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Soniya Sultana; Pulak Sahoo

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# UNIQUENESS RESULTS FOR DIFFERENTIAL POLYNOMIALS SHARING A SET

Soniya Sultana, Berhampore, Pulak Sahoo, Kalyani

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*Abstract.* We investigate the uniqueness results of meromorphic functions if differential polynomials of the form $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share a set counting multiplicities or ignoring multiplicities, where $Q$ is a polynomial of one variable. We give suitable conditions on the degree of $Q$ and on the number of zeros and the multiplicities of the zeros of $Q'$. The results of the paper generalize some results due to T. T. H. An and N. V. Phuong (2017) and that of N. V. Phuong (2021).

*Keywords*: uniqueness; differential polynomials; set sharing; small function

*MSC 2020*: 30D35

## 1. Introduction, definitions and results

Let $f(z)$ be a nonconstant meromorphic function. The term "meromorphic" indicates meromorphic in the entire complex plane $\mathbb{C}$. We denote by $S(r,f)$ any function satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set with finite measure. Here, $T(r,f)$ denotes the Nevanlinna characteristic of $f$, and we use the standard notations of Nevanlinna value distribution theory throughout this work (see [8], [10], [16]). A meromorphic function $\alpha(z)$ is called a small function of some function $f(z)$ if $T(r,\alpha)=S(r,f)$. We say that two meromorphic functions $f$, $g$ share a function $\alpha$ CM (counting multiplicities) if $f-\alpha$ and $g-\alpha$ admit the same zeros with the same multiplicities, and we say that $f$ and $g$ share $\alpha$ IM (ignoring multiplicities) if we do not consider the multiplicities. Let $S$ be either a subset of $\mathbb{C}\cup\{\infty\}$ or a subset of $S(f)\cup\{\infty\}$, where $S(f)$ denotes the set of small functions of $f$. We define

$$E_f(S)=\bigcup_{\alpha\in S}\{z\in\mathbb{C}\colon\ f(z)-\alpha=0\},$$

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where each zero of $f-\alpha$ CM is included in the set, i.e., $E_f(S)$ is a multi-set. In the case we do not count the multiplicities, the collection $\bigcup_{\alpha\in S}\{z\in\mathbb{C}\colon\ f(z)-\alpha=0\}$ of only distinct zeros is denoted by $\overline{E}_f(S)$. Two functions $f$ and $g$ are said to share the set $S$ CM (IM) if $E_f(S)=E_g(S)$ ($\overline{E}_f(S)=\overline{E}_g(S)$). Clearly, in the case when $S$ is singleton, set sharing coincides with value sharing or a single small function sharing.

In 1959, Hayman (see [7]) published one of his significant paper, where the zero distribution of complex differential polynomials was considered, that is, if $f$ is a transcendental meromorphic function and $n\in\mathbb{N}$, then Hayman conjectured that $f'f^n$ takes every finite nonzero value infinitely often.

Hayman conjecture has been proved completely by Hayman in [7] for the case $n\geqslant 3$, by Mues in [11] for $n=2$ and by Bergweiler and Eremenko (see [4]), Chen and Fang (see [6]) and Zalcman (see [17]) for $n=1$.

In 1997, Yang and Hua in [15] studied the unicity problem for meromorphic functions and differential monomials of the form $f'f^n$, when they share only one value.

In 2007, Bhoosnurmath and Dyavanal (see [5]) extended Yang-Hua's result to the case $(f^n)^{(k)}$.

Being inspired by Yang's problem (see [14]) that whether $f^{-1}(S)=g^{-1}(S)$ with $S=\{-1,1\}$ for the two same degree polynomials $f$ and $g$ implies either $f=g$ or $f=-g$, An and Khoai (see [3]) proved a uniqueness result on the meromorphic functions $f$ and $g$ when $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a finite set. In this direction, Khoai and An (see [9]) proved a uniqueness result on meromorphic functions when two differential polynomials of the form $(P(f)^n)^{(k)}$ share a set of roots of unity.

Let $Q(z)$ be a polynomial of degree $q$ in $\mathbb{C}$ and $k$ be a positive integer. Denote the derivative of $Q(z)$ by

$$Q'(z)=b\prod_{i=1}^{l}(z-\zeta_i)^{m_i}$$

with $b\in\mathbb{C}^*$ $(=\mathbb{C}-\{0\})$, and denote by $\nu$ and $h$ the indexes such that $1\leqslant\nu\leqslant h\leqslant l$, and

$$m_1\geqslant m_2\geqslant\ldots\geqslant m_\nu>k\geqslant m_{\nu+1}\geqslant\ldots\geqslant m_l,$$

$$m_1\geqslant m_2\geqslant\ldots\geqslant m_h\geqslant k>m_{h+1}\geqslant\ldots\geqslant m_l.$$

In 2017, An and Phuong (see [1]) proved a uniqueness result on meromorphic functions when $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share a small function $\alpha$ CM. Their result is as follows:

**Theorem A.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a nonzero small function with respect to $f$. Suppose that $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share $\alpha$ CM. If $q>k+6+2\nu(k+1)+2\sum_{i=\nu+1}^{l}m_i$, then one of the following conclusions holds:*

(1) $Q(f) = Q(g) + c$ *for a constant* $c$*;*
(2) $[Q(f)]^{(k)}[Q(g)]^{(k)} = \alpha^2$.

The authors [1] also showed that conclusion (2) of Theorem A can be ruled out by adding more constraints on the multiple zeros of $Q'(z)$ or if $f$ and $g$ share $\infty$ IM and proved the following theorem.

**Theorem B.** *Let* $f$ *and* $g$ *be two nonconstant meromorphic functions, and* $\alpha$ *be a nonzero small function with respect to* $f$*. Assume that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* $\alpha$ *CM. If* $q > k + 6 + 2\nu(k+1) + 2\sum_{i=\nu+1}^{l} m_i$ *and if one of*
(1) $h \geqslant 4$;
(2) $h = 3$ *and* $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, *and* $q \neq 3m_i - 2k + 3$, *for all* $i = 1, 2, 3$; *or*
(3) $h = 2$
*and* $f$ *and* $g$ *share* $\infty$ *IM holds, then*

$$Q(f) = Q(g) + c \quad \text{for a constant } c.$$

In 2021, Phuong (see [12]) proved the following results for sharing the small function $\alpha$ IM.

**Theorem C.** *Let* $f$ *and* $g$ *be two nonconstant meromorphic functions, and* $\alpha$ *be a nonzero small function with respect to* $f$*. Suppose that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* $\alpha$ *IM. If* $q > 4k + 12 + \nu(5k+2) + 5\sum_{i=\nu+1}^{l} m_i$, *then one of the following conclusions holds:*
(1) $Q(f) = Q(g) + c$ *for a constant* $c$*;*
(2) $[Q(f)]^{(k)}[Q(g)]^{(k)} = \alpha^2$.

**Theorem D.** *Let* $f$ *and* $g$ *be two nonconstant meromorphic functions, and* $\alpha$ *be a nonzero small function with respect to* $f$*. Suppose that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* $\alpha$ *IM. If* $q > 4k + 12 + \nu(5k+2) + 5\sum_{i=\nu+1}^{l} m_i$, *and if one of*
(1) $h \geqslant 4$;
(2) $h = 3$ *and* $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, *and* $q \neq 3m_i - 2k + 3$, *for all* $i = 1, 2, 3$; *or*
(3) $h = 2$
*and* $f$ *and* $g$ *share* $\infty$ *IM holds, then*

$$Q(f) = Q(g) + c \quad \text{for a constant } c.$$

Now the following question is inevitable.

**Question 1.1.** *What will happen if sharing a small function $\alpha$ is replaced by sharing a set $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), \ldots, \omega^{d-1}\alpha(z)\}$, with $\omega^d = 1$ in Theorems* A–D*?*

In this regard, we obtain the next main results which answers the above question.

**Theorem 1.1.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a nonzero small function with respect to $f$. Let $d$ be a positive integer such that $q > k+2+4/d+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$ and let $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), \ldots, \omega^{d-1}\alpha(z)\}$, where $\omega^d = 1$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set $S$ CM, then one of the following conclusions holds:*

(1) *$Q(f) = tQ(g) + c$ for a constant $c$ and $t^d = 1$;*
(2) *$[Q(f)]^{(k)}[Q(g)]^{(k)} = t\alpha^{2/d}$ with $t^d = 1$.*

**Theorem 1.2.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a nonzero small function with respect to $f$. Let $d$, $S$ be defined as in Theorem* 1.1 *and $q > k + 2 + 4/d + 2\nu(k+1) + 2\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set $S$ CM and if one of*

(1) *$h \geqslant 4$;*
(2) *$h = 3$ and $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, and $q \neq 3m_i - 2k + 3$, for all $i = 1, 2, 3$; or*
(3) *$h = 2$*

*and $f$ and $g$ share $\infty$ IM holds, then*

$$Q(f) = tQ(g) + c \quad \text{for a constant } c \text{ and } t^d = 1.$$

**Theorem 1.3.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a nonzero small function with respect to $f$. Let $d$, $S$ be defined as in Theorem* 1.1 *and $q > k + 2 + (3k+10)/d + \nu(2k + 2 + 3k/d) + (2 + 3/d)\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set $S$ IM, then one of the conclusions of Theorem* 1.1 *holds.*

**Theorem 1.4.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and $\alpha$ be a nonzero small function with respect to $f$. Let $d$, $S$ be defined as in Theorem* 1.1 *and $q > k+2+(3k+10)/d+\nu(2k+2+3k/d)+(2+3/d)\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set $S$ IM and if one of* (1), (2) *and* (3) *of Theorem* 1.2 *holds, then*

$$Q(f) = tQ(g) + c \quad \text{for a constant } c \text{ and } t^d = 1.$$

R e m a r k 1.1. If we put $d = 1$ in Theorems 1.1–1.4, then we obtain Theorems A–D, respectively.

**Definition 1.1.** Let $a$ be a finite complex number, and let $p$ be a positive integer. We denote by $N_p(r, 1/(f-a))$ the counting function for zeros of $f-a$, where a zero of multiplicity $m$ is counted $m$ times if $m \leqslant p$ and $p$ times if $m > p$.

## 2. Lemmas

We now present some lemmas that will be useful in the next section.

**Lemma 2.1** ([13] Logarithmic derivative lemma). *Let $f$ be a nonconstant meromorphic function on $\mathbb{C}$. Then*

$$m\Big(r, \frac{f'}{f}\Big) = S(r, f)$$

*as $r \to \infty$ outside a subset of finite measure.*

**Lemma 2.2** ([8], [13] First fundamental theorem). *Let $f$ be a meromorphic function, and let $c$ be a complex number. Then*

$$T\Big(r, \frac{1}{f-c}\Big) = T(r, f) + O(1).$$

**Lemma 2.3** ([8], [13] Second fundamental theorem). *Let $f$ be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that $a_i$'s are small functions with respect to $f$ for all $i = 1, \ldots, q$. Then the inequality*

$$(q-2)T(r, f) \leqslant \sum_{j=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) + S(r, f)$$

*holds for all $r$ outside a set $E \subset (0, \infty)$ with finite Lebesgue measure.*

**Lemma 2.4** ([18]). *Let $f$ be a nonconstant meromorphic function, and let $p$ and $k$ be two positive integers. If $f^{(k)} \not\equiv 0$, then*

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant k\overline{N}(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$

*and*

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant k\overline{N}(r, f) + N\Big(r, \frac{1}{f}\Big) + S(r, f).$$

**Lemma 2.5.** *Let $Q$ be a polynomial of degree $q$ in $\mathbb{C}$, and let $k$ be a positive integer. Let*

$$Q'(z) = b\prod_{i=1}^{l}(z-\zeta_i)^{m_i}$$

*with $b \in \mathbb{C}^*$. Let $f$ and $g$ be two nonconstant meromorphic functions. Assume that $([Q(f)]^{(k)})^d = ([Q(g)]^{(k)})^d$. If $q - 2l - 2k - 4 > 0$, then $Q(f) = tQ(g) + c$ for a constant $c$ and $t^d = 1$.*

Proof. Since $([Q(f)]^{(k)})^d = ([Q(g)]^{(k)})^d$, we get $[Q(f)]^{(k)} = t[Q(g)]^{(k)}$ where $t^d = 1$. This gives

$$Q(f) = tQ(g) + \varphi,$$

where $\varphi$ is a polynomial of degree at most $k-1$. Therefore,

$$qT(r,g) \leqslant qT(r,f) + T(r,\varphi) + O(1), \quad \text{and} \quad f'Q'(f) = tg'Q'(g) + \varphi'.$$

If $k = 1$, then $\varphi = c$, a constant.

If $k \geqslant 2$, then proceeding in a similar manner as in the proof of Lemma 3.1 of [1], we can deduce that $\varphi = c$ for a constant $c$. □

**Lemma 2.6.** *Let $f$ and $g$ be two nonconstant meromorphic functions, and let $\alpha$ be a small function with respect to $f$. Let $d$, $S$ be defined as in Theorem 1.1 and $q > 5 + 1/d + \nu(k+1) + \sum\limits_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set $S$ IM, then $T(r,f) = O(T(r,g))$, $T(r,g) = O(T(r,f))$, and $\alpha$ is a small function with respect to $g$.*

Proof. Let

$$F := [Q(f)]^{(k)}, \quad F_1 := Q(f), \quad \widehat{F} := F^d,$$
$$G := [Q(g)]^{(k)}, \quad G_1 := Q(g), \quad \widehat{G} := G^d.$$

It is easy to see that

$$S(r,\widehat{F}) = S(r,F) = S(r,f) \quad \text{and} \quad S(r,\widehat{G}) = S(r,G) = S(r,g).$$

Now we have

$$\begin{aligned} T(r,F_1') &= T(r,f'Q'(f)) \geqslant T\Big(r, f'Q'(f)\frac{1}{f'}\Big) - T\Big(r,\frac{1}{f'}\Big) + O(1) \\ &\geqslant T(r,Q'(f)) - 2T(r,f) + O(1) \geqslant (q-3)T(r,f) + O(1). \end{aligned} \tag{2.1}$$

Applying Lemma 2.3 to $\widehat{F}$, we obtain

$$dT(r,F)=T(r,\widehat{F})\leqslant \overline{N}(r,\widehat{F})+\overline{N}\Big(r,\frac{1}{\widehat{F}}\Big)+\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+S(r,f)$$
$$\leqslant \overline{N}(r,f)+\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+S(r,f). \tag{2.2}$$

Again by Lemma 2.4 with $(F_1')^{(k-1)}=F$, we have

$$T(r,F)\geqslant T(r,F_1')+N_2\Big(r,\frac{1}{F}\Big)-N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f). \tag{2.3}$$

From (2.1), (2.2) and (2.3) we get

$$\begin{aligned}(q-3)T(r,f)&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{F}\Big)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)-N_2\Big(r,\frac{1}{F}\Big)\\&\quad+N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f)\\&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f)\\&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{G}-\alpha}\Big)+N\Big(r,\frac{1}{f'}\Big)+(k+1)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{f-\zeta_i}\Big)\\&\quad+\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{f-\zeta_i}\Big)+S(r,f)\\&\leqslant \Big(2+\frac{1}{d}+\nu(k+1)+\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)+q(k+1)T(r,g)+S(r,f).\end{aligned}$$

Therefore

$$\Big(q-5-\frac{1}{d}-\nu(k+1)-\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)\leqslant q(k+1)T(r,g)+S(r,f),$$

which implies $T(r,f)=O(T(r,g))$ if $q>5+1/d+\nu(k+1)+\sum\limits_{l=\nu+1}^{l}m_i$. Similarly, it can be shown that $T(r,g)=O(T(r,f))$ and hence, $\alpha$ is a small function with respect to $g$. □

**Lemma 2.7** ([2]). *Let $f$ and $g$ be two nonconstant meromorphic functions, and let $\alpha$ be a nonzero small function with respect to both $f$ and $g$. If $f$ and $g$ share $\alpha$ CM, then one of the following three cases holds:*

(1) $T(r,f)\leqslant N_2(r,f)+N_2(r,g)+N_2(r,1/f)+N_2(r,1/g)+S(r,f)+S(r,g)$, *and the same inequality holds for* $T(r,g)$;
(2) $f\equiv g$;
(3) $fg\equiv\alpha^2$.

**Lemma 2.8** ([12]). *Let $f$ and $g$ be two nonconstant meromorphic functions, and let $\alpha$ be a nonzero small function with respect to both $f$ and $g$. If $f$ and $g$ share $\alpha$ IM, then one of the following three cases holds:*

(1) $T(r,f) \leqslant N_2(r,f) + N_2(r,g) + N_2(r,1/f) + N_2(r,1/g) + 2\overline{N}(r,f) + \overline{N}(r,g) + 2\overline{N}(r,1/f) + \overline{N}(r,1/g) + S(r,f) + S(r,g)$, *and the same inequality holds for* $T(r,g)$;

(2) $f \equiv g$;

(3) $fg \equiv \alpha^2$.

**Lemma 2.9.** *Let $f$, $g$ be nonconstant meromorphic functions and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both $f$ and $g$. If*

$$([Q(f)]^{(k)})^d([Q(g)]^{(k)})^d = \alpha^2,$$

*then $h \leqslant 2$ or $h = 3$ and either $q = 2m_1 - 2k + 2$, $q = (3m_1 - 2k + 3)/2$, or $q = 3m_i - 2k + 3$, for $i = 1,2,3$. If we further assume that $f$ and $g$ share $\infty$ IM, then also $h = 1$.*

Proof. From $([Q(f)]^{(k)})^d([Q(g)]^{(k)})^d = \alpha^2$ we have $[Q(f)]^{(k)}[Q(g)]^{(k)} = t\alpha^{2/d}$, where $t^d = 1$. This gives

$$[f'Q'(f)]^{(k-1)}[g'Q'(g)]^{(k-1)} = t\alpha^{2/d}.$$

Since

$$Q'(z) = b\prod_{i=1}^{l}(z-\zeta_i)^{m_i},$$

where $b \in \mathbb{C}^*$ and $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_h \geqslant k > m_{h+1} \geqslant \ldots \geqslant m_l$, we can write

$$\prod_{i=1}^{h}(f-\zeta_i)^{m_i-k+1}\prod_{i=1}^{h}(g-\zeta_i)^{m_i-k+1}R(f,f',\ldots,f^{(k)})\widetilde{R}(g,g',\ldots,g^{(k)}) = t\alpha^{2/d},$$

where $R(f,f',\ldots,f^{(k)})$ and $\widetilde{R}(g,g',\ldots,g^{(k)})$ are polynomials. Then proceeding similarly as in the proof of Lemma 3.4 in [1], we can get the required result. □

## 3. Proof of the theorems

Proof of Theorem 1.1. Let $F$, $G$, $F_1$, $G_1$, $\widehat{F}$ and $\widehat{G}$ be defined as in the proof of Lemma 2.6. Then it is easy to prove that

$$S(r,\widehat{F}) = S(r,F) = S(r,f) \quad \text{and} \quad S(r,\widehat{G}) = S(r,G) = S(r,g).$$

By Lemma 2.6, $\alpha$ is a small function with respect to $g$ also. Since $F$ and $G$ share the set $S$ CM, it follows that $\widehat{F}$ and $\widehat{G}$ share $\alpha$ CM. Therefore by Lemma 2.7, one of the following cases occurs:

(1) $T(r,\widehat{F}) \leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2(r,1/\widehat{F}) + N_2(r,1/\widehat{G}) + S(r,\widehat{F}) + S(r,\widehat{G})$, and the same inequality holds for $T(r,\widehat{G})$;

(2) $\widehat{F} \equiv \widehat{G}$;

(3) $\widehat{F}\widehat{G} \equiv \alpha^2$.

If Case (3) holds, then conclusion (2) of the theorem is proved. If Case (2) holds, then by Lemma 2.5, we get $Q(f) = tQ(g) + c$ for a constant $c$ and $t^d = 1$. So conclusion (1) of the theorem is proved. Now we verify Case (1).

If Case (1) holds, then we have

$$
\begin{aligned}
(3.1)\quad dT(r,F) &= T(r,\widehat{F}) \\
&\leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2\Big(r,\frac{1}{\widehat{F}}\Big) + N_2\Big(r,\frac{1}{\widehat{G}}\Big) + S(r,\widehat{F}) + S(r,\widehat{G}) \\
&\leqslant N_2(r,F) + N_2(r,G) + dN_2\Big(r,\frac{1}{F}\Big) + dN_2\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g).
\end{aligned}
$$

Now using Lemma 2.4, we have

$$
(3.2)\quad N_2\Big(r,\frac{1}{G}\Big) = N_2\Big(r,\frac{1}{(G_1')^{(k-1)}}\Big) \leqslant (k-1)\overline{N}(r,G_1') + N_{k+1}\Big(r,\frac{1}{G_1'}\Big) + S(r,g).
$$

Again, we can write

$$
Q(z) - R(z) = a(z-\beta)Q'(z),
$$

where $a \neq 0$ and $\beta$ are constants and $R(z)$ is a polynomial of degree atmost $q-2$.

Applying Lemma 2.1, we have

$$
\begin{aligned}
m\Big(r,\frac{1}{Q(f)-R(f)}\Big) &= m\Big(r,\frac{(Q(f))'}{Q(f)-R(f)}\cdot\frac{1}{(Q(f))'}\Big) \\
&\leqslant m\Big(r,\frac{f'}{a(f-\beta)}\Big) + m\Big(r,\frac{1}{F_1'}\Big) + O(1) \leqslant m\Big(r,\frac{1}{F_1'}\Big) + S(r,f).
\end{aligned}
$$

From this we get

$$
\begin{aligned}
T(r,F_1') &= m\Big(r,\frac{1}{F_1'}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1) \\
&\geqslant T\Big(r,\frac{1}{Q(f)-R(f)}\Big) - N\Big(r,\frac{1}{Q(f)-R(f)}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1) \\
&\geqslant qT(r,f) - N\Big(r,\frac{1}{Q'(f)}\Big) - N\Big(r,\frac{1}{f-\beta}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1).
\end{aligned}
$$

Therefore, applying Lemma 2.4 to the function $F_1'$ (with the notation $(F_1')^{(k-1)} = F$), we have

$$\begin{aligned} T(r,F) &\geqslant T(r,F_1') + N_2\Big(r,\frac{1}{F}\Big) - N_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r,f) \\ &\geqslant qT(r,f) - N\Big(r,\frac{1}{Q'(f)}\Big) - N\Big(r,\frac{1}{f-\beta}\Big) + N\Big(r,\frac{1}{F_1'}\Big) \\ &\quad + N_2\Big(r,\frac{1}{F}\Big) - N_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r,f). \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) we have

$$\begin{aligned} dqT(r,f) &\leqslant d(k-1)\overline{N}(r,G_1') + dN_{k+1}\Big(r,\frac{1}{G_1'}\Big) + N_2(r,G) \\ &\quad + N_2(r,F) + dN\Big(r,\frac{1}{Q'(f)}\Big) + dN\Big(r,\frac{1}{f-\beta}\Big) \\ &\quad - dN\Big(r,\frac{1}{F_1'}\Big) + dN_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r) \\ &\leqslant (d(k-1)+2)\overline{N}(r,g) + d(k+1)\sum_{i=1}^{\nu} N\Big(r,\frac{1}{g-\zeta_i}\Big) \\ &\quad + dN\Big(r,\frac{1}{g'}\Big) + d\sum_{i=\nu+1}^{l} m_i N\Big(r,\frac{1}{g-\zeta_i}\Big) + 2\overline{N}(r,f) \\ &\quad + d(k+1)\sum_{i=1}^{\nu} N\Big(r,\frac{1}{f-\zeta_i}\Big) + d\sum_{i=\nu+1}^{l} m_i N\Big(r,\frac{1}{f-\zeta_i}\Big) \\ &\quad + dN\Big(r,\frac{1}{f-\beta}\Big) + S(r) \\ &\leqslant \Big(d(k+1) + 2 + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,g) \\ &\quad + \Big(2 + d + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,f) + S(r). \end{aligned}$$

This implies

$$\begin{aligned} &\Big(dq - 2 - d - d\nu(k+1) - d\sum_{i=\nu+1}^{l} m_i\Big)T(r,f) \\ &\qquad \leqslant \Big(d(k+1) + 2 + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,g) + S(r). \end{aligned} \tag{3.4}$$

Similarly, it can be shown that

$$\left(dq - 2 - d - d\nu(k+1) - d\sum_{i=\nu+1}^{l} m_i\right)T(r,g) \leqslant \left(d(k+1) + 2 + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\right)T(r,f) + S(r). \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\left(dq - 4 - d(k+2) - 2d\nu(k+1) - 2d\sum_{i=\nu+1}^{l} m_i\right)(T(r,g) + T(r,f)) \leqslant S(r).$$

Thus, we have $q > k+2+4/d+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$, which is a contradiction. This proves the theorem. □

P r o o f of Theorem 1.2. The proof of this theorem follows from Theorem 1.1 and Lemma 2.9. □

P r o o f of Theorem 1.3. The notations $F$, $G$, $F_1$, $G_1$, $\widehat{F}$ and $\widehat{G}$ are the same as defined in the proof of Lemma 2.6. By Lemma 2.6, $\alpha$ is a small function with respect to $g$ also. Since $F$ and $G$ share the set $S$ IM, $\widehat{F}$ and $\widehat{G}$ share $\alpha$ IM. Therefore by Lemma 2.8, one of the following cases occurs:

(1) $T(r,\widehat{F}) \leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2(r,1/\widehat{F}) + N_2(r,1/\widehat{G}) + 2\overline{N}(r,\widehat{F}) + \overline{N}(r,\widehat{G}) + 2\overline{N}(r,1/\widehat{F}) + \overline{N}(r,1/\widehat{G}) + S(r,\widehat{F}) + S(r,\widehat{G})$, and the same inequality holds for $T(r,\widehat{G})$;
(2) $\widehat{F} \equiv \widehat{G}$;
(3) $\widehat{F}\widehat{G} \equiv \alpha^2$.

Conclusions (1) and (2) of the theorem hold preciously from cases (2) and (3), respectively. Next we assume that Case (1) holds. Then

$$\begin{aligned} dT(r,F) &= T(r,\widehat{F}) \\ &\leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2\Big(r,\frac{1}{\widehat{F}}\Big) + N_2\Big(r,\frac{1}{\widehat{G}}\Big) + 2\overline{N}(r,\widehat{F}) \\ &\quad + \overline{N}(r,\widehat{G}) + 2\overline{N}\Big(r,\frac{1}{\widehat{F}}\Big) + \overline{N}\Big(r,\frac{1}{\widehat{G}}\Big) + S(r,\widehat{F}) + S(r,\widehat{G}) \\ &\leqslant N_2(r,F) + N_2(r,G) + dN_2\Big(r,\frac{1}{F}\Big) + dN_2\Big(r,\frac{1}{G}\Big) + 2\overline{N}(r,F) \\ &\quad + \overline{N}(r,G) + 2\overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g). \end{aligned} \tag{3.6}$$

Now using Lemma 2.4, we have

$$\overline{N}\Big(r,\frac{1}{F}\Big)=N_1\Big(r,\frac{1}{(F_1')^{(k-1)}}\Big)\leqslant (k-1)\overline{N}(r,F_1')+N_k\Big(r,\frac{1}{F_1'}\Big)+S(r,f) \tag{3.7}$$

and

$$\overline{N}\Big(r,\frac{1}{G}\Big)\leqslant (k-1)\overline{N}(r,G_1')+N_k\Big(r,\frac{1}{G_1'}\Big)+S(r,g). \tag{3.8}$$

Again, by similar arguments as in the proof of Theorem 1.1, we can get the inequalities (3.2) and (3.3).

From (3.2), (3.3), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
dqT(r,f) &\leqslant d(k-1)\overline{N}(r,G_1')+dN_{k+1}\Big(r,\frac{1}{G_1'}\Big)+N_2(r,F)+N_2(r,G)\\
&\quad +2\overline{N}(r,F)+\overline{N}(r,G)+2(k-1)N(r,F_1')+2N_k\Big(r,\frac{1}{F_1'}\Big)\\
&\quad +(k-1)\overline{N}(r,G_1')+N_k\Big(r,\frac{1}{G_1'}\Big)+dN\Big(r,\frac{1}{Q'(f)}\Big)\\
&\quad +dN\Big(r,\frac{1}{f-\beta}\Big)-dN\Big(r,\frac{1}{F_1'}\Big)+dN_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r)\\
&\leqslant (d(k-1)+k+2)\overline{N}(r,g)+(d+1)N\Big(r,\frac{1}{g'}\Big)\\
&\quad +(d(k+1)+k)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{g-\zeta_i}\Big)+(d+1)\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{g-\zeta_i}\Big)\\
&\quad +(2k+2)\overline{N}(r,f)+2N\Big(r,\frac{1}{f'}\Big)+(d(k+1)+2k)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{f-\zeta_i}\Big)\\
&\quad +(d+2)\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{f-\zeta_i}\Big)+dN\Big(r,\frac{1}{f-\beta}\Big)+S(r)\\
&\leqslant \Big(d(k+1)+k+4+\nu(d(k+1)+k)+(d+1)\sum_{i=\nu+1}^{l}m_i\Big)T(r,g)\\
&\quad +\Big(d+2k+6+\nu(d(k+1)+2k)+(d+2)\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)+S(r).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big(dq-d-2k-6-\nu(d(k+1)+2k)-(d+2)\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)\\
&\quad\leqslant \Big(d(k+1)+k+4+\nu(d(k+1)+k)+(d+1)\sum_{i=\nu+1}^{l}m_i\Big)T(r,g)+S(r).
\end{aligned} \tag{3.9}$$

Similarly,

(3.10)

$$\Big(dq-d-2k-6-\nu(d(k+1)+2k)-(d+2)\sum_{i=\nu+1}^{l}m_i\Big)T(r,g)$$
$$\leqslant\Big(d(k+1)+k+4+\nu(d(k+1)+k)+(d+1)\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)+S(r).$$

Combining (3.9) and (3.10), we get

$$\Big(dq-d(k+2)-3k-10-\nu(2d(k+1)+3k)-(2d+3)\sum_{i=\nu+1}^{l}m_i\Big)(T(r,f)+T(r,g))\leqslant S(r).$$

Thus, when $q>k+2+(3k+10)/d+\nu(2k+2+3k/d)+(2+3/d)\sum_{i=\nu+1}^{l}m_i$, we have a contradiction. This proves the theorem. □

P r o o f of Theorem 1.4. The proof of this theorem follows from Theorem 1.3 and Lemma 2.9. □

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*Authors' addresses*: *Soniya Sultana*, Department of Mathematics, Berhampore Girls' College, Shankar Mandal Rd, Gora Bazar, Berhampore, West Bengal 742101, India, e-mail: `soniyasultana3@gmail.com`; *Pulak Sahoo* (corresponding author), Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741235, India, e-mail: `sahoopulak1@gmail.com`.