarrow L^2$ be second-order f.s.p.

(a) If $F(t)$ is (i)-m.s. differentiable, then $F_-^r(t)$ and $F_+^r(t)$ are m.s. differentiable as random-value functions for each $r \in [0,1]$ and $F_r'(t) = [(F_-^r)'(t), (F_+^r)'(t)]$, for all $r \in [0,1]$.

(b) If $F(t)$ is (ii)-m.s. differentiable, then $F_-^r(t)$ and $F_+^r(t)$ are m.s differentiable as random-value functions for each $r \in [0,1]$ and $F_r'(t) = [(F_+^r)'(t), (F_-^r)'(t)]$, for all $r \in [0,1]$.

Proof. Case (a). See Example 4.8 in [7].
Case (b). For $h > 0$ and $r \in [0,1]$, we have

$$[F(t-h) \ominus F(t)]^r = [F_-^r(t-h) - F_-^r(t), F_+^r(t-h) - F_+^r(t)].$$

Multiplying with $\frac{1}{-h}$ we have

$$\frac{[F(t-h) \ominus F(t)]^r}{-h} = \Big[\frac{F_+^r(t-h) - F_+^r(t)}{-h}, \frac{F_-^r(t-h) - F_-^r(t)}{-h}\Big].$$

Similarly, we obtain

$$\frac{[F(t) \ominus F(t+h)]^r}{-h} = \Big[\frac{F_+^r(t) - F_+^r(t+h)}{-h}, \frac{F_-^r(t) - F_-^r(t+h)}{-h}\Big].$$

Passing to limit as $h \to 0^+$, we have

$$[F'(t)]^r = [(F_+^r)'(t), (F_-^r)'(t)].$$

Now, by using Theorem 3.19 the proof is complete. □

The following result shows that a second-order f.s.p. $F(t,x)$ for a fixed event $\omega \in \Omega$ has the property that the realization $F(t,x)(\omega)$ is differentiable in the deterministic sense, i. e., it is differentiable.

Theorem 3.21. Let $F(t,\beta)$ be a second-order f.s.p. which depends on a second-order f.r.v. β and the following conditions hold for each $\omega \in \Omega$.

(a) Let the realization $F(t,\beta)(\omega)$ be an (i)-differentiable deterministic fuzzy-valued function on $[t-h, t+h]$ for sufficiently small $h > 0$ with respect to the variable t.

(b) For all $r \in [0,1]$, the r-level sets of $F(t,\beta)(\omega)$ are a twice (i)-differentiable set-valued functions and $\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}[F(t,\beta)(\omega)]^r\| < M < \infty$ on $[t-h, t+h]$, for sufficiently small $h > 0$ with respect to the variable t.

Then, the process $F(t,\beta)$ is (i)-m.s. differentiable and $F'(t,\beta)$ is defined for each $\omega\in\Omega$, by

$$F'(t,\beta)(\omega)=\lim_{h\to 0^+}\frac{F(t+h,\beta)(\omega)\ominus F(t,\beta)(\omega)}{h}=\lim_{h\to 0^+}\frac{F(t,\beta)(\omega)\ominus F(t-h,\beta)(\omega)}{h}.$$

P r o o f. Let $\omega\in\Omega$ be fixed, and consider Taylor's expansion around t of the deterministic crisp twice differentiable functions $F^r_-(t,\beta)(\omega)$ and $F^r_+(t,\beta)(\omega)$,

$$F^r_+(t+h,\beta)(\omega)=F^r_+(t,\beta)(\omega)+h\frac{\mathrm{d}}{\mathrm{d}t}F^r_+(t,\beta)(\omega)(t,\beta)(\omega)+\frac{h^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}F^r_+(t''_\omega,\beta)(\omega),$$

and

$$F^r_-(t+h,\beta)(\omega)=F^r_-(t,\beta)(\omega)+h\frac{\mathrm{d}}{\mathrm{d}t}F^r_-(t,\beta)(\omega)(t,\beta)(\omega)+\frac{h^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}F^r_-(t'_\omega,\beta)(\omega).$$

for some t'_ω, t''_ω between t and $t+h$. Therefore, we have

$$\begin{aligned}
&\rho^2(\frac{F(t+h,\beta)\ominus F(t,\beta)}{h},F'(t,\beta))\\
&=\int_\Omega D^2\Big(\frac{F(t+h,\beta)(\omega)\ominus F(t,\beta)(\omega)}{h},F'(t,\beta)(\omega)\Big)\mathrm{d}P(\omega)\\
&=\int_\Omega\Big(\sup_{0\le r\le 1}\mathrm{d}\Big([\frac{F(t+h,\beta)(\omega)\ominus F(t,\beta)(\omega)}{h}]^r,[F'(t,\beta)(\omega))]^r\Big)\Big)^2\mathrm{d}P(\omega)\\
&=\int_\Omega\Big(\sup_{0\le r\le 1}\mathrm{d}\Big(|\frac{F^r_-(t+h,\beta)(\omega)-F^r_-(t,\beta)(\omega)-h(F^r_-)'(t,\beta)(\omega))}{h}|,\\
&\quad|\frac{F^r_+(t+h,\beta)(\omega)-F^r_+(t,\beta)(\omega)-h(F^r_+)'(t,\beta)(\omega)}{h}|\Big)\Big)^2\mathrm{d}P(\omega)\\
&=\frac{h^2}{4}\int_\Omega\Big(\sup_{0\le r\le 1}\mathrm{d}\big(|(F^r_-)''(t'_\omega,\beta)(\omega)|,|(F^r_+)''(t''_\omega,\beta)(\omega)|\big)\Big)^2\mathrm{d}P(\omega)\\
&\le\frac{h^2}{4}\int_\Omega(\sup_r\mathrm{d}(|(F^r_-)''(t'_\omega,\beta)(\omega)|,|(F^r_+)''(t'_\omega,\beta)(\omega)|)\\
&\qquad+d(|(F^r_+)''(t'_\omega,\beta)(\omega)|,|(F^r_+)''(t''_\omega,\beta)(\omega)|))^2\mathrm{d}P(\omega)\\
&\le\frac{h^2}{4}\int_\Omega(M+\epsilon)^2\mathrm{d}P(\omega)\to 0,\text{as } h\longrightarrow 0^+.
\end{aligned}$$

Note that for deterministic crisp twice differentiable function $F^r_+(t,\beta)(\omega)$, we have $d((F^r_+)''(t'_\omega,\beta)(\omega),(F^r_+)''(t''_\omega,\beta)(\omega))<\epsilon$, when $t<t'_\omega,\ t''_\omega<t+h$ and $h\longrightarrow 0^+$. Therefore,

$$\lim_{h\to 0^+}\rho\Big(\frac{F(t+h,\beta)\ominus F(t,\beta)}{h},F'(t,\beta)\Big)=0.$$

Similarly, we have

$$\lim_{h\to 0^+}\rho\Big(\frac{F(t,\beta)\ominus F(t-h,\beta)}{h},F'(t,\beta)\Big)=0.$$

□

Theorem 3.22. Let $F(t,\beta)$ be a second-order f.s.p. which depends on a second-order f.r.v. β and the following conditions hold for each $\omega \in \Omega$.

(a) Let the realization $F(t,\beta)(\omega)$ an (ii)-differentiable deterministic fuzzy-valued function on $[t-h,t+h]$, for sufficiently small $h>0$ with respect to the variable t.

(b) For all $r \in [0,1]$, the r-level sets of $F(t,\beta)(\omega)$ are a twice (ii)-differentiable set-valued functions and $\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}[F(t,\beta)(\omega)]^r\| < M < \infty$ on $[t-h,t+h]$, for sufficiently small $h>0$ with respect to the variable t.

Then, the process $F(t,\beta)$ is (ii)-m.s. differentiable and $F'(t,\beta)$ is defined for each $\omega \in \Omega$, by

$$F'(t,\beta)(\omega) = \lim_{h\to 0^+} \frac{F(t,\beta)(\omega) \ominus F(t+h,\beta)(\omega)}{-h} = \lim_{h\to 0^+} \frac{F(t-h,\beta)(\omega) \ominus F(t,\beta)(\omega)}{-h}.$$

Proof. The proof is similar to the proof of Theorem 3.21 . □

4. MEAN-SQUARE INTEGRATION

The concepts of mean-square Riemann integral and Riemann–Stieltjes integrals of the f.s.p. and their integrability and differentiability properties have been studied in [7, 9].

Definition 4.1. (Feng [7]) Let $F(t)$ be a second-order f.s.p., defined on $[a,b]$. For each finite partition Δ_n of $[a,b] : \Delta_n : a \le t_0 < t_l < ... < t_n = b$, and for arbitrary points t_i', $t_{i-1} \le t_i' \le t_i$, $i = 1,2,...,n$, let

$$\Delta t_i = t_i - t_{i-1},\ S_n = \sum_{i=1}^{n} \Delta t_i F(t_i') \text{ and } |\Delta_n| = \max_{i\in\{1,\cdots,n\}} \Delta t_i.$$

Then, the mean-square Riemann integral or m.s. integral of F on $[a,b]$ is defined by

$$\int_a^b F(t)\,\mathrm{d}t = \lim_{|\Delta_n|\longrightarrow 0} S_n,$$

provided this limit exists in (L^2,ρ) and it is independent of the partition as well as the selected points t_i'. In this case, we say that $F(t)$ is m.s. integrable on $[a,b]$. Note that in non-random case, this definition degenerates into the definition of integral of Puri and Ralescu [24].

The m.s. integral of F on an infinite interval is defined by

$$\int_{-\infty}^{\infty} F(t)\,\mathrm{d}t = \int_a^b F(t)\,\mathrm{d}t \textit{ as } a \longrightarrow -\infty,\ b \longrightarrow \infty,$$

provided this limit exists. The integrals

$$\int_a^{+\infty} F(t)\,\mathrm{d}t, \text{ and } \int_{-\infty}^{b} F(t)\,\mathrm{d}t$$

can be defined similarly.

Remark 4.2. (Feng [7]) If $F(t)$, $t \in [a,b]$, is non-random, the m.s. convergence coincides with the convergence in D, the m.s. integral is just as one defined by Goetschel and Voxman [16] in the case of E^1. We call it Riemann integral in D (D.R. integral, for short).

In the following, we recall some results on m.s. integrability.

Remark 4.3. (Feng [7])

(a) If $F(t)$, $t \in [a,b]$, is a non-random function, then the m.s. integral is just as one defined by Goetschel and Voxman [16] in the case of E^1.

(b) Let $F(t)$, $t \in [a,b]$, be a second-order f.s.p., then for every $r \in [0,1]$, $[F(t)]^r$ is a second-order set-valued s.p.

(c) If $F(t)$ is m.s. integrable, then $[F(t)]^r$ is m.s. integrable for all $r \in [0,1]$ and

$$\Big[\int_a^b F(t)\,\mathrm{d}t\Big]^r = \int_a^b [F(t)]^r \mathrm{d}t. \tag{14}$$

(d) If A and B are second-order set-valued r.v.'s, then $I_A, I_B \in L^2$.

(e) Let $F : [a,b] \longrightarrow L^2(E^1)$ be m.s. integrable. For $r \in [0,1]$, denote $[F(t)]^r = [F_-^r(t), F_+^r(t)]$. Then $F_-^r(t)$ and $F_+^r(t)$ are m.s. integrable and

$$\int_a^b [F(t)]^r \mathrm{d}t = \Big[\int_a^b F_-^r(t)\,\mathrm{d}t, \int_a^b F_+^r(t)\,\mathrm{d}t\Big], \text{ for all } r \in [0,1]. \tag{15}$$

Theorem 4.4. (Feng [7]) Let $F(t)$ and $G(t)$ be m.s. integrable on $[a,b]$.

(a) For each $\alpha, \beta \in \mathbb{R}$, $\alpha F(t) + \beta G(t)$ is m.s. integrable and

$$\int_a^b (\alpha F(t) + \beta G(t))\,\mathrm{d}t = \alpha \int_a^b F(t)\,\mathrm{d}t + \beta \int_a^b G(t)\,\mathrm{d}t. \tag{16}$$

(b) $F(t)$ is m.s. integrable on any subinterval of $[a,b]$, and

$$\int_a^b F(t)\,\mathrm{d}t = \int_a^c F(t)\,\mathrm{d}t + \int_c^b F(t)\,\mathrm{d}t, \qquad a \leq c \leq b. \tag{17}$$

(c) $E(F(t))$ is $D.R.$ integrable on $[a,b]$ and

$$E(\int_a^b F(t)\,\mathrm{d}t) = \int_a^b E(F(t))\,\mathrm{d}t. \tag{18}$$

(d) If $\rho(F(t), G(t))$ is Riemann integrable on $[a,b]$, then

$$\rho\Big(\int_a^b F(t)\,\mathrm{d}t, \int_a^b G(t)\,\mathrm{d}t\Big) \leq \int_a^b \rho(F(t), G(t))\,\mathrm{d}t. \tag{19}$$

Theorem 4.5. (Feng [7]) Let $F : [a, b] \longrightarrow L^2$. If $F(t)$ is m.s. continuous on $[a, b]$, then $F(t)$ is m.s. integrable on $[a, b]$.

Theorem 4.6. Let $F : [a, b] \longrightarrow L^2$ be m.s. continuous.

(a) The m.s. integral $Y(t) = \int_a^t F(s)\,\mathrm{d}s, t \in [a, b]$, is (i)-m.s. differentiable and

$$Y'(t) = F(t).$$

(b) The m.s. integral $Y(t) = \int_t^b F(s)\,\mathrm{d}s,\ t \in [a, b]$, is (ii)-m.s. differentiable and

$$Y'(t) = -F(t).$$

(c) Let $Y(t) = \gamma \ominus \int_0^t -F(s)\,\mathrm{d}s,\ t \in [0, b]$, where $\gamma \in L^2$ is such that the H-difference exists for $t \in [0, b]$. Then $Y(t)$ is (ii)-m.s. differentiable and

$$Y'(t) = F(t).$$

P r o o f. We present the details for the case (b), the other cases are analogous.

Case(b). Since $F(t)$ is m.s. integrable, then for $h > 0$ we have

$$Y(t) \ominus Y(t + h) = \int_t^{t+h} F(s)\,\mathrm{d}s.$$

Then

$$\frac{Y(t) \ominus Y(t + h)}{-h} = \frac{-1}{h} \int_t^{t+h} F(s)\,\mathrm{d}s.$$

Note that the m.s. continuity of $F(t)$ implies that $g(s) = \rho(F(s), F(t))$ is continuous. Thus, for the given $\epsilon > 0$ and for sufficiently small $h > 0$, we obtain

$$\rho\Big(\frac{Y(t) \ominus Y(t + h)}{-h}, -F(t)\Big) \le \frac{1}{h} \int_t^{t+h} \rho(F(s), F(t))\,\mathrm{d}s < \epsilon.$$

Therefore, we have

$$\lim_{h\to 0^+} \rho\Big(\frac{Y(t) \ominus Y(t + h)}{-h}, -F(t)\Big) = 0.$$

Similarly,

$$\lim_{h\to 0^+} \rho\Big(\frac{Y(t - h) \ominus Y(t)}{-h}, -F(t)\Big) = 0.$$

which proves the case (b). □

Theorem 4.7. Let $G : T \longrightarrow L^2$ be m.s. continuous, $B(t,s) : T \times T \longrightarrow \mathbb{R}$ be differentiable and $\frac{\partial B(t,s)}{\partial t}$ be continuous with respect to t. In addition, let $B(t,s)$ is continuous with respect to s.

(a) If $B(t,s)\frac{\partial B(t,s)}{\partial t} > 0$, then $\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s$ is (i)-m.s. differentiable and

$$\Big(\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s\Big)' = \int_{t_0}^{t} \frac{\partial B(t,s)}{\partial t} G(s)\,\mathrm{d}s + B(t,t)G(t). \tag{20}$$

(b) If $B(t,s)\frac{\partial B(t,s)}{\partial t} < 0$ and the H-difference $\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s$ exists for sufficiently small $h > 0$, then $\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s$ is (ii)-m.s. differentiable and

$$\Big(\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s\Big)' = \int_{t_0}^{t} \frac{\partial B(t,s)}{\partial t} G(s)\,\mathrm{d}s \ominus (-B(t,t)G(t)). \tag{21}$$

Proof. Case (a). See Lemma 3.1 in [10] and Remark 3.4.

Case (b). Since $B(t,s)$ is continuous with respect to t, then for sufficiently small $h > 0$, $B(t,s)$ and $B(t+h,s) - B(t,s)$ have the same sign. By Theorem 2.2 (c) and (d), we get

$$\begin{aligned}
&\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s + \int_{t}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s \\
&= \int_{t_0}^{t} (B(t,s) - B(t+h,s))G(s)\,\mathrm{d}s + \int_{t_0}^{t} B(t+h,s)G(s)\,\mathrm{d}s + \int_{t}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s \\
&= \int_{t_0}^{t} (B(t,s) - B(t+h,s))G(s)\,\mathrm{d}s + \int_{t_0}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s.
\end{aligned}$$

Since the H-difference $\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s$ exists, using Theorem 4.4, we have

$$\begin{aligned}
&\Big(\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t,s)G(s)\,\mathrm{d}s\Big) + \int_{t}^{t+h} B(t,s)G(s)\,\mathrm{d}s \\
= &\int_{t_0}^{t} (B(t,s) - B(t+h,s))G(s)\,\mathrm{d}s.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t,s)G(s)\,\mathrm{d}s \\
= &\int_{t_0}^{t} (B(t,s) - B(t+h,s))G(s)\,\mathrm{d}s \ominus \int_{t}^{t+h} B(t,s)G(s)\,\mathrm{d}s.
\end{aligned}$$

Now, we get

$$\rho\Big(\frac{\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s}{-h}, \int_{t_0}^{t} \frac{\partial B(t,s)}{\partial t} G(s)\,\mathrm{d}s \ominus (-B(t,t)G(t))\Big)$$

$$
\begin{aligned}
&= \rho\Big(\frac{\int_{t_0}^{t}(B(t,s)-B(t+h,s))G(s)\,\mathrm{d}s \ominus \int_{t}^{t+h} B(t,s)G(s)\,\mathrm{d}s}{-h},\\
&\qquad \int_{t_0}^{t}\frac{\partial B(t,s)}{\partial t}G(s)\,\mathrm{d}s \ominus (-B(t,t)G(t))\Big)\\
\leq\ & \rho\Big(\frac{1}{h}\int_{t_0}^{t}(B(t,s)-B(t+h,s))G(s)\,\mathrm{d}s, \int_{t_0}^{t}\frac{\partial B(t,s)}{\partial t}G(s)\,\mathrm{d}s\Big)\\
&+\rho\Big(\frac{1}{h}\int_{t}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s, B(t,t)G(t)\Big)\\
\leq\ & \int_{t_0}^{t}\rho\Big(\frac{(B(t+h,s)-B(t,s))}{h}G(s), \frac{\partial B(t,s)}{\partial t}G(s)\Big)\mathrm{d}s\\
&+\frac{1}{h}\int_{t}^{t+h}\rho\big(B(t+h,s)G(s), B(t,t)G(t)\big)\mathrm{d}s\\
\leq\ & \int_{t_0}^{t}\Big|\frac{(B(t+h,s)-B(t,s))}{h}-\frac{\partial B(t,s)}{\partial t}\Big|\|G(s)\|_2\,\mathrm{d}s\\
&+\frac{1}{h}\int_{t}^{t+h}(B(t+h,s)-B(t,s))\|G(s)\|_2\,\mathrm{d}s\\
&+\frac{1}{h}\int_{t}^{t+h}B(t,s)\rho\big(G(s),G(t)\big)\mathrm{d}s\\
&+\frac{1}{h}\int_{t}^{t+h}\Big|B(t,s)-B(t,t)\Big|\|G(s)\|_2\,\mathrm{d}s \longrightarrow 0, \qquad \text{as } h\longrightarrow 0^+,
\end{aligned}
$$

by the m.s. continuity of $G(t)$ and some simple integral calculations. Hence, we obtain

$$
\rho\Big(\frac{\int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t+h} B(t+h,s)G(s)\,\mathrm{d}s}{-h}, \int_{t_0}^{t}\frac{\partial B(t,s)}{\partial t}G(s)\,\mathrm{d}s \ominus (-B(t,t)G(t))\Big) \longrightarrow 0,
$$

when $h \longrightarrow 0^+$. Similarly,

$$
\rho\Big(\frac{\int_{t_0}^{t-h} B(t-h,s)G(s)\,\mathrm{d}s \ominus \int_{t_0}^{t} B(t,s)G(s)\,\mathrm{d}s}{-h}, \int_{t_0}^{t}\frac{\partial B(t,s)}{\partial t}G(s)\,\mathrm{d}s \ominus (-B(t,t)G(t))\Big) \longrightarrow 0,
$$

when $h \longrightarrow 0^+$. □

Now, we have the following Newton-Leibniz formula.

Theorem 4.8. (Feng [7]) If $F(t)$ is (i)-m.s. differentiable and $F'(t)$ is m.s integrable on $[a,b]$, then for $t \in [a,b]$, we have

$$
F(t) = F(a) + \int_{a}^{t} F'(s)\,\mathrm{d}s. \tag{22}
$$

Theorem 4.9. If $F(t)$ is (ii)-m.s. differentiable and $F'(t)$ is m.s integrable on $[a,b]$, then for $t \in [a,b]$, we have

$$
F(t) = F(a) \ominus (-1)\int_{a}^{t} F'(s)\,\mathrm{d}s. \tag{23}
$$

Proof. We define $G(t) = F(a) \ominus F(t)$. From Corollary 3.12, G is (i)-m.s. differentiable and $G' = (-1)F'(t)$. Then, using Theorem 4.8, we have

$$F(a) \ominus F(t) = G(t) = G(a) + \int_a^t G'(s)\,\mathrm{d}s = (F(a) \ominus F(a)) + \int_a^t (-1)F'(s)\,\mathrm{d}s.$$

Therefore,

$$F(t) = F(a) \ominus (-1)\int_a^t F'(s)\,\mathrm{d}s.$$

□

5. APPLICATIONS AND ILLUSTRATIONS

In this section, we present some examples to illustrate the application of the generalized m.s. differentiability. In the following, we suppose that $(\Omega, \mathcal{A}, P)$ is a complete probability space, where $\Omega = [0, b]$ with $b \in (0, \infty)$, $\mathcal{A}$ is the Borel σ-algebra of subsets of Ω, P is normed Lebesgue measure on $(\Omega, \mathcal{A})$ and $\omega \in \Omega$ is a crisp random variable.

Example 5.1. Let consider a second-order fuzzy stochastic process $y(t, \omega)$ where its membership function is as follows ((t, ω) is fixed)

$$y(t,\omega)(x) = \begin{cases} 1 - (e^{-\omega t}x - 4)^2, & \text{if } x \in [3e^{\omega t}, 5e^{\omega t}], \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the r-level sets of the f.s.p. $y(t, \omega)$ can be considered as

$$[y(t,\omega)]^r = [e^{\omega t}(4 - \sqrt{1-r}), e^{\omega t}(4 + \sqrt{1-r})].$$

If we define $Y(t, \omega) = \omega y(t, \omega)$, then, using cases (c) and (e) in Remark 4.3, for every $(t, \omega) \in [0, \infty) \times \Omega$, we have

$$\Big[\int_0^t Y(s,\omega)\,\mathrm{d}s\Big]^r = [e^{\omega t}(4 - \sqrt{1-r}), e^{\omega t}(4 + \sqrt{1-r})].$$

Thus, the membership function of $\int_0^t Y(s, \omega)\,\mathrm{d}s$ is as follows

$$\int_0^t Y(s,\omega)\,\mathrm{d}s\ (x) = \begin{cases} 1 - (e^{-\omega t}x - 4)^2, & \text{if } x \in [3e^{\omega t}, 5e^{\omega t}], \\ 0, & \text{otherwise.} \end{cases}$$

The graphical representation of the $\int_0^t Y(s, \omega)\,\mathrm{d}s$ can be seen in Figure 1. Note that, we have

$$\int_0^t Y(s,\omega)\,\mathrm{d}s = y(t,\omega).$$

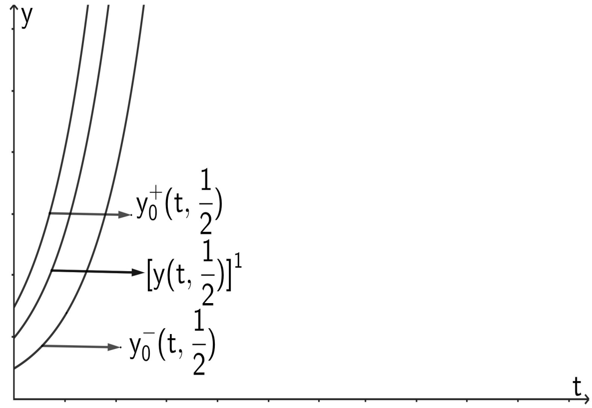

Fig. 1. Graphical representation of the $\int_0^t Y(s,\omega)\,\mathrm{d}s$ for $\omega = \frac{1}{2}$, Example 5.1.

Example 5.2. Consider the following second-order fuzzy stochastic process

$$y(t,\omega) = e^t y_1(t,\omega),$$

where

$$y_1(t,\omega) = (\omega, \omega+1, \omega+2) + \int_0^t e^{-s}\big(((\omega+1)e^{-s} - e^s, (\omega+1)e^{-s}, (\omega+1)e^{-s} + e^s)\big)\mathrm{d}s.$$

By Theorems 3.9 and 4.6, $y_1(t,\omega)$ is (i)-m.s. differentiable and

$$y_1'(t,\omega) = e^{-t}\Big(\big((\omega+1)e^{-t} - e^s, (\omega+1)e^{-t}, (\omega+1)e^{-t} + e^t\big)\Big).$$

If we denote $f(t) = e^t$, then we have $f(t)f'(t) > 0$. Now, using case (a) in Theorem 3.15, for every $(t,\omega) \in [0,\infty) \times \Omega$, we get

$$\begin{aligned} y'(t,\omega) &= e^t\Big((\omega, \omega+1, \omega+2) + \int_0^t e^{-s}\big(((\omega+1)e^{-s} - e^s, (\omega+1)e^{-s}, (\omega+1)e^{-s} + e^s)\big)\mathrm{d}s\Big) \\ &\quad + \big((\omega+1)e^{-t} - e^s, (\omega+1)e^{-t}, (\omega+1)e^{-t} + e^t\big) \\ &= \frac{1}{2}(3e^t + e^{-t})(\omega+1) + (t+2)e^t(-1,0,1). \end{aligned}$$

Example 5.3. Let the following second-order fuzzy stochastic process

$$y(t,\omega) = e^{-t} y_2(t,\omega)$$

where

$$y_2(t,\omega) = (\omega + \int_0^t \omega s e^{-s} \mathrm{d}s, 2\omega + \int_0^t \omega s e^{-s} \mathrm{d}s, 3\omega + \int_0^t \omega s e^{-s} \mathrm{d}s). \tag{24}$$

It is easy to see that $y_2(t,\omega)$ is (ii)-m.s. differentiable and

$$y_2'(t,\omega) = \omega t e^{-t}.$$

If we denote $f(t) = e^{-t}$, then we have $f(t)f'(t) < 0$. Using case (b) in Theorem 3.15, for every $(t,\omega) \in [0,\infty) \times \Omega$, we get

$$y'(t,\omega) = \omega e^{-2t}(-4e^t + 2t + 1, -3e^t + 2t + 1, -2e^t + 2t + 1).$$

Example 5.4. Let consider the second-order fuzzy stochastic process $y(t,\omega)$ given as follows

$$y(t,\omega) = \int_0^t B(t,s)(0, e^{\omega(s-1)^2}, 2e^{\omega(s-1)^2})\,\mathrm{d}s$$

where $B(t,s) = t(1-s)$ for $0 \le s \le t$ and $t \in [0,1]$. Since $B(t,s)\frac{\partial B(t,s)}{\partial t} = t(1-s)^2 > 0$, then, by case (a) in Theorem 4.7, $y(t,\omega)$ is (i)-m.s. differentiable and for every $(t,\omega) \in [0,\infty) \times \Omega$, we have

$$\begin{aligned} y'(t,\omega) &= \int_0^t (1-s)(0, e^{\omega(1-s)^2}, 2e^{\omega(1-s)^2})\,\mathrm{d}s + t(1-t)(0, e^{\omega(t-1)^2}, 2e^{\omega(t-1)^2}) \\ &= \frac{1}{2\omega}\big((2\omega t - 2\omega t^2 - 1)e^{\omega(1-t)^2} + e^{\omega}\big)(0,1,2). \end{aligned}$$

6. CONCLUSION AND FUTURE RESEARCH

In this study, we generalized a mean-square differentiability of the fuzzy stochastic processes and presented some new results on calculus of second-order fuzzy stochastic processes. We propose for further research to study fuzzy stochastic differential equations under the generalized m.s. differentiability.

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Hadi Amirnia, Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan. Iran.
e-mail: hadi.amirnia@iasbs.ac.ir

Alireza Khastan, (Corresponding author.) Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan. Iran.
e-mail: khastan@iasbs.ac.ir