

Abderrahman Hermas; Abdellah Mamouni; Lahcen Oukhtite  
Generalized derivations with power values on rings and Banach algebras

*Mathematica Bohemica*, Vol. 149 (2024), No. 4, 491–502

Persistent URL: <http://dml.cz/dmlcz/152676>

## Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

GENERALIZED DERIVATIONS WITH POWER VALUES  
ON RINGS AND BANACH ALGEBRAS

ABDERRAHMAN HERMAS, Fez, ABDELLAH MAMOUNI, Meknes,  
LAHCEN OUKHTITE, Fez

Received May 23, 2023. Published online February 21, 2024.  
Communicated by Simion Breaz

*Abstract.* Let  $R$  be a prime ring and  $I$  a nonzero ideal of  $R$ . The purpose of this paper is to classify generalized derivations of  $R$  satisfying some algebraic identities with power values on  $I$ . More precisely, we consider two generalized derivations  $F$  and  $H$  of  $R$  satisfying one of the following identities:

- (1)  $aF(x)^m H(y)^m = x^n y^n$  for all  $x, y \in I$ ,
- (2)  $(F(x) \circ H(y))^m = (x \circ y)^n$  for all  $x, y \in I$ ,

for two fixed positive integers  $m \geq 1$ ,  $n \geq 1$  and  $a$  an element of the extended centroid of  $R$ . Finally, as an application, the same identities are studied locally on nonvoid open subsets of a prime Banach algebra.

*Keywords:* prime ring; generalized derivation; Banach algebra; Jacobson radical

*MSC 2020:* 16N60, 46J10, 16W25

## 1. INTRODUCTION

Let  $R$  be a ring with center  $Z(R)$ . Recall that  $R$  is a prime if  $xRy = 0$  implies  $x = 0$  or  $y = 0$ . For any  $x, y \in R$  we write  $[x, y] = xy - yx$  and  $x \circ y = xy + yx$  for the Lie product and Jordan product, respectively. An additive mapping  $d: R \rightarrow R$  is a *derivation* if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . An additive mapping  $F: R \rightarrow R$  is a *generalized derivation* associated to a derivation  $d$  if  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in R$ . A ring  $R$  is called primitive if it has a faithful simple module. An ideal  $P$  of a ring  $R$  is said to be a primitive ideal if  $P$  is the annihilator of a simple  $R$ -module. The Jacobson radical of a ring  $R$ , denoted by  $\text{rad}(R)$ , is the intersection of all primitive ideals of  $R$ . If  $R$  has no primitive ideals (i.e.,  $R$  has no simple modules), then we define  $\text{rad}(R) = R$ . A Banach algebra is a normed algebra whose underlying

vector space is a Banach space. The closure of a subset  $X$  of a Banach algebra  $\mathcal{A}$ , denoted by  $\overline{X}$ , is the intersection of all closed subsets of  $\mathcal{A}$  containing  $X$ . The interior of a subset  $X$  of a Banach algebra  $\mathcal{A}$ , denoted by  $\overset{\circ}{X}$ , is the largest open set contained in  $X$ . Equivalently,  $\overset{\circ}{X}$  is the union of all open subsets of  $\mathcal{A}$  contained in  $X$ .

During the past few decades, there has been an ongoing interest concerning the relationship between the structure of a prime (semi-prime) ring  $R$  and the behavior of generalized derivations of  $R$  satisfying some specific algebraic identities on an appropriate subset of  $R$ . Motivated by various results in this direction, our aim in this paper is to describe generalized derivations satisfying certain functional identities on a nonzero ideal of a prime ring. Moreover, as an application of our results, we investigate continuous generalized derivations satisfying similar algebraic identities locally on open subsets of a prime Banach algebra.

## 2. FUNCTIONAL IDENTITIES ON PRIME RINGS

The main purpose of this section is to prove the following theorems.

**Theorem 2.1.** *Let  $R$  be a prime ring of characteristic different from 2,  $Q_r$  its right Martindale quotient ring,  $C$  its extended centroid,  $I$  a nonzero ideal of  $R$ ,  $a \in C$ ,  $F$  and  $H$  are generalized derivations of  $R$  associated with derivations  $d$  and  $h$ , respectively, such that*

$$aF(x)^m H(y)^m = x^n y^n \quad \forall x, y \in I$$

*for two fixed positive integers  $m \geq 1$  and  $n \geq 1$ . Then  $F(x) = \alpha x$ ,  $H(x) = \beta x$ , for some  $\alpha, \beta \in C$  and  $a(\alpha\beta)^m = 1$ . Moreover, if  $m \neq n$ , then  $m + n$  is even and  $\text{char}(R) = 2^{|m-n|} - 1$ .*

**Theorem 2.2.** *Let  $R$  be a prime ring of characteristic different from 2,  $Q_r$  its right Martindale quotient ring,  $C$  its extended centroid,  $I$  a nonzero ideal of  $R$ ,  $F$  and  $H$  are generalized derivations of  $R$  associated with derivations  $d$  and  $h$ , respectively, such that*

$$(F(x) \circ H(y))^m = (x \circ y)^n \quad \forall x, y \in I$$

*for two fixed positive integers  $m \geq 1$  and  $n \geq 1$ . Then  $F(x) = \alpha x$ ,  $H(x) = \beta x$ , for some  $\alpha, \beta \in C$  and  $(\alpha\beta)^m = 1$ . Moreover, if  $m \neq n$ , then  $m + n$  is even and  $\text{char}(R) = 2^{|m-n|} - 1$ .*

**Proof of Theorem 2.1.** One can suppose that  $a$ ,  $F$  and  $H$  are nonzero, otherwise, the main identity reduces to  $x^n y^n = 0$  for all  $x, y \in I$ . Substituting  $y$  by  $x$ , we get  $x^{2n} = 0$  for all  $x \in I$ . Using [4], Lemma 1.1,  $R$  has a nonzero nilpotent ideal, which contradicts the primeness of  $R$ .

Since  $I$  and  $Q_r$  satisfy the same differential identities (see [8], Theorem 2) we may assume that

$$(2.1) \quad aF(x)^m H(y)^m = x^n y^n \quad \forall x, y \in Q_r.$$

Using [9], Theorem 3, there exist  $\alpha, \beta \in Q_r$  such that  $F(x) = \alpha x + d(x)$  and  $H(x) = \beta x + h(x)$ . Hence, equation (2.1) becomes

$$(2.2) \quad a(\alpha x + d(x))^m (\beta y + h(y))^m = x^n y^n \quad \forall x, y \in Q_r.$$

*Case 1:* If  $d$  and  $h$  are both  $Q_r$ -inner, then there exist  $q_1, q_2 \in Q_r$  such that  $d(x) = [q_1, x]$ ,  $h(x) = [q_2, x]$  for all  $x \in Q_r$ , thus

$$P(x, y) = a(\alpha x + [q_1, x])^m (\beta y + [q_2, y])^m - x^n y^n = 0 \quad \forall x, y \in Q_r.$$

In view of [3], Theorem 2.5 and Theorem 3.5, we know that both  $Q_r$  and  $Q_r \otimes_C \overline{C}$  are centrally closed, where  $\overline{C}$  is the algebraic closure of  $C$ . We may replace  $Q_r$  by itself or  $Q_r \otimes_C \overline{C}$  according whether  $C$  is finite or infinite. Therefore we may assume that  $Q_r$  is centrally closed over  $C$ , which is either finite or algebraically closed. By Martindale's theorem (see [10]),  $Q_r$  is a primitive ring having a nonzero socle  $H$  with  $C$  the associated division ring. In light of Jacobson's theorem (see [5], page 75),  $Q_r$  is isomorphic to a dense ring of linear transformations on a vector space  $V$  over  $C$ .

If  $\dim_C V = k$ , then the density of  $Q_r$  gives  $Q_r \cong M_k(C)$ .

Assume that  $\dim_C V \geq 2$ . We want to show that  $\{u, q_1 u\}$  are linearly  $C$ -dependent for all  $u \in V$ . Indeed, suppose that  $u$  and  $q_1 u$  are linearly  $C$ -independent.

If  $q_2 u \notin \text{Span}_C \{u, q_1 u\}$ , then  $\{u, q_1 u, q_2 u\}$  is  $C$ -independent, invoking [2], Definition 5.11. There exist  $f, g \in Q_r$  such that  $f u = 0$ ,  $f q_1 u = -u$ ,  $f q_2 u = u$ ,  $g u = 0$ ,  $g q_1 u = u$ ,  $g q_2 u = -u$ ,

$$(2.3) \quad P(f, g)u = (a(\alpha f + [q_1, f])^m (\beta g + [q_2, g])^m - f^n g^n)u = 0.$$

It is obvious that  $(\alpha f + [q_1, f])^m u = u$ ,  $(\beta g + [q_2, g])^m u = u$  and  $f^n g^n u = 0$ . Therefore  $P(f, g)u = a u = 0$  for all  $u \in V$ , a contradiction.

Let now  $q_2 u \in \text{Span}_C \{u, q_1 u\}$ . Then  $q_2 u = \lambda u + \mu q_1 u$  for some  $\lambda, \mu \in C$ , hence  $g q_2 u = \lambda g u + \mu g q_1 u = \mu u$ , so  $(\beta g + [q_2, g])^m u = \mu^m u$ , consequently  $P(f, g)u = a \mu^m u = 0$  for all  $u \in V$ , which is absurd.

Then in all cases,  $\{u, q_1 u\}$  are linearly  $C$ -dependent for all  $u \in V$ , that is,  $q_1 u = \lambda_u u$  for some  $\lambda_u \in C$ . Obviously, for any  $v \in V$  such that  $\{u, v\}$  are linearly  $C$ -independent, we have  $q_1(u - v) = \lambda_u u - \lambda_v v = \lambda_{u-v}(u - v)$ , then  $(\lambda_u - \lambda_{u-v})u - (\lambda_v - \lambda_{u-v})v = 0$ , hence  $\lambda_u = \lambda_{u-v} = \lambda_v$ , finally  $q_1 u = \lambda u$  for all  $u \in V$ .

On the other hand, for  $r \in R$  and  $u \in V$  we get

$$(rq_1)u = r(q_1u) = r\lambda u = \lambda(ru) = q_1(ru) = (q_1r)u,$$

then  $[R, q_1]V = 0$ , thus  $q_1 \in C$ . Similarly, we prove that  $q_2 \in C$ . The main equation becomes

$$(2.4) \quad Q(x, y) = a(\alpha x)^m(\beta y)^m - x^n y^n = 0 \quad \forall x, y \in Q_r.$$

Now we aim to prove that  $\{w, \alpha w\}$  are linearly  $C$ -dependent for all  $w \in V$ , indeed, suppose that  $w$  and  $\alpha w$  are linearly  $C$ -independent.

If  $\beta w \notin \text{Span}_C\{w, \alpha w\}$ , then  $\{w, \alpha w, \beta w\}$  are  $C$ -independent,  $Q_r$  being a dense ring of linear transformation of  $V$ . It follows that there exist  $f, g \in Q_r$  such that  $fw = 0$ ,  $f\alpha w = w$ ,  $f\beta w = w$ ,  $gw = w$ ,  $g\alpha w = 0$ ,  $g\beta w = w$ ,

$$(2.5) \quad Q(f, g)w = (a(\alpha f)^m(\beta g)^m - f^n g^n)w = 0.$$

Firstly

$$\begin{aligned} (a(\alpha f)^m(\beta g)^m - f^n g^n)w &= (a(\alpha f)^m(\beta g)^{m-1}(\beta gw) - f^n g^{n-1}(gw)) \\ &= (a(\alpha f)^{n-1}(\alpha f)\beta w - f^{n-1}(fw)) = a\alpha w. \end{aligned}$$

Using relation (2.5), we get  $Q(f, g)w = a\alpha w = 0$  for all  $w \in V$ , a contradiction. Now, if  $\beta w \in \text{Span}_C\{w, \alpha w\}$ , then  $\beta w = \lambda_1 w + \lambda_2 \alpha w$  for some  $\lambda_1, \lambda_2 \in C$ . It follows that  $g\beta w = \lambda_1 w$  and  $f\beta w = \lambda_2 w$ , thus

$$\begin{aligned} Q(f, g)w &= (a(\alpha f)^m(\beta g)^{m-1}(\beta gw) - f^n g^{n-1}(gw)) = (a(\alpha f)^m(\beta g)^{m-2}\beta\lambda_1 w) \\ &= (\lambda_1)^{m-1}(a(\alpha f)^m\beta w) = (\lambda_1)^{m-1}(a(\alpha f)^{m-1}\alpha(f\beta w)) \\ &= (\lambda_1)^{m-1}\lambda_2(a(\alpha f)^{m-1}\alpha w) = (\lambda_1)^{m-1}\lambda_2 a\alpha w. \end{aligned}$$

Using relation (2.5), we get  $Q(f, g)w = (\lambda_1)^{m-1}\lambda_2 a\alpha w = 0$  for all  $w \in V$ , which is absurd. Then in all cases,  $\{w, \alpha w\}$  are linearly  $C$ -dependent for all  $w \in V$ , thus  $\alpha w = \gamma_w w$  for all  $w \in V$  and for some  $\gamma_w \in C$ . It is straightforward that  $\alpha w = \gamma w$ , thus  $[R, \alpha]V = 0$  and  $\alpha \in C$ . Analogously, we prove that  $\beta \in C$ . Then the main equation reduces to

$$(2.6) \quad a(\alpha\beta)^m x^m y^m - x^n y^n = 0 \quad \forall x, y \in I.$$

If  $m = n$ , then  $a(\alpha\beta)^m = 1$  directly follows.

On the other hand, if  $m \neq n$ , invoking [7], Lemma 1,  $I \subseteq M_s(K)$  for a field  $K$  and an integer  $s > 1$ , then  $M_s(K)$  satisfies

$$(2.7) \quad a(\alpha\beta)^m x^m y^m - x^n y^n = 0.$$

Taking  $e_{ii}$  instead of  $x$  and  $y$  in relation (2.7) for a fixed positive integer  $i \leq s$ , we get  $(a(\alpha\beta)^m - 1)e_{ii} = 0$ , then  $a(\alpha\beta)^m = 1$ . Now equation (2.7) becomes

$$(2.8) \quad x^m y^m - x^n y^n = 0 \quad \forall x, y \in M_s(K).$$

Suppose that  $m + n$  is odd, then taking  $-y$  instead of  $y$  in equation (2.8), we obtain

$$(2.9) \quad x^m y^m + x^n y^n = 0 \quad \forall x, y \in M_s(K).$$

Summing relation (2.8) and equation (2.9), we find that  $x^m y^m = 0$ . In particular, for  $x = y = e_{11}$ , the last equation yields a contradiction.

Now if  $m + n$  is even, taking  $2e_{jj}$  instead of  $x$  and  $e_{jj}$  instead of  $y$  in relation (2.8) for a fixed positive integer  $j \leq s$ , we get  $2^m e_{jj} - 2^n e_{jj} = 0$ , that is  $(2^{|m-n|} - 1)e_{jj} = 0$ , which is impossible unless  $\text{char}(R) = 2^{|m-n|} - 1$ .

*Case 2:* If  $d$  and  $h$  are linearly  $C$ -independent modulo inner derivations of  $Q_r(R)$ , then using [6], Theorem 2 along with relation (2.2), we get

$$(2.10) \quad a(\alpha x + z_1)^m (\beta y + z_2)^m = x^n y^n \quad \forall x, y, z_1, z_2 \in Q_r.$$

In particular, for  $x = y = 0$ , equation (2.10) reduces to  $az_1^m z_2^m = 0$  for all  $z_1, z_2 \in Q_r$ , which contradicts [4], Lemma 1.1.

*Case 3:* If  $d$  and  $h$  are linearly  $C$ -dependent modulo inner derivations of  $Q_r(R)$ , then we may suppose that  $d(x) = \delta h(x) + [q, x]$  for all  $x \in R$  with  $\delta \in C \setminus \{0\}$  and  $q \in Q_r(R)$ . Note that  $h$  is  $Q_r$ -outer, otherwise  $d$  and  $h$  are both  $Q_r$ -inner, which has already been treated before in Case 1. The main equation becomes

$$(2.11) \quad a(\alpha x + \delta h(x) + [q, x])^m (\beta y + h(y))^m - x^n y^n = 0 \quad \forall x, y \in Q_r.$$

Theorem 2 of [6] yields

$$a(\alpha x + \delta z_1 + [q, x])^m (\beta y + z_2)^m - x^n y^n = 0 \quad \forall x, y, z_1, z_2 \in Q_r.$$

Arguing as in Case 2, we also get a contradiction. □

**P r o o f** of Theorem 2.2. We may suppose that  $F$  and  $H$  are nonzero. Indeed, otherwise the main identity becomes  $(x \circ y)^n = 0$  for all  $x, y \in I$ . Taking  $x$  instead of  $y$ , we get  $2^n x^{2n} = 0$  for all  $x \in I$ . Invoking  $\text{char}(R) \neq 2$  along with [4], Lemma 1.1,  $R$  has a nonzero nilpotent ideal, which contradicts the primeness of  $R$ .

Now using [8], Theorem 2, our hypothesis yields

$$(2.12) \quad (F(x) \circ H(y))^m = (x \circ y)^n \quad \forall x, y \in Q_r.$$

By view of [9], Theorem 3, there exist  $\alpha, \beta \in Q_r$  such that  $F(x) = \alpha x + d(x)$  and  $H(x) = \beta x + h(x)$ , then relation (2.12) yields

$$((\alpha x + d(x)) \circ (\beta y + h(y)))^m = (x \circ y)^n \quad \forall x, y \in Q_r.$$

*Case 1:*  $d$  and  $h$  are both  $Q_r$ -inner, then there exist  $q_1, q_2 \in Q_r$  such that  $d(x) = [q_1, x]$ ,  $h(x) = [q_2, x]$  for all  $x \in Q_r$ , hence

$$P(x, y) = ((\alpha x + [q_1, x]) \circ (\beta y + [q_2, y]))^m - (x \circ y)^n = 0 \quad \forall x, y \in Q_r.$$

By adopting a similar approach to the one used in Theorem 2.1, it follows that  $Q_r$  is isomorphic to a dense ring of linear transformation of vector space  $V$  over  $C$ .

Assume that  $\dim_C V \geq 2$ , clearly  $\{v, q_1 v\}$  are linearly  $C$ -dependent for all  $v \in V$ , otherwise, we suggest to suppose that  $v$  and  $q_1 v$  are linearly  $C$ -independent.

If  $q_2 v \notin \text{Span}_C\{v, q_1 v\}$ , then  $\{v, q_1 v, q_2 v\}$  is  $C$ -independent. Using the density of  $Q_r$ , there exist  $f, g \in Q_r$  such that  $fv = 0$ ,  $f q_1 v = -v$ ,  $f q_2 v = v$ ,  $gv = 0$ ,  $g q_1 v = -v$ ,  $g q_2 v = v$ ,

$$(2.13) \quad P(f, g)v = (((\alpha f + [q_1, f]) \circ (\beta g + [q_2, g]))^m - (f \circ g)^n)v = 0.$$

The only nonzero terms are  $[q_1, f][q_2, g]v = -v$  and  $[q_2, g][q_1, f]v = -v$ .

$$\begin{aligned} P(f, g)v &= (([q_1, f][q_2, g] + [q_2, g][q_1, f])^m - (f \circ g)^n)v \\ &= ([q_1, f][q_2, g] + [q_2, g][q_1, f])^m v \\ &= ([q_1, f][q_2, g] + [q_2, g][q_1, f])^{m-1}(-2v) = (-2)^m v. \end{aligned}$$

Invoking equation (2.13),  $P(f, g)v = (-2)^m v = 0$  for all  $v \in V$ , a contradiction. Now if  $q_2 v \in \text{Span}_C\{v, q_1 v\}$ , then  $q_2 v = \lambda v + \mu q_1 v$  for some  $\lambda, \mu \in C$ , thus  $g q_2 v = \lambda g v + \mu g q_1 v = -\mu v$ , accordingly  $P(f, g)v = (2\mu)^m v = 0$  for all  $w \in V$ , which is also impossible.

Then generally  $\{v, q_1 v\}$  are linearly  $C$ -dependent for all  $v \in V$ . Simple arguments lead to  $q_1 v = \lambda v$  for all  $v \in V$  with  $\lambda \in C$ , which, as in the proof of Theorem 2.1, forces  $q_1 \in C$ . Using a similar argument, we get  $q_2 \in C$ .

We propose to prove that  $\{w, \alpha w\}$  are linearly  $C$ -dependent for any  $w \in V$ . Suppose that  $\{w, \alpha w\}$  are linearly  $C$ -independent.

If  $\beta w \notin \text{Span}_C\{w, \alpha w\}$ , then  $\{w, \alpha w, \beta w\}$  are  $C$ -independent,  $Q_r$  being a dense ring of linear transformation of  $V$ , it follows that there exist  $f, g \in Q_r$  such that  $fw = w, f\alpha w = 0, f\beta w = w, gw = 0, g\alpha w = w, g\beta w = 0$ ,

$$(2.14) \quad Q(f, g)w = (((\alpha f) \circ (\beta g))^m - (f \circ g)^n)w = 0.$$

Firstly,  $(\alpha f \beta g + \beta g \alpha f)w = \beta g \alpha w = \beta w$  and  $(\alpha f \beta g + \beta g \alpha f)\beta w = \beta w$ . Consequently,  $Q(f, g)w = \beta w = 0$  for all  $w \in V$ , which is impossible.

Let now  $\beta w \in \text{Span}_C\{w, \alpha w\}$ , then  $\beta w = \mu_1 w + \mu_2 \alpha w$  for some  $\mu_1, \mu_2 \in C$ , accordingly, we get  $(\alpha f \beta g + \beta g \alpha f)\beta w = \mu_1(\mu_2 \alpha + \beta)w$ . It is obvious that

$$(\alpha f \beta g + \beta g \alpha f)\mu_1(\mu_2 \alpha + \beta)w = \mu_1(\mu_2 \alpha + \beta)w.$$

Then

$$Q(f, g)w = \mu_1(\mu_2 \alpha + \beta)w$$

for any  $w \in V$ , which is impossible. Then  $\{w, \alpha w\}$  are linearly  $C$ -dependent for any  $w \in V$ . Simple computations lead to  $\alpha w = \gamma w$  for some  $\gamma \in C$ , thus  $[R, \alpha]V = 0$ , then  $\alpha \in C$ . Likewise, we get  $\beta \in C$ . Returning to the main equation, we find that

$$(2.15) \quad (\alpha\beta)^m(x \circ y)^m = (x \circ y)^n \quad \forall x, y \in I.$$

If  $m = n$ , then  $(\alpha\beta)^m = 1$  follows immediately.

Regarding the case where  $m \neq n$ , in light of [7], Lemma 1,  $M_s(K)$  satisfies

$$(2.16) \quad (\alpha\beta)^m(x \circ y)^m = (x \circ y)^n$$

for a field  $K$  and an integer  $s > 1$ . Taking  $e_{ij}$  instead of  $x$  and  $e_{ji}$  instead of  $y$  in relation (2.16) for some fixed positive integers  $i, j \leq s$ , we get  $((\alpha\beta)^m - 1)(e_{ii} + e_{jj}) = 0$ , which implies  $(\alpha\beta)^m = 1$ . Now relation (2.16) reduces to

$$(2.17) \quad (x \circ y)^m - (x \circ y)^n = 0 \quad \forall x, y \in M_s(K).$$

If  $m + n$  is odd, then substituting  $y$  by  $-y$  in equation (2.17), we obtain

$$(2.18) \quad (x \circ y)^m + (x \circ y)^n = 0 \quad \forall x, y \in M_s(K).$$

The summation of relation (2.17) and equation (2.18) gives  $(x \circ y)^m = 0$  for all  $x, y \in M_s(K)$ . Setting  $x = y = I_s$  with  $I_s$  the matrix identity of  $M_s(K)$ , we get to  $2^m I_s = 0$  so that  $I_s = 0$ , a contradiction.



Now if  $m + n$  is even, for  $x = y = e_{ii}$ , equation (2.17) yields

$$(2e_{ii})^m - (2e_{ii})^n = 0.$$

That is,  $(2^{|m-n|} - 1)e_{ii} = 0$ , which is impossible unless  $\text{char}(R) = 2^{|m-n|} - 1$ .

*Case 2:* If  $d$  and  $h$  are linearly  $C$ -independent modulo inner derivations of  $Q_r(R)$ , then the main identity along with [6], Theorem 2 give

$$(2.19) \quad ((\alpha x + z_1) \circ (\beta y + z_2))^m = (x \circ y)^n \quad \forall x, y, z_1, z_2 \in Q_r.$$

Setting  $x = y = 0$ , equation (2.19) reduces to  $(z_1 \circ z_2)^m = 0$  for all  $z_1, z_2 \in Q_r$ ; a contradiction follows directly from [4], Lemma 1.1.

*Case 3:* If  $d$  and  $h$  are linearly  $C$ -dependent modulo inner derivations of  $Q_r(R)$ , then one can show that  $d(x) = \delta h(x) + [q, x]$  for some  $\delta \in C \setminus \{0\}$ ,  $q \in Q_r(R)$  and  $h$  is necessarily  $Q_r$ -outer. The main equation becomes

$$(2.20) \quad ((\alpha x + \delta h(x) + [q, x]) \circ (\beta y + h(y)))^m - (x \circ y)^n = 0 \quad \forall x, y \in Q_r.$$

Theorem 2 of [6] yields

$$((\alpha x + \delta z_1 + [q, x]) \circ (\beta y + z_2))^m - (x \circ y)^n = 0 \quad \forall x, y, z_1, z_2 \in Q_r.$$

An approach similar to that adopted in Case 2 leads to a contradiction. □

### 3. APPLICATION ON PRIME BANACH ALGEBRAS

Throughout this section,  $\mathcal{A}$  denotes a real or complex Banach algebra. To prove our main results we need the following lemma.

**Lemma 3.1** ([1]). *Let  $\mathcal{A}$  be a Banach algebra. If  $P(t) = \sum_{k=0}^n b_k t^k$  is a polynomial in the real variable  $t$  with coefficients in  $\mathcal{A}$ , and if for an infinite set of real values of  $t$ ,  $P(t) \in M$ , where  $M$  is a closed linear subspace of  $\mathcal{A}$ , then every  $b_k$  lies in  $M$ .*

**Theorem 3.1.** *Let  $\mathcal{A}$  be a Banach algebra,  $Q_{\mathcal{A}}$  its right Martindale quotient ring,  $C_{\mathcal{A}}$  its extended centroid,  $F = L_{\alpha} + d$ ,  $H = L_{\beta} + h$  are two continuous generalized derivations with  $L_{\alpha}$  (or  $L_{\beta}$ ) the left multiplication by an element  $\alpha \in \mathcal{A}$  (or  $\beta \in \mathcal{A}$ ),  $d, h$  derivations of  $\mathcal{A}$ ,  $a \in C_{\mathcal{A}}$ ,  $m \geq 1$  and  $n \geq 1$  are two fixed positive integers such that*

$$aF(x)^m H(y)^m - x^n y^n \in \text{rad}(\mathcal{A}) \quad \forall x, y \in \mathcal{A},$$

*then  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$  and  $h(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ . Moreover, if  $\mathcal{A}$  is primitive, then  $F(x) = \alpha x$ ,  $H(x) = \beta x$ , for some  $\alpha, \beta \in C_{\mathcal{A}}$  with  $a(\alpha\beta)^m = 1$  and  $m = n$ .*

**Proof.** Let  $P$  be a primitive ideal, set  $F_P, H_P: \mathcal{A}/P \rightarrow \mathcal{A}/P$  with  $F_P(\bar{x}) = F_P(x + P) = F(x) + P$  and  $H_P(\bar{x}) = H_P(x + P) = H(x) + P$  for all  $\bar{x} \in \mathcal{A}/P$ . Invoking [11], Theorem 2.2 primitive ideals are invariant under  $F$  and  $H$ , then  $F_P$  and  $H_P$  are well defined.  $P$  being primitive, Lemma 5.36 of [2] implies that  $\mathcal{A}/P$  is a primitive ring and thus prime by [2], Lemma 5.4. The main identity becomes

$$aF_P(x)^m H_P(y)^m - x^n y^n = 0 \quad \forall x, y \in \mathcal{A}/P.$$

Using Theorem 2.1, we get  $d_P = 0$  and  $h_P = 0$ , that is,  $d(\mathcal{A}) \subseteq P$  and  $h(\mathcal{A}) \subseteq P$  for any primitive ideal  $P$ . Then  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$  or  $h(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ . Moreover, if  $\mathcal{A}$  is primitive, then  $\text{rad}(\mathcal{A}) = (0)$ . Invoking again Theorem 2.1, we get the required results.  $\square$

Using the same arguments as above, with a suitable modification, application of Theorem 2.2 yields the following result.

**Theorem 3.2.** *Let  $\mathcal{A}$  be a Banach algebra,  $F = L_\alpha + d$ ,  $H = L_\beta + h$  be two continuous generalized derivations with  $L_\alpha$  (or  $L_\beta$ ) the left multiplication by an element  $\alpha \in \mathcal{A}$  (or  $\beta \in \mathcal{A}$ ),  $d, h$  derivations of  $\mathcal{A}$ ,  $m \geq 1$  and  $n \geq 1$  be two fixed positive integers such that*

$$(F(x) \circ H(y))^m - (x \circ y)^n \in \text{rad}(\mathcal{A}) \quad \forall x, y \in \mathcal{A},$$

*then  $d(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$  and  $h(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$ . Moreover, if  $\mathcal{A}$  is primitive, then  $F(x) = \alpha x$ ,  $H(x) = \beta x$ , for some  $\alpha, \beta \in C_{\mathcal{A}}$  with  $(\alpha\beta)^m = 1$  and  $m = n$ .*

**Theorem 3.3.** *Let  $\mathcal{A}$  be a prime Banach algebra,  $O_1, O_2$  nonvoid open subsets on  $\mathcal{A}$ ,  $Q_{\mathcal{A}}$  its right Martindale quotient ring,  $C_{\mathcal{A}}$  its extended centroid,  $a \in C_{\mathcal{A}}$ ,  $F$  and  $H$  are two continuous generalized derivations of  $\mathcal{A}$  associated with derivations  $d$  and  $h$ , respectively, such that*

$$aF(x)^m H(y)^m - x^n y^n = 0 \quad \forall (x, y) \in O_1 \times O_2$$

*for two fixed positive integers  $m \geq 1$  and  $n \geq 1$ . Then  $F(x) = \alpha x$ ,  $H(x) = \beta x$  for some  $\alpha, \beta \in C_{\mathcal{A}}$ . Moreover,  $m = n$  and  $a(\alpha\beta)^m = 1$ .*

**Proof.** By assumption

$$(3.1) \quad F(x)^m H(y)^m - x^n y^n = 0 \quad \forall (x, y) \in O_1 \times O_2.$$

Let  $u \in \mathcal{A}$  and  $x \in O_1$ , then  $x + tu \in O_1$  for a sufficiently small real  $t$ .  $F, H$  being continuous, one can obviously see that  $F(ru) = rF(u)$  and  $H(ru) = rH(u)$  for all  $u \in \mathcal{A}$ ,  $r \in \mathbb{R}$ . Taking  $x + tu$  instead of  $x$  in equation (3.1), we get

$$(3.2) \quad Q(t) = a(F(x) + F(u)t)^m H(y)^m - (x + tu)^n y^n = 0.$$

Setting  $Q(t) = \sum_{k=0}^{\max(m,n)} q_k(u, x, y)t^k$ , if  $m = n$ , invoking Lemma 3.1, we obtain  $q_k(u, x, y) = 0$  for all  $k \in \{0, \dots, m\}$ . In particular,  $q_m(u, x, y) = 0$ , thus

$$aF(u)^m H(y)^m - u^m y^m = 0 \quad \forall (u, y) \in \mathcal{A} \times O_2.$$

Similarly, by acting on  $y$  instead of  $x$ , one can easily get to

$$aF(u)^m H(v)^m - u^m v^m = 0 \quad \forall u, v \in \mathcal{A}.$$

By Theorem 2.1, we get the required results.

Suppose now that  $m < n$ , the right choice of the coefficient yields

$$p_n(u, x, y) = u^n y^n = 0 \quad \forall (u, y) \in \mathcal{A} \times O_2.$$

At the end, we get  $u^n v^n = 0$  for all  $u, v \in \mathcal{A}$ . Substituting  $v$  by  $u$  and invoking [4], Lemma 1.1, it follows that  $\mathcal{A}$  has a nonzero nilpotent ideal, absurd.

Now if  $m > n$ , it follows that  $p_m(u, x, y) = aF(u)^m H(v)^m = 0$  for all  $u, v \in \mathcal{A}$ . The main equation leads to  $u^n v^n = 0$  for all  $u, v \in \mathcal{A}$  and we obtain the same contradiction. Then necessarily  $m = n$ .  $\square$

**Theorem 3.4.** *Let  $\mathcal{A}$  be a prime Banach algebra,  $O_1, O_2$  nonvoid open subsets on  $\mathcal{A}$ ,  $Q_{\mathcal{A}}$  its right Martindale quotient ring,  $C_{\mathcal{A}}$  its extended centroid,  $F$  and  $H$  be two continuous generalized derivations of  $\mathcal{A}$  associated with derivations  $d$  and  $h$ , respectively, such that*

$$(F(x) \circ H(y))^m - (x \circ y)^n = 0 \quad \forall (x, y) \in O_1 \times O_2$$

with  $m \geq 1$  and  $n \geq 1$  be two fixed positive integers. Then  $F(x) = \alpha x$  and  $H(x) = \beta x$  for some  $\alpha, \beta \in C_{\mathcal{A}}$ . Moreover,  $m = n$  and  $(\alpha\beta)^m = 1$ .

*Proof.* Assume that

$$(3.3) \quad (F(x) \circ H(y))^m - (x \circ y)^n = 0 \quad \forall (x, y) \in O_1 \times O_2.$$

Let  $u \in \mathcal{A}$ . For a sufficiently small real  $s$ , one can replace  $x$  by  $x + su$  in equation (3.3)

$$(3.4) \quad P(s) = (F(x) \circ H(y) + (F(u) \circ H(y))s)^m - (x \circ y + (u \circ y)s)^n = 0.$$

Set  $P(s) = \sum_{k=0}^{\max(m,n)} p_k(u, x, y)s^k$ . If  $m = n$ , a direct application of Lemma 3.1 leads to

$$p_m(u, x, y) = (F(u) \circ H(y))^m - (u \circ y)^m = 0 \quad \forall (u, y) \in \mathcal{A} \times O_2.$$

By adopting a similar approach, we get

$$(F(u) \circ H(v))^m - (u \circ v)^m = 0 \quad \forall u, v \in \mathcal{A}$$

and application of Theorem 2.2 gives the required conclusion.

Now if  $m < n$ , a suitable choice of the right coefficient yields

$$p_n(u, x, y) = (u \circ y)^n = 0 \quad \forall (u, y) \in \mathcal{A} \times O_2.$$

Then  $(u \circ v)^n = 0$  for all  $u, v \in \mathcal{A}$ . Replacing  $v$  by  $u$  and using [4], Lemma 1.1, we get a contradiction.

Regarding the case where  $m > n$ , we get  $p_m(u, x, y) = (F(u) \circ H(v))^m = 0$  for all  $u, v \in \mathcal{A}$ . The main equation becomes  $(x \circ y)^n = 0$  for all  $(x, y) \in O_1 \times O_2$ . Arguing as in the last case, we obtain the same contradiction. Accordingly,  $m = n$ .  $\square$

The following example shows that the primeness hypothesis in Theorems 2.1–2.2 is not superfluous.

**Example 3.1.** Let us consider the ring  $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$

and  $I = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{Z} \right\}$  an ideal of  $R$ . Define  $F, H: R \rightarrow R$  with

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & yx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously  $F$  and  $H$  are generalized derivations on  $R$ . Fix  $a \in C \setminus \{0\}$ . It is straightforward that  $aF(X)^m H(Y)^m = X^n Y^n$  and  $(F(X) \circ H(Y))^m = (X \circ Y)^n$  for all  $X, Y \in I$ . However, conclusions of Theorems 2.1 and 2.2 are not satisfied.

### References

- [1] *F. F. Bonsall, J. Duncan: Complete Normed Algebras. Ergebnisse der Mathematik und ihrer Grenzgebiete 80. Springer, New York, 1973.* [zbl](#) [MR](#) [doi](#)
- [2] *M. Brešar: Introduction to Noncommutative Algebra. Universitext. Springer, Cham, 2014.* [zbl](#) [MR](#) [doi](#)
- [3] *T. S. Erickson, W. S. Martindale III, J. M. Osborn: Prime nonassociative algebras. Pac. J. Math. 60 (1975), 49–63.* [zbl](#) [MR](#) [doi](#)
- [4] *I. N. Herstein: Topics in Ring Theory. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, 1969.* [zbl](#) [MR](#)
- [5] *N. Jacobson: Structure of Rings. Colloquium Publications 37. AMS, Providence, 1964.* [zbl](#) [MR](#)
- [6] *V. K. Kharchenko: Differential identities of prime rings. Algebra Logic 17 (1979), 155–168.* [zbl](#) [MR](#) [doi](#)

- [7] *C. Lanski*: An Engel condition with derivation. *Proc. Am. Math. Soc.* *118* (1993), 731–834. [zbl](#) [MR](#) [doi](#)
- [8] *T.-K. Lee*: Semiprime rings with differential identities. *Bull. Inst. Math., Acad. Sin.* *20* (1992), 27–38. [zbl](#) [MR](#)
- [9] *T.-K. Lee*: Generalized derivations of left faithful rings. *Commun. Algebra* *27* (1999), 4057–4073. [zbl](#) [MR](#) [doi](#)
- [10] *W. S. Martindale III*: Prime rings satisfying a generalized polynomial identity. *J. Algebra* *12* (1969), 576–584. [zbl](#) [MR](#) [doi](#)
- [11] *A. M. Sinclair*: Continuous derivations on Banach algebras. *Proc. Am. Math. Soc.* *29* (1969), 166–170. [zbl](#) [MR](#) [doi](#)

*Authors' addresses:* *Abderrahman Hermas*, Department of Mathematics, Faculty of Science and Technology, Sidi Mohamed Ben Abdellah University, Fez, Morocco, e-mail: [Abde.hermas@gmail.com](mailto:Abde.hermas@gmail.com); *Abdellah Mamouni*, Department of Mathematics, Faculty of Science, Moulay Ismail University, Meknes, Morocco, e-mail: [a.mamouni.fste@gmail.com](mailto:a.mamouni.fste@gmail.com); *Lahcen Oukhtite* (corresponding author), Department of Mathematics, Faculty of Science and Technology, Sidi Mohamed Ben Abdellah University, Fez, Morocco, e-mail: [oukhtitel@hotmail.com](mailto:oukhtitel@hotmail.com).