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NONLINEAR FOURTH ORDER PROBLEMS WITH ASYMPTOTICALLY LINEAR NONLINEARITIES

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Abstract. We investigate some nonlinear elliptic problems of the form

$$\Delta^2 v + \sigma(x)v = h(x,v) \quad \text{in } \Omega, \quad v = \Delta v = 0 \quad \text{on } \partial\Omega, \tag{P}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, $\sigma(x)$ a positive function in $L^\infty(\Omega)$, and the nonlinearity $h(x,t)$ is indefinite. We prove the existence of solutions to the problem (P) when the function $h(x,t)$ is asymptotically linear at infinity by using variational method but without the Ambrosetti-Rabinowitz condition. Also, we consider the case when the nonlinearities are superlinear and subcritical.

Keywords: asymptotically linear; mountain pass theorem; biharmonic equation; Cerami sequence

MSC 2020: 35A15, 35J35, 35J60, 35J91

1. Introduction and main results

In this paper, we consider the quasilinear elliptic problem

$$\begin{cases} \Delta^2 v + \sigma(x)v = h(x,v) & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a regular bounded domain in $\mathbb{R}^N$ and σ is a positive function in $L^\infty(\Omega)$. We are interested in this type of equations because biharmonic problems have many applications in micro-electro-mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells and other fields of science, see for example [5], [6], [9]. Also, asymptotically linear nonlineari-

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ties at infinity attract attention since they approximate linear cases. First in 1996, Mironescu and Rădulescu in [18] investigated the harmonic problem

$$
\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where the reaction term $h(x,t)$ did not depend on x and $h(x,t) = \mu k(t)$ precisely for $\mu > 0$ and $k(t)$ satisfying

$$
k \text{ is a positive, nondecreasing and convex function in } C^1[0,\infty), \tag{1.3}
$$

and

$$
\lim_{t\to\infty} \frac{k(t)}{t} = \omega \in (0,\infty). \tag{1.4}
$$

With this type of nonlinearities and harmonic extension problems, we can cite [1], [7], [10], [14], [16]–[19], [21]. The generalization of the problem (1.1) to biharmonic operators and the asymptotically linear nonlinearity satisfying (1.3) and (1.4) was investigated in many works, we can cite [2], [11], [22] and the references therein.

In order to consider a larger class of asymptotically linear nonlinearities, Zhou in 2007 studied the problem (1.2) (see [28]) but he made the following assumptions:

(F1) $h(x,t)$ is in $C(\overline{\Omega}\times\mathbb{R})$, $h(x,t) \geqslant 0$ for all $x\in\Omega$ and $t\geqslant 0$ and $h(x,t) = 0$ for all $x\in\Omega$ and $t \geqslant 0$.

(F2) $\lim\limits_{t\to 0} h(x,t)/t = p(x)$ and $\|p(x)\|_\infty < \theta_1$, where $\theta_1 > 0$ is the first eigenvalue of the operator $(-\Delta, H_0^1(\Omega))$ and $\lim\limits_{t\to\infty} h(x,t)/t = \alpha$ uniformly for $x\in\Omega$ and $0 < \alpha \leqslant \infty$.

(F3) The function $h(x,t)/t$ is nondecreasing with respect to t in $(0,\infty)$ for a.e. $x\in\Omega$.

With these conditions, many second and fourth order partial differential equations have been treated [3], [12], [15], [26]–[28].

In this paper, we suppose that the nonlinearities change sign and satisfy some conditions of the same kind like those introduced in [28]. We suppose that:

(H1) $h(x,t)$ is in $C(\overline{\Omega}\times\mathbb{R})$, $h(x,t)\geqslant 0$ for all $t>0$ and $h(x,t) = 0$ for all $t\leqslant 0$.

(H2) $\lim\limits_{t\to 0} h(x,t)/t = p(x)$ and $\|p(x)\|_\infty < \theta_1$, where $\theta_1 > 0$ is the first eigenvalue of the operator $(\Delta^2 + \sigma(x), H^2(\Omega)\cap H_0^1(\Omega))$.

(H3) $\lim\limits_{t\to\infty} h(x,t)/t = \alpha$ uniformly for $x\in\Omega$ and $0<\alpha\leqslant\infty$.

(H4) $\lim\limits_{t\to\infty} h(x,t)/t^{p-1} = 0$ uniformly in $x\in\Omega$ for some $p\in(2,2_*)$, here and hereafter

$$
2_* = \begin{cases} \dfrac{2N}{N-4} & \text{if } N > 4, \\ \infty & \text{if } N \leqslant 4. \end{cases}
$$

(H5) The function $h(x,t)/t$ is nondecreasing with respect to t in $(0,\infty)$ for a.e. $x \in \Omega$.

The condition (H4) will be used in the investigation of the case where $h(x,t)$ is superlinear.

Henceforth, we suppose that $\sigma(x)$ is a positive function in $L^\infty(\Omega)$ satisfying

(A) $0 < a_0 \leqslant \sigma(x) \leqslant a_1$ for all $x \in \Omega$.

Also, we denote φ_1 a normalised positive eigenfunction associated to θ_1, the first eigenvalue of the operator $\Delta^2 + \sigma(x)$ with Navier boundary conditions on the open domain Ω, that is,

$$\begin{cases} \Delta^2\varphi_1 + \sigma(x)\varphi_1 = \theta_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = \Delta\varphi_1 = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \varphi_1^2 \,\mathrm{d}x = 1. \end{cases} \tag{1.5}$$

Our results read as follows.

Theorem 1.1. *If* (H1), (H2) *and* (H3) *hold and* $\alpha \in (0,\infty)$, *then we get:*

(i) *When* $0 < \alpha < \theta_1$ *and the function* $h(x,t)$ *satisfies the assumption* (H5), *there is no positive solution for the problem* (1.1).

(ii) *When* $\alpha > \theta_1$, *there exists a positive solution to the problem* (1.1).

(iii) *When* $\alpha = \theta_1$ *and the function* $h(x,t)$ *satisfies the assumption* (H5), *there exists a positive solution* v *to the problem* (1.1) *if and only if* $v = c_0\varphi_1$ *and* $h(x,v) = \theta_1 v$ *for some constant* $c_0 > 0$.

Theorem 1.2. *Suppose that* (H1)–(H5) *hold and* $\alpha = \infty$. *Then, the problem* (1.1) *has a positive solution.*

In order to prove the existence of solution in variational methods, we often suppose some complementary condition in order to prove the compactness property [4], [8], [9], [13], [20], [23]–[25] and the references therein.

One of these conditions is the Ambrosetti-Rabinowitz condition ((AR) for short) (see [4], [20]). Let

$$H(x,t) = \int_0^t h(x,s)\,\mathrm{d}s,$$

the (AR) condition is:

(AR) There exists a constant $\lambda > N$ and a constant $\delta > 0$ such that

$$0 < \lambda H(x,t) \leqslant h(x,t)t$$

for all $|t| \geqslant \delta$ and $x \in \Omega$.

In fact, the condition (AR) gives

$$\lim_{t\to\infty}\frac{H(x,t)}{t^2}=\infty,$$

hence $\lim\limits_{t\to\infty} h(x,t)/t=\infty$. That is, $h(x,t)$ must be superlinear with respect to t at infinity but in this work, we deal with asymptotically linear nonlinearity in Theorem 1.1 and so we prove the existence of nontrivial solutions without use of the (AR) condition or any of its refinements.

This paper is organized as follows: in Section 2, we introduce some notations and the variational setting. Section 3 is assigned to prove Theorem 1.1 and Section 4 proves Theorem 1.2. In the sequel, a constant C may change from line to another.

2. Preliminaries

Let Ω be a bounded open domain in $\mathbb{R}^N$, $N\geqslant 2$. For $v\in L^p(\Omega)$, $1\leqslant p<\infty$, we recall the Lebesgue norm

$$\|v\|_p=\Bigl(\int_\Omega |v|^p\,\mathrm{d}x\Bigr)^{1/p}.$$

In this work, we consider the space $\mathcal{H}=H^2(\Omega)\cap H^1_0(\Omega)$ endowed with the inner product

$$\langle u,v\rangle=\int_\Omega(\Delta u\Delta v+\sigma(x)uv)\,\mathrm{d}x \tag{2.1}$$

and the induced norm

$$\|v\|=\Bigl(\int_\Omega(|\Delta v|^2+\sigma(x)v^2)\,\mathrm{d}x\Bigr)^{1/2}. \tag{2.2}$$

We consider the following definition of solution (weak solution) for the problem (1.1).

Definition 2.1. A function $v\in\mathcal{H}$ is called the solution of the problem (1.1) if

$$\int_\Omega \Delta v\Delta\varphi\,\mathrm{d}x+\int_\Omega\sigma(x)v\varphi\,\mathrm{d}x=\int_\Omega h(x,v)\varphi\,\mathrm{d}x\quad\forall\,\varphi\in\mathcal{H}. \tag{2.3}$$

Since the equation (1.1) has a variational form, let ψ be the functional defined on $\mathcal{H}$ by

$$\psi(v)=\frac12\int_\Omega(|\Delta v|^2+\sigma(x)v^2)\,\mathrm{d}x-\int_\Omega H(x,v)\,\mathrm{d}x \tag{2.4}$$

where

$$H(x,s)=\int_0^s h(x,t)\,\mathrm{d}t.$$

To prove the existence of a nonzero critical point of ψ, we use a different version of the mountain pass theorem given in [9].

Theorem 2.1 ([9]). *Let X be a real Banach space and $\psi \in C^1(X, \mathbb{R})$ a functional satisfying:*

(i) *There exist $\delta, \tau > 0$ such that for all $v \in \partial B(0, \delta)$, $\psi(v) \geqslant \tau$.*
(ii) *There exists $x_1 \in X$ such that $\|x_1\| > \delta$ and $\psi(x_1) < 0$.*
(iii) $\max\{\psi(0), \psi(x_1)\} < \tau$.

Let c be the number characterized by

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \psi(\gamma(t)),$$

where $\Gamma := \{\eta \in C([0,1], X) \colon \eta(0) = 0$ and $\eta(1) = x_1\}$ is the set of continuous paths joining 0 and x_1 in X. Then, $c \geqslant \tau$ and there exists a sequence (v_n) in X satisfying the Cerami conditions

$$\psi(v_n) \to c \quad \text{as } n \to \infty \tag{2.5}$$

and

$$(1 + \|v_n\|)\|\psi(v_n)\|_* \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

3. Proof of Theorem 1.1

We begin this section by the following elementary result.

Lemma 3.1. $(\mathcal{H}, \|\cdot\|)$ *is a Hilbert space.*

Proof. We know that $(\mathcal{H}, \|\cdot\|_{H^2})$ is a Banach space where

$$\|u\|_{H^2} = (\|u\|_2^2 + \|\Delta u\|_2^2)^{1/2}$$

is the standard norm in $H^2(\Omega)$, see [15]. From the condition (A), we get

$$C_0\|u\|_{H^2} \leqslant \|u\| \leqslant C_1\|u\|_{H^2},$$

where $C_1 = \max\{1, \sqrt{a_1}\}$ and $C_0 = \min\{1, \sqrt{a_0}\}$. So, the norms $\|u\|_{H^2}$ and $\|u\|$ are equivalent on $\mathcal{H}$ and thus $(\mathcal{H}, \|\cdot\|)$ is a Banach space. Therefore, $(\mathcal{H}, \|\cdot\|)$ is a Hilbert space. □

Next, we prove the first geometric property of the functional ψ.

Lemma 3.2. *Assume that the function $h(x,t)$ satisfies the conditions* (H1)–(H3). *Then, we have:*

(i) *There exist $\delta > 0$ and $\tau > 0$ such that for all $v \in \partial B(0, \delta)$, $\psi(v) \geqslant \tau$.*
(ii) *When $\theta_1 < \alpha$, $\psi(t\varphi_1) \to -\infty$ as $t \to \infty$.*

P r o o f. (i) Let $\varepsilon > 0$, there exists $C = C(\varepsilon) \geqslant 0$ such that for all $t \geqslant 0$ and for all $q \geqslant 1$, we have

$$H(x,t) \leqslant \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)t^2 + C|t|^{q+1}. \tag{3.1}$$

Since

$$\psi(v) = \frac{1}{2}\|v\|^2 - \int_\Omega H(x,v)\,\mathrm{d}x, \tag{3.2}$$

then

$$\psi(v) \geqslant \frac{1}{2}\|v\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|v\|_2^2 - C\|v\|_{q+1}^{q+1}.$$

If we choose $1 < q < 2_* - 1$, by the Sobolev embedding theorem we get

$$\|v\|_{q+1}^{q+1} \leqslant C_1\|v\|^{q+1}.$$

Therefore

$$\psi(v) \geqslant \frac{1}{2}\|v\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|v\|_2^2 - C_2\|v\|^{q+1}.$$

From the property of θ_1, we get

$$\psi(v) \geqslant \frac{1}{2}\Big(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\theta_1}\Big)\|v\|^2 - C_2\|v\|^{q+1}. \tag{3.3}$$

If we consider $\varepsilon > 0$ such that $\|p(x)\|_\infty + \varepsilon < \theta_1$, then we can choose $\|v\| = \delta$ small enough in order to have $\psi(v) \geqslant \tau$ for some $\tau > 0$ sufficiently small.

(ii) Suppose that $\theta_1 < \alpha < \infty$. Let $t > 0$, we have

$$\psi(t\varphi_1) = \frac{t^2}{2}\int_\Omega (|\Delta\varphi_1|^2 + \sigma(x)|\varphi_1|^2)\,\mathrm{d}x - \int_\Omega H(x,t\varphi_1)\,\mathrm{d}x. \tag{3.4}$$

From (1.5), we obtain

$$\psi(t\varphi_1) = \frac{t^2}{2}\theta_1 - \int_\Omega H(x,t\varphi_1)\,\mathrm{d}x. \tag{3.5}$$

Then, by the use of Fatou's lemma,

$$\lim_{t\to\infty}\frac{\psi(t\varphi_1)}{t^2} \leqslant \frac{1}{2}\theta_1 - \int_\Omega \lim_{t\to\infty}\frac{H(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\,\mathrm{d}x.$$

Since $h(x,t)$ is asymptotically linear, we get

$$\lim_{t\to\infty}\frac{H(x,t)}{t^2} = \frac{\alpha}{2}. \tag{3.6}$$

Therefore

$$\lim_{t\to\infty}\frac{\psi(t\varphi_1)}{t^2} \leqslant \frac{1}{2}(\theta_1 - \alpha). \tag{3.7}$$

Then, $\lim\limits_{t\to\infty}\psi(t\varphi_1) = -\infty$. $\square$

P r o o f of Theorem 1.1. (i) Assume that $0 < \alpha < \theta_1$. Suppose that $v \in \mathcal{H}$ is a positive solution of the problem (1.1). In this case, from the conditions (H1)–(H3) and (H5), we get

$$\int_\Omega (|\Delta v|^2 + \sigma(x)v^2)\,\mathrm{d}x = \int_\Omega h(x,v)v\,\mathrm{d}x \leqslant \int_\Omega \alpha v^2\,\mathrm{d}x. \tag{3.8}$$

So, $\theta_1 \leqslant \alpha$ and this contradicts the hypothesis of this first case. Then, Theorem 1.1 (i) is proved.

(ii) Assume that $\theta_1 < \alpha$ and (H1)–(H3) hold. The functional ψ introduced by (2.4) is C^1 and satisfies $\psi(0) = 0$. By Lemma 3.2, there exist $\delta > 0$, $\tau > 0$ and $x_1 \in \mathcal{H}$ such that $\|x_1\| > \delta$ and $\psi(x_1) < 0$. Since $\max\{\psi(0), \psi(x_1)\} < \tau$, from Theorem 2.1, there exists a sequence $(v_n) \subset \mathcal{H}$ satisfying (2.5) and (2.6). It follows that

$$\psi(v_n) = \frac{1}{2}\|v_n\|^2 - \int_\Omega H(x,v_n)\,\mathrm{d}x \to c \quad \text{as } n \to \infty \tag{3.9}$$

and

$$\|\psi'(v_n)\|_* \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Suppose that (v_n) is bounded in $\mathcal{H}$. Since $(\mathcal{H}, \|\cdot\|)$ is a reflexive space, then there exist $v \in \mathcal{H}$ and a subsequence of (v_n), still denoted (v_n), satisfying

$$\begin{aligned} v_n &\rightharpoonup v && \text{weakly in } \mathcal{H} && \text{as } n \to \infty,\\ v_n &\to v && \text{strongly in } L^2(\Omega) && \text{as } n \to \infty,\\ v_n(x) &\to v(x) && \text{a.e. in } \Omega && \text{as } n \to \infty. \end{aligned}$$

From (3.10), for all $\varphi \in \mathcal{H}$ we have

$$\int_\Omega (\Delta v_n \Delta\varphi + \sigma(x)v_n\varphi)\,\mathrm{d}x - \int_\Omega h(x,v_n)\varphi\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty, \tag{3.11}$$

that is,

$$\Delta^2 v_n + \sigma(x)v_n - h(x,v_n) \to 0 \quad \text{in } \mathcal{H}', \tag{3.12}$$

where $\mathcal{H}'$ is the dual space of $\mathcal{H}$.

Note that by (H1), $h(x,v_n) \to h(x,v)$ in $L^2(\Omega)$ and since the dual space of $L^2(\Omega)$ is the space $L^2(\Omega)$ and $L^2(\Omega) \hookrightarrow \mathcal{H}'$, we have

$$\Delta^2 v_n + \sigma(x)v_n \to h(x,v) \quad \text{in } \mathcal{H}'. \tag{3.13}$$

Therefore, by using the fact that the operator $L = \Delta^2 + \sigma(x)$ is an isomorphism from $\mathcal{H}$ to $\mathcal{H}'$, we get

$$v_n \to L^{-1}(h(x,v)) \quad \text{in } \mathcal{H}. \tag{3.14}$$

From (3.14) and the uniqueness of the limit, we deduce that the sequence (v_n) converges to the function v in $\mathcal{H}$. So, v is a critical point of the functional ψ and then a nontrivial solution of the problem (1.1).

In order to finish the proof, we have to demonstrate that the sequence (v_n) is bounded in $\mathcal{H}$. For this, we argue by contradiction. Suppose that (v_n) is not bounded in $\mathcal{H}$. So, up to a subsequence, $\|v_n\| \to \infty$. Let

$$z_n = \frac{v_n}{\|v_n\|}, \quad t_n = \|v_n\|. \tag{3.15}$$

Since (z_n) is bounded in $\mathcal{H}$, up to a subsequence, there exists $z \in \mathcal{H}$ such that

$$\begin{aligned} z_n &\rightharpoonup z && \text{weakly in } \mathcal{H},\\ z_n &\to z && \text{strongly in } L^2(\Omega),\\ z_n(x) &\to z(x) && \text{a.e. in } \Omega. \end{aligned}$$

We claim that

$$\Delta^2 z + \sigma(x) z = \alpha z^+ \quad \text{in } \Omega. \tag{3.16}$$

For the proof of the claim (3.16), we divide (3.11) by $t_n = \|v_n\|$. We get

$$\int_\Omega (\Delta z_n \Delta\varphi + \sigma(x) z_n \varphi)\,\mathrm{d}x - \int_\Omega \frac{h(x,v_n)}{\|v_n\|}\varphi\,\mathrm{d}x \to 0 \quad \forall\,\varphi \in \mathcal{H}. \tag{3.17}$$

That is

$$\Delta^2 z_n + \sigma(x) z_n - \frac{h(x,v_n)}{\|v_n\|} \to 0 \quad \text{in } (\mathcal{H})'. \tag{3.18}$$

We have

$$\frac{h(x,v_n)}{\|v_n\|} = \frac{h(x,v_n)}{v_n} z_n.$$

In $\Omega_1 = \{x \in \Omega\colon\ z_n(x) \to z(x) \text{ and } z(x) > 0\}$, since $v_n = \|v_n\| z_n$, we obtain $\lim\limits_{n\to\infty} v_n = \infty$. So by using condition (H2), we get

$$\lim_{n\to\infty} \frac{h(x,v_n)}{v_n} z_n = \alpha z^+.$$

In $\Omega_2 = \{x \in \Omega\colon z_n(x) \to z(x) \text{ and } z(x) < 0\}$, since $v_n = \|v_n\| z_n$, we obtain $\lim_{n\to\infty} v_n = -\infty$. Since $z^+ = 0$, by using conditions (H1), we get

$$\lim_{n\to\infty} \frac{h(x, v_n)}{v_n} z_n = 0 z(x) = 0 = \alpha z^+.$$

In $\Omega_3 = \{x \in \Omega\colon z_n(x) \to z(x) \text{ and } z(x) = 0\}$, we obtain $\lim_{n\to\infty} v_n/\|v_n\| = 0$. Then $\lim_{n\to\infty} v_n = c_1 \in \mathbb{R}$. From (H1), h is continuous so admits a limit. We deduce that

$$\lim_{n\to\infty} \frac{h(x, v_n)}{v_n} z_n = b_1 z = 0 = \alpha z^+$$

for a constant b_1 in $\mathbb{R}$.

Now, the sequence $z_n \to z$ in $L^2(\Omega)$. By Theorem IV.9 in [5] z_n is dominated in $L^2(\Omega)$, up to a subsequence. Therefore, $(h(x, v_n)/v_n) z_n$ is dominated and then converges to αz^+ in $L^2(\Omega)$.

Since $L^2(\Omega) \hookrightarrow \mathcal{H}'$, from (3.18), we get the equation (3.16) and so the claim is proved. Therefore,

$$\begin{cases} \Delta^2 z + \sigma(x) z = \alpha z^+ & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

By applying the maximum principle we obtain that $z > 0$ and then $z = z^+$ solves the problem (3.19).

It follows that $z = c\varphi_1$ for some constant $c > 0$ and $\alpha = \theta_1$, which contradicts the fact that $\theta_1 < \alpha < \infty$.

(iii) Let $\alpha = \theta_1$. Suppose that v is a positive solution for the problem (1.1). On one hand, if we take $\varphi = \varphi_1$ in (2.3), we obtain

$$\int_\Omega (\Delta v \Delta \varphi_1 + \sigma(x) v \varphi_1)\, \mathrm{d}x = \int_\Omega h(x, v) \varphi_1 \, \mathrm{d}x. \tag{3.20}$$

On the other hand, consider the equation (1.5) and take v as a test function, we then obtain

$$\int_\Omega (\Delta v \Delta \varphi_1 + \sigma(x) v \varphi_1)\, \mathrm{d}x = \alpha \int_\Omega v \varphi_1 \, \mathrm{d}x. \tag{3.21}$$

So,

$$\int_\Omega (h(x, v) - \alpha v) \varphi_1 \, \mathrm{d}x = 0.$$

Now, from (H1)–(H3) and (H5) and the fact that $\varphi_1 > 0$, we get $h(x, v) = \alpha v$ a.e. in Ω. Hence, $h(x, v) = \theta_1 v$ a.e. in Ω and the result follows from the fact that the eigenvalue θ_1 is simple. Conversely, if $\alpha = \theta_1$ for $v = c_0 \varphi_1$ with some constant $c_0 > 0$ and $h(x, v) = \theta_1 v$, then that v is an eigenfunction satisfying (1.5) and so a solution of the problem (1.1). □

4. Proof of Theorem 1.2

We start by proving the geometric properties for the functional ψ introduced by (2.4).

Lemma 4.1. *Suppose that* (H1)–(H3) *and* (H5) *hold,* $\alpha = \infty$ *and the function* $h(x,t)$ *is subcritical at* $t = \infty$ *uniformly on* x *a.e. in* Ω. *We have*

(i) *There exist positive constants* $\delta, \beta > 0$ *such that* $\psi(v) \geqslant \beta$ *for all* $v \in \mathcal{H}$ *with* $\|v\| = \delta$.

(ii) $\lim\limits_{t\to\infty} \psi(t\varphi_1) = -\infty$.

Proof. (i) In this subcritical case, the condition (H4), which is

$$\lim_{t\to\infty} \frac{h(x,t)}{t^{r-1}} = 0 \quad \text{for some } r \in (2, 2^*), \tag{4.1}$$

and the condition (H2) give that for any $\varepsilon > 0$, there exists $C = C(\varepsilon) \geqslant 0$ such that for all $t \in \mathbb{R}$ and $x \in \Omega$,

$$H(x,t) \leqslant \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)t^2 + C|t|^r, \tag{4.2}$$

and so

$$\psi(v) \geqslant \frac{1}{2}\|v\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|v\|_2^2 - C\|v\|_r^r.$$

Since $2 < r < 2^*$, by the Sobolev embedding theorem we have $\|v\|_r^r \leqslant C_1\|v\|^r$ for some constant $C_1 > 0$ and then

$$\psi(v) \geqslant \frac{1}{2}\|v\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|v\|_2^2 - C_2\|v\|^r. \tag{4.3}$$

We use a characterization of θ_1, which is $\theta_1\|v\|_2^2 \leqslant \|v\|^2$ for all $v \in \mathcal{H}$ and so

$$\psi(v) \geqslant \frac{1}{2}\Big(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\theta_1}\Big)\|v\|^2 - C_2\|v\|^r. \tag{4.4}$$

Now, we can choose $\varepsilon > 0$ in (4.4) such that $\|p(x)\|_\infty + \varepsilon < \theta_1$ and $\|v\| = \delta$ small enough in order to have $\psi(v) \geqslant \beta$ for $\beta > 0$ sufficiently small.

(ii) Since the positive function φ_1 is in $C(\Omega)$. Let $\Omega_0 \subset \mathbb{R}^N$ be an open domain such that $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$ and let $\gamma > 0$ be a number satisfying $\varphi_1(x) \geqslant \gamma > 0$ for all $x \in \Omega_0$. From the condition (H5), we obtain

$$0 \leqslant 2H(x,t) \leqslant th(x,t) \tag{4.5}$$

and then the function $H(x,t)/t^2$ is nondecreasing with respect to $t > 0$ for a.e. $x \in \Omega_0$. We have that when $\alpha = \infty$, it implies that

$$\lim_{t\to\infty} \frac{H(x,t)}{t^2} = \infty.$$

So, for all $x \in \Omega_0$ and $t > 0$,

$$\frac{H(x, t\varphi_1(x))}{t^2\varphi_1^2(x)} \geqslant \frac{H(x, t\gamma)}{t^2\gamma^2}. \tag{4.6}$$

For all $B > 0$, there exists t_1 satisfying for all $t \geqslant t_1$ and for all $x \in \Omega_0$

$$\frac{H(x, t\varphi_1(x))}{t^2\varphi_1^2(x)} \geqslant B, \tag{4.7}$$

$$\frac{\psi(t\varphi_1)}{t^2} = \frac{1}{2}\int_\Omega (|\Delta\varphi_1|^2 + \sigma(x)\varphi_1^2)\,\mathrm{d}x - \int_\Omega \frac{H(x, t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\,\mathrm{d}x. \tag{4.8}$$

So,

$$\frac{\psi(t\varphi_1)}{t^2} \leqslant \frac{1}{2}\int_\Omega \varphi_1^2\,\mathrm{d}x - \int_{\Omega_0} \frac{H(x, t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\,\mathrm{d}x. \tag{4.9}$$

From (4.7) and (4.9) we get

$$\frac{\psi(t\varphi_1)}{t^2} \leqslant \frac{1}{2}\theta_1 \int_\Omega \varphi_1^2\,\mathrm{d}x - B\int_{\Omega_0} \varphi_1^2\,\mathrm{d}x$$

and so

$$\frac{\psi(t\varphi_1)}{t^2} \leqslant \frac{1}{2}\theta_1 - B\gamma^2|\Omega_0|.$$

We can choose $B > 0$ large enough so that

$$\frac{\psi(t\varphi_1)}{t^2} \leqslant -C < 0,$$

where $C > 0$ is a positive constant. Therefore

$$\lim_{t\to\infty} \psi(t\varphi_1) = -\infty.$$

□

Before we start the proof of the second existence result, Theorem 1.2, we recall the following result whose proof is similar to [25], Lemma 2.3.

Lemma 4.2. *Let ψ be the functional defined by* (2.4)*. Suppose that* (H5) *holds and*

$$\langle \psi'(v_n), v_n \rangle \to 0 \quad \text{as } n \to \infty.$$

Then, (v_n) has a subsequence, still denoted (v_n), satisfying for all $t > 0$ and for all $n > 0$

$$\psi(tv_n) \leqslant \frac{1 + t^2}{2n} + \psi(v_n).$$

P r o o f of Theorem 1.2. Suppose that $\alpha=\infty$, the conditions (H1)–(H3) and (H5) hold and $h(x,t)$ is subcritical at ∞ uniformly a.e. on $x\in\Omega$. From Lemma 4.1 and Theorem 2.1, there exists a sequence (v_n) satisfying the Cerami conditions (2.5) and (2.6) and so (3.9) and (3.10) hold.

We only need to prove that the sequence (v_n) is bounded in $\mathcal{H}$ and the proof we will use is the same as the proof of Theorem 1.1 (ii). We proceed by contradiciton, which means that, we suppose that (v_n) is not bounded in $\mathcal{H}$, then, up to a subsequence, $\|v_n\|\to\infty$ when $n\to\infty$.

Let $d>0$ be a positive number and set

$$z_n=\frac{v_n}{d\|v_n\|},\quad t_n=\frac{1}{d\|v_n\|}. \tag{4.10}$$

Since the sequence (z_n) is bounded in $\mathcal{H}$, there exists $z\in\mathcal{H}$ such that, up to a subsequence, $z_n\rightharpoonup z$ weakly in $\mathcal{H}$, $z_n\to z$ strongly in $L^2(\Omega)$ and $z_n(x)\to z(x)$ a.e. in Ω. As a consequence,

$$z_n^+\to z^+\quad\text{in } L^2(\Omega),$$

where $z_n^+=\frac{1}{2}(z_n+|z_n|)$ and

$$z_n^+\to z^+\quad\text{a.e. in } \Omega.$$

From the formula (2.4), we obtain

$$\psi(z_n)=\frac{1}{2}\|z_n\|^2-\int_\Omega H(x,z_n)\,\mathrm{d}x.$$

From the condition (H1), we get

$$\psi(z_n)=\frac{1}{2}\|z_n\|^2-\int_\Omega H(x,z_n^+)\,\mathrm{d}x. \tag{4.11}$$

Let $\Omega_+=\{x\in\Omega\colon\ z^+(x)>0\}$. For $x\in\Omega_+$,

$$v_n^+(x)=dz_n^+(x)\|v_n\|\to\infty$$

and so, for any $B>0$, there exists $n_1=n_1(x)>0$ such that for all $n\geqslant n_1$, we have

$$\frac{h(x,v_n^+(x))}{v_n^+(x)}\geqslant B. \tag{4.12}$$

Also, $z_n^+(x)\to z^+(x)$ and then there exists $n_2=n_2(x)>0$ such that for all $n\geqslant n_2$, we have

$$z_n^+(x)\geqslant\frac{z^+(x)}{2}. \tag{4.13}$$

From (4.11) and (4.12), we get

$$\frac{h(x,v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2 \geqslant B\frac{(z^+(x))^2}{4}.$$

So, for n large enough and for all $x \in \Omega_+$,

$$\lim_{n\to\infty} \frac{h(x,v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2 \geqslant B\frac{(z^+(x))^2}{4}. \tag{4.14}$$

From (3.11), by taking the test function $\varphi = v_n$, we get

$$\|v_n\|^2 - \int_\Omega h(x,v_n)v_n \, \mathrm{d}x \to 0,$$

hence

$$\frac{1}{d^2} - \int_\Omega \frac{h(x,v_n)}{v_n}(z_n)^2 \, \mathrm{d}x \to 0. \tag{4.15}$$

From (4.15) and the condition (H1), we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{h(x,v_n^+)}{v_n^+}(z_n^+)^2 \, \mathrm{d}x = \frac{1}{d^2}. \tag{4.16}$$

Therefore, by Fatou's lemma

$$\frac{1}{d^2} \geqslant \lim_{n\to\infty} \int_{\Omega_+} \frac{h(x,v_n^+)}{v_n^+}(z_n^+)^2 \, \mathrm{d}x \geqslant \int_{\Omega_+} \lim_{n\to\infty} \frac{h(x,v_n^+(x))}{v_n^+(x)}(z_n^+(x))^2.$$

By (4.14), we have then

$$\frac{1}{d^2} \geqslant \frac{B}{4}\int_{\Omega_+} (z^+(x))^2 \, \mathrm{d}x \tag{4.17}$$

and this holds for all $B > 0$. So, $|\Omega_+| = 0$ and then $z^+ \equiv 0$. From (4.11), we obtain

$$\lim_{n\to\infty} \psi(z_n) = \frac{1}{2d^2}. \tag{4.18}$$

On the other hand, by Lemma 4.2 and up to a subsequence, we get

$$\psi(z_n) = \psi(t_n v_n) \leqslant \frac{1}{2n}(1+t_n^2) + \psi(v_n). \tag{4.19}$$

From (3.9), (4.10), (4.18) and (4.19)

$$\frac{1}{2d^2} \leqslant c$$

for all $d > 0$ which is impossible and so the sequence (v_n) is bounded in $\mathcal{H}$ and Theorem 1.2 follows. □

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