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Kybernetika, Vol. 60 (2024), No. 6, 797–818

Persistent URL: http://dml.cz/dmlcz/152860

KYBERNETIKA — VOLUME *60* (2024), NUMBER 6, PAGES 797–818

INVERSE OPTIMAL DYNAMIC BOUNDARY CONTROL FOR UNCERTAIN KORTEWEG–DE VRIES–BURGERS EQUATION

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We investigate Korteweg–de Vries–Burgers (KdVB) equation, where the dissipation and dispersion coefficients are unknown, but their lower bounds are known. First, we establish dynamic boundary controls with update laws to globally exponentially stabilize this uncertain system. Secondly, we demonstrate that the dynamic boundary control design is suboptimal to a meaningful functional after some minor modifications of the dynamic boundary controls. In addition, we also consider dynamic boundary controls for the case of unknown dissipation or dispersion coefficients, and obtain corresponding results. Finally, three examples are used to demonstrate the effectiveness of the proposed control design.

Keywords: Korteweg–de Vries–Burgers equation, dynamic boundary control, uncertainty, globally exponential stabilization

Classification: 93Cxx, 93Dxx

1. INTRODUCTION

The KdVB is a partial differential equation (PDE) model that integrates dispersion, dissipation, and nonlinearity simultaneously. It is used to model physical phenomena, such as, nonlinear waves [1, 14], traffic flow [7, 23], and blood pressure fluctuations [12]. Many results of KdVB have been reported, such as [2, 3, 13, 15, 21, 22, 24, 25, 27].

The inverse optimal method is presented by Kalman and is introduced into robust nonlinear control by Freeman [17] based on the control Lyapunov function [28]. The inverse optimal method is to design a control law first and then prove that the control law has some optimality for a meaningful functional [6, 9, 11]. So it is called inverse optimal. The inverse optimal concept is of practical importance since it allows the design of optimal control laws, which may possess certain robustness margins, without the need to solve a Hamilton–Jacobi–Isaacs PDE that may not be possible to solve [20].

Inverse optimal control for linear systems with time-varying input delay and additive disturbances are presented in [4, 5]. For stochastic delayed-switched positive systems, intermittent static output feedback control is appeared in [16]. Inverse optimal control for strict-feedforward nonlinear systems with input delays is shown in [8]. Inverse optimal stabilization for Burgers' equation is solved in [18]. Inverse optimal control is presented

DOI: 10.14736/kyb-2024-6-0797

for a heat PDE equation with an unknown constant parameter in [19]. Regarding the KdVB equation, the boundary control design in [2]is powerful, we prove that it is inverse optimal to a meaningful functional in [10]. If the dispersion and dissipation of the KdVB equation are unknown, how can we design inverse optimal control for it? This will be a challenging problem, but from the authors' knowledge, it has not been published. Compared with the deterministic setting, the main difficulty lies in how to use the information of unknown dispersion and dissipation parameters to design boundary control to stabilize the uncertain KdVB system, and how to prove that it has some optimality for a meaningful functional.

In this paper, we consider inverse optimal control for the uncertain KdVB equation where dispersion and dissipation coefficients are unknown, but their bounds are known. The main contributions of this paper are as follows:

1. We design two dynamic parameters in the boundary controls, and update them by two update laws. When the dynamic boundary controls force the closed-loop system to converge to the equilibrium in $L^2(0,1)$-sense, they also force both dynamic parameters to converge.

2. We prove that the closed-loop system globally exponentially converges to the equilibrium in $L^2(0,1)$-sense under the proposed boundary controls.

3. We show that the proposed dynamic boundary controls are suboptimal to some meaningful functionals.

4. Compared to the exist work from simulation, it is clear that the convergence speed under the proposed boundary control is faster than that in it.

This paper is organized as follows: System description and some Lemmas are in section 2. Dynamic boundary control designs are in section 3. Inverse optimal control is in section 4, and simulation results are shown in section 5. Concluding remarks are in section 6.

Notation. For a scalar function $u(x,t) \in H^3([0,1] \times [0,\infty))$, we denote with $u_t(x,t) = \frac{\partial u(x,t)}{\partial t}$, $u_x(x,t) = \frac{\partial u(x,t)}{\partial x}$, $u_{xx}(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $u_{xxx}(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3}$, and $\|u(t)\|^2 = \int_0^1 u(x,t)^2 \,\mathrm{d}x$. For any $v(x) \in C^2[0,1]$, we denote $v' = \frac{dv}{dx}$, $v'' = \frac{d^2 v}{dx^2}$.

2. SYSTEM DESCRIPTION AND SOME LEMMAS

Consider the uncertain KdVB equation

$$u_t(x,t) = \varepsilon u_{xx}(x,t) - \delta u_{xxx}(x,t) - u_x(x,t)u(x,t), \tag{1}$$
$$u(x,0) = u_0(x), \tag{2}$$

where $t \geq 0$, $0 \leq x \leq 1$ and $u(x,t)$ is a real valued function representing the system state, $u_0(x)$ is its initial value, and $\varepsilon > 0$, $\delta > 0$ describe dissipation and dispersion coefficients, respectively, where ε or δ, or both of them are unknown.

The following Lemmas 2.1–2.2 are from [18].

Lemma 2.1. (Poincare's inequality) For any $u \in C^1[0,1]$, it holds

$$\int_0^1 u(x)^2 \, \mathrm{d}x \le 2u(0)^2 + 4 \int_0^1 u_x(x)^2 \, \mathrm{d}x, \tag{3}$$

$$\int_0^1 u(x)^2 \, \mathrm{d}x \le 2u(1)^2 + 4 \int_0^1 u_x(x)^2 \, \mathrm{d}x. \tag{4}$$

Lemma 2.2. (Young's inequality) For $a, b \ge 0, \lambda > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$, it holds

$$ab \le \frac{\lambda^p}{p} a^p + \frac{1}{q\lambda^q} b^q. \tag{5}$$

The following Lemmas 2.3–2.4 are from [26].

Lemma 2.3. Let $\beta < 0$. If $u(x,t) \in L_2(0,\infty)$, then it holds

$$\int_0^\infty e^{\beta(t-\tau)} u(1,\tau)^2 \, \mathrm{d}\tau \to 0, \tag{6}$$

as $t \to \infty$.

Lemma 2.4. Let $\beta < 0$. If $u(x,t) \in L_{2\alpha+2}(0,\infty)$, where α is a positive integer, then it holds

$$\int_0^\infty e^{\beta(t-\tau)} u(1,\tau)^{2\alpha+2} \, \mathrm{d}\tau \to 0, \tag{7}$$

as $t \to \infty$.

The definition of globally exponentially stable is from [28].

Definition 2.5. The origin of system $\dot{x}(t) = f(x(t)), x(t) \in R^n$ is said to be globally exponentially stable if there exist $k, \eta > 0$ such that, for any initial value $x(0) \in R^n$, the corresponding solution of system satisfies $|x(t)| \le k|x(0)|e^{-\eta t}$ for all $t \ge 0$.

3. DYNAMIC BOUNDARY CONTROL DESIGN FOR UNCERTAIN KDVB EQUATION

In this section, we consider dynamic boundary control for uncertain KdVB equation (1), (2), where $\varepsilon > 0$, $\delta > 0$ are unknown, but their lower bounds $\underline{\varepsilon} > 0$ and $\underline{\delta} > 0$ are known, dynamic boundary controllers are designed as

$$u(0,t) = 0, \tag{8}$$

$$u_x(1,t) = -u(1,t), \tag{9}$$

$$u_{xx}(1,t) = (\eta_1(t) + 1)u(1,t) + \eta_2(t)u(1,t)^3, \tag{10}$$

where $\eta_1(t), \eta_2(t) \in C^1[0,\infty)$. The problem of uncertain KdVB equation (1), (2), where $\varepsilon > 0$, $\delta > 0$ are unknown under the dynamic boundary controllers (8)–(10) can be described as an abstract initial problem.

Consider Hilbert spaces $X = L^2(0,1)$, $H = H^1(0,1)$, the operator $A : (D(A) \subset X) \to X^*$ given by

$$Au = -\varepsilon u_{xx} + \delta u_{xxx} + (0.5u^2)_x, \tag{11}$$

where the domain

$$D(A) = \{u \in H^3(0,1) | u(0) = 0, u^{'}(1) = -u(1), u^{''}(1) = (\eta_1 + 1)u(1) + \eta_2 u(1)^3\}.$$

The closed-loop system (1), (2), (8)–(10) can be written as

$$\frac{du}{dt} + Au = 0, u(0) = u_0. \tag{12}$$

If $u_0 \in D(A)$, then system (12) posses a unique solution from [2].

Theorem 3.1. For any initial value $u_0 \in D(A)$, consider KdVB equation (1), (2) where $\varepsilon > 0$, $\delta > 0$ are unknown, but their lower bounds $\underline{\varepsilon} > 0$ and $\underline{\delta} > 0$ are known, then dynamic boundary controllers (8)–(10) with the update laws

$$\dot{\eta}_1(t) = \gamma_1 u(1,t)^2, \tag{13}$$

$$\dot{\eta}_2(t) = \gamma_2 u(1,t)^4, \tag{14}$$

with $\gamma_1 > 0, \gamma_2 > 0$, and $\eta_1(0) \geq 0, \eta_2(0) \geq 0$, are such that the closed-loop system is globally exponential stable in the $L^2(0,1)$-sense.

P r o o f. Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2}\int_0^1 u(x,t)^2 \,\mathrm{d}x, \tag{15}$$

and computing its time derivative along trajectory of KdVB equation (1), (2), it holds

$$\begin{aligned} \dot{V}(t) = & \frac{\delta}{2}u_x(1,t)^2 + \varepsilon u(1,t)u_x(1,t) - \delta u_{xx}(1,t)u(1,t) \\ & - \frac{1}{3}u(1,t)^3 - u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\ & - \varepsilon \int_0^1 u_x(x,t)^2 \,\mathrm{d}x - \frac{\delta}{2}u_x(0,t)^2. \end{aligned} \tag{16}$$

Note that the update laws $\dot{\eta}_1(t)$, $\dot{\eta}_2(t)$ are given by (13), (14), respectively, and $\eta_1(0) \geq 0, \eta_2(0) \geq 0$, it implies $\eta_1(t) \geq 0$, $\eta_2(t) \geq 0$, for all $t \geq 0$. In addition, from Young's inequality, it holds

$$\frac{1}{3}u(1,t)^3 \leq \frac{1}{2}u(1,t)^2 + \frac{1}{18}u(1,t)^4. \tag{17}$$

Using Poincare's inequality, with $u(0,t) = 0$, it yields

$$-\int_0^1 u_x(x,t)^2 \,\mathrm{d}x \leq -\frac{1}{4}\int_0^1 u(x,t)^2 \,\mathrm{d}x \leq -\frac{1}{2}V(t). \tag{18}$$

Under boundary controls (8)–(10), with the help of (17), (18), from (16), we achieve

$$\begin{aligned}\dot{V}(t) &\le \frac{\delta}{2}u(1,t)^2 - \varepsilon u(1,t)^2 - \delta(\eta_1(t)+1)u(1,t)^2 \\ &\quad - \delta\eta_2(t)u(1,t)^4 + \frac{1}{2}u(1,t)^2 + \frac{1}{18}u(1,t)^4 - \varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x \\ &\le -\frac{\underline{\varepsilon}V(t)}{2} + (-\underline{\varepsilon} - \underline{\delta}\eta_1(t) - 0.5\underline{\delta} + 0.5)u(1,t)^2 \\ &\quad + \left(-\underline{\delta}\eta_2(t) + \frac{1}{18}\right)u(1,t)^4. \end{aligned} \tag{19}$$

Define the energy function as follows

$$S_1(t) = V(t) + \frac{1}{2\underline{\delta}\gamma_1}(\underline{\varepsilon} + \underline{\delta}\eta_1(t) + 0.5\underline{\delta} - 0.5)^2 + \frac{1}{2\underline{\delta}\gamma_2}\left(\underline{\delta}\eta_2(t) - \frac{1}{18}\right)^2, \tag{20}$$

where $\gamma_1 > 0, \gamma_2 > 0$ are given by (13), (14), respectively. Computing its time derivative along trajectory of KdVB equation (1), (2), we have

$$\dot{S}_1(t) = \dot{V}(t) + \frac{(\underline{\varepsilon} + \underline{\delta}\eta_1(t) + 0.5\underline{\delta} - 0.5)\dot{\eta}_1(t)}{\gamma_1} + \frac{(\underline{\delta}\eta_2(t) - \frac{1}{18})\dot{\eta}_2(t)}{\gamma_2}. \tag{21}$$

Using (19), and the update laws (13), (14), we get

$$\dot{S}_1(t) \le -\frac{\underline{\varepsilon}V(t)}{2}, \tag{22}$$

then $S_1(t) \le S_1(0)$, for $t > 0$. Thus $\eta_1(t), \eta_2(t)$ are bounded functions for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty) \bigcap L_4(0,+\infty)$. Using Gronwall's inequality, from (19), it can be deduced that

$$V(t) \le V(0)e^{-\frac{\underline{\varepsilon}t}{2}} + \int_0^t M_1 e^{-\frac{\underline{\varepsilon}(t-\tau)}{2}}(u(1,\tau)^2 + u(1,\tau)^4)\,\mathrm{d}\tau,$$

with $M_1 = \max\{\sup|\underline{\varepsilon} + \underline{\delta}\eta_1(t) + 0.5\underline{\delta} - 0.5|, \sup|\underline{\delta}\eta_2(t) - \frac{1}{18}|\}$. Thus $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3 and Lemma 2.4. We can conclude that $\int_0^1 u(x,t)^2\,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$ under boundary controllers (8)–(10) with the update laws (13), (14). The proof is completed. □

Remark 1. Dynamic boundary controllers (8)–(10) with the update laws (13), (14) aren't dependent on ε and δ, which are assumed to be unknown.

Remark 2. $\eta_1(t)$, $\eta_2(t)$ are monotonically increasing by the update laws (13), (14), further, $\eta_1(t), \eta_2(t)$ are bounded for $t \ge 0$ from (20), (22). Thus $\eta_1(t)$, $\eta_2(t)$ converge to a constant, respectively.

Remark 3. We design two dynamic parameters in the boundary controls (8)–(10), and update them by two update laws (13), (14). When the dynamic boundary controls force the closed-loop system to converge to the equilibrium in $L^2(0,1)$-sense, they also force the two dynamic parameters to converge separately.

When $\varepsilon > 0$ or $\delta > 0$ is known, the boundary control design will be simple, and only a parameter needs to be updated. The results are provided in the following Corollaries 3.2 and 3.3.

Corollary 3.2. For any initial value $u_0 \in D(A)$, consider KdVB equation (1), (2) where $\varepsilon > 0$ is unknown, but its lower bound $\underline{\varepsilon} > 0$ and $\delta > 0$ are known, then dynamic boundary controllers

$$u(0,t) = 0, \tag{23}$$

$$u_x(1,t) = -u(1,t), \tag{24}$$

$$u_{xx}(1,t) = (\eta_1(t)+1)u(1,t) + \frac{1}{18\delta}u(1,t)^3, \tag{25}$$

and the update law

$$\dot{\eta}_1(t) = \gamma_1 u(1,t)^2, \tag{26}$$

with $\gamma_1 > 0$, $\eta_1(0) \geq 0$, are such that the closed-loop system globally exponential stable in the $L^2(0,1)$-sense.

P r o o f. Consider the Lyapunov function candidate (15), and computing its time derivative along trajectory of KdVB equation (1), (2), it holds (16). Using boundary controls (23)–(25), and Young's inequality and Lemma 2.1, from (16), we have

$$\begin{aligned}\dot{V}(t) &\leq -\left(\delta\eta_1(t) - \frac{1}{2}\right)u(1,t)^2 - \varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x \\ &\leq -\frac{\underline{\varepsilon}V(t)}{2} - \left(\underline{\delta}\eta_1(t) - \frac{1}{2}\right)u(1,t)^2. \end{aligned}\tag{27}$$

Define the energy function as follows

$$S_2(t) = V(t) + \frac{\left(\underline{\delta}\eta_1(t) - \frac{1}{2}\right)^2}{2\underline{\delta}\gamma_1}, \tag{28}$$

where $\gamma_1 > 0$ is given by (26). Computing its time derivative along trajectory of KdVB equation (1), (2), and using (27), we have

$$\dot{S}_2(t) \leq -\frac{\underline{\varepsilon}V(t)}{2}, \tag{29}$$

then $S_2(t) \leq S_2(0)$, for $t > 0$. Thus $\eta_1(t)$ is bounded function for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty)$. Using Gronwall's inequality, from (27), it can be deduced that

$$V(t) \leq V(0)e^{-\frac{\underline{\varepsilon}t}{2}} + \int_0^t M_2 e^{-\frac{\underline{\varepsilon}(t-\tau)}{2}}u(1,\tau)^2\,\mathrm{d}\tau,$$

with $M_2 = \sup|\underline{\delta}\eta_1(t) - \frac{1}{2}|$. Thus $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3. We conclude that $\int_0^1 u(x,t)^2\,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$. The proof is completed. □

Remark 4. Dynamic boundary controllers (23)–(25) aren't dependent on dissipation coefficient ε, which is unknown.

Corollary 3.3. For any initial value $u_0 \in D(A)$, consider KdVB equation (1), (2) where $\delta > 0$ is unknown, but its lower bound $\underline{\delta} > 0$ and $\varepsilon > 0$ are known, then dynamic boundary controllers

$$u(0,t) = 0, \tag{30}$$

$$u_x(1,t) = -u(1,t), \tag{31}$$

$$u_{xx}(1,t) = (\eta_1(t)+1)u(1,t) + \frac{1}{18\underline{\delta}}u(1,t)^3, \tag{32}$$

and the update law

$$\dot{\eta}_1(t) = \gamma_1 u(1,t)^2, \tag{33}$$

with $\gamma_1 > 0$, $\eta_1(0) \geq 0$, are such that the closed-loop system globally exponential stable in the $L^2(0,1)$-sense.

Proof. Consider the Lyapunov function candidate (15), and computing its time derivative along trajectory of KdVB equation (1)–(2), it holds (16). Using boundary controls (30)–(32), and Young's inequality, and Lemma 2.1, from (16), we have

$$\begin{aligned}\dot{V}(t) \leq & -\left(\underline{\delta}\eta_1(t) + \frac{\delta}{2} - \frac{1}{2}\right)u(1,t)^2 - \left(\frac{1}{18}\frac{\delta}{\underline{\delta}} - 1\right)u(1,t)^4 \\ & - \varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x \\ \leq & -\frac{\varepsilon}{2}V(t) - (\underline{\delta}\eta_1(t) + \frac{\delta}{2} - \frac{1}{2})u(1,t)^2. \end{aligned}\tag{34}$$

Define the energy function as follows

$$S_3(t) = V(t) + \frac{1}{2\underline{\delta}\gamma_1}\left(\underline{\delta}\eta_1(t) + \frac{\delta}{2} - \frac{1}{2}\right)^2, \tag{35}$$

where $\gamma_1 > 0$ is given by (33). Computing its time derivative along trajectory of KdVB equation (1)–(2), we have

$$\dot{S}_3(t) = \dot{V}(t) + \frac{(\underline{\delta}\eta_1(t) + \frac{\delta}{2} - \frac{1}{2})\dot{\eta}_1(t)}{\gamma_1}. \tag{36}$$

Using (34), (36), and the update law (33), we get

$$\dot{S}_3(t) \leq -\frac{\varepsilon V(t)}{2}, \tag{37}$$

then $S_3(t) \le S_3(0)$, for $t > 0$. Thus $\eta_1(t)$ is bounded function for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty)$. Using Gronwall's inequality, from (34), it can be deduced that

$$V(t) \le V(0)e^{-\frac{\varepsilon t}{2}} + \int_0^t M_3 e^{-\frac{\varepsilon(t-\tau)}{2}} u(1,\tau)^2 \,\mathrm{d}\tau,$$

with $M_3 = \sup|\underline{\delta}\eta_1(t) + \frac{\delta}{2} - \frac{1}{2}|$. Thus $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3. We obtain that $\int_0^1 u(x,t)^2 \,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$. The proof is completed. □

4. INVERSE OPTIMAL DYNAMIC BOUNDARY CONTROL

In this section, we consider inverse optimal dynamic boundary control for system (1)–(2).

Theorem 4.1. For any initial value $u_0 \in D(A)$, consider KdVB equation (1)–(2) where $\varepsilon > 0$, $\delta > 0$ are unknown, but their lower bounds $\underline{\varepsilon} > 0$, $\underline{\delta} > 0$ are known, then dynamic boundary controllers

$$u(0,t) = 0, \tag{38}$$

$$u_x(1,t) = -2u(1,t), \tag{39}$$

$$u_{xx}(1,t) = 2\left(\eta_1(t) + 2\right) u(1,t) + 2\eta_2(t) u(1,t)^3, \tag{40}$$

and the update laws (13), (14) with $\gamma_1 > 0, \gamma_2 > 0$, and $\eta_1(0) \ge 0$, $\eta_2(0) \ge 0$, are such that cost functional

$$J_1 = \int_0^\infty (l_1(t) + \frac{\varepsilon}{2} u_x(1,t)^2 + u(0,t)^2) \,\mathrm{d}t, \tag{41}$$

with

$$l_1(t) = 2\underline{\varepsilon} \int_0^1 u_x(x,t)^2 \,\mathrm{d}x + \underline{\delta} u_x(0,t)^2 \tag{42}$$

is suboptimal.

Proof. It is clear $\eta_1(t) \ge 0, \eta_2(t) \ge 0$ for $t > 0$ under the update laws (13), (14) with $\gamma_1 > 0, \gamma_2 > 0$, and $\eta_1(0) \ge 0$, $\eta_2(0) \ge 0$. Consider the Lyapunov function candidate (15), and computing its time derivative along trajectory of KdVB equation (1), (2), it holds (16). Using boundary controls (38)–(40), and Young's inequality, and Lemma 2.1, we have

$$\begin{aligned}\dot{V}(t) &\le -(2\varepsilon + 2\delta\eta_1(t) + 2\delta - 0.5)u(1,t)^2 - \left(2\delta\eta_2(t) - \frac{1}{18}\right) u(1,t)^4 \\ &\quad - \varepsilon \int_0^1 u_x(x,t)^2 \,\mathrm{d}x - \frac{\delta}{2} u_x(0,t)^2 \\ &\le -\frac{\underline{\varepsilon} V(t)}{2} - 2(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)u(1,t)^2 - 2\left(\underline{\delta}\eta_2(t) - \frac{1}{18}\right) u(1,t)^4. \end{aligned} \tag{43}$$

Define the energy function as follows

$$\bar{S}_1(t) = V(t) + \frac{(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)^2}{\underline{\delta}\gamma_1} + \frac{(\underline{\delta}\eta_2(t) - \frac{1}{18})^2}{\underline{\delta}\gamma_2}, \tag{44}$$

where $\gamma_1 > 0, \gamma_2 > 0$ are given by (13), (14), respectively. Computing its time derivative along trajectory of KdVB equation (1), (2), and using (43), (13), (14), we have

$$\begin{aligned}\dot{\bar{S}}_1(t) &= \dot{V}(t) + \frac{2(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)\dot{\eta}_1(t)}{\gamma_1} + \frac{2(\underline{\delta}\eta_2(t) - \frac{1}{18})\dot{\eta}_2(t)}{\gamma_2}\\ &\le -\frac{\underline{\varepsilon}V(t)}{2},\end{aligned} \tag{45}$$

then $\bar{S}_1(t) \le \bar{S}_1(0)$ for $t > 0$. Thus $\eta_1(t)$ and $\eta_2(t)$ are bounded function for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty) \bigcap L_4(0,+\infty)$. Using Gronwall's inequality, from (43), it can be deduced that

$$V(t) \le V(0)e^{-\frac{\underline{\varepsilon} t}{2}} + \int_0^t \bar{M}_1 e^{-\frac{\underline{\varepsilon}(t-\tau)}{2}} (u(1,\tau)^2 + u(1,\tau)^4)\,\mathrm{d}\tau, \tag{46}$$

with $\bar{M}_1 = \max\{\sup|2\underline{\varepsilon} + 2\underline{\delta}\eta_1(t) - 1|, \sup|2\underline{\delta}\eta_2(t) - \frac{1}{9}|\}$. Then $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3 and Lemma 2.4. Thus boundary controls (38)–(40) and the update laws (13), (14), with $\gamma_1 > 0, \gamma_2 > 0$, $\eta_1(0) \ge 0$, $\eta_2(0) \ge 0$, are such that $\int_0^1 u(x,t)^2\,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$.

It is easy to know $l_1(t) \ge 0$. By (16), (43), (45), it holds

$$\begin{aligned}l_1(t) = {}& 2\underline{\varepsilon}\int_0^1 u_x(x,t)^2\,\mathrm{d}x + \underline{\delta}u_x(0,t)^2\\ &+ 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\\ &- \delta u_x(1,t)^2 - 2\varepsilon u(1,t)u_x(1,t) + 2\delta u_{xx}(1,t)u(1,t)\\ &+ \frac{2}{3}u(1,t)^3 + 2\varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x + \delta u_x(0,t)^2\\ &- \frac{2(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)\dot{\eta}_1(t)}{\gamma_1} - \frac{2(\underline{\delta}\eta_2(t) - \frac{1}{18})\dot{\eta}_2(t)}{\gamma_2}\\ &- 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\\ &+ \delta u_x(1,t)^2 + 2\varepsilon u(1,t)u_x(1,t) - 2\delta u_{xx}(1,t)u(1,t)\\ &- \frac{2}{3}u(1,t)^3 - 2\varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x - \delta u_x(0,t)^2\\ &+ \frac{2(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)\dot{\eta}_1(t)}{\gamma_1} + \frac{2(\underline{\delta}\eta_2(t) - \frac{1}{18})\dot{\eta}_2(t)}{\gamma_2}\\ ={}& -2\dot{\bar{S}}_1(t) + \delta u_x(1,t)^2 + 2\varepsilon u(1,t)u_x(1,t) - 2\delta u_{xx}(1,t)u(1,t)\end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{3}u(1,t)^3 - 2(\varepsilon - \underline{\varepsilon})\int_0^1 u_x(x,t)^2\,\mathrm{d}x - (\delta - \underline{\delta})u_x(0,t)^2 \\
&+ \frac{2(\underline{\varepsilon} + \underline{\delta}\eta_1(t) - 0.5)\dot{\eta}_1(t)}{\gamma_1} + \frac{2(\underline{\delta}\eta_2(t) - \frac{1}{18})\dot{\eta}_2(t)}{\gamma_2} \\
&- 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right).
\end{aligned} \tag{47}
$$

Using the update laws (13), (14), from (47), and Young's inequality, we get

$$
\begin{aligned}
l_1(t) \le\ & -2\dot{\bar{S}}_1(t) + 2\varepsilon u(1,t)u_x(1,t) + 2\underline{\varepsilon}u(1,t)^2 \\
&- 2\delta u_{xx}(1,t)u(1,t) - 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&+ 2\underline{\delta}\eta_1(t)u(1,t)^2 + 2\underline{\delta}\eta_2(t)u(1,t)^4 + \delta u_x(1,t)^2 \\
\le\ & -2\dot{\bar{S}}_1(t) + 2\varepsilon u(1,t)u_x(1,t) + 2\underline{\varepsilon}u(1,t)^2 \\
&+ \frac{\delta}{m(t)}\Bigg(m(t) - u_{xx}(1,t)u(1,t) - \frac{u(0,t)}{\delta} \\
&\times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\Bigg)^2,
\end{aligned} \tag{48}
$$

where $m(t) = 2\eta_1(t)u(1,t)^2 + 2\eta_2(t)u(1,t)^4 + u_x(1,t)^2$. From (41), using (48), we have

$$
\begin{aligned}
0 \le J_1 =\ & \int_0^\infty (l_1(t) + \frac{\varepsilon}{2}u_x(1,t)^2 + u(0,t)^2)\,\mathrm{d}t \\
\le\ & 2\bar{S}_1(0) + \frac{\varepsilon}{2}\int_0^\infty (u_x(1,t) + 2u(1,t))^2\,\mathrm{d}t \\
&+ \int_0^\infty \frac{\delta}{m(t)}\Bigg(\ m(t) - u_{xx}(1,t)u(1,t) \\
&- \frac{u(0,t)}{\delta}\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\Bigg)^2\,\mathrm{d}t \\
&+ \int_0^\infty u(0,t)^2\,\mathrm{d}t.
\end{aligned} \tag{49}
$$

From (49), it is clear that

$$
u(0,t) = 0, \tag{50}
$$

$$
u_x(1,t) = -2u(1,t), \tag{51}
$$

$$
\begin{aligned}
u_{xx}(1,t) =\ & \frac{m(t)}{u(1,t)} - \frac{u(0,t)}{\delta u(1,t)} \\
& \times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)
\end{aligned} \tag{52}
$$

are such that the functional $J_1 \le 2\bar{S}_1(0)$. Note that (50)– (52) is just (38)–(40). This completes the proof. □

Remark 5. After some minor modifications of the dynamic boundary control in Theorem 1, it is shown that boundary controls (38)–(40) with update laws (13), (14) globally exponentially stabilize system (1)–(2) in the $L^2(0,1)$-sense.

Remark 6. Boundary controls (38)–(40) with update laws (13), (14) are such that functional (41) is suboptimal, that is $J_1 \leq 2\bar{S}_1(0)$, since $l_1(t) = 2\underline{\varepsilon}\int_0^1 u_x(x,t)^2\,\mathrm{d}x + \underline{\delta}u_x(0,t)^2$, not $l_1(t) = 2\varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x + \delta u_x(0,t)^2$, where ε, δ are unknown, we can't have $J_1 = 2\bar{S}_1(0)$.

If $\varepsilon > 0$ is unknown, but its lower bound $\underline{\varepsilon} > 0$ and $\delta > 0$ are known, we have the following Corollary 4.2.

Corollary 4.2. For any initial value $u_0 \in D(A)$, consider KdVB equation (1), (2) where $\varepsilon > 0$ is unknown, but its lower bound $\underline{\varepsilon} > 0$ and $\delta > 0$ are known, then dynamic boundary controllers

$$u(0,t) = 0, \tag{53}$$

$$u_x(1,t) = -2u(1,t), \tag{54}$$

$$u_{xx}(1,t) = 2(\eta_1(t) + 2)u(1,t) + \frac{1}{9\delta}u(1,t)^3, \tag{55}$$

and update law (26) with $\gamma_1 > 0$, $\eta_1(0) \geq 0$, are such that cost functional

$$J_2 = \int_0^\infty (l_2(t) + \frac{\underline{\varepsilon}}{2}u_x(1,t)^2 + 2\underline{\varepsilon}u(1,t)^2 + u(0,t)^2)\,\mathrm{d}t, \tag{56}$$

with

$$l_2(t) = 2\underline{\varepsilon}\int_0^1 u_x(x,t)^2\,\mathrm{d}x + \delta u_x(0,t)^2, \tag{57}$$

is suboptimal.

P r o o f. It is clear $\eta_1(t) \geq 0$ for $t > 0$. Consider the Lyapunov function candidate (15), and computing its time derivative along trajectory of KdVB equation (1), (2), it holds (16). Using boundary controls (53)–(55), and Young's inequality, and Lemma 2.1, we have

$$\begin{aligned}\dot{V}(t) &\leq -2\varepsilon u(1,t)^2 - \frac{u(1,t)^4}{18} + \frac{u(1,t)^2}{2} - 2\delta u(1,t)^2 \\ &\quad - 2\delta\eta_1(t)u(1,t)^2 - \varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x - \frac{\delta}{2}u_x(0,t)^2 \\ &\leq -\frac{\underline{\varepsilon}V(t)}{2} - 2\left(\delta\eta_1(t) - \frac{1}{2}\right)u(1,t)^2. \end{aligned}\tag{58}$$

Define the energy function as follows

$$\bar{S}_2(t) = V(t) + \frac{\left(\delta\eta_1(t) - \frac{1}{2}\right)^2}{\delta\gamma_1}, \tag{59}$$

where $\gamma_1 > 0$ is given by (26). Computing its time derivative along trajectory of KdVB equation (1), (2), and using (27), we have

$$\dot{\bar{S}}_2(t) \leq -\frac{\underline{\varepsilon} V(t)}{2}, \tag{60}$$

then $\bar{S}_2(t) \leq \bar{S}_2(0)$, for $t > 0$. Thus $\eta_1(t)$ is bounded function for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty)$. Using Gronwall's inequality, from (58), it can be deduced that

$$V(t) \leq V(0)e^{-\frac{\underline{\varepsilon} t}{2}} + \int_0^t \bar{M}_2 e^{-\frac{\underline{\varepsilon}(t-\tau)}{2}} u(1,\tau)^2 \,\mathrm{d}\tau,$$

with $\bar{M}_2 = \sup|2\delta\eta_1(t) - 1|$. Thus $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3. We can conclude that $\int_0^1 u(x,t)^2 \,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$.

Next, from (16) and (57), we know

$$\begin{aligned}
l_2(t) &= 2\underline{\varepsilon} \int_0^1 u_x(x,t)^2 \,\mathrm{d}x + \delta u_x(0,t)^2 \\
&\quad + 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&\quad - \delta u_x(1,t)^2 - 2\varepsilon u(1,t)u_x(1,t) + 2\delta u_{xx}(1,t)u(1,t) \\
&\quad + \frac{2}{3}u(1,t)^3 + 2\varepsilon \int_0^1 u_x(x,t)^2 \,\mathrm{d}x - \frac{(2\delta\eta_1(t) - 1)\dot{\eta}_1(t)}{\gamma_1} \\
&\quad - 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&\quad + \delta u_x(1,t)^2 + 2\varepsilon u(1,t)u_x(1,t) - 2\delta u_{xx}(1,t)u(1,t) \\
&\quad - \frac{2}{3}u(1,t)^3 - 2\varepsilon \int_0^1 u_x(x,t)^2 \,\mathrm{d}x + \frac{(2\delta\eta_1(t) - 1)\dot{\eta}_1(t)}{\gamma_1} \\
&\leq -2\dot{\bar{S}}_2(t) + 2\varepsilon u(1,t)u_x(1,t) \\
&\quad + \frac{\delta}{m_1(t)}\Bigg(m_1(t) - u_{xx}(1,t)u(1,t) - \frac{u(0,t)}{\delta} \\
&\quad \times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\Bigg)^2,
\end{aligned} \tag{61}$$

where $m_1(t) = \frac{1}{9\delta}u(1,t)^4 + 2\delta\eta_1(t)u(1,t)^2 + u_x(1,t)^2$. From (56), using (61), we have

$$\begin{aligned}
0 \leq J_2 &= \int_0^\infty (l_2(t) + \frac{\underline{\varepsilon}}{2}u_x(1,t)^2 + 2\underline{\varepsilon} u(1,t)^2 + u(0,t)^2) \,\mathrm{d}t \\
&\leq 2\bar{S}_2(0) + \frac{\varepsilon}{2}\int_0^\infty (u_x(1,t) + 2u(1,t))^2 \,\mathrm{d}t + \int_0^\infty u(0,t)^2 \,\mathrm{d}t \\
&\quad + \delta \int_0^\infty \frac{1}{m_1(t)}\Bigg(m_1(t) - u_{xx}(1,t)u(1,t) - \frac{u(0,t)}{\delta} \\
&\quad \times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\Bigg)^2 \,\mathrm{d}t.
\end{aligned} \tag{62}$$

From (62), it is clear that

$$u(0,t) = 0, \tag{63}$$
$$u_x(1,t) = -2u(1,t), \tag{64}$$
$$u_{xx}(1,t) = \frac{m_1(t)}{u(1,t)} - \frac{u(0,t)}{\delta u(1,t)} \times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right), \tag{65}$$

are such that the functional $J_2 \le 2\bar{S}_2(0)$. Note that (63)–(65) is just (53)–(55). Therefore dynamic boundary controllers (53)–(55) and update law (26) with $\gamma_1 > 0$, $\eta_1(0) \ge 0$ are such that cost functional (56) is suboptimal. □

If $\delta > 0$ is unknown, but its lower bound $\underline{\delta} > 0$ and $\varepsilon > 0$ are known, we have the following Corollary 4.3.

Corollary 4.3. For any initial value $u_0 \in D(A)$, consider KdVB equation (1), (2) where $\delta > 0$ is unknown, but its lower bound $\underline{\delta} > 0$ and $\varepsilon > 0$ are known, then dynamic boundary controllers

$$u(0,t) = 0, \tag{66}$$
$$u_x(1,t) = -2u(1,t), \tag{67}$$
$$u_{xx}(1,t) = 2(\eta_1(t) + 2)u(1,t) + \frac{1}{9\underline{\delta}}u(1,t)^3, \tag{68}$$

and update law (33) with $\gamma_1 > 0$, $\eta_1(0) \ge 0$, are such that the cost functional

$$J_3 = \int_0^\infty (l_3(t) + \frac{\varepsilon}{2}u_x(1,t)^2 + 2\varepsilon u(1,t)^2 + u(0,t)^2)\,\mathrm{d}t, \tag{69}$$

with

$$l_3(t) = 2\varepsilon \int_0^1 u_x(x,t)^2\,\mathrm{d}x + \underline{\delta} u_x(0,t)^2, \tag{70}$$

is suboptimal.

Proof. It is clear $\eta_1(t) \ge 0$, $t > 0$. Consider the Lyapunov function candidate (15), it holds (16). Using boundary controls (66)–(68), and Young's inequality, and Lemma 2.1, we have

$$\begin{aligned}\dot{V}(t) &\le -2\varepsilon u(1,t)^2 - \frac{1}{9}\left(\frac{\delta}{\underline{\delta}} - \frac{1}{2}\right)u(1,t)^4 + \frac{u(1,t)^2}{2} - 2\delta u(1,t)^2 \\ &\quad - 2\delta\eta_1(t)u(1,t)^2 - \varepsilon\int_0^1 u_x(x,t)^2\,\mathrm{d}x - \frac{\delta}{2}u_x(0,t)^2 \\ &\le -\frac{\varepsilon V(t)}{2} - 2\left(\underline{\delta}\eta_1(t) - \frac{1}{2}\right)u(1,t)^2. \end{aligned}\tag{71}$$

Define the energy function as follows

$$\bar{S}_3(t) = V(t) + \frac{\left(\underline{\delta}\eta_1(t) - \frac{1}{2}\right)^2}{\underline{\delta}\gamma_1}, \tag{72}$$

where $\gamma_1 > 0$ is given by (33). Computing its time derivative along trajectory of KdVB equation (1), (2), and using (71), (72), we have

$$\dot{\bar{S}}_3(t) \leq -\frac{\varepsilon V(t)}{2}, \tag{73}$$

then $\bar{S}_3(t) \leq \bar{S}_3(0)$, for $t > 0$. Thus $\eta_1(t)$ is bounded function for $t > 0$, it yields $u(1,t) \in L_2(0,+\infty)$. Using Gronwall's inequality, from (71), it can be deduced that

$$V(t) \leq V(0)e^{-\frac{\varepsilon t}{2}} + \int_0^t \bar{M}_3 e^{-\frac{\varepsilon(t-\tau)}{2}} u(1,\tau)^2 \,\mathrm{d}\tau,$$

with $\bar{M}_3 = \sup|2\underline{\delta}\eta_1(t) - 1|$. Thus $V(t)$ converges to zero as $t \to \infty$ by Lemma 2.3. We have that $\int_0^1 u(x,t)^2 \,\mathrm{d}x$ globally exponentially converges to zero as $t \to \infty$.

Next, from (16) and (71), we know

$$\begin{aligned}
l_3(t) &= 2\varepsilon \int_0^1 u_x(x,t)^2 \,\mathrm{d}x + \underline{\delta} u_x(0,t)^2 \\
&\quad + 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&\quad - \delta u_x(1,t)^2 - 2\varepsilon u(1,t)u_x(1,t) + 2\delta u_{xx}(1,t)u(1,t) \\
&\quad + \frac{2}{3}u(1,t)^3 + \delta u_x(0,t)^2 - \frac{(2\underline{\delta}\eta_1(t) - 1)\dot{\eta}_1(t)}{\gamma_1} \\
&\quad - 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&\quad + \delta u_x(1,t)^2 + 2\varepsilon u(1,t)u_x(1,t) - 2\delta u_{xx}(1,t)u(1,t) \\
&\quad - \frac{2}{3}u(1,t)^3 - \delta u_x(0,t)^2 + \frac{(2\underline{\delta}\eta_1(t) - 1)\dot{\eta}_1(t)}{\gamma_1} \\
&\leq -2\dot{\bar{S}}_3(t) + \delta u_x(1,t)^2 + 2\varepsilon u(1,t)u_x(1,t) - 2\delta u_{xx}(1,t)u(1,t) \\
&\quad + \frac{1}{9}u(1,t)^4 + 2\underline{\delta}\eta_1(t)u(1,t)^2 \\
&\quad - 2u(0,t)\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right) \\
&\leq -2\dot{\bar{S}}_3(t) + 2\varepsilon u(1,t)u_x(1,t) \\
&\quad + \frac{\delta}{m_2(t)}\Bigg(m_2(t) - u_{xx}(1,t)u(1,t) - \frac{u(0,t)}{\delta} \\
&\quad \times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\Bigg)^2,
\end{aligned} \tag{74}$$

where $m_2(t) = 2\eta_1(t)u(1,t)^2 + \frac{1}{9\delta}u(1,t)^4 + u_x(1,t)^2$.

From (69), using (74), we have

$$\begin{aligned} 0 \le J_3 = &\int_0^\infty (l_3(t) + \frac{\varepsilon}{2}u_x(1,t)^2 + 2\varepsilon u(1,t)^2 + u(0,t)^2)\,\mathrm{d}t \\ &\le 2\bar{S}_3(0) + \frac{\varepsilon}{2}\int_0^\infty (u_x(1,t) + 2u(1,t))^2\,\mathrm{d}t + \int_0^\infty u(0,t)^2\,\mathrm{d}t \\ &+ \delta\int_0^\infty \frac{1}{m_2(t)}\left(m_2(t) - u_{xx}(1,t)u(1,t) - \frac{u(0,t)}{\delta}\right. \\ &\times \left.\left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right)\right)^2\,\mathrm{d}t. \end{aligned} \tag{75}$$

From (75), it is clear that

$$u(0,t) = 0, \tag{76}$$

$$u_x(1,t) = -2u(1,t), \tag{77}$$

$$\begin{aligned} u_{xx}(1,t) = &\frac{m_2(t)}{u(1,t)} - \frac{u(0,t)}{\delta u(1,t)} \\ &\times \left(\varepsilon u_x(0,t) - \delta u_{xx}(0,t) - \frac{1}{3}u(0,t)^2\right), \end{aligned} \tag{78}$$

are such that the functional $J_3 \le 2\bar{S}_3(0)$. Note that (76)–(78) is just (66)–(68). Therefore dynamic boundary controllers (66)–(68) and update law (33) with $\gamma_1 > 0$, $\eta_1(0) \ge 0$ are such that cost functional (69) is suboptimal. □

5. SIMULATION RESULTS

Example 1. Consider the uncertain KdVB equation in [21] as follows

$$u_t(x,t) = \left(1.5 + \frac{1}{3}\sin(3\pi x)e^{\frac{-t}{2}}\right)u_{xx}(x,t) - 2.7u_{xxx}(x,t) - u_x(x,t)u(x,t), \tag{79}$$

$$u(x,0) = 1.1 - 0.3\cos(\pi x), \tag{80}$$

where $1.5 + \frac{1}{3}\sin(3\pi x)e^{\frac{-t}{2}}$ is uncertain, but its lower bound is $\frac{7}{6}$. Using Corollary 4.2, dynamic boundary controls are designed as

$$u(0,t) = 0, \tag{81}$$

$$u_x(1,t) = -2u(1,t), \tag{82}$$

$$u_{xx}(1,t) = \frac{1}{9}u(1,t)^3 + 2(\eta_1(t) + 2)u(1,t), \tag{83}$$

with the update law

$$\dot{\eta}_1(t) = 0.1u(1,t)^2, \tag{84}$$

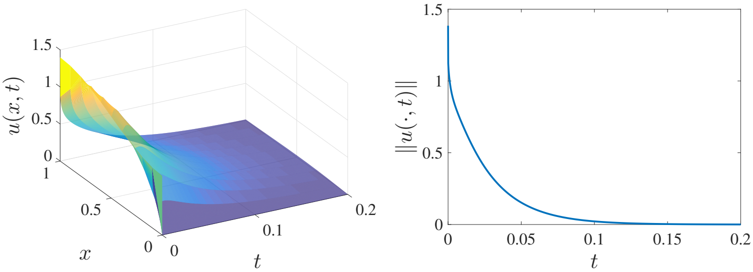

Fig. 1. State $u(x,t)$ and its norm $\|u(\cdot,t)\|$ under boundary controls (81)–(83), with the update law (84).

with $\eta_1(0) \geq 0$.

Responses of the PDE state $u(x,t)$ with the state norm $\|u(\cdot,t)\|$ under boundary controls (81)–(83), and the update law (84) with $\eta_1(0) = 0.1$ are in Figure 1.

In [21], for system (79), (80), boundary controls are designed as

$$u(0,t) = 0, \tag{85}$$
$$u_x(1,t) = 0, \tag{86}$$
$$u_{xx}(1,t) = U(t), \tag{87}$$

where

$$U(t) = \begin{cases} -0.7u^2(1,t) + \frac{1.1}{u(1,t)}(\int_0^1 u^2(x,t)\,\mathrm{d}x)^{0.006}, & u(1,t) \neq 0, \\ 0, & u(1,t) = 0. \end{cases}$$

Responses of the state and its norm $\|u(\cdot,t)\|$ of the closed-loop system (79), (80) with (85)–(87) are indicated in Figure 2.

Compared to Figure 1 and Figure 2, one can see that boundary controls (81)–(83), and the update law (84) with $\eta_1(0) = 0.1$ are such that the closed-loop system converges to zero without exceeding $t = 0.15$. From Figure 2, boundary controls (85)–(87) robust finite time stabilize system (79), (80). The settling time is $t^* \leq T = 0.2341$. It is clear that the convergence speed under the proposed boundary control is faster than that in [21].

Example 2. Consider the uncertain KdVB equation

$$u_t(x,t) = \varepsilon u_{xx}(x,t) - \delta u_{xxx}(x,t) - u_x(x,t)u(x,t), \tag{88}$$
$$u(x,0) = \sin(1.49\pi x) + \cos(1.51\pi x), \tag{89}$$

where ε and δ are unknown, but their lower bounds $\underline{\varepsilon} > 0$ and $\underline{\delta} > 0$ are known. Using Theorem 4.1, dynamic boundary controls are designed as

$$u(0,t) = 0, \tag{90}$$
$$u_x(1,t) = -2u(1,t), \tag{91}$$

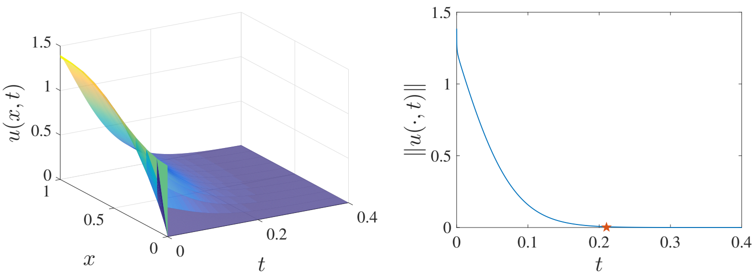

Fig. 2. State $u(x,t)$ and its norm $\|u(\cdot,t)\|$ under boundary controls (85)–(87).

$$u_{xx}(1,t) = 2\left(\eta_1(t) + 2\right) u(1,t) + 2\eta_2(t) u(1,t)^3, \tag{92}$$

and the update laws

$$\dot{\eta}_1(t) = 0.1u(1,t)^2, \tag{93}$$
$$\dot{\eta}_2(t) = 0.1u(1,t)^4, \tag{94}$$

with $\eta_1(0) \geq 0$, $\eta_2(0) \geq 0$. A simulation study is performed with $\underline{\varepsilon} = 1.5$, $\underline{\delta} = 1$, responses of the PDE state $u(x,t)$ together with the state norm $\|u(\cdot,t)\|$ under boundary controls (90)–(92), and the update laws (93), (94) with $\eta_1(0) = 0.1, \eta_2(0) = 0.2$ are in Figure 3. Boundary controls (91), (92) are in Figure 4, and the update laws (93), (94) are in Figure 5. From Figure 5, the update laws $\eta_1(t), \eta_2(t)$ are increasing and tend towards a constant, respectively. Further, boundary controls (90)–(92), and the update laws (93), (94) are such that the cost functional

$$J_1 = \int_0^\infty (l_1(t) + \frac{1.5}{2} u_x(1,t)^2 + u(0,t)^2)\, \mathrm{d}t, \tag{95}$$

with

$$l_1(t) = 3\int_0^1 u_x(x,t)^2\, \mathrm{d}x + u_x(0,t)^2, \tag{96}$$

is suboptimal.

Example 3. Consider the uncertain KdVB equation

$$u_t(x,t) = 1.5u_{xx}(x,t) - \delta u_{xxx}(x,t) - u_x(x,t)u(x,t), \tag{97}$$
$$u(x,0) = \cos(1.5\pi x), \tag{98}$$

where δ is unknown, but its lower bound $\underline{\delta} > 0$ is known. Using Corollary 4.3, dynamic boundary controls are designed as

$$u(0,t) = 0, \tag{99}$$

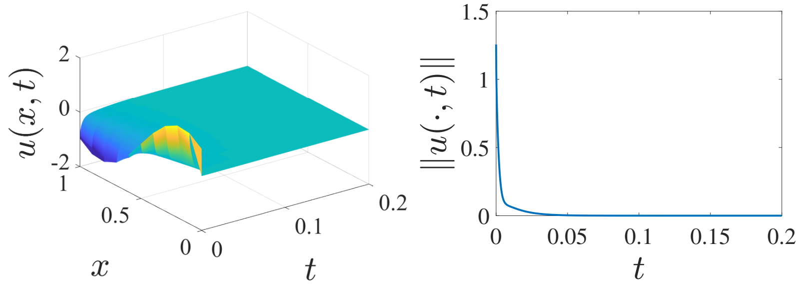

Fig. 3. State $u(x,t)$ and its norm $\|u(\cdot,t)\|$ under boundary controls (90)–(92), and the update laws (93), (94).

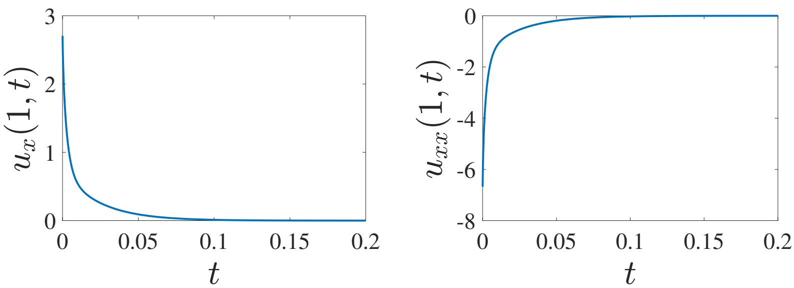

Fig. 4. Boundary controls (91), (92).

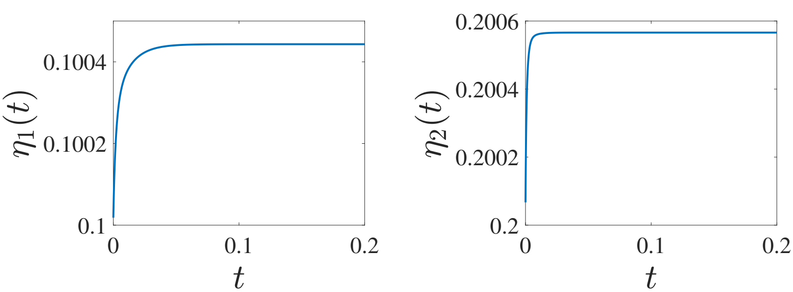

Fig. 5. Response of the update laws (93), (94).

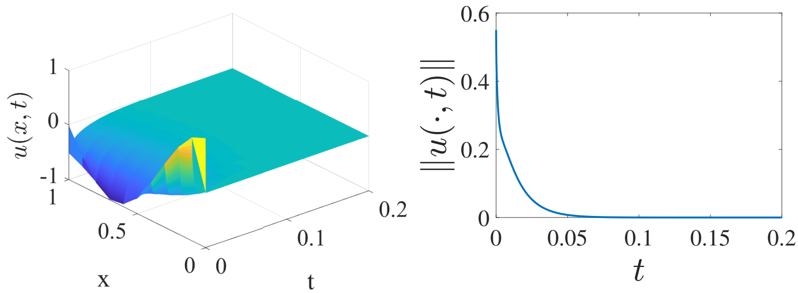

Fig. 6. State $u(x,t)$ and its norm $\|u(\cdot,t)\|$ under boundary controls (99)–(101), with the update law (102).

$$u_x(1,t) = -2u(1,t), \tag{100}$$

$$u_{xx}(1,t) = 2(\eta_1(t)+2)u(1,t) + \frac{1}{9\underline{\delta}}u(1,t)^3, \tag{101}$$

and the update law

$$\dot{\eta}_1(t) = 0.1u(1,t)^2, \tag{102}$$

with $\eta_1(0) \geq 0$.

A simulation study is given with $\underline{\delta} = 1.2$, responses of the PDE state $u(x,t)$ together with the state norm $\|u(\cdot,t)\|$ under boundary controls (99)–(101), and the update law (102) with $\eta_1(0) = 0.2$ are in Figure 6. Boundary controls (100), (101) and the update law (102) are in Figure 7. From Figure 7, the update law $\eta_1(t)$ is increasing and tends towards a constant. Further, boundary controls (99)–(101) and the update law (102) are such that the cost functional

$$J_2 = \int_0^\infty (l_3(t) + \frac{1.5u_x(1,t)^2}{2} + 3u(1,t)^2 + u(0,t)^2)\,\mathrm{d}t, \tag{103}$$

with

$$l_2(t) = 3\int_0^1 u_x(x,t)^2\,\mathrm{d}x + 1.2u_x(0,t)^2, \tag{104}$$

is suboptimal.

6. CONCLUSIONS

We have considered the KdVB equation where dissipation and dispersion coefficients are unknown, but their lower bounds are known. First, dynamic boundary controls and update laws have been achieved to globally exponentially stabilize this kind of system. Next, inverse optimal control for the uncertain KdVB equation has been explored. After some minor modifications, it has been shown that the dynamic boundary control is

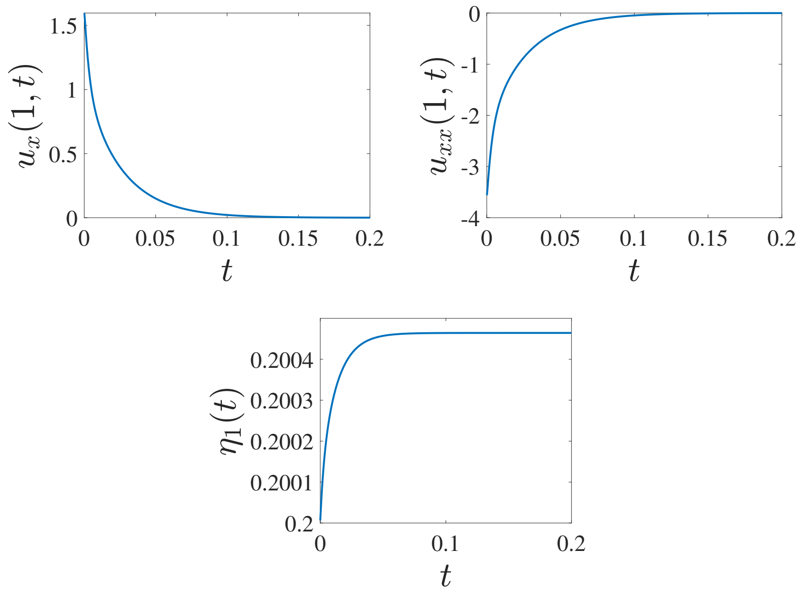

Fig. 7. Boundary controls (100), (101) and update law (102).

suboptimal to a meaningful functional. In addition, dynamic boundary control has also been studied in the case of unknown dissipation or dispersion coefficient. Finally, the validity of the proposed boundary controls have been illustrated by three examples. In future, we will explore dynamic boundary control design of uncertain KdVB equations with time delays and disturbances.

ACKNOWLEDGMENTS

The authors thank the National Natural Science Foundation of China (grant nos. 62173131). The authors thank the anonymous reviewers for providing valuable suggestions to improve the paper.

(Received November 11, 2023)

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