Nayna Govindbhai Kalsariya; Bhikha Lila Ghodadra

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GENERALIZED ABSOLUTE CONVERGENCE OF SINGLE AND DOUBLE VILENKIN-FOURIER SERIES AND RELATED RESULTS

Nayna Govindbhai Kalsariya, Bhikha Lila Ghodadra, Vadodara

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Abstract. We consider the Vilenkin orthonormal system on a Vilenkin group G and the Vilenkin-Fourier coefficients $\hat{f}(n)$, $n \in \mathbb{N}$, of functions $f \in L^p(G)$ for some $1 < p \leqslant 2$. We obtain certain sufficient conditions for the finiteness of the series $\sum\limits_{n=1}^{\infty} a_n|\hat{f}(n)|^r$, where $\{a_n\}$ is a given sequence of positive real numbers satisfying a mild assumption and $0 < r < 2$. We also find analogous conditions for the double Vilenkin-Fourier series. These sufficient conditions are in terms of (either global or local) moduli of continuity of f and give multiplicative analogue of some results due to Móricz (2010), Móricz and Veres (2011), Golubov and Volosivets (2012), and Volosivets and Kuznetsova (2020).

Keywords: generalized absolute convergence; Vilenkin-Fourier series; modulus of continuity; multiplicative system

MSC 2020: 42C10

1. Introduction

In 2006, Gogoladge and Meskhia (see [4]) considered the convergence of the series $\sum\limits_{n=1}^{\infty} \gamma_n \varrho_n^r(f)$, $0 < r < 2$, where $\varrho_n(f) = (a_n^2(f) + b_n^2(f))^{1/2}$, $a_n(f)$, $b_n(f)$ are the coefficients of the Fourier trigonometric series of the function f, and $\{\gamma_n\}$ is a sequence of positive numbers satisfying certain definite conditions. In 2010, Móricz (see [9]) considered the Walsh orthonormal system on the interval $[0,1)$ in the Paley enumeration and the Walsh-Fourier coefficients $\hat{f}(n)$, $n \in \mathbb{N}$, of functions $f \in L^p[0,1)$

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for some $1 < p \leqslant 2$. He found certain best possible sufficient conditions for the finiteness of the series $\sum\limits_{n=1}^{\infty} a_n|\hat{f}(n)|^r$, where $\{a_n\}$ is a given sequence of nonnegative real numbers satisfying a mild assumption considered by Gogoladge and Meskhia in [4], and $0 < r < 2$. Those sufficient conditions were in terms of (either global or local) dyadic moduli of continuity of f. In 2011, Móricz and Veres (see [10]) proved analogues of the results proved in [9] for the double Walsh-Fourier series. In 2012, Golubov and Volosivets (see [6]) obtained several sufficient conditions for generalized absolute convergence of bounded type single and double Vilenkin-Fourier series. Those conditions gave a multiplicative analogue of results due to Gogoladze and Meskhia (see [4]), and Izumi and Izumi (see [8]). They noticed that their results are analogous of the results obtained by Móricz in [9], and Móricz and Veres in [10]. They also discussed the sharpness of some of their results. In 1966, Walker (see [14]) proved Bernstein's original theorem for Lipschitz functions on Vilenkin groups without bounded property. In 1992, Yonis (see [15]) used the Walker's technique to prove a result for the β-absolute convergence of Vilenkin-Fourier series on an arbitrary Vilenkin group. Using the technique of Walker, we prove the analogues of some results of [9], [10], [6], and [13], for single and double Vilenkin-Fourier series on an arbitrary Vilenkin group

2. Notations and definitions

2.1. Single Vilenkin-Fourier series. Let G be a compact, metrizable, 0-dimensional, abelian group. Then the dual group X of G is a countable, discrete, abelian, torsion group. In 1947, Vilenkin developed a part of the Fourier theory on G. He proved the existence of an increasing sequence $\{X_n\}$ of finite subgroups of X and of a sequence $\{\varphi_n\}$ of characters in X such that the following hold.

(1) $X_0 = \{\chi_0\}$, where $\chi_0(x) = 1$ for all $x \in G$.
(2) For each $n \geqslant 1$, X_n/X_{n-1} is of prime order p_n.
(3) $X = \bigcup\limits_{n=0}^{\infty} X_n$.
(4) $\varphi_n \in X_{n+1} \setminus X_n$ for all $n \geqslant 0$.
(5) $\varphi_n^{p_{n+1}} \in X_n$ for all $n \geqslant 0$.

Using these φ_n one can enumerate the elements of X as follows. Let $m_0 = 1$ and let $m_k = \prod\limits_{i=1}^{k} p_i$ for $k \geqslant 1$. If $l \geqslant 1$ and if $l = \sum\limits_{i=0}^{s} a_i m_i$, with $0 \leqslant a_i < p_{i+1}$ if $0 \leqslant i \leqslant s$, then $\chi_l = \varphi_0^{a_0} \dots \varphi_s^{a_s}$. Then $X_k = \{\chi_i\colon\ 0 \leqslant i < m_k\}$. Next, if G_k is the annihilator of X_k, that is,

$$G_k = \{x \in G\colon\ \chi(x) = 1 \text{ for all } \chi \in X_k\},$$

then obviously $G = G_0 \supset G_1 \supset G_2 \supset \ldots, \bigcap_{k=0}^{\infty} G_k = \{0\}$, and the G_k's form a fundamental system of neighborhoods of zero in G. Further, the index of G_k in G is m_k, and since the Haar measure is translation invariant with $m(G) = 1$, one has $m(G_k) = 1/m_k$. The metric on G is then given by

$$d(x, y) = |x - y| \quad \text{for } x, y \in G,$$

where $|x| = 0$ if $x = 0$, and $|x| = 1/m_{k+1}$ if $x \in G_k \setminus G_{k+1}$ for $k = 0, 1, 2, \ldots$

Furthermore, for each $k \geqslant 0$ there exists an $x_k \in G_k \setminus G_{k+1}$ such that $\chi_{m_k}(x_k) = \exp(2\pi i/p_{k+1})$, and each $x \in G$ can be represented uniquely by $x = \sum_{i=0}^{\infty} b_i x_i$ with $0 \leqslant b_i < p_{i+1}$ for all $i \geqslant 0$. Also,

$$G_k = \Big\{ x \in G\colon\ x = \sum_{i=0}^{\infty} b_i x_i, b_0 = \ldots = b_{k-1} = 0 \Big\}.$$

Consequently, each coset of G_k in G can be represented as $z + G_k$, where $z = \sum_{i=0}^{k-1} b_i x_i$ for some choice of the b_i, $0 \leqslant b_i < p_{i+1}$. We shall denote these z, ordered lexicographically, by $z_{q,k}^G$, $0 \leqslant q < m_k$.

Next, let $\mathrm{d}x$ or m denote the normalized Haar measure on G. In this section, f denotes a function from G to $\mathbb{C}$. For $f \in L^1(G)$ the Fourier series of f is the series

$$S[f](x) = \sum_{k=0}^{\infty} \hat{f}(k) \chi_k(x), \tag{2.1}$$

where

$$\hat{f}(k) = \int_G f(t) \overline{\chi}_k(t)\, \mathrm{d}t, \quad k \in \mathbb{N},$$

is the k-th Vilenkin-Fourier coefficient of f.

If $\sup_k p_k = p_0 < \infty$, we refer to G as a bounded group. A group G is said to be primary if $p_i = p$ for all i. If $p_k = 2$ for all k, G is the so-called dyadic group or Walsh group and the elements of its character group X are the Walsh functions (see [1]). We denote this group by W. Note that in this case $m_k = 2^k$ and $G_k = [0, 1/2^k) = W_k$, say. As usual, the space $L^p(G)$, $1 \leqslant p < \infty$, is endowed with the norm $\|f\|_p = \left(\int_G |f(t)|^p \,\mathrm{d}t\right)^{1/p}$.

If $S \subset G$, then *oscillation* of f over S (see, e.g. [11], Definition 1) is defined as

$$\operatorname{osc}(f, S) = \sup\{|f(x) - f(y)|\colon\ x, y \in S\}. \tag{2.2}$$

For $k \in \mathbb{N} \cup \{0\}$, the k-th *modulus of continuity* of f (see, e.g. [11], Definition 2) is defined as

(2.3)
$$\omega(f,k) = \sup\{|(T_h f - f)(x)| \colon x \in G, h \in G_k\}, \quad (T_h f)(x) := f(x+h), \quad x \in G.$$

For $k \in \mathbb{N} \cup \{0\}$, the k-th *local modulus of continuity* of f over the coset $I = y_0 + G_K$ (see, e.g. [3], Definition 2.1) is defined as

$$\omega(f,k,I) = \sup\{|(T_h f - f)(x)| \colon x \in I, h \in G_k\}. \tag{2.4}$$

Note that if $I = y_0 + G_K$, then $x \in I$ if and only if $x - y_0 \in G_K$, and hence for each $k \in \mathbb{N} \cup \{0\}$, it is clear that

$$\omega(f,k,I) \leqslant \omega(f,k) \quad \text{and} \quad \operatorname{osc}(f, z_{q,k}^G + G_k) = \omega(f,k,z_{q,k}^G + G_k). \tag{2.5}$$

For $k \in \mathbb{N} \cup \{0\}$, $f \in L^p(G)$, and $1 \leqslant p < \infty$, the k-th *integral modulus of continuity of order* p (see, e.g. [12], Definition 2.2) is defined as

$$\omega^{(p)}(f,k) = \sup\{\|T_h f - f\|_p \colon h \in G_k\}. \tag{2.6}$$

It is clear that

$$\omega^{(p)}(f,k) \leqslant \omega(f,k), \quad k \in \mathbb{N} \cup \{0\},\ 1 \leqslant p < \infty. \tag{2.7}$$

For $f \in L^p(G)$, $1 \leqslant p < \infty$, the best approximation of f (see [6]) is defined as

$$E^{(p)}(f,n) = \inf\{\|f - Q\|_p \colon Q \in \mathcal{P}_n\}, \quad n \in \mathbb{N}, \tag{2.8}$$

where $\mathcal{P}_n = \{f \in L^1(G) \colon \hat{f}(i) = 0,\ i \geqslant n\}$, $n \in \mathbb{N}$. The best approximation and the modulus of continuity are connected by the inequalities of Efimov (see, e.g. [6], page 107 or [5], §10.5):

$$2^{-1}\omega^{(p)}(f,n) \leqslant E^{(p)}(f,m_n) \leqslant \omega^{(p)}(f,n). \tag{2.9}$$

For a function $f \in L^p(G)$, $k \in \mathbb{N} \cup \{0\}$, and $1 \leqslant p < \infty$, the k-th *local integral modulus of continuity of order* p of f over the coset $I = y_0 + G_K$ (see, e.g. [3], Definition 2.2) is defined as

$$\omega^{(p)}(f,k,I) = \sup\Big\{\Big(\frac{1}{m(I)}\int_I |(T_h f - f)(x)|^p \,\mathrm{d}x\Big)^{1/p} \colon h \in G_k\Big\}. \tag{2.10}$$

For $k \in \mathbb{N} \cup \{0\}$ and $1 \leqslant p < \infty$, it is clear that

$$\omega^{(p)}(f, k, I) \leqslant \omega(f, k, I). \tag{2.11}$$

For $\alpha > 0$ if $\omega(f, k) = O(m_k^{-\alpha})$, then f is said to satisfy a *Lipschitz condition of order* α and this class is denoted by $\mathrm{Lip}\,(\alpha, G)$ (see, e.g. [11], Definition 3). The class $\mathrm{Lip}\,(\alpha, p, G)$ of functions satisfying *Lipschitz condition of order* α, $0 < \alpha \leqslant 1$, *in the mean of order* p, $1 \leqslant p < \infty$ (see [12], Definition 2.3), is defined by

$$\mathrm{Lip}\,(\alpha, p, G) = \{f \in \mathrm{L}^p(G)\colon\ \omega^{(p)}(f, k) = O(m_k^{-\alpha})\}. \tag{2.12}$$

It is clear that

$$\mathrm{Lip}\,(\alpha, G) \subset \mathrm{Lip}\,(\alpha, p, G), \quad 1 \leqslant p < \infty. \tag{2.13}$$

Following Móricz (see [9], page 278) we define the s-bounded fluctuation as follows.

Definition 2.1. A function f is of *s-bounded fluctuation* for some $0 < s < \infty$ on G (in symbols: $f \in \mathrm{BF}_s(G)$) if

$$\mathcal{F}l_s(f, G) := \sup_{k \geqslant 0} \left(\sum_{q=0}^{m_k - 1} (\omega(f, k, z_{q,k}^G + G_k))^s \right)^{1/s} < \infty$$

and $\mathcal{F}l_s(f, G)$ is called the *total s-fluctuation* of f on G.

In view of the equality in (2.5), for $s \geqslant 1$, our Definition 2.1 is equivalent to [11], Definition 4. Also, it is clear that if $f \in \mathrm{BF}_s(G)$, $0 < s < \infty$, then f is bounded on G.

Following the definition of Gogoladge and Meskhia (see [4]), Golubov and Volosivets (see [6], page 108) considered the following definition (see also [9], page 279).

Definition 2.2. A sequence $\{a_k\}$ of positive numbers is said to belong to the *class* $\mathfrak{A}_\gamma(G)$ for some $\gamma \geqslant 1$ if the inequality

$$\left(\sum_{k \in D_\mu^G} a_k^\gamma \right)^{1/\gamma} \leqslant \kappa m_\mu^{(1-\gamma)/\gamma} \sum_{k \in D_{\mu-1}^G} a_k := \kappa m_\mu^{(1-\gamma)/\gamma} A_{\mu-1}^G, \quad \mu \in \mathbb{N} \cup \{0\}, \tag{2.14}$$

is satisfied, where

$$D_\mu^G := \{m_\mu, m_\mu + 1, \ldots, m_{\mu+1} - 1\} \quad \text{for } \mu \in \mathbb{N} \cup \{0\}, \quad \text{and} \quad D_{-1}^G := \{1\}, \tag{2.15}$$

and the constant $\kappa \geqslant 1$ does not depend on μ.

We note that for any bounded group G we have (see, e.g. [13], page 220)

$$\mathfrak{A}_{\gamma_1}(G) \subset \mathfrak{A}_{\gamma_2}(G) \quad \text{for } \gamma_1 > \gamma_2. \tag{2.16}$$

However, this is not true if G is unbounded (see Lemma 3.1).

2.2. Double Vilenkin-Fourier series. Let G be a Vilenkin group as in Section 2.1. Let H be another such Vilenkin group and let the corresponding sequence of primes be $\{q_l\}$. Let Y be the dual group of H with characters ψ_i, $i = 0, 1, 2, \dots$ and $\{Y_l\}$ be the increasing sequence of finite subgroups of Y as in Section 2.1. Then $Y_l = \{\psi_i : 0 \leqslant i < n_l\}$, where $n_l := \prod_{i=1}^{l} q_i$. Let H_l be annihilator of Y_l, that is,

$$H_l = \{y \in H\colon\ \psi(y) = 1 \text{ for all } \psi \in Y_l\}. \tag{2.17}$$

The group $G \times H$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by $\mathrm{d}m(x, y)$. In this section, f will denote a function from $G \times H$ to $\mathbb{C}$. Also, for $y \in H$, $f(\cdot, y)$ denotes the function on G defined by $f(\cdot, y)(x) = f(x, y)$ and for $x \in G$, $f(x, \cdot)$ denotes the function on H defined by $f(x, \cdot)(y) = f(x, y)$.

The two-dimensional Fourier coefficients of $f \in L^1(G \times H)$ are defined as

$$\hat{f}(m, n) := \int_{G\times H} f(x, y)\bar{\chi}_m(x)\overline{\psi}_n(y)\,\mathrm{d}m(x, y), \quad m, n \in \mathbb{N}.$$

We recall the difference operators $\Delta_{1,0}$, $\Delta_{0,1}$, and $\Delta_{1,1}$, which are defined in the usual way as follows:

$$\Delta_{1,0} f(x, y; h_1) := f(x + h_1, y) - f(x, y), \quad \Delta_{0,1} f(x, y; h_2) := f(x, y + h_2) - f(x, y),$$

and

$$\Delta_{1,1} f(x, y; h_1, h_2) := f(x + h_1, y + h_2) - f(x, y + h_2) - f(x + h_1, y) + f(x, y).$$

If $W \times Z \subset G \times H$, then *oscillation of* f over $W \times Z$ (see, e.g. [13], page 220) is defined as

$$\operatorname{osc}(f, W \times Z) = \sup\{|f(x, y) - f(w, y) - f(x, z) + f(w, z)|\colon\ x, w \in W,\ y, z \in Z\}.$$

For $k, l \in \mathbb{N} \cup \{0\}$, the (k, l)-th *modulus of continuity* of f (see, e.g. [6], page 107) is defined as

$$\omega(f, k, l) = \sup\{|\Delta_{1,1} f(x, y; h_1, h_2)|\colon\ h_1 \in G_k, h_2 \in H_l\}, \quad (x, y) \in G \times H. \tag{2.18}$$

For $k, l \in \mathbb{N} \cup \{0\}$, the (k,l)-th *local modulus of continuity* of f over the set $I \times J$, $I = z_0 + G_K$, $J = w_0 + H_L$ is defined as

$$\omega(f,k,l;I\times J) = \sup\{|\Delta_{1,1} f(x,y;h_1,h_2)|\colon\ (x,y)\in I\times J,\ h_1\in G_k,\ h_2\in H_l\}.$$

For $k, l \in \mathbb{N}\cup\{0\}$ and each set $I\times J$, it is easy to verify that $\omega(f,k,l;I\times J) \leqslant \omega(f,k,l)$ and

$$\operatorname{osc}(f,(z^G_{q_1,k}+G_k)\times(z^G_{q_2,l}+H_l)) = \omega(f,k,l,(z^G_{q_1,k}+G_k)\times(z^G_{q_2,l}+H_l)). \tag{2.19}$$

For $k, l \in \mathbb{N}\cup\{0\}$, $f \in L^p(G\times H)$, and $1 \leqslant p < \infty$, the (k,l)-th *integral modulus of continuity of order* p (see [6], page 107) is defined as

$$\omega^{(p)}(f,k,l) = \sup\{\|\Delta_{1,1}(x,y;h_1,h_2)\|_p\colon\ h_1\in G_k,\ h_2\in H_l\}. \tag{2.20}$$

It is clear from the definitions that

$$\omega^{(p)}(f,k,l) \leqslant \omega(f,k,l), \quad k,l\in\mathbb{N}\cup\{0\},\ 1\leqslant p<\infty. \tag{2.21}$$

Following Móricz and Veres (see [10], page 125), the (k,l)-th *local integral modulus of continuity of order* p ($1 \leqslant p < \infty$), of a function $f \in L^p(G\times H)$, over the set $I\times J$, $I = z_0 + G_K$, $J = w_0 + H_L$, $k,l\in\mathbb{N}\cup\{0\}$, is defined as

$$\begin{aligned}&\omega^{(p)}(f,k,l;I\times J)\\ &\quad= \sup\biggl\{\biggl(\frac{1}{m(I\times J)}\int_{I\times J}|\Delta_{1,1}f(x,y;h_1,h_2)|^p\,\mathrm{d}m(x,y)\biggr)^{1/p}\colon\ h_1\in G_k,\ h_2\in H_l\biggr\}.\end{aligned} \tag{2.22}$$

As in the case of one variable, for $k,l\in\mathbb{N}\cup\{0\}$ and $1\leqslant p<\infty$, we have $\omega^{(p)}(f,k,l;I\times J)\leqslant\omega(f,k,l;I\times J)$.

Now, analogously to one variable we introduce the following. For $\alpha,\beta>0$ if $\omega(f,k,l) = O(m_k^{-\alpha}n_l^{-\beta})$, we say that f satisfies a *Lipschitz condition of order* (α,β) and this class is denoted by $\operatorname{Lip}(\alpha,\beta;G\times H)$. We define the class $\operatorname{Lip}(\alpha,\beta,p;G\times H)$ of functions satisfying *Lipschitz condition of order* (α,β), $0<\alpha,\beta\leqslant 1$, in the mean of order p, $1\leqslant p<\infty$, as

$$\operatorname{Lip}(\alpha,\beta,p;G\times H) = \{f\in \mathrm{L}^p(G\times H)\colon\ \omega^{(p)}(f,k,l) = O(m_k^{-\alpha}n_l^{-\beta})\}. \tag{2.23}$$

It is clear that

$$\operatorname{Lip}(\alpha,\beta;G\times H)\subset\operatorname{Lip}(\alpha,\beta,p;G\times H),\quad 0<\alpha,\beta\leqslant 1,\ 1\leqslant p<\infty. \tag{2.24}$$

Similarly to the case of one variable, following Móricz and Veres (see [10]), we have the following definition.

Definition 2.3. Let $0 < s < \infty$. We say that a function f is of *s-bounded fluctuation* (in symbols: $f \in \mathrm{BF}_s(G \times H)$) if the *total s-fluctuation* of f on $G \times H$,

$$\mathcal{F}l_s(f, G \times H) := \sup_{k,l \geqslant 0} \left(\sum_{q_1=0}^{m_k-1} \sum_{q_2=0}^{n_l-1} (\omega(f, k, l, (z_{q_1,k}^G + G_k) \times (z_{q_2,l}^G + H_l)))^s \right)^{1/s} < \infty,$$

and $\mathcal{F}l_s(f, G \times H)$ is called the total s-fluctuation of f on $G \times H$. In view of (2.19), we can replace $\omega(f, k, l, (z_{q_1,k}^G + G_k) \times (z_{q_2,l}^G + H_l))$ by $\operatorname{osc}(f, (z_{q_1,k}^G + G_k) \times (z_{q_2,l}^G + H_l))$ in the above definition of $\mathcal{F}l_s(f, G \times H)$.

Remark 2.1. Likewise the functions on rectangles, if f is such that $\mathcal{F}l_s(f, G \times H) < \infty$, then it is not necessary that f be measurable or bounded. For example, let E be a non-measurable subset of G (such a non-measurable set always exists for any infinite compact abelian group (see [7], 16.13) and hence, in particular, for a Vilenkin group), and define $f(x, y) = \chi_E(x)$, $(x, y) \in G \times H$. Then $\mathcal{F}l_s(f, G \times H) = 0 < \infty$ but f is not measurable, as the set $\{(x, y) \in G \times H \colon f(x, y) \geqslant 1\} = E \times H$ is a non-measurable set, because E is a non-measurable set. Further, let $f(x, y) = 1/|x|$ for $0 \neq x \in G$ and $f(0, y) = 0$ for all $y \in H$, where $|\cdot|$ is as defined in Section 2.1. Then $\mathcal{F}l_s(f, G \times H) = 0 < \infty$, but f is not bounded as $f(x, y) \to \infty$ as $k \to \infty$ for $x \in G_k \setminus G_{k+1}$. However, if f is such that $\mathcal{F}l_s(f, G \times H) < \infty$ and for a fixed $(x_0, y_0) \in G \times H$, $\mathcal{F}l_s(f(x_0, \cdot), H) < \infty$ and $\mathcal{F}l_s(f(\cdot, y_0), H) < \infty$, then f is bounded. Indeed, for $(x, y) \in G \times H$ we have

$$\begin{aligned} (2.25) \qquad |f(x, y)| &\leqslant |f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)| + |f(x_0, y_0)| \\ &\quad + |f(x_0, y) - f(x_0, y_0)| + |f(x, y_0) - f(x_0, y_0)| \\ &\leqslant \operatorname{osc}(f, G_0 \times H_0) + \operatorname{osc}(f(x_0, \cdot), H_0) \\ &\quad + \operatorname{osc}(f(\cdot, y_0), G_0) + |f(x_0, y_0)| \\ &\leqslant \mathcal{F}l_s(f, G \times H) + \mathcal{F}l_s(f(x_0, \cdot), H) \\ &\quad + \mathcal{F}l_s(f(\cdot, y_0), G) + |f(x_0, y_0)| \\ &< \infty. \end{aligned}$$

Therefore f is bounded on $G \times H$.

Volosivets and Kuznetsova (see [13]) gave an analogue of Waterman's well-known definition of bounded Λ-variation as follows.

Definition 2.4 ([13]). Let $p \geqslant 1$ and $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ and $\Psi = \{\psi_j\}_{j=1}^{\infty}$ be two nondecreasing sequences of positive numbers such that $\Lambda_n = \sum\limits_{i=1}^{n} \lambda_i^{-1}$ and $\Psi_n = \sum\limits_{i=1}^{n} \psi^{-1}$ tend to infinity as $n \to \infty$.

Let $f(x,y)$ be bounded on $G\times H$. For fixed $k,l\in\mathbb{N}$, let

$$(2.26)\quad \mathcal{V}_{\Lambda,\Psi,p}(f,k,l):=\sup\Biggl(\sum_{i=0}^{m_k-1}\sum_{j=0}^{n_l-1}\frac{(\operatorname{osc}(f,(z^G_{q_{\alpha_i},k}+G_k)\times(z^H_{q_{\beta_j},l}+H_l)))^p}{\lambda_{i+1}\psi_{j+1}}\Biggr)^{1/p},$$

where the supremum in the formula for $\mathcal{V}$ is taken over all permutations $\{\alpha_i\}_{i=1}^{m_k}$ and $\{\beta_j\}_{j=1}^{n_l}$ of the index sets $\{0,1,\dots,m_k-1\}$ and $\{0,1,\dots,n_l-1\}$. If

$$(2.27)\qquad V_{\Lambda,\Psi,p}(f,G\times H):=\sup\{\mathcal{V}_{\Lambda,\Psi,p}(f,k,l)\colon\ k,l\in\mathbb{N}\}<\infty,$$

then we say that $f\in(\Lambda,\Psi)\mathcal{F}l_p(G\times H)$.

For $G=W$, a two-dimensional analogue of the class $\mathfrak{A}_\gamma(G)$ (see Definition 2.2) was defined by Móricz and Veres in [10], page 127. Their definition is a particular case of the following definition given by Golubov and Volosivets (see [6]) in the case when $G=W$.

Definition 2.5. Let $\{a_{kl}\colon\ k,l=1,2,\dots\}$ be a double sequence of positive numbers and $\gamma\geqslant 1$. If for arbitrary $\mu,\nu\in\mathbb{N}\cup\{0\}$ the inequality

$$(2.28)\qquad \Biggl(\sum_{k\in D^G_\mu}\sum_{l\in D^G_\nu}a^\gamma_{kl}\Biggr)^{1/\gamma}\leqslant C(m_\mu m_\nu)^{(1-\gamma)/\gamma}\sum_{k\in D^G_{\mu-1}}\sum_{l\in D^G_{\nu-1}}a_{kl}$$

is satisfied, where D^G_μ is as in (2.15) and the constant $\kappa\geqslant 1$ does not depend on μ or ν, then $\{a_{kl}\}$ is said to belong to the class $A^*(\gamma,2)$.

Analogously to the class $A^*(\gamma,2)$, defined above, we define the class $\mathfrak{A}^*_\gamma(G\times H)$ as follows.

Definition 2.6. A sequence $\{a_{kl}\}$ of positive numbers is said to belong to the class $\mathfrak{A}^*_\gamma(G\times H)$ for some $\gamma\geqslant 1$ if the inequality

$$(2.29)\qquad \Biggl(\sum_{k\in D^G_\mu}\sum_{l\in D^H_\nu}a^\gamma_{kl}\Biggr)^{1/\gamma}\leqslant \kappa(m_\mu n_\nu)^{(1-\gamma)/\gamma}\sum_{k\in D^G_{\mu-1}}\sum_{l\in D^H_{\nu-1}}a_{kl}$$
$$:=\kappa(m_\mu n_\nu)^{(1-\gamma)/\gamma}A^*_{\mu-1,\nu-1},\quad \mu,\nu\in\mathbb{N}\cup\{0\},$$

is satisfied, where D^G_μ is as in (2.15),

$$(2.30)\quad D^H_\nu:=\{n_\nu,n_\nu+1,\dots,n_{\nu+1}-1\}\quad\text{for }\nu\in\mathbb{N}\cup\{0\},\quad\text{and}\quad D^H_{-1}:=\{1\},$$

and the constant $\kappa\geqslant 1$ does not depend on μ or ν.

We note that the class $\mathfrak{A}^*_\gamma(G \times H)$ is a generalization of the class $\mathfrak{A}_\gamma(G)$, $\gamma \geqslant 1$, defined by Móricz and Veres (see [10], page 127).

In this paper, we shall prove certain results analogous to the results proved by Móricz in [9], and Móricz and Veres in [10] for the single and double Vilenkin-Fourier series, respectively. We shall also prove some results analogous to the results proved by Golubov and Volosivets in [6], and Volosivets and Kuznetsova in [13] for arbitrary, bounded or unbounded Vilenkin group. In what follows, C denote a positive constant, which may not have the same value at each occurrence.

3. Results

3.1. Single Vilenkin-Fourier series. Our first result is the following example, which shows that (2.16) does not hold if we replace a bounded Vilenkin group by an unbounded Vilenkin group.

Example 3.1. If G is unbounded, then there exists $\{a_n\} \in \mathfrak{A}_2(G)$ such that $\{a_n\} \notin \mathfrak{A}_1(G)$.

Our next result is a Vilenkin group analogue of a result of Móricz, see [9], Theorem 1. Our theorem also gives an analogue of a result of Golubov and Volosivets (see [6], Corollary 1) for any Vilenkin group.

Theorem 3.1. *If $f \in L^p(G)$ for some $1 < p \leqslant 2$ and*

$$\{a_n\} \in \mathfrak{A}_{p/(p-rp+r)}(G) \quad \text{for some } 0 < r < q, \tag{3.1}$$

where $1/p + 1/q = 1$, then

$$\sum_{n=1}^{\infty} a_n |\hat{f}(n)|^r \leqslant 2^{-r/2} \kappa \sum_{\mu=0}^{\infty} m_\mu^{-r/q} A^G_{\mu-1} (\omega^{(p)}(f,\mu))^r, \tag{3.2}$$

where κ is from (2.14) corresponding to $\gamma = p/(p-rp+r)$. In particular, if the series on the right-hand side of (3.2) converges, then

$$\sum_{n=1}^{\infty} a_n |\hat{f}(n)|^r < \infty. \tag{3.3}$$

Remark 3.1. Theorem 3.1 is proved in a similar way, except for a few steps, by Golubov and Volosivets (see [6], Corollary 1, and the proof of Theorem 1). They used the boundedness of G to prove this result. However, our proof will work for any group, whether it is bounded or unbounded. To prove this result for arbitrary group, we will use the technique used by Walker in [14].

Corollary 3.1. *If the hypotheses of Theorem* 3.1 *hold and the series*

$$\sum_{n=0}^{\infty} a_n n^{-r/q} (E^{(p)}(f,n))^r \tag{3.4}$$

converges, then (3.3) *holds.*

Corollary 3.2. *If* $f \in L^2(G)$, G *is bounded, and* $\sum\limits_{\mu=1}^{\infty} m_\mu^{1/2} \omega^{(2)}(f,\mu) < \infty$, *then* $\sum\limits_{n=1}^{\infty} |\hat{f}(n)| < \infty$.

We note that Corollary 3.2 is not true for an unbounded Vilenkin group G (see [12], Corollary 4.2 for $p = 2$).

Remark 3.2. Our Corollary 3.1 is an analogue of Corollary 2 of [6] for any Vilenkin group. Since Theorem 2 of [6] shows the unimprovability of Corollary 2 of [6], it shows that our Corollary 3.1 is also unimprovable for any Vilenkin group.

It is worth formulating Theorem 3.1 in the particular case when $f \in \mathrm{Lip}\,(\alpha, p, G)$ and $a_n \equiv 1$.

Corollary 3.3. *If* $f \in \mathrm{Lip}\,(\alpha, p, G)$ *for some* $\alpha > 0$, $1 < p \leqslant 2$, $1/p + 1/q = 1$, G *is bounded, and if*

$$\frac{q}{1+\alpha q} < r < q, \tag{3.5}$$

then

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^r < \infty. \tag{3.6}$$

For an unbounded group G, Corollary 3.3 is already known due to Younis, see [15], Theorem 3.1 (Actually only the condition $p/(p + \alpha p - 1) < q$ was used in the proof of Theorem 3.1 of [15]).

Next, we formulate Theorem 3.1 in the particular case when $f \in \mathrm{Lip}\,(\alpha, p, G)$, $a_n = n^\delta$ and $r = 1$.

Corollary 3.4. *If* $f \in \mathrm{Lip}\,(\alpha, p, G)$ *for some* $\alpha > 0$, $1 < p \leqslant 2$, G *is bounded, and if* $\delta \in \mathbb{R}$ *is such that*

$$\delta < \alpha - \frac{1}{p}, \tag{3.7}$$

then

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| < \infty. \tag{3.8}$$

For functions of the narrower class Lip (α, G) and $p = 2$, Corollaries 3.3 and 3.4 are proved by Onneweer in [11]. For an unbounded group G, a proof of Corollary 3.4 can be given similarly to the proof of Theorem 3.1 in [15], now considering $\varphi(k) = \sum_{T_k} n^{\delta}|\hat{f}|$ and applying Hölder's inequality.

Our next theorem is formulated in terms of the n-th integral modulus of continuity of order p over the cosets, which is a Vilenkin group analogous of Theorem 2 of [9].

Theorem 3.2. *Let f and $\{a_n\}$ be as in Theorem 3.1. Then we have*

$$\sum_{n=1}^{\infty} a_n|\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-r} A_{\mu-1}^G \Bigg(\sum_{k=0}^{m_\mu - 1} (\omega^{(p)}(f, \mu, z_{k,\mu}^G + G_\mu))^p \Bigg)^{r/p}, \tag{3.9}$$

where κ is from (2.14) corresponding to $\gamma = p/(p - rp + r)$.

Our next result is formulated in the following theorem, which is an analogue of Theorem 3 of [6] for any (unbounded) Vilenkin group.

Theorem 3.3. *Let f be a measurable function on G. If $1 < p' < \infty$, $1/p' + 1/q' = 1$, $1 \leqslant \beta < 2p'$, $\mathcal{F}l_\beta(f, G) < \infty$, and*

$$\{a_n\} \in \mathfrak{A}_{2/(2-r)}(G) \quad \text{for some } 0 < r < 2, \tag{3.10}$$

then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n|\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa(\mathcal{F}l_\beta(f, G))^{\beta r/(2p')} \\ \times \sum_{\mu=0}^{\infty} m_\mu^{-r/2 - r/(2p')} (\omega^{(\beta + (2-\beta)q')}(f, \mu))^{r - \beta r/(2p')} A_{\mu-1}^G. \end{aligned} \tag{3.11}$$

where κ is from (2.14) corresponding to $\gamma = 2/(2 - r)$. In particular, if the series on the right-hand side of (3.11) converges, then (3.3) holds.

Our next result is a Vilenkin group analogue of Theorem 3 of [9].

Theorem 3.4. *If f is a measurable function on G, $f \in \mathrm{BF}_s(G)$ for some $0 < s < 2$, and if $\{a_n\}$ satisfies (3.10), then*

$$\sum_{n=1}^{\infty} a_n|\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f, G))^{rs/2} \sum_{\mu=0}^{\infty} m_\mu^{-r} A_{\mu-1}^G (\omega(f, \mu))^{(2-s)r/2}, \tag{3.12}$$

where κ is from (2.14) corresponding to $\gamma = 2/(2 - r)$ and $\mathcal{F}l_s(f, G)$ is as in Definition 2.1. In particular, if the series on the right-hand side of (3.12) converges, then (3.3) holds.

We formulate Theorem 3.4 in the particular case when $f \in \mathrm{Lip}\,(\alpha, G) \cap \mathrm{BF}_s(G)$, G is bounded, and $a_n \equiv 1$, and obtain a Vilenkin group analogue of [9], Corollary 3.

Corollary 3.5. *If* $f \in \mathrm{Lip}\,(\alpha, G) \cap \mathrm{BF}_s(G)$ *for some* $\alpha > 0$, $0 < s < 2$, G *is bounded, and if*

$$r > \frac{2}{2 + \alpha(2 - s)}, \tag{3.13}$$

then (3.6) *is satisfied.*

Finally, we formulate Theorem 3.4 in the particular case when G is bounded, $r = 1$, and $a_n = n^{\delta}$, and obtain a Vilenkin group analogue of [9], Corollary 4.

Corollary 3.6. *If* $f \in \mathrm{Lip}\,(\alpha, G) \cap \mathrm{BF}_s(G)$ *for some* $\alpha > 0$, $0 < s < 2$, G *is bounded, and if* $\delta \in \mathbb{R}$ *is such that*

$$\delta < \frac{\alpha(2 - s)}{2}, \tag{3.14}$$

then (3.8) *is satisfied.*

3.2. Double Vilenkin-Fourier series. For a double Vilenkin-Fourier series, our first result is the following theorem which is a Vilenkin-Fourier series analogue of a result of Móricz and Veres (see [10], Theorem 1) and a two-dimensional analogue of Theorem 1 of Section 3.1.

Theorem 3.5. *Suppose* $f \in L^p(G \times H)$ *for some* $1 < p \leqslant 2$. *If*

$$\{a_{mn}\} \in \mathfrak{A}^*_{p/(p-rp+r)}(G \times H) \tag{3.15}$$

for some $0 < r < q$, *where* $1/p + 1/q = 1$, *then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} |\hat{f}(m,n)|^r \leqslant 2^{-r} \kappa \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} (m_\mu n_\nu)^{-r/q} A^*_{\mu-1,\nu-1} (\omega^{(p)}(f, \mu, \nu))^r, \tag{3.16}$$

where κ *is from* (2.29) *corresponding to* $\gamma = p/(p - rp + r)$. *In particular, if the series on the right-hand side of* (3.16) *converges, then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} |\hat{f}(m,n)|^r < \infty. \tag{3.17}$$

When G and H are bounded, we have the following corollaries, analogous to [10], Corollaries 1 and 2.

Corollary 3.7. *Suppose G and H are bounded, $f \in \mathrm{Lip}\,(\alpha, \beta, p; G \times H)$ for some $\alpha, \beta > 0$, and $1 < p \leqslant 2$. If*

$$\frac{q}{1 + q\min\{\alpha, \beta\}} < r < q, \tag{3.18}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\hat{f}(m,n)|^r < \infty. \tag{3.19}$$

Corollary 3.8. *Suppose G and H are bounded, $f \in \mathrm{Lip}\,(\alpha, \beta, p; G \times H)$ for some $\alpha, \beta > 0$, and $1 < p \leqslant 2$. If $\delta_1, \delta_2 \in \mathbb{R}$ are such that*

$$\delta_1 < \alpha - \frac{1}{p} \quad \text{and} \quad \delta_2 < \beta - \frac{1}{p}, \tag{3.20}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\delta_1} n^{\delta_2} |\hat{f}(m,n)| < \infty. \tag{3.21}$$

Our next result is the following theorem, which is a Vilenkin group analogue of a result of Móricz and Veres (see [10], Theorem 2) and a two-dimensional analogue of Theorem 3.2 of Section 3.1.

Theorem 3.6. *Let f and $\{a_{mn}\}$ be as in Theorem 3.5. Then we have*

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} |\hat{f}(m,n)|^r \\ &\qquad \leqslant 2^{-r}\kappa \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} (m_\mu n_\nu)^{-r} A^*_{\mu-1,\nu-1} \\ &\qquad\quad \times \left(\sum_{k_1=0}^{m_\mu - 1} \sum_{k_2=0}^{n_\nu - 1} (\omega^{(p)}(f, \mu, \nu; (z^G_{k_1,\mu} + G_\mu) \times (z^H_{k_2,\nu} + H_\nu)))^p \right)^{r/p}, \end{aligned} \tag{3.22}$$

where κ is from (2.29) corresponding to $\gamma = p/(p - rp + r)$. In particular, if the series on the right-hand side of (3.22) converges, then (3.17) holds.

Our next result is a two-dimensional analogue of Theorem 3.3 of Section 3.1.

Theorem 3.7. *Let f be a measurable function on $G \times H$. If $1 < p' < \infty$, $1/p' + 1/q' = 1$, $1 \leqslant \beta < 2p'$, $\mathcal{F}l_\beta(f, G\times H) < \infty$, $\mathcal{F}l_\beta(f(\cdot,0), G) < \infty$, $\mathcal{F}l_\beta(f(0,\cdot), H) < \infty$, and*

$$\{a_{mn}\} \in \mathfrak{A}^*_{2/(2-r)}(G \times H) \quad \text{for some } 0 < r < 2, \tag{3.23}$$

then

$$\begin{aligned}\sum_{m=1}^\infty \sum_{n=1}^\infty & a_{mn} |\hat{f}(m,n)|^r \\ &\leqslant 2^{-r} \kappa (\mathcal{F}l_\beta(f, G \times H))^{\beta r/(2p')} \\ &\quad \times \sum_{\mu=0}^\infty \sum_{\nu=0}^\infty (m_\mu n_\nu)^{-r/(2p')-r/2} (\omega^{(\beta+(2-\beta)q')}(f,\mu,\nu))^{r-\beta r/(2p')} A^*_{\mu-1,\nu-1},\end{aligned} \tag{3.24}$$

where κ is from (2.29) *corresponding to $\gamma = 2/(2-r)$. In particular, if the series on the right-hand side of* (3.24) *converges, then* (3.17) *holds.*

Our next result is a Vilenkin group analogue of Theorem 3 of [10] and a two-dimensional analogue of Theorem 3.4 of Section 3.1.

Theorem 3.8. *If f is a measurable function on $G \times H$, $f \in \mathrm{BF}_s(G \times H)$, $f(\cdot, 0) \in \mathrm{BF}_s(G)$, and $f(0,\cdot) \in \mathrm{BF}_s(H)$ for some $0 < s < 2$, and if $\{a_{mn}\}$ satisfies* (3.23), *then*

$$\begin{aligned}\sum_{m=1}^\infty \sum_{n=1}^\infty & a_{mn} |\hat{f}(m,n)|^r \\ &\leqslant 2^{-r} \kappa (\mathcal{F}l_s(f, G \times H))^{rs/2} \sum_{\mu=0}^\infty \sum_{\nu=0}^\infty (m_\mu n_\nu)^{-r} (\omega(f,\mu,\nu))^{(2-s)r/2} A^*_{\mu-1,\nu-1},\end{aligned} \tag{3.25}$$

where κ is from (2.29) *corresponding to $\gamma = 2/(2-r)$. In particular, if the series on the right-hand side of* (3.25) *converges, then* (3.17) *holds.*

We formulate Theorem 3.8 in the particular case when G and H are bounded, $f \in \mathrm{Lip}\,(\alpha, \beta; G \times H) \cap \mathrm{BF}_s(G \times H)$, and $a_{mn} \equiv 1$, and obtain a Vilenkin group analogue of [10], Corollary 3.

Corollary 3.9. *If $f \in \mathrm{Lip}\,(\alpha, \beta; G \times H) \cap \mathrm{BF}_s(G \times H)$, $f(\cdot,0) \in \mathrm{BF}_s(G)$, and $f(0,\cdot) \in \mathrm{BF}_s(H)$ for some $\alpha, \beta > 0$, $0 < s < 2$, G and H are bounded, and if*

$$r > \frac{2}{2 + \min\{\alpha,\beta\}(2-s)}, \tag{3.26}$$

then (3.19) *is satisfied.*

Finally, we formulate Theorem 3.8 in the particular case when G and H are bounded, $r=1$, and $a_{mn}=m^{\delta_1}n^{\delta_2}$, and obtain a Vilenkin group analogue of [10], Corollary 4.

Corollary 3.10. *If $f\in \operatorname{Lip}(\alpha,\beta;G\times H)\cap \mathrm{BF}_s(G\times H)$, $f(\cdot,0)\in \mathrm{BF}_s(G)$, and $f(0,\cdot)\in \mathrm{BF}_s(H)$ for some $\alpha,\beta>0$, $0<s<2$, G and H are bounded, and if*

$$\delta_1<\frac{\alpha(2-s)}{2}\quad\text{and}\quad \delta_2<\frac{\beta(2-s)}{2}, \tag{3.27}$$

then (3.21) is satisfied.

Our last result is the following theorem, which is an analogue of [13], Theorem 6 for any (unbounded) Vilenkin group.

Theorem 3.9. *Let $p',q'>1$, $1/p'+1/q'=1$, $1\leqslant\beta<p'+1$, f be measurable on $G\times H$, $f\in(\Lambda,\Psi)\mathcal{F}l_{p'}(G\times H)$, $0<r<2$, and $\{a_{mn}\}\in\mathfrak{A}^*(2/(2-r),2)$. If the series*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\Bigl(\frac{(\omega^{((2-\beta)q'+\beta)}(f,k,l))^{2p'-\beta}}{\Lambda_{m_k}\Psi_{n_l}}\Bigr)^{r/(2p')}(m_kn_l)^{-r/2}A^*_{k-1,l-1} \tag{3.28}$$

converges, then the series $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty}a_{mn}|\hat f(m,n)|^r$ also converges.

4. Proof of results

4.1. Single Vilenkin-Fourier series. We need the following lemma, which gives examples of certain sequences in $\mathfrak{A}_\gamma(G)$. This lemma is already known (see, e.g. [13]).

Lemma 4.1. *If G is bounded, then $\{k^\beta\}\in\mathfrak{A}_\gamma(G)$ for all $\beta\in\mathbb{R}$ and $\gamma\geqslant 1$.*

P r o o f of Example 3.1. Let G be any unbounded group. Then there is an increasing sequence $\{r_k\}$ of natural numbers such that $p_{r_k}\to\infty$. Now, we consider the ordered sets $A=\{r_k\in\mathbb{N}\colon r_k \text{ is even}\}$ and $B=\{r_k\in\mathbb{N}\colon r_k\text{ is odd}\}$. Then either A is infinite or B is infinite.

Case I. A is infinite. Rename the elements of A by $n_1,n_2,\dots$ Then each n_k is even and $p_{n_k}\to\infty$ as $k\to\infty$. Let $\{a_n\}$ be defined as follows. For $m_k\leqslant n<m_{k+1}$, that is, for $n\in D_k^G$, $k\in\mathbb{N}\cup\{0\}$, let

$$a_n=\begin{cases}\dfrac{1}{(p_{k+2}-1)^{1/2}(m_{k+1}-m_k)} & \text{if } k \text{ is even,}\\[2ex] \dfrac{1}{m_{k+1}-m_k} & \text{if } k \text{ is odd.}\end{cases} \tag{4.1}$$

Note that for any $\mu \in \mathbb{N} \cup \{0\}$ we have

$$\sum_{k \in D^G_{2\mu}} a_k = \sum_{k=m_{2\mu}}^{m_{2\mu+1}-1} \frac{1}{(p_{2\mu+2}-1)^{1/2}(m_{2\mu+1}-m_{2\mu})} \tag{4.2}$$
$$= \frac{1}{(p_{2\mu+2}-1)^{1/2}(m_{2\mu+1}-m_{2\mu})} \sum_{k=m_{2\mu}}^{m_{2\mu}+1} 1$$
$$= \frac{1}{(p_{2\mu+2}-1)^{1/2}(m_{2\mu+1}-m_{2\mu})}(m_{2\mu+1}-m_{2\mu})$$
$$= \frac{1}{(p_{2\mu+2}-1)^{1/2}}$$

and

$$\sum_{k \in D^G_{2\mu+1}} a_k = \sum_{k=m_{2\mu+1}}^{m_{2\mu+2}-1} a_k = \sum_{k=m_{2\mu+1}}^{m_{2\mu+2}-1} \frac{1}{m_{2\mu+2}-m_{2\mu+1}} \tag{4.3}$$
$$= \frac{1}{m_{2\mu+2}-m_{2\mu+1}} \sum_{k=m_{2\mu+1}}^{m_{2\mu+2}-1} 1$$
$$= \frac{1}{m_{2\mu+2}-m_{2\mu+1}}(m_{2\mu+2}-m_{2\mu+1}) = 1.$$

As n_μ is even for each $\mu \in \mathbb{N}$ and $p_{n_\mu} \to \infty$ as $\mu \to \infty$, in view of (4.2) and (4.3), we have

$$\frac{\sum_{k \in D^G_{2(n_\mu/2-1)+1}} a_k}{\sum_{k \in D^G_{2(n_\mu/2-1)}} a_k} = \frac{1}{1/(p_{2(n_\mu/2-1)+2}-1)^{1/2}} = (p_{n_\mu}-1)^{1/2} \to \infty \quad \text{as } \mu \to \infty.$$

Hence, there cannot exist any κ such that $\sum\limits_{k \in D^G_\mu} a_k \leqslant \kappa \sum\limits_{k \in D^G_{\mu-1}} a_k$ for all $\mu \geqslant 0$. Thus, $\{a_n\} \notin \mathfrak{A}_1(G)$. Now, we show that $\{a_n\} \in \mathfrak{A}_2(G)$. Note that

$$\Big(\sum_{k \in D^G_0} a_k^2\Big)^{1/2} = \Big(\sum_{k=m_0}^{m_1-1} \frac{1}{(p_2-1)^{1/2}(m_1-m_0)^2}\Big)^{1/2} \tag{4.4}$$
$$= \Big(\frac{1}{(p_2-1)(m_1-m_0)^2}(m_1-m_0)\Big)^{1/2}$$
$$= \frac{1}{(p_2-1)^{1/2}(m_1-m_0)^{1/2}}$$
$$= (m_1-m_0)^{1/2}\frac{1}{(p_2-1)^{1/2}(m_1-m_0)}$$
$$= (p_1-1)^{1/2}a_1 = (p_1-1)^{1/2} \sum_{k \in D^G_{-1}} a_k.$$

Next, for $\mu \in \mathbb{N}$, in view of (4.3), we have

$$\begin{aligned}(4.5)\qquad \Big(\sum_{k\in D^G_{2\mu}} a_k^2\Big)^{1/2} &= \Big(\sum_{k=m_{2\mu}}^{m_{2\mu+1}-1} \frac{1}{((p_{2\mu+2}-1)^{1/2}(m_{2\mu+1}-m_{2\mu}))^2}\Big)^{1/2}\\ &= \Big(\frac{1}{(p_{2\mu+2}-1)(m_{2\mu+1}-m_{2\mu})^2}\,(m_{2\mu+1}-m_{2\mu})\Big)^{1/2} \times 1\\ &= \frac{1}{(p_{2\mu+2}-1)^{1/2} m_{2\mu}^{1/2}(p_{2\mu+1}-1)^{1/2}} \sum_{k\in D^G_{2\mu-1}} a_k\\ &\leqslant \frac{1}{m_{2\mu}^{1/2}} \sum_{k\in D^G_{2\mu-1}} a_k \leqslant (p_1-1)^{1/2} m_{2\mu}^{(1-2)/2} \sum_{k\in D^G_{2\mu-1}} a_k\end{aligned}$$

and for $\mu \in \mathbb{N}\cup\{0\}$, in view of (4.2), we have

$$\begin{aligned}(4.6)\qquad \Big(\sum_{k\in D^G_{2\mu+1}} a_k^2\Big)^{1/2} &= \Big(\sum_{k=m_{2\mu+1}}^{m_{2\mu+2}-1} \frac{1}{(m_{2\mu+2}-m_{2\mu+1})^2}\Big)^{1/2}\\ &= \Big(\frac{1}{(m_{2\mu+2}-m_{2\mu+1})^2}(m_{2\mu+2}-m_{2\mu+1})\Big)^{1/2} \times 1\\ &= \frac{1}{m_{2\mu+1}^{1/2}(p_{2\mu+2}-1)^{1/2}}(p_{2\mu+2}-1)^{1/2} \sum_{k\in D^G_{2\mu}} a_k\\ &= \frac{1}{m_{2\mu+1}^{1/2}} \sum_{k\in D^G_{2\mu}} a_k \leqslant (p_1-1)^{1/2} m_{2\mu+1}^{(1-2)/2} \sum_{k\in D^G_{2\mu}} a_k.\end{aligned}$$

From (4.4)–(4.6) it follows that $\{a_n\} \in \mathfrak{A}_2(G)$.

Case II. B is infinite. Rename the elements of B as $n_1, n_2, \dots$ Then each n_k is odd and $p_{n_k} \to \infty$ as $k \to \infty$. Let $\{a_n\}$ be defined as follows. For $m_k \leqslant n < m_{k+1}$, that is, for $n \in D^G_k$, $k \in \mathbb{N}\cup\{0\}$, let

$$(4.7)\qquad a_n = \begin{cases} \dfrac{1}{m_{k+1}-m_k} & \text{if } k \text{ is even,}\\[2ex] \dfrac{1}{(p_{k+2}-1)^{1/2}(m_{k+1}-m_k)} & \text{if } k \text{ is odd.}\end{cases}$$

Note that for any $\mu \in \mathbb{N}\cup\{0\}$ we have

$$(4.8)\qquad \sum_{k\in D^G_{2\mu}} a_k = \sum_{k=m_{2\mu}}^{m_{2\mu+1}-1} a_k = \sum_{k=m_{2\mu}}^{m_{2\mu+1}-1} \frac{1}{m_{2\mu+1}-m_{2\mu}} = 1$$

and

$$(4.9)\qquad \sum_{k\in D^G_{2\mu+1}} a_k = \sum_{k=m_{2\mu+1}}^{m_{2\mu+2}-1} \frac{1}{(p_{2\mu+3}-1)^{1/2}(m_{2\mu+2}-m_{2\mu+1})} = \frac{1}{(p_{(2\mu+1)+2}-1)^{1/2}}.$$

As n_μ is odd for each $\mu\in\mathbb{N}$ and $p_{n_\mu}\to\infty$ as $\mu\to\infty$, in view of (4.8) and (4.9), for $\mu\geqslant 2$, we have

$$\frac{\sum_{k\in D^G_{2((n_\mu-1)/2)}} a_k}{\sum_{k\in D^G_{2((n_\mu-1)/2)-1}} a_k} = \frac{1}{(p_{2((n_\mu-1)/2)-1+2}-1)^{-1/2}} = (p_{n_\mu}-1)^{1/2}\to\infty \quad \text{as } \mu\to\infty.$$

Hence, there cannot exist any κ such that $\sum\limits_{k\in D^G_\mu} a_k \leqslant \kappa \sum\limits_{k\in D^G_{\mu-1}} a_k$ for all $\mu\geqslant 0$. Thus, $\{a_n\}\notin \mathfrak{A}_1(G)$. Now, proceeding as in Case I, we can show that $\{a_n\}\in\mathfrak{A}_2(G)$. Thus, in any case, we have a sequence $\{a_n\}\in\mathfrak{A}_2(G)$ such that $\{a_n\}\notin\mathfrak{A}_1(G)$. $\square$

P r o o f of Theorem 3.1. Fix $\mu\in\mathbb{N}\cup\{0\}$, $h_1\in G_\mu\setminus G_{\mu+1}$, and set

$$(4.10)\qquad g(x) := f(x+h_1)-f(x),\quad x\in G.$$

Then for $n\in\mathbb{N}$ we have

$$(4.11)\qquad \begin{aligned}\hat{g}(n) &= \int_G g(x)\overline{\chi}_n(x)\,\mathrm{d}x = \int_G (f(x+h_1)-f(x))\,\overline{\chi}_n(x)\,\mathrm{d}x\\ &= \int_G f(x)\overline{\chi}_n(x-h_1)\,\mathrm{d}x - \hat{f}(n) = \int_G f(x)\overline{\chi}_n(x)\overline{\chi}_n(-h_1)\,\mathrm{d}x - \hat{f}(n)\\ &= \chi_n(h_1)\hat{f}(n)-\hat{f}(n) = (\chi_n(h_1)-1)\hat{f}(n).\end{aligned}$$

Note that

$$o(G_\mu/G_{\mu+1}) = \frac{o(G/G_{\mu+1})}{o(G/G_\mu)} = \frac{m_{\mu+1}}{m_\mu} = p_{\mu+1}.$$

Since $p_{\mu+1}$ is prime, it follows that $G_\mu/G_{\mu+1}$ is cyclic and that every element other than the identity element is its generator. That is,

$$G_\mu/G_{\mu+1} = \langle h_0+G_{\mu+1}\rangle \quad \forall\, h_0\in G_\mu\setminus G_{\mu+1}.$$

Since $h_1\in G_\mu\setminus G_{\mu+1}$, $h_1+G_{\mu+1}$ is a generator of the group $G_\mu/G_{\mu+1}$. We shall show that if $\chi\in X_{\mu+1}\setminus X_\mu$, then $\chi(h_1)\neq 1$. Let $\chi\in X_{\mu+1}\setminus X_\mu$. If possible, suppose $\chi(h_1)=1$. Let $z\in G_\mu$. Then $z+G_{\mu+1}\in G_\mu/G_{\mu+1}=\langle h_1+G_{\mu+1}\rangle$. So, there is an integer k depending on z such that $z+G_{\mu+1}=k(h_1+G_{\mu+1})=kh_1+G_{\mu+1}$. That is, $z-kh_1\in G_{\mu+1}$. Hence, there exists $z'\in G_{\mu+1}$ such that $z-kh_1=z'$. Therefore

$$\chi(z)=\chi(kh_1+z')=\chi(h_1)^k\chi(z')=1^k\chi(z')=\chi(z').$$

Now, as $\chi \in X_{\mu+1}$ and $z' \in G_{\mu+1}$, by definition of $G_{\mu+1}$, $\chi(z') = 1$. Therefore $\chi(z) = 1$. Since $z \in G_\mu$ was arbitrary, $\chi(z) = 1$ for all $z \in G_\mu$. Hence, by definition of G_μ, $\chi \in X_\mu$. This is a contradiction. So, if $\chi \in X_{\mu+1} \setminus X_\mu$, then $\chi(h_1) \neq 1$.

Note that for $\chi \in X_{\mu+1}$, $\chi X_\mu \in X_{\mu+1}/X_\mu$. Also, $o(X_{\mu+1}/X_\mu) = m_{\mu+1}/m_\mu = p_{\mu+1}$. Therefore

$$\chi^{p_{\mu+1}} X_\mu = (\chi X_\mu)^{p_{\mu+1}} = (\chi X_\mu)^{o(X_{\mu+1}/X_\mu)} = \chi_0 X_\mu = X_\mu$$

and hence $\chi^{p_{\mu+1}} \in X_\mu$. Since $h_1 \in G_\mu$, by definition of G_μ, we have $\chi^{p_{\mu+1}}(h_1) = 1$. Therefore

$$\chi(h_1) = \mathrm{e}^{2\pi \mathrm{i} k/p_{\mu+1}} \tag{4.12}$$

for some $1 \leqslant k < p_{\mu+1}$ and k depends on χ. Let

$$T^G_{\mu+1} := X_{\mu+1} \setminus X_\mu = \{\chi_{m_\mu}, \chi_{m_\mu+1}, \ldots, \chi_{m_{\mu+1}-1}\} = \{\chi_n \colon\ n \in D^G_\mu\}$$

and m be such that

$$\frac{1}{2^{m+1}} < \frac{1}{p_{\mu+1}} \leqslant \frac{1}{2^m}, \quad \text{i.e.,}\ m = \frac{\log p_{\mu+1}}{\log 2}.$$

Then for any $\chi \in T^G_{\mu+1}$, as $1 \leqslant k \leqslant p_{\mu+1} - 1$, we have

$$\frac{1}{2^{m+1}} < \frac{1}{p_{\mu+1}} \leqslant \frac{k}{p_{\mu+1}} \leqslant \frac{p_{\mu+1}-1}{p_{\mu+1}} = 1 - \frac{1}{p_{\mu+1}} < 1 - \frac{1}{2^{m+1}}. \tag{4.13}$$

Now define

$$T^G_{\mu+1,1} = \Big\{\chi \in T^G_{\mu+1} \colon\ \frac{1}{4} < \frac{k}{p_{\mu+1}} < \frac{3}{4}\Big\}$$

and for $l = 2, 3, \ldots, m$,

$$T^G_{\mu+1,l} = \Big\{\chi \in T^G_{\mu+1} \colon\ \frac{1}{2^{l+1}} < \frac{k}{p_{\mu+1}} < \frac{1}{2^l} \text{ or } 1 - \frac{1}{2^l} < \frac{k}{p_{\mu+1}} < 1 - \frac{1}{2^{l+1}}\Big\}.$$

Also, for $l = 1, 2, \ldots, m$, let

$$D^G_{\mu,l} := \{n \in D^G_\mu \colon\ \chi_n \in T^G_{\mu+1,l}\}.$$

Then D^G_μ is the disjoint union

$$D^G_\mu = \bigcup_{l=1}^m D^G_{\mu,l}. \tag{4.14}$$

Since $p_{\mu+1}$ is prime, $k/p_{\mu+1}$ cannot be equal to $1/2^i$ or $1-1/2^i$ for any $i = 2, 3, \ldots, m$, and hence in view of (4.13), $T^G_{\mu+1}$ is the disjoint union

$$T^G_{\mu+1} = \bigcup_{j=1}^m T^G_{\mu+1,j}.$$

Since $h_1 \in G_\mu \setminus G_{\mu+1}$, it follows that $2h_1, 3h_1, \dots, (p_{\mu+1}-1)h_1 \in G_\mu \setminus G_{\mu+1}$. Indeed, if possible, suppose for some $2 \leqslant t \leqslant p_{\mu+1}-1$, $th_1 \notin G_\mu \setminus G_{\mu+1}$. Since $h_1 \in G_\mu$ and G_μ is a group, $th_1 \in G_\mu$. Since $th_1 \notin G_\mu \setminus G_{\mu+1}$, it follows that $th_1 \in G_{\mu+1}$. Therefore for $\chi \in X_{\mu+1} \setminus X_\mu$, $1 = \chi(th_1) = \chi^t(h_1) = \mathrm{e}^{2\pi\mathrm{i}kt/p_{\mu+1}}$. But then $p_{\mu+1}$ divides kt. Since $p_{\mu+1}$ is a prime, either $p_{\mu+1}$ divides k or $p_{\mu+1}$ divides t, which is not true as $1 \leqslant k, t < p_{\mu+1}$. Thus, $th_1 \in G_\mu \setminus G_{\mu+1}$ for $t = 2, 3, \dots, p_{\mu+1}-1$. For $l = 1, 2, \dots, m$, put $t_l := 2^{l-1}$. Then $1 \leqslant t_l = 2^{l-1} \leqslant 2^{m-1} \leqslant 2^m - 1 \leqslant p_{\mu+1}-1$, that is, $t_l \in \{1, 2, \dots, p_{\mu+1}-1\}$. Therefore, as seen above, $t_l h_1 \in G_\mu \setminus G_{\mu+1}$. Thus

$$\chi_n(t_l h_1) \neq 1 \quad \text{for any } \chi_n \in X_{\mu+1} \setminus X_\mu. \tag{4.15}$$

So, using (4.11) replacing h_1 by $t_l h_1$, we get

$$\hat{f}(n) = \frac{\hat{g}(n)}{\chi_n(t_l h_1) - 1}. \tag{4.16}$$

Also, in view of (4.12), for $\chi \in X_{\mu+1} \setminus X_\mu$ we have

$$\begin{aligned} |\chi(t_l h_1) - 1| &= |\mathrm{e}^{2\pi\mathrm{i}t_l k/p_{\mu+1}} - 1| \\ &= \Big| 2\mathrm{i}\mathrm{e}^{\pi\mathrm{i}t_l k/p_{\mu+1}} \frac{\mathrm{e}^{\pi\mathrm{i}t_l k/p_{\mu+1}} - \mathrm{e}^{-\pi\mathrm{i}t_l k/p_{\mu+1}}}{2\mathrm{i}} \Big| = 2\Big| \sin \frac{\pi t_l k}{p_{\mu+1}} \Big|. \end{aligned} \tag{4.17}$$

Note that for $1 \leqslant l \leqslant m$ we have

$$\begin{aligned} \chi \in T^G_{\mu+1,l} &\Rightarrow \frac{1}{2^{l+1}} < \frac{k}{p_{\mu+1}} < \frac{1}{2^l} \\ &\text{or} \quad 1 - \frac{1}{2^l} < \frac{k}{p_{\mu+1}} < 1 - \frac{1}{2^{l+1}} \Rightarrow \frac{\pi 2^{l-1}}{2^{l+1}} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \frac{\pi 2^{l-1}}{2^l} \\ &\text{or} \quad \pi 2^{l-1} - \frac{\pi 2^{l-1}}{2^l} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \pi 2^{l-1} - \frac{\pi 2^{l-1}}{2^{l+1}} \Rightarrow \frac{\pi}{4} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \frac{\pi}{2} \\ &\text{or} \quad \pi 2^{l-1} - \frac{\pi}{2} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \pi 2^{l-1} - \frac{\pi}{4}. \end{aligned}$$

Observe that

$$\frac{\pi}{4} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \frac{\pi}{2} \Rightarrow \sin \frac{k\pi 2^{l-1}}{p_{\mu+1}} \geqslant \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Next, for $l = 1$ we have

$$\pi 2^{l-1} - \frac{\pi}{2} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \pi 2^{l-1} - \frac{\pi}{4} \Rightarrow \frac{\pi}{2} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \frac{3\pi}{4} \Rightarrow \sin \frac{k\pi 2^{l-1}}{p_{\mu+1}} > \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}},$$

and for $l \geqslant 2$, as sine is increasing in $(\pi 2^{l-1} - \frac{1}{2}\pi, \pi 2^{l-1} - \frac{1}{4}\pi)$, we have

$$\pi 2^{l-1} - \frac{\pi}{2} < \frac{k\pi 2^{l-1}}{p_{\mu+1}} < \pi 2^{l-1} - \frac{\pi}{4} \Rightarrow \sin\frac{k\pi 2^{l-1}}{p_{\mu+1}} < \sin\Big(\pi 2^{l-1} - \frac{\pi}{4}\Big) = -\frac{1}{\sqrt{2}}$$
$$\Rightarrow \Big|\sin\frac{k\pi 2^{l-1}}{p_{\mu+1}}\Big| > \frac{1}{\sqrt{2}}.$$

Therefore for $\chi \in T^G_{\mu+1,l}$ we have

$$\Big|\sin\frac{\pi t_l k}{p_{\mu+1}}\Big| \geqslant \frac{1}{\sqrt{2}}. \tag{4.18}$$

(We note that instead of this inequality, Golubov and Volosivets use the inequality $|\chi_j(1/m_{k+1}) - 1| \geqslant 2\sin\pi/N$, where N is such that $p_i \leqslant N$ for all $i = 1, 2, \ldots$, which actually depends on boundedness of $\{p_i\}$ and hence the corresponding bound appear in the final conclusion, too.)

In view of (4.16), (4.17) and (4.18), we have

$$\sum_{n\in D^G_{\mu,l}} |\hat{f}(n)|^q = \sum_{n\in D^G_{\mu,l}} \frac{1}{|\chi_n(t_l h_1) - 1|^q}|\hat{g}(n)|^q = \sum_{n\in D^G_{\mu,l}} \frac{1}{2^q|\sin(\pi t_l k/p_{\mu+1})|^q}|\hat{g}(n)|^q$$
$$\leqslant \sum_{n\in D^G_{\mu,l}} \frac{2^{q/2}}{2^q}|\hat{g}(n)|^q = \frac{1}{2^{q/2}}\sum_{n\in D^G_{\mu,l}} |\hat{g}(n)|^q.$$

So, for $\mu \in \mathbb{N}\cup\{0\}$ we have

$$\Big(\sum_{n\in D^G_{\mu}} |\hat{f}(n)|^q\Big)^{1/q} = \Big(\sum_{l=1}^{m}\sum_{n\in D^G_{\mu,l}} |\hat{f}(n)|^q\Big)^{1/q} \leqslant \Big(\sum_{l=1}^{m}\frac{1}{2^{q/2}}\sum_{n\in D^G_{\mu,l}} |\hat{g}(n)|^q\Big)^{1/q}$$
$$= \frac{1}{\sqrt{2}}\Big(\sum_{l=1}^{m}\sum_{n\in D^G_{\mu,l}} |\hat{g}(n)|^q\Big)^{1/q} = \frac{1}{\sqrt{2}}\Big(\sum_{n\in D^G_{\mu}} |\hat{g}(n)|^q\Big)^{1/q}. \tag{4.19}$$

Therefore, for $1 < p \leqslant 2$, by virtue of the Hausdorff-Young inequality (see, e.g. [2], equation (4.28)), and (4.10), (4.19) becomes

$$\Big(\sum_{n\in D^G_{\mu}} |\hat{f}(n)|^q\Big)^{1/q} \leqslant \frac{1}{\sqrt{2}}\Big(\sum_{n=1}^{\infty} |\hat{g}(n)|^q\Big)^{1/q} \leqslant \frac{1}{\sqrt{2}}\Big(\int_G |g(x)|^p\,\mathrm{d}x\Big)^{1/p}$$
$$= \frac{1}{\sqrt{2}}\Big(\int_G |f(x+h_1) - f(x)|^p\,\mathrm{d}x\Big)^{1/p} \tag{4.20}$$
$$\leqslant \frac{1}{\sqrt{2}}\omega^{(p)}(f,\mu) \tag{4.21}$$

for all $\mu \in \mathbb{N} \cup \{0\}$. Since $1/(q/r) + 1/(q/(q-r)) = r/q + (q-r)/q = 1$, applying Hölder's inequality with exponents

$$\frac{q}{r} = \frac{p}{r(p-1)} \quad \text{and} \quad \frac{q}{q-r} = \frac{p}{p-rp+r}, \tag{4.22}$$

it follows from (2.14), (3.1) and (4.21) that

$$\begin{aligned} \sum_{n\in D_\mu^G} a_n|\hat{f}(n)|^r &= \sum_{n\in D_\mu^G} |\hat{f}(n)|^r a_n \\ &\leqslant \Big(\sum_{n\in D_\mu^G} |\hat{f}(n)|^q\Big)^{r/q} \Big(\sum_{n\in D_\mu^G} a_n^{p/(p-rp+r)}\Big)^{(p-rp+r)/p} \\ &\leqslant \frac{1}{2^{r/2}}(\omega^{(p)}(f,\mu))^r \kappa m_\mu^{(1-p/(p-rp+r))/(p/(p-rp+r))} \sum_{n\in D_{\mu-1}^G} a_n \\ &= \frac{1}{2^{r/2}}(\omega^{(p)}(f,\mu))^r \kappa m_\mu^{-r/q} A_{\mu-1}^G \end{aligned} \tag{4.23}$$

for all $\mu \in \mathbb{N} \cup \{0\}$. Summing (4.23) over $\mu \in \mathbb{N} \cup \{0\}$ yields

$$\begin{aligned} \sum_{n=1}^\infty a_n|\hat{f}(n)|^r &= \sum_{\mu=0}^\infty \sum_{n\in D_\mu^G} a_n|\hat{f}(n)|^r \leqslant \sum_{\mu=0}^\infty \frac{1}{2^{r/2}}(\omega^{(p)}(f,\mu))^r \kappa m_\mu^{-r/q} A_{\mu-1}^G \\ &= 2^{-r/2}\kappa \sum_{\mu=0}^\infty m_\mu^{-r/q} A_{\mu-1}^G (\omega^{(p)}(f,\mu))^r, \end{aligned}$$

which is (3.2) to be proved. This completes the proof of Theorem 3.1. □

Proof of Corollary 3.1. Since the hypotheses of Theorem 3.1 hold, we have equation (4.23). Therefore by (2.9) and the fact that for $n \leqslant m_\mu$, $E^{(p)}(f,n) \geqslant E^{(p)}(f,m_\mu)$, we have

$$\begin{aligned} \sum_{n\in D_\mu^G} a_n|\hat{f}(n)|^r &\leqslant \frac{1}{2^{r/2}}(\omega^{(p)}(f,\mu))^r \kappa m_\mu^{-r/q} A_{\mu-1}^G \\ &\leqslant \frac{1}{2^{r/2}}(2E^{(P)}(f,m_\mu))^r \kappa m_\mu^{-r/q} \sum_{n\in D_{\mu-1}^G} a_n \\ &\leqslant C(E^{(p)}(f,m_\mu))^r \sum_{n\in D_{\mu-1}^G} a_n n^{-r/q} \\ &= \sum_{n\in D_{\mu-1}^G} a_n n^{-r/q}(E^{(p)}(f,n))^r. \end{aligned} \tag{4.24}$$

Summing up the inequality in (4.24) over μ, we get the statement of Corollary 3.1.

□

P r o o f of Corollary 3.2. We shall put $a_n \equiv 1$ in Theorem 3.1. Since G is bounded, setting $\beta = 0$ in Lemma 4.1, we see that $\{a_n\} \in \mathfrak{A}_\gamma(G)$ for every $\gamma \geqslant 1$. In particular, $\{a_n\}$ satisfies (3.1) for $1 < p \leqslant 2$ and $0 < r < q$, so for $p = 2$ and $r = 1$. Also, as $f \in L^2(G)$, all the conditions of Theorem 3.1 hold. Therefore by Theorem 3.1, we have (3.2) with $a_n \equiv 1$, $p = q = 2$ and $r = 1$. This means we have the inequality

$$\sum_{n=1}^{\infty} |\hat{f}(n)| \leqslant 2^{-1/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-1/2} A_{\mu-1}^G \omega^{(2)}(f,\mu), \tag{4.25}$$

where κ is from (2.14) corresponding to $\gamma = 2/(2-2+1) = 2$. Further, in this case, we have

$$A_{\mu-1}^G = \sum_{n\in D_{\mu-1}^G} a_n = \sum_{n\in D_{\mu-1}^G} 1 = m_\mu - m_{\mu-1} < m_\mu, \quad \mu \in \mathbb{N}, \quad A_{-1}^G = a_1 = 1 \leqslant m_0. \tag{4.26}$$

Therefore (4.25) becomes

$$\sum_{n=1}^{\infty} |\hat{f}(n)| < 2^{-1/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-1/2} m_\mu \omega^{(2)}(f,\mu) = 2^{-1/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{1/2} \omega^{(2)}(f,\mu).$$

Hence, if $\sum\limits_{\mu=0}^{\infty} m_\mu^{1/2}\omega^{(2)}(f,\mu) < \infty$ then, $\sum\limits_{n=1}^{\infty} |\hat{f}(n)| < \infty$. This completes the proof. $\square$

P r o o f of Corollary 3.3. We shall put $a_n \equiv 1$ in Theorem 3.1. Since G is bounded, setting $\beta = 0$ in Lemma 4.1, we see that $\{a_n\} \in \mathfrak{A}_\gamma(G)$ for every $\gamma \geqslant 1$. In particular, $\{a_n\}$ satisfies (3.1) for $1 < p \leqslant 2$ and $0 < r < q$. Also, as $f \in \operatorname{Lip}(\alpha, p, G)$, $f \in L^p(G)$ for $1 < p \leqslant 2$. Therefore, by Theorem 3.1, we have (3.2) with $a_n \equiv 1$. This means we have the inequality

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-r/q} A_{\mu-1}^G (\omega^{(p)}(f,\mu))^r, \tag{4.27}$$

where the constant κ is from (2.14) corresponding to $\gamma = p/(p - rp + r)$, $0 < r < q$. Finally, as $f \in \operatorname{Lip}(\alpha, p, G)$, we have

$$\omega^{(p)}(f,\mu) \leqslant C m_\mu^{-\alpha}, \quad \mu \in \mathbb{N} \cup \{0\}. \tag{4.28}$$

Using (4.26) and (4.28) in (4.27), we have

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^r < 2^{-r/2}\kappa C^r \sum_{\mu=0}^{\infty} m_\mu^{-r/q} m_\mu m_\mu^{-r\alpha} = 2^{-r/2}\kappa C^r \sum_{\mu=0}^{\infty} m_\mu^{-r/q+1-r\alpha}. \tag{4.29}$$

Now in view of (3.5), we have $q < r(1+\alpha q)$, so $1 < r/q + r\alpha$, and hence $-r/q + 1 - r\alpha < 0$. Also, as $m_\mu \geqslant 2^\mu$, it follows that $m_\mu^{-r/q+1-r\alpha} \leqslant 2^{\mu(-r/q+1-r\alpha)}$. So, from (4.29) we get

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa C^r \sum_{\mu=0}^{\infty} \Big(\frac{1}{2^{r/q-1+r\alpha}}\Big)^\mu. \tag{4.30}$$

Since $r/q - 1 + r\alpha > 0$, $0 < 1/2^{r/q-1+r\alpha} < 1$ and hence the geometric series on the right-hand side of (4.30) converges. So we get (3.6), completing the proof of Corollary 3.3. □

P r o o f of Corollary 3.4. Suppose $f \in \mathrm{Lip}\,(\alpha, p, G)$ for some $\alpha > 0$ and $1 < p \leqslant 2$, and $\delta < \alpha - 1/p$. Since G is bounded, in view of Lemma 4.1, $\{n^\delta\} \in \mathfrak{A}_\gamma(G)$ for all $\gamma \geqslant 1$. So, we can put $a_n = n^\delta$ in Theorem 3.1 to get (3.2) with $a_n = n^\delta$, that is,

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-r/q} A_{\mu-1}^G (\omega^{(p)}(f,\mu))^r,$$

where the constant κ is from (2.14) corresponding to $\gamma = p/(p - rp + r)$, $0 < r < q$. Now, setting $r = 1$, in the above inequality we get

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| \leqslant 2^{-1/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-1/q} A_{\mu-1}^G \omega^{(p)}(f,\mu). \tag{4.31}$$

Also, when $\delta \geqslant 0$, we have

$$\begin{aligned} A_{-1}^G &= a_1 = 1^\delta \leqslant m_0^{\delta+1},\\ A_{\mu-1}^G &= \sum_{n\in D_{\mu-1}^G} n^\delta \leqslant \sum_{n\in D_{\mu-1}^G} m_\mu^\delta = m_\mu^\delta(m_\mu - m_{\mu-1}) < m_\mu^\delta m_\mu = m_\mu^{\delta+1}, \quad \mu \in \mathbb{N}, \end{aligned}$$

and for $\delta < 0$ we have

$$\begin{aligned} A_{-1}^G &= a_1 = 1^\delta \leqslant m_0^{1+\delta},\\ A_{\mu-1}^G &= \sum_{n\in D_{\mu-1}^G} n^\delta \leqslant \sum_{n\in D_{\mu-1}^G} m_{\mu-1}^\delta = m_{\mu-1}^\delta(m_\mu - m_{\mu-1}) = m_{\mu-1}^\delta m_{\mu-1}(p_\mu - 1)\\ &\leqslant m_{\mu-1}^{\delta+1} p_\mu = m_{\mu-1}^{\delta+1} p_\mu^{\delta+1} p_\mu^{-\delta} = m_\mu^{\delta+1} p_\mu^{-\delta} \leqslant p_0^{-\delta} m_\mu^{\delta+1}, \quad \mu \in \mathbb{N}. \end{aligned}$$

So, in either case,

$$A_{\mu-1}^G = \sum_{n\in D_{\mu-1}^G} n^\delta \leqslant C m_\mu^{\delta+1}. \tag{4.32}$$

Since $f \in \mathrm{Lip}\,(\alpha, p, G)$, we have (4.28). Using (4.32) and (4.28) in (4.31), we get

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| \leqslant 2^{-1/2}\kappa \sum_{\mu=0}^{\infty} m_\mu^{-1/q} C m_\mu^{\delta+1} C m_\mu^{-\alpha} \leqslant C \sum_{\mu=0}^{\infty} m_\mu^{-1/q+\delta+1-\alpha}. \tag{4.33}$$

Now, in view of (3.7), we have $\delta < \alpha - 1/p$, so $\alpha > \delta + 1/p = \delta + 1 - 1/q$, and hence $-1/q + \delta + 1 - \alpha < 0$. Also, as $m_\mu \geqslant 2^\mu$, it follows that $m_\mu^{-1/q+\delta+1-\alpha} \leqslant 2^{\mu(-1/q+\delta+1-\alpha)}$. So, from (4.33) we get

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| \leqslant C \sum_{\mu=0}^{\infty} \Big(\frac{1}{2^{1/q-\delta-1+\alpha}}\Big)^\mu. \tag{4.34}$$

Since $1/q - \delta - 1 + \alpha > 0$, $0 < 1/2^{1/q-\delta-1+\alpha} < 1$, and hence the geometric series on the right-hand side of (4.34) converges. So we get (3.8) to be proved. □

Proof of Theorem 3.2. Proceeding as in the proof of Theorem 3.1, for $h_1 \in G_\mu \setminus G_{\mu+1}$, $\mu \geqslant 0$, we get (4.20). So, in view of the fact that G is the disjoint union of the cosets $z^G_{k,\mu} + G_\mu$, $k = 0, 1, \ldots, m_\mu - 1$, each of measure $1/m_\mu$, we get

$$\begin{aligned}\Big(\sum_{n\in D^G_\mu} |\hat{f}(n)|^q\Big)^{1/q} &\leqslant \frac{1}{\sqrt{2}}\Big(\int_G |f(x+h_1) - f(x)|^p \,\mathrm{d}x\Big)^{1/p} \\ &\leqslant \frac{1}{\sqrt{2}}\Big(\sum_{k=0}^{m_\mu-1} \int_{z^G_{k,\mu}+G_\mu} |f(x+h_1) - f(x)|^p \,\mathrm{d}x\Big)^{1/p} \\ &\leqslant \frac{1}{\sqrt{2}}\Big(\sum_{k=0}^{m_\mu-1} \frac{1}{m_\mu}(\omega^{(p)}(f,\mu,z^G_{k,\mu}+G_\mu))^p\Big)^{1/p},\end{aligned} \tag{4.35}$$

by definition of $\omega^{(p)}(f,\mu,z^G_{k,\mu}+G_\mu)$. Now, applying Hölder's inequality with the exponents in (4.22), it follows from (2.14), (3.1), and (4.35) that

$$\begin{aligned}\sum_{n\in D^G_\mu} a_n|\hat{f}(n)|^r &= \sum_{n\in D^G_\mu} |\hat{f}(n)|^r a_n \leqslant \Big(\sum_{n\in D^G_\mu} |\hat{f}(n)|^q\Big)^{r/q}\Big(\sum_{n\in D^G_\mu} a_n^{p/(p-rp+r)}\Big)^{(p-rp+r)/p} \\ &\leqslant 2^{-r/2}\Big(\sum_{k=0}^{m_\mu-1}\frac{1}{m_\mu}(\omega^{(p)}(f,\mu,z^G_{k,\mu}+G_\mu))^p\Big)^{r/p} \\ &\quad\times \kappa m_\mu^{(1-p/(p-rp+r))/(p/(p-rp+r))} \sum_{n\in D^G_{\mu-1}} a_n \\ &= 2^{-r/2} m_\mu^{-r/p}\Big(\sum_{k=0}^{m_\mu-1}(\omega^{(p)}(f,\mu,z^G_{k,\mu}+G_\mu))^p\Big)^{r/p} \kappa m_\mu^{-r/q} A^G_{\mu-1} \\ &= 2^{-r/2}\kappa m_\mu^{-r} A^G_{\mu-1}\Big(\sum_{k=0}^{m_\mu-1}(\omega^{(p)}(f,\mu,z^G_{k,\mu}+G_\mu))^p\Big)^{r/p}\end{aligned} \tag{4.36}$$

for all $\mu \geqslant 0$. Summing (4.36) over $\mu \in \mathbb{N}\cup\{0\}$ yields (3.9). This completes the proof of Theorem 3.2. □

P r o o f of Theorem 3.3. We prove this theorem by proceeding similarly to the proof of Theorem 3 of [6]. Since $\mathcal{F}l_\beta(f,G) < \infty$, it follows that f is bounded. Therefore, as f is measurable, it follows that $f \in L^2(G)$. So, proceeding as in Theorem 3.1, for $h_1 \in G_\mu \setminus G_{\mu+1}$, $\mu \geqslant 0$, we have (4.20) with $p = q = 2$. Therefore, for $1 < p' < \infty$ we have

$$\Big(\sum_{n\in D_\mu^G} |\hat f(n)|^2\Big)^{p'} \leqslant \frac{1}{2^{p'}}\Big(\int_G |f(x+h_1)-f(x)|^2\,\mathrm{d}x\Big)^{p'}. \tag{4.37}$$

Now, writing

$$2 = \frac{\beta}{p'} + \frac{(2-\beta)q'+\beta}{q'}, \tag{4.38}$$

and applying the integral form of Hölder's inequality with the exponents p' and q' yields

$$\begin{aligned}\Big(\sum_{n\in D_\mu^G} |\hat f(n)|^2\Big)^{p'} &\leqslant \frac{1}{2^{p'}}\Big(\int_G |f(x+h_1)-f(x)|^{\beta/p'+((2-\beta)q'+\beta)/q'}\,\mathrm{d}x\Big)^{p'} \\ &\leqslant \frac{1}{2^{p'}}\Big(\Big(\int_G |f(x+h_1)-f(x)|^{(\beta/p')p'}\,\mathrm{d}x\Big)^{1/p'} \\ &\quad\times\Big(\int_G |f(x+h_1)-f(x)|^{(((2-\beta)q'+\beta)/q')q'}\,\mathrm{d}x\Big)^{1/q'}\Big)^{p'} \\ &= \frac{1}{2^{p'}}\int_G |f(x+h_1)-f(x)|^{\beta}\,\mathrm{d}x \\ &\quad\times\Big(\int_G |f(x+h_1)-f(x)|^{(2-\beta)q'+\beta}\,\mathrm{d}x\Big)^{p'/q'}.\end{aligned} \tag{4.39}$$

Now, in view of the fact that G is the disjoint union of the cosets $z_{q,\mu}^G + G_\mu$, $1 \leqslant q \leqslant m_\mu - 1$, each of measure $1/m_\mu$, (4.39) becomes

$$\begin{aligned}\Big(\sum_{n\in D_\mu^G} |\hat f(n)|^2\Big)^{p'} &\leqslant 2^{-p'}\Big(\sum_{q=0}^{m_\mu-1}\int_{z_{q,\mu}^G+G_\mu} |f(x+h_1)-f(x)|^{\beta}\,\mathrm{d}x\Big) \\ &\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{(p'/q')((2-\beta)q'+\beta)} \\ &\leqslant 2^{-p'}\Big(\sum_{q=0}^{m_\mu-1}\int_{z_{q,\mu}^G+G_\mu} (\operatorname{osc}(f, z_{q,\mu}^G+G_\mu))^{\beta}\,\mathrm{d}x\Big) \\ &\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{2p'-\beta}\end{aligned} \tag{4.40}$$

$$
\begin{aligned}
&= 2^{-p'}\Big(\sum_{q=0}^{m_\mu-1}(\operatorname{osc}(f, z^G_{q,\mu}+G_\mu))^\beta m_\mu^{-1}\Big)\\
&\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{2p'-\beta}\\
&= 2^{-p'}m_\mu^{-1}\Big(\sum_{q=0}^{m_\mu-1}(\operatorname{osc}(f, z^G_{q,\mu}+G_\mu))^\beta\Big)\\
&\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{2p'-\beta}\\
&\leqslant 2^{-p'}m_\mu^{-1}(\mathcal{F}l_\beta(f,G))^\beta(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{2p'-\beta}.
\end{aligned}
$$

Since $\frac{1}{2}r+\frac{1}{2}(2-r)=1$, applying Hölder's inequality with the exponents $2/r$ and $2/(2-r)$, in view of (4.40), (3.10) and (2.14), we have

$$
\begin{aligned}
\text{(4.41)}\quad \sum_{n\in D^G_\mu} a_n|\hat f(n)|^r &\leqslant \Big(\sum_{n\in D^G_\mu}|\hat f(n)|^2\Big)^{r/2}\Big(\sum_{n\in D^G_\mu}a_n^{2/(2-r)}\Big)^{(2-r)/2}\\
&\leqslant (2^{-1}m_\mu^{-1/p'}(\mathcal{F}l_\beta(f,G))^{\beta/p'}(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{2-\beta/p'})^{r/2}\\
&\quad\times\kappa m_\mu^{(1-2/(2-r))/(2/(2-r))}\sum_{n\in D^G_{\mu-1}}a_n\\
&= 2^{-r/2}m_\mu^{-r/(2p')}(\mathcal{F}l_\beta(f,G))^{\beta r/(2p')}\\
&\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{r-\beta r/(2p')}\kappa m_\mu^{-r/2}A^G_{\mu-1}\\
&= 2^{-r/2}\kappa(\mathcal{F}l_\beta(f,G))^{\beta r/(2p')}m_\mu^{-r/2-r/(2p')}\\
&\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu))^{r-\beta r/(2p')}A^G_{\mu-1}
\end{aligned}
$$

for all $\mu\geqslant 0$. Summing (4.41) over all $\mu\in\mathbb{N}\cup\{0\}$, we get (3.11). This completes the proof of Theorem 3.3. □

P r o o f of Theorem 3.4. Suppose f is measurable on G, $f\in \mathrm{BF}_s(G)$ for some $0<s<2$, and $\{a_n\}$ satisfies (3.10). As $\mathcal{F}l_s(f,G)<\infty$, it follows that f is bounded. Since f is measurable, it follows that $f\in L^2(G)$. Now, proceeding as in Theorem 3.1, for $h_1\in G_\mu\setminus G_{\mu+1}$, $\mu\geqslant 0$, we get (4.20). Setting $p=q=2$ in (4.20), in view of Parseval's formula (see [7], Chapter VI, §23), (4.10) of the definition of g, and the fact that G is the disjoint union of the cosets $z^G_{k,\mu}+G_\mu$, $k=0,1,\ldots,m_\mu-1$, each of measure $1/m_\mu$, we have

$$
\begin{aligned}
\text{(4.42)}\quad \Big(\sum_{n\in D^G_\mu}|\hat f(n)|^2\Big)^{1/2} &\leqslant \frac{1}{\sqrt2}\Big(\int_G|f(x+h_1)-f(x)|^2\,\mathrm{d}x\Big)^{1/2}\\
&= \frac{1}{\sqrt2}\Big(\sum_{k=0}^{m_\mu-1}\int_{z^G_{k,\mu}+G_\mu}|f(x+h_1)-f(x)|^2\,\mathrm{d}x\Big)^{1/2}
\end{aligned}
$$

$$\leqslant \frac{1}{\sqrt{2}}\Bigl(\sum_{k=0}^{m_\mu-1}\frac{1}{m_\mu}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^2\Bigr)^{1/2}$$
$$=\frac{1}{\sqrt{2m_\mu}}\Bigl(\sum_{k=0}^{m_\mu-1}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^2\Bigr)^{1/2}.$$

Now, applying Hölder's inequality with the exponents $2/r$ and $2/(2-r)$, it follows from (2.14), (3.10) and (4.42) that

(4.43)
$$\sum_{n\in D_\mu^G} a_n|\hat f(n)|^r$$
$$=\sum_{n\in D_\mu^G}|\hat f(n)|^r a_n \leqslant \Bigl(\sum_{n\in D_\mu^G}|\hat f(n)|^2\Bigr)^{r/2}\Bigl(\sum_{n\in D_\mu^G}a_n^{2/(2-r)}\Bigr)^{(2-r)/2}$$
$$\leqslant \frac{1}{2^{r/2}m_\mu^{r/2}}\Bigl(\sum_{k=0}^{m_\mu-1}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^2\Bigr)^{r/2}\kappa m_\mu^{(1-2/(2-r))/(2/(2-r))}\sum_{n\in D_{\mu-1}^G}a_n$$
$$=\frac{1}{2^{r/2}m_\mu^{r/2}}\Bigl(\sum_{k=0}^{m_\mu-1}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^{2-s}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^s\Bigr)^{r/2}\kappa m_\mu^{-r/2}A_{\mu-1}^G$$
$$\leqslant 2^{-r/2}\kappa m_\mu^{-r}A_{\mu-1}^G\Bigl(\sum_{k=0}^{m_\mu-1}(\omega(f,\mu))^{2-s}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^s\Bigr)^{r/2}$$
$$\leqslant 2^{-r/2}\kappa m_\mu^{-r}A_{\mu-1}^G(\omega(f,\mu))^{(2-s)r/2}\Bigl(\sum_{k=0}^{m_\mu-1}(\omega(f,\mu,z_{k,\mu}^G+G_\mu))^s\Bigr)^{r/2}$$
$$\leqslant 2^{-r/2}\kappa m_\mu^{-r}A_{\mu-1}^G(\omega(f,\mu))^{(2-s)r/2}(\mathcal{F}l_s(f,G))^{rs/2}.$$

Summing (4.43) over $\mu\in\mathbb{N}\cup\{0\}$ yields (3.12). This completes the proof of Theorem 3.4. □

P r o o f of Corollary 3.5. We shall put $a_n\equiv 1$ in Theorem 3.4. Since G is bounded, setting $\beta=0$ in Lemma 4.1, we see that $\{a_n\}\in\mathfrak{A}_\gamma(G)$ for every $\gamma\geqslant 1$. In particular, $\{a_n\}$ satisfies (3.10) for $0<r<2$. Also, by our assumption, $f\in \mathrm{BF}_s(G)$ for $0<s<2$. Therefore, by Theorem 3.4, we have (3.12) with $a_n\equiv 1$. This means we have

$$\sum_{n=1}^{\infty}|\hat f(n)|^r\leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2}\sum_{\mu=0}^{\infty}m_\mu^{-r}A_{\mu-1}^G(\omega(f,\mu))^{(2-s)r/2}, \tag{4.44}$$

where the constant κ is from (2.14) corresponding to $\gamma=2/(2-r)$. Since $a_n\equiv 1$, from (4.26) we have $A_{\mu-1}=m_{\mu-1}(p_\mu-1)$. Further, by our hypothesis, $f\in \mathrm{Lip}\,(\alpha,G)$

and hence $\omega(f,\mu) \leqslant Cm_\mu^{-\alpha}$. So, (4.44) becomes

$$(4.45)\quad \sum_{n=1}^{\infty} |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} \sum_{\mu=0}^{\infty} m_\mu^{-r} m_{\mu-1}(p_\mu - 1)(Cm_\mu^{-\alpha})^{(2-s)r/2}$$
$$< 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} \sum_{\mu=0}^{\infty} m_\mu^{-r} m_{\mu-1} p_\mu (Cm_\mu^{-\alpha})^{(2-s)r/2}$$
$$\leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} C \sum_{\mu=0}^{\infty} m_\mu^{-r} m_\mu (m_\mu^{-\alpha})^{(2-s)r/2}$$
$$= 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} C \sum_{\mu=0}^{\infty} m_\mu^{-r+1-\alpha r(2-s)/2}.$$

Now, in view of (3.13), we have $2r + \alpha r(2-s) > 2$. So, $r + \frac{1}{2}\alpha r(2-s) > 1$, and hence $-r+1-\frac{1}{2}\alpha r(2-s) < 0$. Also, as $m_\mu \geqslant 2^\mu$, it follows that $m_\mu^{-r+1-\alpha r(2-s)/2} \leqslant 2^{\mu(-r+1-\alpha r(2-s)/2)}$. So, from (4.45) we get

$$(4.46)\qquad \sum_{n=1}^{\infty} |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} C \sum_{\mu=0}^{\infty} \Big(\frac{1}{2^{r-1+\alpha r(2-s)/2}}\Big)^\mu.$$

Since $r-1+\frac{1}{2}\alpha r(2-s) > 0$, $0 < 1/2^{r-1+\alpha r(2-s)/2} < 1$, and hence the geometric series on the right-hand side of (4.46) converges. This completes the proof of Corollary 3.5. $\square$

P r o o f of Corollary 3.6. Suppose $f \in \mathrm{Lip}\,(\alpha, G) \cap \mathrm{BF}_s(G)$ for some $\alpha > 0$, $0 < s < 2$, G is bounded, and $\delta < \frac{1}{2}\alpha(2-s)$. Then, in view of Lemma 4.1, $\{n^\delta\} \in \mathfrak{A}_\gamma(G)$ for all $\gamma \geqslant 1$. So, we can put $a_n = n^\delta$ in Theorem 3.4 to get (3.12) with $a_n = n^\delta$, that is,

$$\sum_{n=1}^{\infty} n^\delta |\hat{f}(n)|^r \leqslant 2^{-r/2}\kappa(\mathcal{F}l_s(f,G))^{rs/2} \sum_{\mu=0}^{\infty} m_\mu^{-r} A_{\mu-1}^G (\omega(f,\mu))^{(2-s)r/2},$$

where the constant κ is from (2.14) corresponding to $\gamma = 2/(2-r)$. Now, setting $r = 1$, we get

$$(4.47)\qquad \sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| \leqslant 2^{-1/2}\kappa(\mathcal{F}l_s(f,G))^{s/2} \sum_{\mu=0}^{\infty} m_\mu^{-1} A_{\mu-1}^G (\omega(f,\mu))^{(2-s)/2}.$$

Since $a_n = n^\delta$, from (4.32) we have $A^G_{\mu-1} \leqslant Cm_\mu^{\delta+1}$. Further, as $f \in \operatorname{Lip}(\alpha, G)$, we have $\omega(f,\mu) \leqslant Cm_\mu^{-\alpha}$, $\mu \in \mathbb{N} \cup \{0\}$. Therefore, from (4.47) we get

$$\begin{aligned}(4.48)\qquad \sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| &\leqslant 2^{-1/2}\kappa(\mathcal{F}l_s(f,G))^{s/2}\sum_{\mu=0}^{\infty} m_\mu^{-1} C m_\mu^{\delta+1}(Cm_\mu^{-\alpha})^{(2-s)/2}\\ &\leqslant 2^{-1/2}\kappa(\mathcal{F}l_s(f,G))^{s/2} C \sum_{\mu=0}^{\infty} m_\mu^{\delta-\alpha(2-s)/2}.\end{aligned}$$

Since $\delta < \frac{1}{2}\alpha(2-s)$, we have $\delta - \frac{1}{2}\alpha(2-s) < 0$. Also, as $m_\mu \geqslant 2^\mu$, it follows that $m_\mu^{\delta-\alpha(2-s)/2} \leqslant 2^{\mu(\delta-\alpha(2-s)/2)}$. So from (4.48) we get

$$(4.49)\qquad \sum_{n=1}^{\infty} n^\delta |\hat{f}(n)| \leqslant 2^{-1/2}\kappa(\mathcal{F}l_s(f,G))^{s/2} C \sum_{\mu=0}^{\infty}\Big(\frac{1}{2^{-\delta+\alpha(2-s)/2}}\Big)^\mu.$$

Since $-\delta + \frac{1}{2}\alpha(2-s) > 0$, $0 < 1/2^{-\delta+\alpha(2-s)/2} < 1$, and hence the geometric series on the right-hand side of (4.49) converges. So, we get (3.8). This proves Corollary 3.6.

□

4.2. Double Vilenkin-Fourier series. Almost all results of Section 3.2 can be proved using similar techniques to the case of one variable. For the readers' convenience, we shall give a complete proof of Theorem 3.5 and an outline of proofs for the remaining results. First, we state the following lemma, which is a two-dimensional analogue of Lemma 4.1 and easily follows from it.

Lemma 4.2. *If G and H are bounded, then $\{k^{\beta_1} l^{\beta_2}\} \in \mathfrak{A}^*_\gamma(G \times H)$ for all $\beta_1, \beta_2 \in \mathbb{R}$ and $\gamma \geqslant 1$.*

Proof of Theorem 3.5. Fix $\mu, \nu \in \mathbb{N} \cup \{0\}$, $(h_1, h_2) \in (G_\mu \setminus G_{\mu+1}) \times (H_\nu \setminus H_{\nu+1})$, and set

$$(4.50)\qquad g(x,y) := \Delta_{1,1} f(x,y;h_1,h_2), \quad (x,y) \in G \times H.$$

Then for $m, n \in \mathbb{N}$ we have

(4.51)

$$\hat{g}(m,n) = \int_{G\times H} g(x,y)\bar{\chi}_m(x)\overline{\psi}_n(y)\,\mathrm{d}m(x,y) = (\chi_m(h_1)-1)(\psi_n(h_2)-1)\hat{f}(m,n).$$

Since $h_1 \in G_\mu \setminus G_{\mu+1}$, in view of (4.12), for $\chi \in X_{\mu+1} \setminus X_\mu$ we have

$$(4.52)\qquad \chi(h_1) = \mathrm{e}^{2\pi i k_1/p_{\mu+1}} \quad \text{for some } 1 \leqslant k_1 < p_{\mu+1}.$$

Similarly as $h_2 \in H_\nu \setminus H_{\nu+1}$, using (4.12) for H, for $\psi \in Y_{\nu+1} \setminus Y_\nu$ we have

$$(4.53)\qquad \psi(h_2) = \mathrm{e}^{2\pi i k_2/q_{\nu+1}} \quad \text{for some } 1 \leqslant k_2 < q_{\nu+1}.$$

As in the case of one variable, let $m_1 = [\log p_{\mu+1}/\log 2]$ and $m_2 = [\log q_{\nu+1}/\log 2]$. Now, using the notations of the proof of Theorem 3.1 for groups G and H, in view of (4.51), (4.15), (4.17) and (4.18), for $1 \leqslant l_i \leqslant m_i$, $i = 1, 2$ we have

$$
\begin{aligned}
(4.54)\quad & \sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} |\hat{f}(m,n)|^q \\
&= \sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} \frac{1}{|\chi_m(t_{l_1}h_1)-1|^q|\psi_n(t_{l_2}h_2)-1|^q}|\hat{g}(m,n)|^q \\
&= \sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} 2^{-q}\left|\sin\frac{\pi t_{l_1}k_1}{p_{\mu+1}}\right|^{-q} 2^{-q}\left|\sin\frac{\pi t_{l_2}k_2}{q_{\nu+1}}\right|^{-q}|\hat{g}(m,n)|^q \\
&\leqslant \sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} \frac{2^{q/2}2^{q/2}}{2^q2^q}|\hat{g}(m,n)|^q = \frac{1}{2^q}\sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} |\hat{g}(m,n)|^q.
\end{aligned}
$$

Using (4.14) for G and H, and (4.54) for $\mu, \nu \in \mathbb{N}\cup\{0\}$ we have

$$
\begin{aligned}
(4.55)\quad \Big(\sum_{m\in D^G_{\mu}} \sum_{n\in D^H_{\nu}} |\hat{f}(m,n)|^q\Big)^{1/q} &= \Big(\sum_{l_1=1}^{m_1}\sum_{l_2=1}^{m_2}\sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} |\hat{f}(m,n)|^q\Big)^{1/q} \\
&\leqslant \Big(\sum_{l_1=1}^{m_1}\sum_{l_2=1}^{m_2}\frac{1}{2^q}\sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} |\hat{g}(m,n)|^q\Big)^{1/q} \\
&= \frac{1}{2}\Big(\sum_{l_1=1}^{m_1}\sum_{l_2=1}^{m_2}\sum_{m\in D^G_{\mu,l_1}} \sum_{n\in D^H_{\nu,l_2}} |\hat{g}(m,n)|^q\Big)^{1/q} \\
&= \frac{1}{2}\Big(\sum_{m\in D^G_{\mu}} \sum_{n\in D^H_{\nu}} |\hat{g}(m,n)|^q\Big)^{1/q}.
\end{aligned}
$$

Therefore, for $1 < p \leqslant 2$, by virtue of the Hausdorff-Young inequality (see, e.g. [2], equation (4.28)) and (4.50), (4.55) becomes

$$
\begin{aligned}
\Big(\sum_{m\in D^G_{\mu}} \sum_{n\in D^H_{\nu}} |\hat{f}(m,n)|^q\Big)^{1/q} &\leqslant \frac{1}{2}\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\hat{g}(m,n)|^q\Big)^{1/q} \\
&\leqslant \frac{1}{2}\Big(\int_{G\times H}|g(x,y)|^p\,\mathrm{d}m(x,y)\Big)^{1/p} \\
(4.56)\quad &= \frac{1}{2}\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^p\,\mathrm{d}m(x,y)\Big)^{1/p} \\
(4.57)\quad &\leqslant \frac{1}{2}\omega^{(p)}(f,\mu,\nu)
\end{aligned}
$$

for all $\mu, \nu \in \mathbb{N} \cup \{0\}$. As in the case of one variable, applying Hölder's inequality with exponents in (4.22), it follows from (2.29), (3.15) and (4.57) that

$$\begin{aligned}(4.58)\qquad &\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} a_{mn}|\hat{f}(m,n)|^r \\ &\leqslant \Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat{f}(m,n)|^q\Big)^{r/q}\Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} a_{mn}^{p/(p-rp+r)}\Big)^{(p-rp+r)/p} \\ &\leqslant \frac{1}{2^r}(\omega^{(p)}(f,\mu,\nu))^r\kappa(m_\mu n_\nu)^{-r/q}A^*_{\mu-1,\nu-1}\end{aligned}$$

for all $\mu, \nu \in \mathbb{N} \cup \{0\}$. Summing (4.58) over $\mu, \nu \in \mathbb{N} \cup \{0\}$ yields

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}|\hat{f}(m,n)|^r &= \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} a_{mn}|\hat{f}(m,n)|^r \\ &\leqslant 2^{-r}\kappa\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}(m_\mu n_\nu)^{-r/q}A^*_{\mu-1,\nu-1}(\omega^{(p)}(f,\mu,\nu))^r,\end{aligned}$$

which is (3.16) to be proved. This completes the proof of Theorem 3.5. □

Substituting $a_{mn} \equiv 1$ and $a_{mn} = m^{\delta_1}n^{\delta_2}$ in Theorem 3.5 and proceeding as in the proofs of Corollaries 3.3 and 3.4, respectively, we can write proofs of Corollaries 3.7 and 3.8. Also, the way we have proved Theorem 3.2 using Theorem 3.1 allows us to prove Theorem 3.6 using Theorem 3.5.

Proof of Theorem 3.7. Since $\mathcal{F}l_\beta(f, G\times H)$, $\mathcal{F}l_\beta(f(\cdot,0), G)$ and $\mathcal{F}l_\beta(f(0,\cdot), H)$ are finite, in view of Remark 2.1, f is bounded. Therefore, as f is measurable on $G \times H$, it follows that $f \in L^2(G \times H)$. So, proceeding as in Theorem 3.5, for $(h_1, h_2) \in (G_\mu \setminus G_{\mu+1}) \times (H_\nu \setminus H_{\nu+1})$, $\mu, \nu \geqslant 0$, we have (4.56) with $p = q = 2$, that is, we have

$$(4.59)\qquad \Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat{f}(m,n)|^2\Big)^{1/2} \leqslant \frac{1}{2}\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^2\,\mathrm{d}m(x,y)\Big)^{1/2}.$$

So, for $1 < p' < \infty$ we have

$$(4.60)\qquad \Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat{f}(m,n)|^2\Big)^{p'} \leqslant \frac{1}{4^{p'}}\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^2\,\mathrm{d}m(x,y)\Big)^{p'}.$$

Now, proceeding as in the proof of Theorem 3.3, starting from (4.60) instead of (4.37), we can complete the proof of Theorem 3.7. □

P r o o f of Theorem 3.8. Suppose f is measurable on $G \times H$, $f \in \mathrm{BF}_s(G \times H)$, $f(\cdot, 0) \in \mathrm{BF}_s(G)$, $f(0, \cdot) \in \mathrm{BF}_s(H)$ for some $0 < s < 2$, and $\{a_{mn}\}$ satisfies (3.23). As $V_s(f, G \times H) \leqslant \mathcal{F}l_s(f, G \times H) < \infty$, in view of Remark 2.1, it follows that f is bounded. Since f is measurable on $G \times H$, it follows that $f \in L^2(G \times H)$. So, proceeding as in the proof of Theorem 3.7, we get (4.59). Now, proceeding as in the proof of Theorem 3.4, starting from (4.59) instead of (4.20) with $p = q = 2$, we can complete the proof of this theorem. □

P r o o f of Corollary 3.9. We shall put $a_{mn} \equiv 1$ in Theorem 3.8. Since G and H are bounded, setting $\beta = 0$ in Lemma 4.2, we see that $\{a_{mn}\} \in \mathfrak{A}^*_\gamma(G \times H)$ for every $\gamma \geqslant 1$. In particular, $\{a_{mn}\}$ satisfies (3.23) for $0 < r < 2$. Also, by our assumption, $f \in \mathrm{BF}_s(G \times H)$, $f(\cdot, 0) \in \mathrm{BF}_s(G)$, and $f(0, \cdot) \in \mathrm{BF}_s(H)$ for $0 < s < 2$. Therefore, by Theorem 3.8, we have (3.25) with $a_{mn} \equiv 1$. This means we have

$$(4.61) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\hat{f}(m,n)|^r \leqslant 2^{-r}\kappa(\mathcal{F}l_s(f, G \times H))^{rs/2} \times \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} (m_\mu n_\nu)^{-r}(\omega(f,\mu,\nu))^{(2-s)r/2} A^*_{\mu-1,\nu-1},$$

where κ is from (2.29) corresponding to $\gamma = 2/(2-r)$. Now, proceeding as in the proof of Corollary 3.5, starting with (4.61) instead of (4.44), we can complete the proof of this corollary. □

P r o o f of Corollary 3.10. Suppose $f \in \mathrm{Lip}\,(\alpha, \beta; G \times H) \cap \mathrm{BF}_s(G \times H)$, $f(\cdot, 0) \in \mathrm{BF}_s(G)$, $f(0, \cdot) \in \mathrm{BF}_s(H)$ for some $\alpha, \beta > 0$, $0 < s < 2$, G and H are bounded, $\delta_1 < \frac{1}{2}\alpha(2-s)$, and $\delta_2 < \frac{1}{2}\beta(2-s)$. Then, in view of Lemma 4.2, $\{m^{\delta_1} n^{\delta_2}\} \in \mathfrak{A}^*_\gamma(G \times H)$ for all $\gamma \geqslant 1$. So, we can put $a_{mn} = m^{\delta_1} n^{\delta_2}$ and $r = 1$ in Theorem 3.8 to get (3.25) with $a_{mn} = m^{\delta_1} n^{\delta_2}$ and $r = 1$, that is,

$$(4.62) \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\delta_1} n^{\delta_2} |\hat{f}(m,n)| \leqslant 2^{-1}\kappa(\mathcal{F}l_s(f, G \times H))^{s/2} \times \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} (m_\mu n_\nu)^{-1}(\omega(f,\mu,\nu))^{(2-s)/2} A^*_{\mu-1,\nu-1},$$

where the constant κ is from (2.29) corresponding to $\gamma = 2$. Now, proceeding as in the proof of Corollary 3.6, starting from (4.62) instead of (4.47), we can complete the proof of Corollary 3.10. □

The proof of Theorem 3.9 is similar to the proof of [13], Theorem 6. However, we note that in the statement as well as in the proof of [13], Theorem 6, the authors have identified G with $[0, 1)$.

P r o o f of Theorem 3.9. Since $f \in (\Lambda,\Psi)\mathcal{F}l_p(G\times H)$, it is bounded and hence, being measurable, it follows that $f \in L^2(G\times H)$. Proceeding as in the proof of Theorem 3.7, we get (4.59), that is, we have

$$\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat f(m,n)|^2 \leqslant \frac14\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^2\,\mathrm{d}m(x,y).$$

Now writing 2 as in (4.38) and applying the integral form of Hölder's inequality with the exponents p' and q' yields

(4.63)

$$\begin{aligned}\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat f(m,n)|^2 &\leqslant \frac14\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta/p'+((2-\beta)q'+\beta)/q'}\,\mathrm{d}m(x,y)\\ &= \frac14\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta}\,\mathrm{d}m(x,y)\Big)^{1/p'}\\ &\quad\times\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{((2-\beta)q'+\beta)}\,\mathrm{d}m(x,y)\Big)^{1/q'}\\ &\leqslant \frac14\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta}\,\mathrm{d}m(x,y)\Big)^{1/p'}\\ &\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{(1/q')((2-\beta)q'+\beta)}.\end{aligned}$$

Therefore

$$\begin{aligned}(4.64)\quad \Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat f(m,n)|^2\Big)^{p'} &\leqslant \frac{1}{4^{p'}}\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta}\,\mathrm{d}m(x,y)\Big)^{p'/p'}\\ &\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{p'(2-\beta+\beta/q')}\\ &= \frac{1}{4^{p'}}\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta}\,\mathrm{d}m(x,y)\Big)\\ &\quad\times(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}.\end{aligned}$$

Now multiplying the above inequality by $(\lambda_{i+1}\psi_{j+1})^{-1}$ and, after that, summing the resulting inequalities over $i=0,1,\ldots,m_\mu-1$ and $j=0,1,\ldots,n_\nu-1$, we get

$$\begin{aligned}\Lambda_{m_\mu}\Psi_{n_\nu}\Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat f(m,n)|^2\Big)^{p'} &= \sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{1}{\lambda_{i+1}\psi_{j+1}}\Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat f(m,n)|^2\Big)^{p'}\\ \leqslant \sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{1}{\lambda_{i+1}\psi_{j+1}}\frac{1}{4^{p'}}&\Big(\int_{G\times H}|\Delta_{1,1}f(x,y;h_1,h_2)|^{\beta}\,\mathrm{d}m(x,y)\Big)\\ \times(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}&\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4^{p'}}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{1}{\lambda_{i+1}\psi_{j+1}}\\
&\quad\times\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}\int_{(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)}|\Delta_{1,1}f(x,y;h_1,h_2)|^\beta\,\mathrm{d}m(x,y)\\
&\leqslant \frac{1}{4^{p'}}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{1}{\lambda_{i+1}\psi_{j+1}}\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}1\\
&\quad\times\int_{(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)}(\operatorname{osc}(f,(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)))^\beta\,\mathrm{d}m(x,y)\\
&= \frac{1}{4^{p'}}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\\
&\quad\times\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}\sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{(\operatorname{osc}(f,(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)))^\beta}{\lambda_{i+1}\psi_{j+1}}\\
&\quad\times\int_{(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)}1\,\mathrm{d}m(x,y)\\
&= \frac{1}{4^{p'}}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\\
&\quad\times\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}\Bigg(\sum_{i=0}^{m_\mu-1}\sum_{j=0}^{n_\nu-1}\frac{(\operatorname{osc}(f,(z^G_{q_k,\mu}+G_\mu)\times(z^H_{q_l,\nu}+H_\nu)))^\beta}{\lambda_{i+1}\psi_{j+1}}\Bigg)\frac{1}{m_\mu n_\nu}\\
&\leqslant \frac{1}{4^{p'}m_\mu n_\nu}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,\mu,\nu))^\beta\\
&\leqslant \frac{1}{4^{p'}m_\mu n_\nu}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,\mu,\nu))^\beta\sum_{k=0}^{m_\mu-1}\sum_{l=0}^{n_\nu-1}1\\
&= \frac{1}{4^{p'}m_\mu n_\nu}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,\mu,\nu))^\beta m_\mu n_\nu\\
&\leqslant \frac{1}{4^{p'}}(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,G\times H))^\beta,
\end{aligned}
$$

whence we obtain the inequality

(4.65)

$$
\Bigg(\sum_{m\in D^G_\mu}\sum_{n\in D^H_\nu}|\hat{f}(m,n)|^2\Bigg)^{p'}\leqslant\frac{(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,G\times H))^\beta}{\Lambda_{m_\mu}\Psi_{n_\nu}},
$$

which implies that

$$
\sum_{m\in D^G_\mu}\sum_{n\in D^H_\nu}|\hat{f}(m,n)|^2\leqslant\Big(\frac{(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}(\mathcal{V}_{\Lambda,\Psi,\beta}(f,G\times H))^\beta}{\Lambda_{m_\mu}\Psi_{n_\nu}}\Big)^{1/p'}.
$$

Now for $0 < r < 2$, as $1 = \frac{1}{2}r + \frac{1}{2}(2-r)$, in view of (2.29), (4.65), and Hölder's inequality, we have

(4.66)
$$\begin{aligned}\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} & a_{mn}|\hat{f}(m,n)|^r \\ & \leqslant \Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} a_{mn}^{2/(2-r)}\Big)^{(2-r)/2}\Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat{f}(m,n)|^2\Big)^{r/2} \\ & \leqslant \kappa(m_\mu n_\nu)^{(1-2/(2-r))/(2/(2-r))}\sum_{m\in D_{\mu-1}^G}\sum_{n\in D_{\nu-1}^H} a_{mn}\Big(\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H}|\hat{f}(m,n)|^2\Big)^{r/2} \\ & \leqslant \kappa(m_\mu n_\nu)^{-r/2}A^*_{\mu-1,\nu-1}\Big(\frac{(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\mathcal{V}_{\Lambda,\Psi,\beta}(f,G\times H)}{\Lambda_{m_\mu}\Psi_{n_\nu}}\Big)^{r/(2p')}.\end{aligned}$$

Summing (4.66) over $\mu,\nu \in \mathbb{N}\cup\{0\}$, we get

(4.67)
$$\begin{aligned}\sum_{m=1}^\infty\sum_{n=1}^\infty a_{mn}|\hat{f}(m,n)|^r &= \sum_{\mu=0}^\infty\sum_{\nu=0}^\infty\sum_{m\in D_\mu^G}\sum_{n\in D_\nu^H} a_{mn}|\hat{f}(m,n)|^r \\ &\leqslant \sum_{\mu=0}^\infty\sum_{\nu=0}^\infty \kappa(m_\mu n_\nu)^{-r/2}A^*_{\mu-1,\nu-1} \\ &\quad\times\Big(\frac{(\omega^{((2-\beta)q'+\beta)}(f,\mu,\nu))^{2p'-\beta}\mathcal{V}_{\Lambda,\Psi,\beta}(f,G\times H)}{\Lambda_{m_\mu}\Psi_{n_\nu}}\Big)^{r/(2p')}.\end{aligned}$$

This completes the proof. □

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Authors' address: *Nayna Govindbhai Kalsariya* (corresponding author), *Bhikha Lila Ghodadra*, Department of Mathematics, Faculty of Science, The M. S. University of Baroda, Vadodara 390 002, India, e-mail: `kalsariyanayna.g125@gmail.com`, `bhikhu_ghodadra@yahoo.com`.