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ALMOST COMPLETE CONVERGENCE OF A RECURSIVE KERNEL ESTIMATOR OF THE DENSITY WITH COMPLETE AND CENSORED INDEPENDENT DATA

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In this paper, we firstly introduce a recursive kernel estimator of the density in the censored data case. Then, we establish its pointwise and uniform almost complete convergences, with rates, in both complete and censored independent data. Finally, we illustrate the accuracy of the proposed estimators throughout a simulation study.

Keywords: recursive kernel estimator, density, almost complete convergence, censored indepented data, right censored data, rate of convergence

Classification: 62G20, 62N01, 62G05, 62G07

1. INTRODUCTION

Nonparametric density function estimation has been the subject of intense investigation in several inference problems. A classical density kernel estimator has been studied in literature by many authors, such as [4, 13] and [6].

However, the recursive method allows us to update the estimation whenever new observations are obtained which is not the case of non recursive kernel one. This provides an important saving in computational time, memory and adaptability, especially for large sequential datasets, making it a more practical and scalable solution for modern data analysis tasks. In contrast to the classical kernel estimator, these methods enable iterative updates without requiring full recalculations, making them both practical and easy to implement. This efficiency makes the recursive method is better suitable for handling large, streaming, or dynamically changing datasets, offering superior efficiency and scalability than classical method. Another important advantage of the recursive estimator is that the variance of the estimator defined in [1] reduces the variance compared to the variance of classical estimator.

In this field, [16] is the forerunners to introduce the kernel recursive estimators. Inspired by these works, [1] introduced the generalized form of the recursive estimator and established its almost sure convergence for complete independent data. Then, [12] extended the result for complete weak dependent data. The asymptotic normality and

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mean square error of the recursive estimator were investigated by [2] for strongly mixing data.

In the reliability and survival time studies, the data are often unavailable due to the lag of information, which makes the setting of censored data more suitable in practice. [9] studied the almost sure convergence of the conditional density recursive kernel estimator for censored data. Recently, [10] established the asymptotic normality and mean square error for adaptive recursive kernel conditional density estimator under censoring data.

Our work is motivated by the fact that the previous results focused on the almost sure convergence of recursive kernel estimators, which is weaker than the almost complete one. Furthermore, to the best of our knowledge the recursive kernel density estimator for right censored independent data has not been studied yet in literature.

To this aim, the paper is organized as follows. Firstly, in section 2, we introduce a new recursive kernel estimator of density which extends the estimator studied in [1] to right censored data case. Then, we devote the sections 3 and 4 to derive the pointwise and the uniform almost complete convergences under mild conditions in our context for complete and right censored data cases. The performance of our estimator is validated by a simulation study. Finally, the proofs of the theoretical results are developed in the Appendix.

2. ESTIMATION

Let T be a nonnegative lifetime which has an unknown probability density function (p.d.f.) f with respect to Lebesgue measure. In the case of complete data, [1] introduced the recursive kernel estimator of the density f based on a sample of independent identically distributed (i.i.d.) observations (T_i) of T by the following expression

$$\bar{f}_n^{\ell}(t) = \frac{1}{\sum_{i=1}^{n} h_i^{1-\ell}} \sum_{i=1}^{n} h_i^{-\ell} K\left(\frac{t-T_i}{h_i}\right), \quad t \in \mathbb{R}, \tag{1}$$

where $\ell \in [0,1]$ is a smoothing parameter, h_n is a given positive sequence decreasing to 0 and K is a density function.

Regrettably, T_i is unavailable in practice. We can only observe a sample $(Y_i, \delta_i)_{1\leq i\leq n}$ of i.i.d., observations of $(Y = T \wedge C, \delta)$, where the nonnegative censoring variables C_i are i.i.d. with unknown continuous survival function G and $\delta = 1_{\{T\leq C\}}$ (where 1_A denotes the indicator function of the set A).

We need the following assumption.

- (C.1) C and T are independent.

Combining the ideas in [1] and [7] and the fact that

$$\begin{aligned} &E\left(K\left(\frac{t-Y_i}{h_i}\right)\frac{\delta_i}{G(Y_i)}\right) \\ &= E\left(\frac{K\left(\frac{t-T_i}{h_i}\right)}{G(T_i)} E(\delta_i/T_i)\right) \end{aligned}$$

$$= E\left(K\left(\frac{t - T_i}{h_i}\right)\right),$$

we can construct a pseudo recursive kernel estimator of f by

$$\widetilde{f}_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} K\left(\frac{t - Y_i}{h_i}\right) \frac{\delta_i}{G(Y_i)}, \quad t \in \mathbb{R}. \tag{2}$$

Since G is usually unknown, we will replace it by its [8] estimator G_n defined as

$$G_n(t) = \begin{cases} \prod_{i=1}^n \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{1_{\{Y_{(i)} \leq t\}}} & \text{if } t < Y_{(n)} \\ 0 & \text{if } t \geqslant Y_{(n)}, \end{cases}$$

where $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ are the order statistics of Y_i and $\delta_{(i)}$ the noncensoring indicator corresponding to $Y_{(i)}$.

Therefore, a feasible estimator of $f(t)$ is given by

$$\widehat{f}_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{-\ell} K\left(\frac{t - Y_i}{h_i}\right) \frac{\delta_i}{G_n(Y_i)}, \quad t \in \mathbb{R}. \tag{3}$$

The recursive formulation (3) facilitates the updating of estimates without the need to recalculate the entire dataset as new observations are introduced. Indeed the estimator (3) fullfils

$$f_{n+1}^\ell(t) = \frac{\sum_{i=1}^n h_i^{1-\ell}}{\sum_{i=1}^{n+1} h_i^{1-\ell}} f_n^\ell(t) + \frac{1}{\sum_{i=1}^{n+1} h_i^{1-\ell}} K_{n+1}(t - Y_{n+1}) \frac{\delta_{n+1}}{G_{n+1}(Y_{n+1})},$$

with $K_{n+1}(t - Y_{n+1}) = \frac{1}{h_{n+1}^\ell} K\left(\frac{t - Y_{n+1}}{h_{n+1}}\right)$.

3. POINTWISE ALMOST COMPLETE CONVERGENCE

3.1. Complete data

To establish the pointwise almost complete convergence of $\bar{f}_n^\ell(t)$ for a fixed point $t \in \mathbb{R}$, we need the following standard conditions.

- (H.1) $\lim_{n\to\infty} h_n = 0, \quad \lim_{n\to\infty} nh_n = \infty, \quad \lim_{n\to\infty} \frac{\ln n}{nh_n} = 0.$
- (H.2) $\beta_{n,r} = \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n}\right)^r \to \beta_r < \infty$, as $n \to \infty, r \leq 3$.
- (F.1) f is continuous.
- (F.2) f is twice continuously differentiable around t.
- (K.1) K is bounded and integrable function, with compact support.
- (K.2) $\int vK(v)\,\mathrm{d}v = 0$.

Remark 3.1. Since K has a compact support and h_n converges to 0, we can take in both sums (2) and (3) $Y_i \in [0,\theta]$ (for all large enough n), where $\theta < T_Y$, where $T_Y := T_{F_Y} = \sup\{t \in \mathbb{R}; F_Y(t) < 1\}$ denote the upper endpoint of the support of Y.

The following theorem gives the pointwise almost complete convergence of the recursive estimator $\bar{f}_n^\ell(t)$.

Theorem 3.2. Assume that the assumptions $(H.1)$, $(H.2)$ and $(K.1)$ are fulfilled.

(i) If $(F.1)$ is satisfied, we get

$$\bar{f}_n^\ell(t) - f(t) = o_{a.co}(1).$$

(ii) If $(F.2)$ and $(K.2)$ are satisfied, we obtain

$$\bar{f}_n^\ell(t) - f(t) = O(h_n^2) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

The proof of theorem 3.2 follows directly from the decomposition given by

$$\bar{f}_n^\ell(t) - f(t) = \left(\bar{f}_n^\ell(t) - E\bar{f}_n^\ell(t)\right) + \left(E\bar{f}_n^\ell(t) - f(t)\right), \tag{4}$$

and the accompanying lemmas, which proofs are deferred to the appendix.

Lemma 3.3. Under assumptions $(H.1)$, $(H.2)$ and $(K.1)$, we get

$$\bar{f}_n^\ell(t) - E\bar{f}_n^\ell(t) = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Lemma 3.4. Under assumptions $(H.1)$, $(H.2)$ and $(K.1)$.

(i) If $(F.1)$ is satisfied, we have

$$E\bar{f}_n^\ell(t) \to f(t) \text{ as } n \to \infty.$$

(ii) If $(F.2)$ and $(K.2)$ are satisfied, we obtain

$$E\bar{f}_n^\ell(t) - f(t) = O(h_n^2).$$

3.2. Censored data

In this section, we study the pointwise almost complete convergence of $\widehat{f}_n^\ell(t)$ for a fixed point $t \in \mathbb{R},\ t \le \theta$. For this aim, we need the same conditions as in complete data in addition to the right censoring condition $(C.1)$.

Theorem 3.5. Assume that the assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$ are fullfilled.

(i) If $(F.1)$ is satisfied, we have

$$\widehat{f}_n^{\ell}(t) - f(t) = o_{a.co}(1).$$

(ii) If $(F.2)$ and $(K.2)$ are satisfied, we obtain

$$\widehat{f}_n^{\ell}(t) - f(t) = O(h^2) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

The proof of the Theorem 3.5 is a direct consequence of the decomposition given by

$$\widehat{f}_n^{\ell}(t) - f(t) = (\widehat{f}_n^{\ell}(t) - \widetilde{f}_n^{\ell}(t)) + (\widetilde{f}_n^{\ell}(t) - E\widetilde{f}_n^{\ell}(t)) + (E\widetilde{f}_n^{\ell}(t) - f(t)), \tag{5}$$

and the following lemmas, which proofs are postponed to the appendix.

Lemma 3.6. Under assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$, we get

$$\widetilde{f}_n^{\ell}(t) - E\widetilde{f}_n^{\ell}(t) = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Lemma 3.7. Assume that the assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$ are fulfilled.

(i) If $(F.1)$ is satisfied, we have

$$E\widetilde{f}_n^{\ell}(t) \to f(t) \text{ as } n \to \infty.$$

(ii) If $(F.2)$ and $(K.2)$ are satisfied, we get

$$E\widetilde{f}_n^{\ell}(t) - f(t) = O(h_n^2).$$

Lemma 3.8. Under assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$ we have

$$\widehat{f}_n^{\ell}(t) - \widetilde{f}_n^{\ell}(t) = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

4. UNIFORM ALMOST COMPLETE CONVERGENCE

In the sequel, we extend the previous results to the uniform convergence with both complete and censored data. For this purpose, we need the following conditions in addition to the conditions (H.1), (H.2), (K.1), (K.2) and (C.1). Throughout this section, let $S \in [0, T_Y[$ be a fixed compact subset of $\mathbb{R}$ and let us $\theta < T_Y$.

- (F'.1) f is continuous on a compact subset S.
- (F'.2) f is twice continuously differentiable on S.
- (K.3) K is Lipschitzian, $\exists\beta > 0$, $\exists C > 0$, $\forall x, y \in S$, $|K(x) - K(y)| \leq C|x - y|^{\beta}$.

4.1. Complete data

Theorem 4.1. Assume that the assumptions $(H.1)$, $(H.2)$ $(K.1)$ and $(K.3)$ are satisfied.

(i) If $(F'.1)$ is satisfied, we obtain

$$\sup_{t\in S} |\bar{f}_n^\ell(t) - f(t)| = o_{a.co}(1).$$

(ii) If $(F'.2)$ and $(K.2)$ are satisfied, we get

$$\sup_{t\in S} |\bar{f}_n^\ell(t) - f(t)| = O(h_n^2) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Remark 4.2. Notice that this rate of convergence is the same as that of the classical density estimator in the complete data case, see Theorem 2 in [11], where $L = 0$ and $R = +\infty$.

The proof of Theorem 4.1 is based on the decomposition (4) and the following Lemmas for which the proofs are given in the Appendix.

Lemma 4.3. Under assumptions $(H.1)$, $(H.2)$, $(K.1)$ and $(K.3)$, we have

$$\sup_{t\in S} |\bar{f}_n^\ell(t) - E\bar{f}_n^\ell(t)| = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Lemma 4.4. Assume that hypotheses $(H.1)$, $(H.2)$ and $(K.1)$ hold,

(i) If $(F'.1)$ is fulfilled, we get

$$\sup_{t\in S} |E\bar{f}_n^\ell(t) - f(t)| \to 0, \text{ as } n \to \infty.$$

(ii) If $(F'.2)$ and $(K.2)$ are fulfilled, we have

$$\sup_{t\in S} |E\bar{f}_n^\ell(t) - f(t)| = O(h_n^2).$$

4.2. Censored data

Theorem 4.5. Assume that the assumptions $(C.1)$, $(H.1)$, $(H.2)$, $(K.1)$ and $(K.3)$ are fulfilled.

(i) If $(F'.1)$ is satisfied, we have

$$\sup_{t\in S} |\widehat{f}_n^\ell(t) - f(t)| = o_{a.co}(1).$$

(ii) If $(F'.2)$ and $(K.2)$ are satisfied, we obtain

$$\sup_{t\in S} |\widehat{f}_n^\ell(t) - f(t)| = O(h^2) + O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Remark 4.6. It is worth noting that this rate of convergence is the same as that of the classical density estimator in the right censored data case, see Theorem 2 in [11], where $L = 0$.

The proof of Theorem 4.5 is a direct consequence of the decomposition (5) and the following lemmas, the proofs of which are relegated to the Appendix.

Lemma 4.7. Under assumptions $(C.1)$, $(H.1)$, $(H.2)$ $(K.1)$ and $(K.3)$, we get

$$\sup_{t\in S}|\widetilde{f}_n^{\ell}(t) - E\widetilde{f}_n^{\ell}(t)| = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

Lemma 4.8. Assume that the assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$ are fulfilled.

(i) If $(F'.1)$ is satisfied, we obtain

$$\sup_{t\in S}|E\widetilde{f}_n^{\ell}(t) - f(t)| \to 0, \text{ as } n \to \infty.$$

(ii) If $(F'.2)$ and $(K.2)$ are satisfied, we have

$$\sup_{t\in S}|E\widetilde{f}_n^{\ell}(t) - f(t)| = O(h_n^2).$$

Lemma 4.9. Under assumptions $(C.1)$, $(H.1)$, $(H.2)$ and $(K.1)$ we have

$$\sup_{t\in S}|\widehat{f}_n^{\ell}(t) - \widetilde{f}_n^{\ell}(t)| = O_{a.co}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right).$$

5. SIMULATION STUDY

This section aims to examine the behavior of our estimators (RKD) for different sample sizes ($n = 100,\ 300,\ 500,\ 1000$) in both complete and censored data cases.

For the computation of our estimator(RKD) and the classical kernel density estimator (CKD), we use Epanechnikov kernel $K(u) = \frac{3}{4}(1-u)^2 1_{|u|\le 1}$ and the bandwidth of the form $h_i = c\, i^{-\frac{1}{5}}$, $c > 0$, $i = \overline{1, n}$, for RKD estimator and $h_n = c'\, n^{-\frac{1}{5}}$, $c' > 0$ for CKD estimator.

The results are plotted in Figures 3−4, where, the continuous curve (resp. the dashed ones) correspond to the true density (resp. the estimated ones obtained by the mentioned method).

To be more precise, we assess their empirical mean square errors (EMSE), with

$$EMSE = \frac{1}{n}\sum_{1}^{n}(\widehat{f}(X_i) - f(X_i))^2$$

where $f(X_i)$ (resp.$\widehat{f}(X_i)$) is the real (resp. the estimated) value.

5.1. Effect of the parameters ℓ and n

5.1.1. Complete data

We generate an i.i.d. sample of the variable T_i from Weibull distribution $W(12, 10)$. The obtained results are in Table 1.

	$n = 100$	$n = 300$	$n = 500$	$n = 1000$
RKD $\ell = 0$	0.00084	0.00073	0.000275	0.00014
RKD $\ell = 0.5$	0.00087	0.00073	0.00026	0.00012
RKD $\ell = 1$	0.00092	0.00073	0.00026	0.00010
CKD	0.00097	0.00069	0.000275	0.00022

Tab. 1. EMSE for density estimators according to sample sizes for complete data.

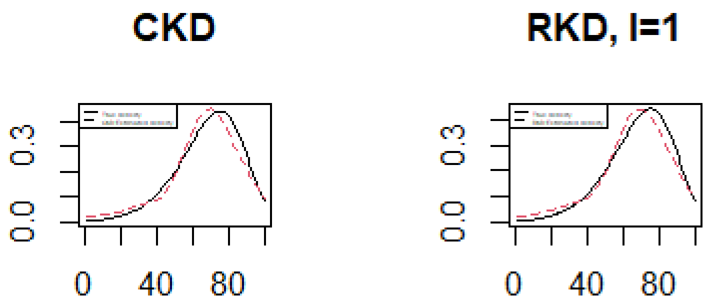

Fig. 1. Representation of the CKD and RKD estimator of the density with $l = 1$ and $n = 100$ for complete data.

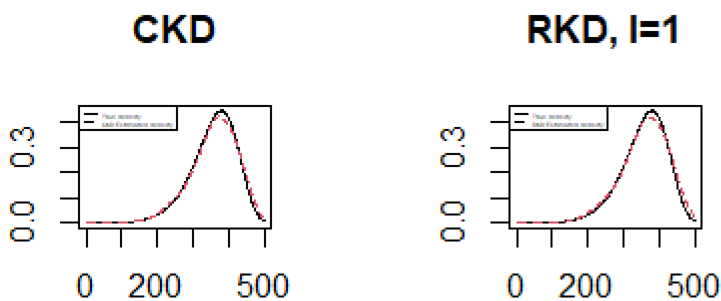

Fig. 2. Representation of the CKD and RKD estimator of the density with $l = 1$ and $n = 500$ for complete data.

From Figures (1) − (2), as well as the CKD estimator, it is clear that the estimators RKD (dashed curve) converge to the true density (continuous curve). Notice that the shapes are improved when n increases. This is comfirmed by the results of EMSE for different values of ℓ in Table 1. Also, we see that the choice of ℓ has non influence.

5.1.2. Censored data

Now, we adopt the censored mechanism $(\min(T_i, C_i), \delta_i = 1_{T_i \leq C_i})_{i=1,\cdots,n}$, where the values of T_i and C_i are generated independently from Weibull distribution $T \sim W(12, 10)$ and $C \sim W(12, 14)$. The obtained results are in Table 2.

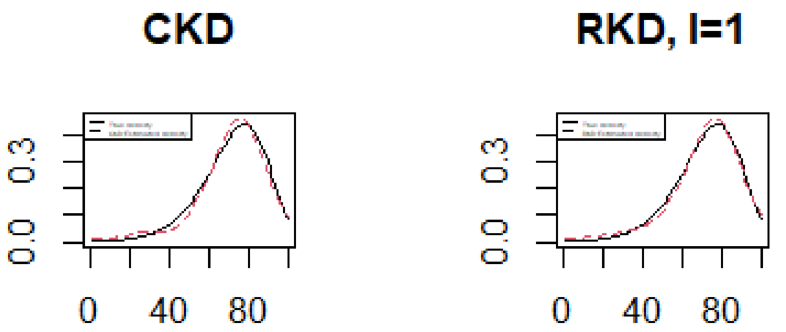

Fig. 3. Representation of the CKD and the RKD estimator of the density with $l = 1$ and $n = 100$ for censored data $CR = 30\%$.

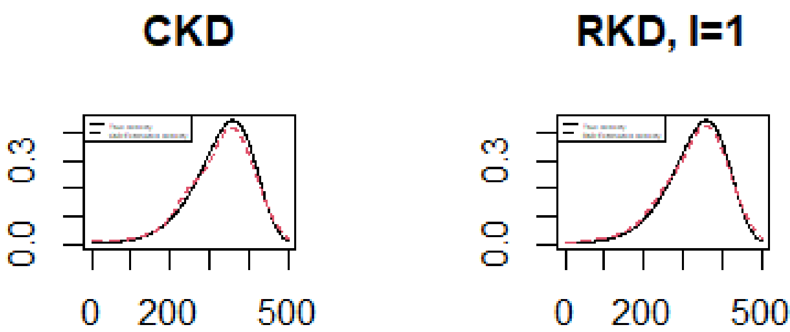

Fig. 4. Representation of the CKD and the RKD estimator of the density with $l = 1$ and $n = 500$ for censored data $CR = 30\%$.

	$n = 100$	$n = 300$	$n = 500$	$n = 1000$
RKD $\ell = 0$	0.00108	0.00091	0.00045	0.0003561
RKD $\ell = 0.5$	0.00095	0.00085	0.00044	0.00031
RKD $\ell = 1$	0.00094	0.00083	0.00034	0.00021
CKD	0.00108	0.00069	0.00043	0.00022

Tab. 2. EMSE for density estimators according to sample sizes for censored data.

We conclude from Figures (3) – (4) and Table 2, without surprise, that the quality of all estimators is better for large sample size n, even for censored data. Additionally, we observe that the choice of ℓ has no influence in the censored case either.

5.2. Computational time

This subsection highlights a key advantage of the recursive estimator over the classical estimator defined by (1), specifically in terms of the computational time taken (in seconds) for each method.

The results obtained are presented in Tables 3 and 4.

The time of computation in seconds is given for different sample sizes $n = 300$, $n = 500$, $n = 1000$, $n = 2000$ and $n = 5000$ and a fixed $\ell = 1$.

5.2.1. Complete data

	$n = 300$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
RKD	0.13	0.16	0.3	0.74	3.39
CKD	0.31	0.37	0.89	1.53	7.23

Tab. 3. Comparison of the computational time of the RKD and CKD for complete data.

The computational duration is clearly shorter when utilizing the RKD in comparison to the CKD.

5.2.2. Censored data

	$n = 300$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
RKD	0.27	0.3	0.43	1.03	3.62
CKD	1.33	0.45	20.94	30.36	35.43

Tab. 4. Comparison of the computational time of the RKD and CKD for censored data.

While computational time increases with sample size, there are significant differences in computation time between the two estimators. In particular, the recursive estimator (RKD) requires substantially less time compared to the classical kernel estimator (CKD) (see Table (3) and Table (4).

6. CONCLUSION

In conclusion, both our theoretical and practical analyses highlight the robust performance of the density recursive kernel estimator for both complete and right censored data.

7. APPENDIX

Proof of Lemma 3.3

Let us set,

$$\bar{f}_n^\ell(t) - E\bar{f}_n^\ell(t) = \frac{1}{n}\sum_{i=1}^{n} \xi_i,$$

such that

$$\xi_i = \frac{n h_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}} \left[K\left(\frac{t-T_i}{h_i}\right) - EK\left(\frac{t-T_i}{h_i}\right)\right]. \tag{6}$$

We need first to prove that $|\xi_i|$ is bounded.

Indeed, since h_i is decreasing and under $(K.1)$, we obtain

$$0 \leq \frac{n h_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}} K\left(\frac{t-Y_i}{h_i}\right) \leq 2\|K\|_\infty \frac{n h_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}} \leq \frac{C}{h_n}.$$

Finally, according to Theorem 2 in [17], with $\epsilon = \eta\sqrt{\frac{\ln n}{n h_n^2}}$, we get

$$P\left(\left|\widetilde{f}_n^\ell(t) - E\widetilde{f}_n^\ell(t)\right| > \sqrt{\frac{\ln n}{n h_n^2}}\right) \leq \exp\left\{\frac{-2\eta^2 \ln n}{C}\right\} \leq n^{\frac{-2\eta^2}{C}}, \tag{7}$$

thus, the right term of this inequality is a general term of Riemann series for an appropriate choice of η and C. □

Proof of Lemma 3.4

Let us set

$$\begin{aligned} E\bar{f}_n^\ell(t) =& \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^{n} h_i^{-\ell} E\left(K\left(\frac{t-T_i}{h_i}\right)\right) \\ =& \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^{n} h_i^{-\ell} \int K\left(\frac{t-u}{h_i}\right) f(u)\, du. \end{aligned}$$

(i) With a change of variable $z = t - u$, we get

$$E\bar{f}_n^\ell(t) = \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{1-\ell} \int \frac{1}{h_i} K\left(\frac{z}{h_i}\right) f(t-z)\,\mathrm{d}z$$
$$= \frac{1}{\beta_{n,1-\ell}} \frac{1}{n} \sum_{i=1}^n \left(\frac{h_i}{h_n}\right)^{1-\ell} \int \frac{1}{h_i} K\left(\frac{z}{h_i}\right) f(t-z)\,\mathrm{d}z.$$

Under conditions $(H.1)$, $(F.1)$ and $(K.1)$, Bochner theorem is applicable and gives

$$\lim_{i\longrightarrow\infty} \int \frac{1}{h_i} K\left(\frac{z}{h_i}\right) f(t-z)\,\mathrm{d}z = f(t),$$

hence, under $(H.2)$, we can use Lemma 2.4.1. in [1] to obtain

$$\lim_{n\longrightarrow\infty} E\bar{f}_n^\ell(t) = \frac{1}{\beta_{1-\ell}} \beta_{1-\ell} f(t) = f(t).$$

(ii) With a change of variable $z = \frac{t-u}{h_i}$, we have

$$|E\bar{f}_n^\ell(t) - f(t)| \leq \frac{1}{\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{1-\ell} \int K(x)|f(t-h_i x) - f(t)|\,\mathrm{d}x.$$

Because of hypothesis $(F.2)$, Taylor formula gives

$$f(t - xh_i) = f(t) + xh_i f'(t) + \frac{x^2 h_i^2}{2!} f^{(2)}(t - \theta h_i x),$$

where $\theta \in]0,1[$.
Into account $(K.1)$ and $(K.2)$, we obtain

$$|E\bar{f}_n^\ell(t) - f(t)| \leq \frac{1}{2\sum_{i=1}^n h_i^{1-\ell}} \sum_{i=1}^n h_i^{3-\ell} \int x^2 K(x) f^{(2)}(t - \theta h_i x)\,\mathrm{d}x.$$

The assumptions $(F.2)$ and the dominate convergence theorem imply that

$$\lim_{i\to\infty} \int x^2 K(x) f^{(2)}(t-\theta h_i x)\,\mathrm{d}x = f^{(2)}(t) \int x^2 K(x)\,\mathrm{d}x.$$

Thus, by applying the Lemma 2.4.1. in [1] under $(H.2)$, we have

$$\lim_{n\to\infty} h_n^{-2} |E\bar{f}_n^\ell(t) - f(t)| = \frac{\beta_{3-\ell}}{2\beta_{1-\ell}} < \infty.$$

□

P r o o f of Lemma 3.6

Let us set

$$\widetilde{f}_n^\ell(t) - E\widetilde{f}_n^\ell(t) = \frac{1}{n}\sum_{i=1}^n \zeta_i,$$

where

$$\zeta_i = \frac{nh_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}}\left[K\left(\frac{t-Y_i}{h_i}\right)\frac{\delta_i}{G(Y_i)} - E\left(K\left(\frac{t-Y_i}{h_i}\right)\frac{\delta_i}{G(Y_i)}\right)\right].$$

Under assumptions $(C.1)$ and $(K.1)$, we have

$$0 \leq \frac{nh_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}}K\left(\frac{t-Y_i}{h_i}\right)\frac{\delta_i}{G(Y_i)} \leq \frac{2\|K\|_\infty}{G(\theta)}\frac{nh_i^{-\ell}}{\sum_{i=1}^n h_i^{1-\ell}} \leq \frac{C}{h_n}.$$

According to Theorem 2 in [17], with $\epsilon = \eta'\sqrt{\frac{\ln n}{nh_n^2}}$, we get

$$P\left(\left|\widetilde{f}_n^\ell(t) - E\widetilde{f}_n^\ell(t)\right| > \sqrt{\frac{\ln n}{nh_n^2}}\right) \leq \exp\left\{\frac{-2{\eta'}^2\ln n}{C}\right\} \leq n^{\frac{-2{\eta'}^2}{C}}, \tag{8}$$

thus, the right term of this inequality is a general term of Riemann series for an appropriate choice of η' and C. □

P r o o f of Lemma 3.7

As $E\left(\delta/T\right) = G(T)$, we get

$$\begin{aligned} E\widetilde{f}_n^\ell(t) =& \frac{1}{\sum_{i=1}^n h_i^{1-\ell}}\sum_{i=1}^n h_i^{-\ell}E\left(\frac{\delta_i}{G(Y_i)}K\left(\frac{t-Y_i}{h_i}\right)\right) \\ =& \frac{1}{\sum_{i=1}^n h_i^{1-\ell}}\sum_{i=1}^n h_i^{-\ell}E\left(K\left(\frac{t-T_i}{h_i}\right)\right). \end{aligned}$$

So, it is clear that the rest of the proof is similar to the proof of Lemma 3.4. □

P r o o f of Lemma 3.8

Because of the definitions of $\widehat{f}_n^\ell(t)$ and $\widetilde{f}_n^\ell(t)$ and the assumption $(C.1)$, we can write

$$|\widehat{f}_n^\ell(t) - \widetilde{f}_n^\ell(t)| \leq C\frac{\sup_{t<\theta}|G_n(t)-G(t)|}{G_n(\theta)G(\theta)}|\bar{f}_n^\ell(t)|.$$

Moreover, by Theorem 1 of [3], we obtain, under assumption $(C.1)$

$$\sup_{t<\theta}|G_n(t)-G(t)| = O_{a.s}\left(\sqrt{\frac{\ln n}{n}}\right),$$

which is equals to $O_{a.s}\left(\sqrt{\frac{\ln n}{nh_n^2}}\right)$. The proof is then ended. □

Proof of Lemma 4.3

To study the first term, using the compactness of the interval S, we have

$$S \subset \cup_{i=1}^{\lambda_n}]x_i - \gamma_n, x_i + \gamma_n[, \quad \text{with} \quad \gamma_n = c\lambda_n^{-1}, c > 0. \tag{9}$$

Taking first $x_t = \arg\min_{x \in \{x_1, \ldots, x_{\lambda_n}\}} |t - x|$, we can write

$$\begin{aligned} P\left(\sup_{x\in S} |\bar{f}_n^\ell(t) - \mathbb{E}\bar{f}_n^\ell(t)| > \epsilon\right) \leq \quad & P\left(\sup_{x\in S} |\bar{f}_n^\ell(t) - f_n^\ell(x_t)| > \epsilon\right) \\ & + P\left(\sup_{x\in S} |\mathbb{E}\bar{f}_n^\ell(x_t) - \mathbb{E}\bar{f}_n^\ell(t)| > \epsilon\right) \\ & + P\left(\sup_{x\in S} |\bar{f}_n^\ell(x_t) - \mathbb{E}\bar{f}_n^\ell(x_t)| > \epsilon\right) := I_1 + I_2 + I_3. \end{aligned}$$

The first and the second term are treated similarly, since K is Lipschitzian in the condition $(K.3)$, we get

$$|\bar{f}_n^\ell(t) - \bar{f}_n^\ell(x_t)| \leq \frac{C\gamma_n^\beta}{h_n^{\beta+1}}.$$

Choosing now $\gamma_n = n^{-\frac{\beta+1}{\beta}}$, we find

$$I_1 < \frac{C}{(nh_n)^{\beta+1}}$$

and we can derive

$$I_2 < \frac{C}{(nh_n)^{\beta+1}}.$$

Hence, the assumption $(H.1)$ and the choice $\epsilon = \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}$ give $P\left(\frac{C}{(nh_n)^{\beta+1}} > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) = 0$, for n sufficiently large. Thus, we get

$$\sum_n P\left(I_1 > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty \quad \text{and} \quad \sum_n P\left(I_2 > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty.$$

Finally, for the term I_3, we have

$$\begin{aligned} P\left(I_3 > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) \quad & = P\left(\max_{i=1:\lambda_n} |\bar{f}_n^\ell(x_i) - \mathbb{E}\bar{f}_n^\ell(x_i)| > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) \\ & \leq \sum_{i=1}^{\lambda n} P\left(|\bar{f}_n^\ell(x_i) - \mathbb{E}\bar{f}_n^\ell(x_i)| > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) \\ & \leq \lambda_n n^{-p}, \end{aligned} \tag{10}$$

thanks to the relation (7), with $p = \frac{2\epsilon_0^2}{C}$. Using the fact that $\lambda_n = n^{\frac{\beta+1}{\beta}}$, we find, for $p > \frac{\beta+1}{\beta} + 1$,

$$\sum_{n\geq 0} P\left(I_3 > \epsilon_0\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty.$$

Proof of Lemma 4.4

Taking into account hypotheses $(F'.1)$ and $(F'.2)$, along with condition $(K.3)$, the proof follows the same structure as that of Lemma 3.4. □

Proof of Lemma 4.7

The compactness of S permits us to write that

$$\begin{aligned} P\left(\sup_{x\in S}|\widetilde{f}_n^{\ell}(t)-\mathbb{E}\widetilde{f}_n^{\ell}(t)|)>\epsilon\right) &\leq P\left(\sup_{x\in S}|\widetilde{f}_n^{\ell}(t)-\widetilde{f}_n^{\ell}(x_t)|>\epsilon\right) \\ &+P\left(\sup_{x\in S}|\mathbb{E}\widetilde{f}_n^{\ell}(x_t)-\mathbb{E}\widetilde{f}_n^{\ell}(t)|>\epsilon\right) \\ &+P\left(\sup_{x\in S}|\widetilde{f}_n^{\ell}(x_t)-\mathbb{E}\widetilde{f}_n^{\ell}(x_t)|>\epsilon\right) := T_1+T_2+T_3. \end{aligned}$$

First, as K is Lipschitzian and G is decreasing, we have

$$|\widetilde{f}_n^{\ell}(t)-\widetilde{f}_n^{\ell}(x_t)| \leq \frac{C\gamma_n^{\beta}}{h_n^{\beta+1}G(\theta)},$$

This, together with $\gamma_n = n^{-\frac{\beta+1}{\beta}}$ allows us to obtain

$$T_1 < \frac{C}{(nh_n)^{\beta+1}},$$

and we can also deduce that

$$T_2 < \frac{C}{(nh_n)^{\beta+1}}.$$

Thus, the assumption $(H.1)$ and the choice $\epsilon = \epsilon_0'\sqrt{\frac{\ln n}{nh_n^2}}$ give

$$\sum_n P\left(T_1 > \epsilon_0'\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty \text{ and } \sum_n P\left(T_2 > \epsilon_0'\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty, \quad \epsilon_0' > 0.$$

Finally, for the term T_3, we proceed as in (10) and use the relation (8) instead of (7) to obtain

$$\sum_{n\geq 0} P\left(T_3 > \epsilon_0'\sqrt{\frac{\ln n}{nh_n^2}}\right) < \infty.$$

□

Proof of Lemma 4.8

Considering hypotheses $(F'.1)$ and $(F'.2)$, together with condition $(K.3)$, the proof proceeds in the same manner as in Lemma 3.7. □

Proof of Lemma 4.9

The proof is analogous to the one in Lemma 3.8. □

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