

Alois Kufner; Komil Kuliev; Gulchehra Kulieva; Mohlaroyim Eshimova  
New equivalent conditions for Hardy-type inequalities

*Mathematica Bohemica*, Vol. 149 (2024), No. 1, 57–73

Persistent URL: <http://dml.cz/dmlcz/152293>

## Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

## NEW EQUIVALENT CONDITIONS FOR HARDY-TYPE INEQUALITIES

ALOIS KUFNER, Prague, KOMIL KULIEV, GULCHEHRA KULIEVA,  
MOHLAROYIM ESHIMOVA, Samarkand

Received June 23, 2022. Published online March 3, 2023.  
Communicated by Dagmar Medková

*Abstract.* We consider a Hardy-type inequality with Oinarov's kernel in weighted Lebesgue spaces. We give new equivalent conditions for satisfying the inequality, and provide lower and upper estimates for its best constant. The findings are crucial in the study of oscillation and non-oscillation properties of differential equation solutions, as well as spectral properties.

*Keywords:* integral operator; norm; weight function; Lebesgue space; Hardy-type inequality; kernel

*MSC 2020:* 26D10, 26D15, 47B01, 47B34, 47B37, 47B93, 47G10

### 1. INTRODUCTION

Let  $(a, b) \subset \mathbb{R}$  and  $u, v$  be weight functions on  $(a, b)$ , i.e., positive measurable functions defined a.e. on  $(a, b)$ . Let  $1 < p \leq q < \infty$  and consider the Hardy-type inequality

$$(1.1) \quad \left( \int_a^b \left( \int_a^x k(x, t) f(t) dt \right)^q u(x) dx \right)^{1/q} \leq C \left( \int_a^b f^p(x) v(x) dx \right)^{1/p}$$

for functions  $f \geq 0$  a.e. in  $(a, b)$ , where  $k(x, t)$  is called a kernel of the inequality, which is a nonnegative measurable function defined a.e. on  $(a, b) \times (a, b)$ .

If  $k(x, t) \equiv 1$ , then (1.1) takes the form

$$(1.2) \quad \left( \int_a^b \left( \int_a^x f(t) dt \right)^q u(x) dx \right)^{1/q} \leq C \left( \int_a^b f^p(x) v(x) dx \right)^{1/p},$$

---

The first author was supported by RVO: 67985840.

which holds if and only if the condition

$$(1.3) \quad A_M := \sup_{t \in (a,b)} \left( \int_t^b u(\tau) \, d\tau \right)^{1/q} \left( \int_a^t v^{1-p'}(\tau) \, d\tau \right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1}$$

holds, and the best constant  $C$  (the least constant for which the inequality holds) in (1.2) satisfies the estimates

$$(1.4) \quad A_M \leq C \leq l(p, q) A_M,$$

where  $l(p, q)$  are values of  $p$  and  $q$  discovered by scholars in different times—in the range  $p = q > 1$  in 1969 by Tomaselli (see [23]) and Talenti (see [22]); in 1972 Muckenhoupt (see [16]) derived the expression. The case  $1 < p \leq q < \infty$  was investigated independently in 1978 by Bradley (see [2]) and in 1979 by Kokilashvili (see [7]) and obtained also by Maz'ja (see [15]) etc. (see also [10], [11] and [18] for more details). Let us present Opic's expression

$$(1.5) \quad l(p, q) = \left( 1 + \frac{q}{p'} \right)^{1/q} \left( 1 + \frac{p'}{q} \right)^{1/p'},$$

which will further be used in the proofs of our results.

In 2001, Persson and Stepanov (see [19]) also gave a new equivalent condition

$$(1.6) \quad A_{PS} = \sup_{t \in (a,b)} \left( \int_a^t v^{1-p'}(\tau) \, d\tau \right)^{-1/p} \left( \int_a^t u(\tau) \left( \int_a^\tau v^{1-p'}(\xi) \, d\xi \right)^q \, d\tau \right)^{1/q} < \infty$$

and for the best constant the estimates

$$A_{PS} \leq C \leq p' A_{PS}.$$

Therefore, conditions (1.3) and (1.6) are mutually equivalent, that is, if one condition is satisfied, the other is also. Later, in 2003 (see [24]) Wedestig gave parameter-dependent conditions, where it was shown in examples that it is a better approximation for the best constant of (1.2). As a result, any new equivalent condition is extremely useful for investigating the Hardy inequalities. In [12] Kufner, Persson and Wedestig, in [5] Gogatishvili, Kufner, Persson and Wedestig and in [4] Gogatishvili, Kufner and Persson gave a family of conditions ensuring the fulfillment of inequality (1.2).

The problem becomes significantly more difficult if the kernel  $k(x, t)$  is not a constant function. The satisfying of (1.1) began to be studied in the last decades of the twentieth century. Let us now give some scientific conclusions concerning

inequalities of this type. For example, Martin-Reyes and Sawyer in [14], Prokhorov and Stepanov in [21], Prokhorov in [20] considered (1.1) with the kernel  $k(x, t) = (x - t)^{\alpha-1}$ ,  $\alpha > 0$ . Bloom and Kerman in [1] and Oinarov in [17] gave equivalent conditions with kernels  $k(x, t)$  being a continuous nonnegative function increasing in the first argument, decreasing in the second argument and satisfying the following condition: There exists a number  $h \geq 1$  such that  $k(x, s) \leq h(k(x, t) + k(t, s))$  for all  $a < s \leq t \leq x < b$ . Functions  $k(x, t)$  satisfying the above conditions are also called Oinarov's kernel. It was proved that the conditions

$$A_1 := \sup_{x \in (a, b)} \left( \int_x^b k^q(t, x) u(t) dt \right)^{1/q} \left( \int_a^x v^{1-p'}(t) dt \right)^{1/p'} < \infty$$

and

$$A_2 := \sup_{x \in (a, b)} \left( \int_x^b u(t) dt \right)^{1/q} \left( \int_a^x k^{p'}(x, t) v^{1-p'}(t) dt \right)^{1/p'}$$

are equivalent to (1.1). Kufner, Kuliev and Oinarov in [8] and Kufner, Kuliev and Persson in [9] also considered (1.1) with the kernel  $k(x, t) = \sum_{i=1}^n a_i(x) b_i(t)$ . In the foregoing works, the authors focused mainly on finding equivalent conditions for the fulfillment of the inequality. But, in the theory of differential equations and spectral theory it is very important to find the lower and upper estimates for the best constant. Let us note that in [3] Drábek, Kuliev, Marletta also gave an estimate for the best constant for a special choice of the kernel  $k(x, t) = x - t$  which appears from the fourth order nonlinear Shturm-Liouville problem, where they significantly used these estimates. In 2021, Kalybay and Baiarystanov (see [6]) proved the estimates

$$(1.7) \quad A \leq C \leq (h + 1)^3 p^{1/q} (p')^{1/p'} A,$$

where  $A = \max\{A_1, A_2\} < \infty$ . Recently, Kuliev, Kulieva and Eshimova in [13] also obtained the estimate

$$A \leq C \leq X,$$

where  $X$  is a positive solution of the equation

$$(1.8) \quad X^{q'} - h q^{1/(q-1)} (q')^{1/(q-1)} A^{1/(q-1)} X = h q^{1/(q-1)} (p')^{q'/p'} A^{q'}, \quad q' = \frac{q}{q-1}.$$

In this work, we present new equivalent conditions. What can also be seen is the convenience of these conditions in the estimates obtained for the best constant of inequality (1.1), and for the kernel chosen as a constant function, the lower values of the estimates are better from the previously obtained values. In the second section we present the main results of the work, the proofs of which are given in the third section.

## 2. MAIN PART

We give some new couple conditions which are equivalent to Oinarov's couple condition  $A_1$  and  $A_2$ . The upper estimates are given by solutions of some nonlinear algebraic equations. Let us denote

$$\begin{aligned}
 A_1(s) &= \left( \int_s^b k^q(t, s) u(t) dt \right)^{1/q} \left( \int_a^s v^{1-p'}(t) dt \right)^{1/p'} ; \\
 A_2(s) &= \left( \int_s^b u(t) dt \right)^{1/q} \left( \int_a^s k^{p'}(s, t) v^{1-p'}(t) dt \right)^{1/p'} ; \\
 B_1(s) &= \left( \int_s^b v^{1-p'}(t) \left( \int_t^b k^q(x, t) u(x) dx \right)^{p'} dt \right)^{1/p'q} \left( \int_a^s v^{1-p'}(t) dt \right)^{1/p'q'} ; \\
 B_2(s) &= \left( \int_s^b v^{1-p'}(t) \left( \int_t^b k(x, t) u(x) dx \right)^{p'} dt \right)^{1/p'} \left( \int_s^b u(t) dt \right)^{-1/q'} ; \\
 B_3(s) &= \left( \int_a^s u(t) \left( \int_a^t k(t, x) v^{1-p'}(x) dx \right)^q dt \right)^{1/q} \left( \int_a^s v^{1-p'}(t) dt \right)^{-1/p} ; \\
 B_4(s) &= \left( \int_a^s u(t) \left( \int_a^t k^{p'}(t, x) v^{1-p'}(x) dx \right)^q dt \right)^{1/q p'} \left( \int_s^b u(t) dt \right)^{1/q p} .
 \end{aligned}$$

Our first main result reads:

**Theorem 2.1.** *Let  $1 < p \leq q < \infty$  and the kernel  $k(x, t)$  satisfy Oinarov's condition. Then inequality (1.1) holds if and only if*

$$B_1 := \sup_{s \in (a, b)} B_1(s) < \infty \quad \text{and} \quad B_2 := \sup_{s \in (a, b)} B_2(s) < \infty.$$

The best constant of inequality (1.1) satisfies

$$(2.1) \quad \max \left\{ \sup_{s \in (a, b)} [A_1^{p'q}(s) + qB_1^{p'q}(s)]^{1/p'q}, \sup_{s \in (a, b)} [A_2^{p'}(s) + B_2^{p'}(s)]^{1/p'} \right\} \leq C \leq X,$$

where  $X$  is a positive solution of the nonlinear equation

$$(2.2) \quad X^{q'} - h(qB_2)^{1/(q-1)} X = hq^{p'+1/p'(q-1)} (q')^{1/p'} B_1^{q'}.$$

**Remark 2.2.** Equation (2.2) has a unique positive solution since the function

$$h(x) = \frac{x^{q'}}{q^{(p'+1)/(p'(q-1))} (q')^{1/p'} B_1^{q'} + (qB_2)^{1/(q-1)} x}$$

is a continuous and monotone increasing function of  $x$  in the half line  $(0, \infty)$ ,  $h(0) = 0$  and  $h(\infty) = \infty$ .

**Example 2.3.** Let  $1 < p \leq q = 2$  and  $h \geq 1$ . Then equation (2.2) and its positive solution take the forms

$$X^2 - 2hB_2X = 2^{(p'+2)/p'}hB_1^2 \quad \text{and} \quad X = (h + \sqrt{h^2 + 2^{(p'+2)/p'}h})B,$$

respectively. Then corresponding estimates take the form

$$\begin{aligned} \max \left\{ \sup_{s \in (a,b)} [A_1^{2p'}(s) + 2B_1^{2p'}(s)]^{1/2p'}, \sup_{s \in (a,b)} [A_2^{p'}(s) + B_2^{p'}(s)]^{1/p'} \right\} \\ \leq C \leq (hB_2 + \sqrt{h^2B_2^2 + 2^{(p'+2)/p'}hB_1^2}). \end{aligned}$$

Our second main result reads:

**Theorem 2.4.** *Let  $1 < p \leq q < \infty$  and the kernel  $k(x, t)$  satisfy Oinarov's condition. Then inequality (1.1) holds if and only if*

$$B_3 := \sup_{s \in (a,b)} B_3(s) < \infty \quad \text{and} \quad B_4 := \sup_{s \in (a,b)} B_4(s) < \infty.$$

The best constant of the inequality satisfies

$$(2.3) \quad \max \left\{ \sup_{s \in (a,b)} [A_1^q(s) + B_3^q(s)]^{1/q}, \sup_{s \in (a,b)} [A_2^{p'q}(s) + p'B_4^{p'q}(s)]^{1/p'q} \right\} \leq C \leq X,$$

where  $X$  is a positive solution of the nonlinear equation

$$X^p - h(p'B_3)^{p-1}X = hp^{1/q}(p')^{(q+1)/(p'-1)q}B_4^p.$$

**Corollary 2.5.** *For (1.2) the corresponding  $B_1, B_2$  and  $B_3, B_4$  conditions take the forms:*

$$\begin{aligned} B_1 &= \sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( \int_t^b u(x) dx \right)^{p'} dt \right)^{1/p'q} \left( \int_a^s v^{1-p'}(t) dt \right)^{1/p'q'}; \\ B_2 &= \sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( \int_t^b u(x) dx \right)^{p'} dt \right)^{1/p'} \left( \int_s^b u(x) dx \right)^{-1/q'}; \\ B_3 &= \sup_{s \in (a,b)} \left( \int_a^s u(t) \left( \int_a^t v^{1-p'}(x) dx \right)^q dt \right)^{1/q} \left( \int_a^s v^{1-p'}(t) dt \right)^{-1/p}; \\ B_4 &= \sup_{s \in (a,b)} \left( \int_a^s u(t) \left( \int_a^t v^{1-p'}(x) dx \right)^q dt \right)^{1/q} \left( \int_s^b u(t) dt \right)^{1/q}. \end{aligned}$$

It is known from [4] that the conditions are mutually equivalent since  $B_1 = A_9(1/p'q')$ ,  $B_2 = A_4(1/q')$ ,  $B_3 = A_3(1/p) = A_{PS}$  and  $B_4 = A(1/pq)$ , which

means that each condition is equivalent to validity of (1.2). Therefore, the lower estimate for the best constant of (1.2) is

$$\begin{aligned} & \max\{A_{M,1}, A_{M,2}, A_{M,3}, A_{M,4}\} \leq C, \\ A_{M,1} & := \sup_{s \in (a,b)} [A_M^{p'q}(s) + qB_1^{p'q}(s)]^{1/p'q}, \quad A_{M,2} := \sup_{s \in (a,b)} [A_M^{p'q}(s) + B_2^{p'q}(s)]^{1/p'}, \\ A_{M,3} & := \sup_{s \in (a,b)} [A_M^q(s) + B_3^q(s)]^{1/q}, \quad A_{M,4} := \sup_{s \in (a,b)} [A_M^{p'q}(s) + p'B_4^{p'q}(s)]^{1/p'q}. \end{aligned}$$

As a special case we derive that

$$\begin{aligned} & \sup_{s \in (a,b)} [A_M^q(s) + A_{\text{PS}}^q(s)]^{1/q} \leq C, \\ A_{\text{PS}}(s) & = \left( \int_a^s u(t) \left( \int_a^t v^{1-p'}(x) dx \right)^q dt \right)^{1/qp'} \left( \int_s^b u(t) dt \right)^{1/qp}. \end{aligned}$$

It is seen that the lower estimate is better than the estimates given in [11], Theorem 1.2 and in [10], Theorem 5.

### 3. PROOFS

In this section we present proofs of the main theorems. First we deal with the duality principle, which will further be used.

**Duality principle.** When it comes to creating new conditions and working with conjugate inequality, the concept of duality is critical.

**Lemma 3.1.** *Let  $1 < p, q < \infty$ ,  $0 < C < \infty$  and  $k(x, t)$  be a nonnegative measurable function on  $(a, b) \times (a, b)$ . Then inequality (1.1) holds for all positive measurable functions  $f$  on  $(a, b)$  if and only if “its dual inequality”*

$$(3.1) \quad \left( \int_a^b \left( \int_x^b k(t, x)g(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'} \leq C \left( \int_a^b g^{q'}(x)u^{1-q'}(x) dx \right)^{1/q'}$$

holds for all nonnegative functions  $g$  with the same best constant  $C$ .

*Proof.* For the proof see page 78 in [11]. □

*Proof of Theorem 2.1. Necessity.* Here we first assume that Hardy-type inequality (1.1) holds, where  $C$  is the best constant (the smallest constant in which the inequality holds). It is known from Lemma 3.1 that inequality (1.1) holds if and only if dual inequality (3.1) holds. Now by choosing the function  $g_\tau$  in (3.1) as

$$g_\tau(x) = k^{q-1}(x, \tau)u(x)\chi_{[\tau, b)}(x)$$

we get for the right-hand side of (3.1)

$$\left( \int_a^b g_\tau^{q'}(x) u^{1-q'}(x) dx \right)^{1/q'} = \left( \int_\tau^b k^q(x, \tau) u(x) dx \right)^{1/q'}$$

and for the left-hand side

$$\begin{aligned} S &:= \left( \int_a^b \left( \int_x^b k(t, x) g_\tau(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'} \\ &= \left( \int_a^b \left( \int_x^b k(t, x) k^{q-1}(t, \tau) u(t) \chi_{[\tau, b]}(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'} \\ &= \left( \int_a^\tau \left( \int_\tau^b k(t, x) k^{q-1}(t, \tau) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right. \\ &\quad \left. + \int_\tau^b \left( \int_x^b k(t, x) k^{q-1}(t, \tau) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'}. \end{aligned}$$

Using monotonicity of  $k$  in the first integral  $k(t, x) \geq k(t, \tau)$  for  $x \geq \tau$  and in the second integral as  $k(t, \tau) \geq k(t, x)$  for  $\tau \geq x$ , we have

$$\begin{aligned} S &\geq \left( \int_a^\tau \left( \int_\tau^b k^q(t, \tau) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right. \\ &\quad \left. + \int_\tau^b \left( \int_x^b k^q(t, x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'}. \end{aligned}$$

From these and (3.1) we obtain

$$\begin{aligned} \left( \int_\tau^b k^q(t, \tau) u(t) dt \right)^{p'} \left( \int_a^\tau v^{1-p'}(x) dx \right) + \int_\tau^b \left( \int_x^b k^q(t, x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \\ \leq C^{p'} \left( \int_\tau^b k^q(x, \tau) u(x) dx \right)^{p'/q'}. \end{aligned}$$

This implies that

$$A_1^{p'}(\tau) + \int_\tau^b \left( \int_x^b k^q(t, x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \left( \int_\tau^b k^q(x, \tau) u(x) dx \right)^{-p'/q'} \leq C^{p'},$$

i.e.,

$$A_1^{p'q}(\tau) + B_1^{p'q}(\tau) \leq C^{p'} A_1^{p'(q-1)}(\tau).$$

Using Young's inequality to the right-hand side of the estimate we obtain

$$A_1^{p'q}(\tau) + B_1^{p'q}(\tau) \leq \frac{C^{p'q}}{q} + \frac{A_1^{p'q}(\tau)}{q'}$$

and then

$$\sup_{\tau \in (a, b)} \{A_1^{p'q}(\tau) + qB_1^{p'q}(\tau)\} \leq C^{p'q}.$$



This proves the first part of (2.1) . Now we prove the second part. Choosing the function  $g_\tau$  in (3.1) as

$$g_\tau(x) = u(x)\chi_{[\tau,b)}(x)$$

we get

$$\left( \int_a^b g_\tau^{q'}(x) u^{1-q'}(x) dx \right)^{1/q'} = \left( \int_\tau^b u(x) dx \right)^{1/q'}$$

and the left-hand side is estimated as follows:

$$\begin{aligned} & \left( \int_a^b \left( \int_x^b k(t,x) g_\tau(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'} \\ &= \left( \int_a^b \left( \int_x^b k(t,x) u(t) \chi_{[\tau,b)}(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'} \\ &= \left( \int_a^\tau \left( \int_\tau^b k(t,x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right. \\ & \quad \left. + \int_\tau^b \left( \int_x^b k(t,x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right)^{1/p'}. \end{aligned}$$

From these and (3.1) we obtain

$$\begin{aligned} & \int_\tau^b \left( \int_x^b k(t,x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \left( \int_\tau^b u(x) dx \right)^{-p'/q'} \\ & \quad + \left( \int_a^\tau \left( \int_\tau^b k(t,x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \right) \left( \int_\tau^b u(x) dx \right)^{-p'/q'} \leq C^{p'}, \end{aligned}$$

i.e.,

$$B_2^{p'}(\tau) + \int_a^\tau \left( \int_\tau^b k(t,x) u(t) dt \right)^{p'} v^{1-p'}(x) dx \left( \int_\tau^b u(x) dx \right)^{-p'/q'} \leq C^{p'}.$$

Using monotonicity of  $k$ — $k(t,x) \geq k(\tau,x)$  for  $t \geq \tau$ —we obtain

$$C^{p'} \geq B_2^{p'}(\tau) + \int_a^\tau k^{p'}(\tau,x) v^{1-p'}(x) dx \left( \int_\tau^b u(x) dx \right)^{p'/q'},$$

i.e.,

$$B_2^{p'}(\tau) + A_2^{p'}(\tau) \leq C^{p'}.$$

The second part of (2.1) is also proved.

*Sufficiency.* Let us denote

$$I = \int_a^b \left( \int_a^x k(x,t) f(t) dt \right)^q u(x) dx,$$

then consequently using Fubini's theorem and Hölder's inequality we get

$$\begin{aligned}
(3.2) \quad I &= q \int_a^b \left( \int_a^x k(x,t) f(t) \left( \int_a^t k(x,s) f(s) ds \right)^{q-1} dt \right) u(x) dx \\
&= q \int_a^b f(t) \left( \int_t^b k(x,t) u(x) \left( \int_a^t k(x,s) f(s) ds \right)^{q-1} dx \right) dt \\
&\leq q \|f\|_{p,v} \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t) u(x) \left( \int_a^t k(x,s) f(s) ds \right)^{q-1} dx \right)^{p'} dt \right)^{1/p'} \\
&= q \|f\|_{p,v} J^{1/p'}.
\end{aligned}$$

We now proceed to the proof by estimating  $J$ . To do this, we estimate its inner integral separately. Using Hölder's inequality with exponents  $[q]/(q-1)$  and  $[q]/(1-\{q\})$ , we have

$$\begin{aligned}
J_1 &:= \int_t^b k(x,t) u(x) \left( \int_a^t k(x,s) f(s) ds \right)^{q-1} dx \\
&= \int_t^b \left( k^{\{q\}}(x,t) u(x) \left( \int_a^t k(x,s) f(s) ds \right)^{[q]} \right)^{(q-1)/[q]} \left( k^q(x,t) u(x) \right)^{(1-\{q\})/[q]} dx \\
&\leq \left( \int_t^b k^{\{q\}}(x,t) u(x) \left( \int_a^t k(x,s) f(s) ds \right)^{[q]} dx \right)^{(q-1)/[q]} \\
&\quad \times \left( \int_t^b k^q(x,t) u(x) dx \right)^{(1-\{q\})/[q]}.
\end{aligned}$$

We now derive the following estimate from the condition of Oinarov's kernel and then Newton's binomial formulae

$$\begin{aligned}
(3.3) \quad J_1 &\leq h^{q-1} \left( \int_t^b k^{\{q\}}(x,t) u(x) \left( k(x,t) \int_a^t f(s) ds \right. \right. \\
&\quad \left. \left. + \int_a^t k(t,s) f(s) ds \right)^{[q]} dx \right)^{(q-1)/[q]} \left( \int_t^b k^q(x,t) u(x) dx \right)^{(1-\{q\})/[q]} \\
&= h^{q-1} \left( \int_t^b k^{\{q\}}(x,t) u(x) \left( \sum_{n=0}^{[q]} C_{[q]}^n k^n(x,t) \left( \int_a^t f(s) ds \right)^n \right. \right. \\
&\quad \left. \left. \times \left( \int_a^t k(t,s) f(s) ds \right)^{[q]-n} \right) dx \right)^{(q-1)/[q]} \left( \int_t^b k^q(x,t) u(x) dx \right)^{(1-\{q\})/[q]} \\
&= h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^{\{q\}+n}(x,t) u(x) dx \right) \left( \int_a^t f(s) ds \right)^n \right. \\
&\quad \left. \times \left( \int_a^t k(t,s) f(s) ds \right)^{[q]-n} \right)^{(q-1)/[q]} \left( \int_t^b k^q(x,t) u(x) dx \right)^{(1-\{q\})/[q]}.
\end{aligned}$$

Using Hölder's inequality to the first integral of the sum for  $0 < n < [q]$  we have

$$\begin{aligned} & \int_t^b k^{\{q\}+n}(x, t)u(x) \, dx \\ &= \int_t^b (k^q(x, t)u(x))^{(n-1+\{q\})/(q-1)} (k(x, t)u(x))^{([q]-n)/(q-1)} \, dx \\ &\leq \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{(n-1+\{q\})/(q-1)} \left( \int_t^b k(x, t)u(x) \, dx \right)^{([q]-n)/(q-1)}. \end{aligned}$$

Then (3.3) is estimated as follows:

$$\begin{aligned} J_1 &\leq h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{(n-1+\{q\})/(q-1)} \right. \\ &\quad \times \left( \int_t^b k(x, t)u(x) \, dx \right)^{([q]-n)/(q-1)} \left( \int_a^t f(s) \, ds \right)^n \\ &\quad \times \left. \left( \int_a^t k(t, s)f(s) \, ds \right)^{[q]-n} \right)^{(q-1)/[q]} \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{(1-\{q\})/[q]} \\ &= h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{n/(q-1)} \left( \int_t^b k(x, t)u(x) \, dx \right)^{([q]-n)/(q-1)} \right. \\ &\quad \times \left. \left( \int_a^t f(s) \, ds \right)^n \left( \int_a^t k(t, s)f(s) \, ds \right)^{[q]-n} \right)^{(q-1)/[q]} \\ &= h^{q-1} \left( \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{1/(q-1)} \int_a^t f(s) \, ds \right. \\ &\quad \left. + \left( \int_t^b k(x, t)u(x) \, dx \right)^{1/(q-1)} \int_a^t k(t, s)f(s) \, ds \right)^{q-1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & \int_t^b k(x, t)u(x) \left( \int_a^t k(x, s)f(s) \, ds \right)^{q-1} \, dx \\ &\leq h^{q-1} \left( \left( \int_t^b k^q(x, t)u(x) \, dx \right)^{1/(q-1)} \int_a^t f(s) \, ds \right. \\ &\quad \left. + \left( \int_t^b k(x, t)u(x) \, dx \right)^{1/(q-1)} \int_a^t k(t, s)f(s) \, ds \right)^{q-1}. \end{aligned}$$

From this we get

$$J = \int_a^b v^{1-p'}(t) \left( \int_t^b k(x, t)u(x) \left( \int_a^t k(x, s)f(s) \, ds \right)^{q-1} \, dx \right)^{p'} \, dt$$

$$\begin{aligned} &\leq h^{(q-1)p'} \int_a^b v^{1-p'}(t) \left( \left( \int_t^b k^q(x,t)u(x) dx \right)^{1/(q-1)} \int_a^t f(s) ds \right. \\ &\quad \left. + \left( \int_t^b k(x,t)u(x) dx \right)^{1/(q-1)} \int_a^t k(t,s)f(s) ds \right)^{(q-1)p'} dt. \end{aligned}$$

Using Minkowski's integral inequality, we obtain

$$\begin{aligned} J &\leq h^{(q-1)p'} \left( \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x) dx \right)^{p'} \right. \right. \\ &\quad \left. \left. \times \left( \int_a^t f(s) ds \right)^{(q-1)p'} dt \right)^{1/(q-1)p'} \right. \\ &\quad \left. + \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x) dx \right)^{p'} \right. \right. \\ &\quad \left. \left. \times \left( \int_a^t k(t,s)f(s) ds \right)^{(q-1)p'} dt \right)^{1/(q-1)p'} \right)^{(q-1)p'} \\ &= h^{(q-1)p'} \left( I_1^{1/(q-1)p'} + I_2^{1/(q-1)p'} \right)^{(q-1)p'}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x) dx \right)^{p'} \left( \int_a^t f(s) ds \right)^{(q-1)p'} dt, \\ I_2 &= \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x) dx \right)^{p'} \left( \int_a^t k(t,s)f(s) ds \right)^{(q-1)p'} dt. \end{aligned}$$

From this and (3.2) we have that

$$I \leq q \|f\|_{p,v} J^{1/p'} \leq q \|f\|_{p,v} h^{q-1} (I_1^{1/(q-1)p'} + I_2^{1/(q-1)p'})^{q-1},$$

then we raise both sides of this inequality to the power of  $1/(q-1)$ , i.e.,

$$I^{1/(q-1)} \leq h q^{1/(q-1)} \|f\|_{p,v}^{1/(q-1)} (I_1^{1/(q-1)p'} + I_2^{1/(q-1)p'}).$$

Further, estimate  $I_1$  and  $I_2$ , separately. To estimate  $I_1$ , we use Hardy inequality (1.2) of the exponents  $\bar{p} := p$ ,  $\bar{q} := (q-1)p'$  and weight functions  $\bar{u} := v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x) dx \right)^{p'}$  and  $\bar{v}(t) := v(t)$ , i.e.,

$$\begin{aligned} (3.4) \quad I_1^{1/(q-1)p'} &= \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x) dx \right)^{p'} \left( \int_a^t f(s) ds \right)^{(q-1)p'} dt \right)^{1/(q-1)p'} \\ &\leq C_{p,(q-1)p'} \left( \int_a^b f^p(t)v(t) dt \right)^{1/p}, \end{aligned}$$

which holds since its characteristic condition (1.3) is satisfied for  $\bar{p} \leq \bar{q}$ :

$$B_1^{q'} = \sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( \int_t^b k^q(x,t) u(x) dx \right)^{p'} dt \right)^{1/(q-1)p'} \left( \int_a^s v^{1-p'}(t) dt \right)^{1/p'} < \infty.$$

Using (1.5) and (1.4) we get the following upper estimate for the best constant  $C_{p,(q-1)p'}$  in (3.4):

$$C_{p,(q-1)p'} \leq q^{1/p'(q-1)} (q')^{1/p'} B_1^{q'}.$$

Then  $I_1^{1/(q-1)p'} \leq q^{1/p'(q-1)} (q')^{1/p'} B_1^{q'} \|f\|_{p,v}$ .

Let us estimate  $I_2$ :

$$\begin{aligned} I_2 &= \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t) u(x) dx \right)^{p'} \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} dt \\ &= \int_a^b \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} d \left( - \int_t^b \left( \int_s^b k(x,s) u(x) dx \right)^{p'} v^{1-p'}(s) ds \right) \\ &= \int_a^b \left( \int_t^b \left( \int_s^b k(x,s) u(x) dx \right)^{p'} v^{1-p'}(s) ds \right) d \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} \\ &= \int_a^b B_2^{p'}(t) \left( \int_t^b u(x) dx \right)^{p'/q'} d \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} \\ &\leq B_2^{p'} \int_a^b \left( \int_t^b u(x) dx \right)^{p'/q'} d \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} \\ &= B_2^{p'} \left( \left( \int_a^b \left( \int_t^b u(x) dx \right)^{p'/q'} d \left( \int_a^t k(t,s) f(s) ds \right)^{(q-1)p'} \right)^{q'/p'} \right)^{p'/q'} \\ &\leq B_2^{p'} \left( \int_a^b u(x) \left( \int_a^x k(x,s) f(s) ds \right)^q dx \right)^{p'/q'} = B_2^{p'} I^{p'/q'}. \end{aligned}$$

To get the last estimate we used Minkowski's integral inequality. Therefore,

$$I_2^{1/(q-1)p'} \leq B_2^{1/q-1} I^{1/q}.$$

From the above estimates we have finally obtained

$$\begin{aligned} (3.5) \quad I^{1/(q-1)} &\leq h q^{1/(q-1)} \|f\|_{p,v}^{1/(q-1)} (q^{1/p'(q-1)} (q')^{1/p'} B_1^{q'} \|f\|_{p,v} + B_2^{1/(q-1)} I^{1/q}) \\ &= h q^{(p'+1)/p'(q-1)} (q')^{1/p'} B_1^{q'} \|f\|_{p,v}^{q'} + h (q B_2 \|f\|_{p,v})^{1/(q-1)} I^{1/q}. \end{aligned}$$

Using Young's inequality  $ab \leq a^q/q + b^{q'}/q'$  to the second term of the last line as

$$h (q B_2 \|f\|_{p,v})^{1/(q-1)} I^{1/q} \leq \frac{h^q (q B_2 \|f\|_{p,v})^{q'}}{q} + \frac{I^{1/(q-1)}}{q'},$$

then inserting this estimate into (3.5), we find that

$$I^{1/(q-1)} \leq hq^{(p'+1)/p'(q-1)}(q')^{1/p'} B_1^{q'} \|f\|_{p,v}^{q'} + h^q q^{q'-1} B_2^{q'} \|f\|_{p,v}^{q'} + \frac{I^{1/(q-1)}}{q'}$$

and then

$$I^{1/(q-1)} \leq hq^{1/p'(q-1)+q'}(q')^{1/p'} B_1^{q'} \|f\|_{p,v}^{q'} + h^q q^{q'} B_2^{q'} \|f\|_{p,v}^{q'}.$$

From this we obtain that

$$I^{1/q} \leq q(hq^{1/p'(q-1)}(q')^{1/p'} + h^q)^{1/q'} B \|f\|_{p,v}, \quad B = \max\{B_1, B_2\}.$$

This implies that the conditions  $B_1 < \infty$  and  $B_2 < \infty$  are sufficient for the inequality to hold. Further, we obtain the upper estimate for the best constant  $C$ . Using  $I^{1/q} \leq C\|f\|_{p,v}$  in (3.5) we get

$$I^{1/(q-1)} \leq hq^{1/(q-1)} \|f\|_{p,v}^{q/(q-1)} (q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'} + B_2^{1/(q-1)} C),$$

i.e.,

$$\left( \frac{I^{1/q}}{\|f\|_{p,v}} \right)^{q'} \leq hq^{1/(q-1)} (q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'} + B_2^{1/(q-1)} C).$$

Then we have the estimate for the best constant

$$C^{q'} \leq hq^{1/(q-1)} (q^{1/p'(q-1)}(q')^{1/p'} B_1^{q'} + B_2^{1/(q-1)} C).$$

Consequently, we obtain

$$\frac{C^{q'}}{q^{(p'+1)/p'(q-1)}(q')^{1/p'} B_1^{q'} + (qB_2)^{1/(q-1)} C} \leq h.$$

Let us consider the function

$$f(x) = \frac{x^{q'}}{q^{(p'+1)/p'(q-1)}(q')^{1/p'} B_1^{q'} + (qB_2)^{1/(q-1)} x}$$

corresponding to the left-hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in  $(0, \infty)$ ,  $f(0) = 0$  and  $f(\infty) = \infty$ , which implies that the equation  $f(x) = h$  has exactly one positive solution in  $(0, \infty)$ . If  $X$  is a solution of the equation, which means that

$$X^{q'} = h(q^{(p'+1)/p'(q-1)}(q')^{1/p'} B_1^{q'} + (qB_2)^{1/(q-1)} X),$$

then  $C \leq X$ . The proof is complete. □

Proof of Theorem 2.4. *Necessity.* Let Hardy-type inequality (1.1) hold and  $C$  be the best constant. Then choosing the function  $f_\tau$  in (1.1) as

$$f_\tau(x) = v^{1-p'}(x)\chi_{(a,\tau]}(x),$$

we get the right-hand side

$$\left( \int_a^b f_\tau^p(x)v(x) \, dx \right)^{1/p} = \left( \int_a^\tau v^{1-p'}(x) \, dx \right)^{1/p}$$

and the left-hand side is estimated as

$$\begin{aligned} & \left( \int_a^b \left( \int_a^x k(x,t)f_\tau(t) \, dt \right)^q u(x) \, dx \right)^{1/q} \\ &= \left( \int_a^b \left( \int_a^x k(x,t)v^{1-p'}(t)\chi_{(a,\tau]}(t) \, dt \right)^q u(x) \, dx \right)^{1/q} \\ &= \left( \int_a^\tau \left( \int_a^x k(x,t)v^{1-p'}(t) \, dt \right)^q u(x) \, dx + \int_\tau^b \left( \int_a^\tau k(x,t)v^{1-p'}(t) \, dt \right)^q u(x) \, dx \right)^{1/q}. \end{aligned}$$

From these and (1.1) we obtain

$$\begin{aligned} & \int_a^\tau \left( \int_a^x k(x,t)v^{1-p'}(t) \, dt \right)^q u(x) \, dx \left( \int_a^\tau v^{1-p'}(x) \, dx \right)^{-q/p} \\ &+ \int_\tau^b \left( \int_a^\tau k(x,t)v^{1-p'}(t) \, dt \right)^q u(x) \, dx \left( \int_a^\tau v^{1-p'}(x) \, dx \right)^{-q/p} \leq C^q, \end{aligned}$$

i.e.,

$$B_3^q(\tau) + \int_\tau^b \left( \int_a^\tau k(x,t)v^{1-p'}(t) \, dt \right)^q u(x) \, dx \left( \int_a^\tau v^{1-p'}(x) \, dx \right)^{-q/p} \leq C^q.$$

Using the monotonicity of  $k$ — $k(x,t) \geq k(x,\tau)$  for all  $t \geq \tau$ —we have

$$C^q \geq B_3^q(\tau) + \int_\tau^b k^q(x,\tau)u(x) \, dx \left( \int_a^\tau v^{1-p'}(x) \, dx \right)^{q/p'},$$

which implies

$$\sup_{\tau \in (a,b)} (B_3^q(\tau) + A_1^q(\tau)) \leq C^q.$$

The first part of (2.3) is proved.

Now we prove the second part. By choosing the function  $f_\tau$  in (1.1) as

$$f_\tau(x) = k^{p'-1}(\tau, x)v^{1-p'}(x)\chi_{(a, \tau]}(x)$$

we get

$$\left( \int_a^b f_\tau^p(x)v(x) dx \right)^{1/p} = \left( \int_a^\tau k^{p'}(\tau, x)v^{1-p'}(x) dx \right)^{1/p}$$

and

$$\begin{aligned} & \left( \int_a^b \left( \int_a^x k(x, t)f_\tau(t) dt \right)^q u(x) dx \right)^{1/q} \\ &= \left( \int_a^b \left( \int_a^x k(x, t)k^{p'-1}(\tau, t)v^{1-p'}(t)\chi_{(a, \tau]}(t) dt \right)^q u(x) dx \right)^{1/q} \\ &= \left( \int_a^\tau \left( \int_a^x k(x, t)k^{p'-1}(\tau, t)v^{1-p'}(t) dt \right)^q u(x) dx \right. \\ & \quad \left. + \int_\tau^b \left( \int_a^\tau k(x, t)k^{p'-1}(\tau, t)v^{1-p'}(t) dt \right)^q u(x) dx \right)^{1/q}. \end{aligned}$$

Using the monotonicity of  $k$  in the first integral  $k(\tau, t) \geq k(x, t)$  for  $\tau \geq x$  and in the second integral as  $k(x, t) \geq k(\tau, t)$  for  $x \geq \tau$  we get

$$\begin{aligned} & \geq \left( \int_a^\tau \left( \int_a^x k^{p'}(x, t)v^{1-p'}(t) dt \right)^q u(x) dx \right. \\ & \quad \left. + \int_\tau^b \left( \int_a^\tau k^{p'}(\tau, t)v^{1-p'}(t) dt \right)^q u(x) dx \right)^{1/q}. \end{aligned}$$

From these and (1.1) we obtain

$$\begin{aligned} & \left( \int_a^\tau \left( \int_a^x k^{p'}(x, t)v^{1-p'}(t) dt \right)^q u(x) dx + \int_\tau^b \left( \int_a^\tau k^{p'}(\tau, t)v^{1-p'}(t) dt \right)^q u(x) dx \right)^{1/q} \\ & \leq C \left( \int_a^\tau k^{p'}(\tau, x)v^{1-p'}(x) dx \right)^{1/p}, \end{aligned}$$

i.e.,

$$\begin{aligned} & \int_a^\tau \left( \int_a^x k^{p'}(x, t)v^{1-p'}(t) dt \right)^q u(x) dx + \int_\tau^b \left( \int_a^\tau k^{p'}(\tau, t)v^{1-p'}(t) dt \right)^q u(x) dx \\ & \leq C^q \left( \int_a^\tau k^{p'}(\tau, x)v^{1-p'}(x) dx \right)^{q/p}, \end{aligned}$$

or

$$B_4^{p'q}(\tau)A_2^{q/(1-p)}(\tau) + A_2^q(\tau) \leq C^q,$$

and then

$$B_4^{p'q}(\tau) + A_2^{p'q}(\tau) \leq C^q A_2^{q/(p-1)}(\tau).$$



Using Young's inequality to the right-hand side of the estimate we obtain

$$A_2^{p'q}(\tau) + B_4^{p'q}(\tau) \leq \frac{C^{p'q}}{p'} + \frac{A_2^{p'q}(\tau)}{p}$$

and then

$$\sup_{\tau \in (a,b)} (A_2^{p'q}(\tau) + p' B_4^{p'q}(\tau)) \leq C^{p'q}.$$

The second part of (2.3) is also proved.

*Sufficiency.* The sufficiency of the conditions is analogously proved as Theorem 2.1. In this instance, dual inequality (3.1) is used instead of inequality (1.1). The proof is given step by step, with the exception of the difference occurring at integration's orientation. The proof is complete.  $\square$

*Acknowledgement.* The authors thank the reviewer for his/her valuable comments and suggestions, which really improved the quality of comprehensibility of the paper.

#### References

- [1] *S. Bloom, R. Kerman:* Weighted norm inequalities for operators of Hardy type. Proc. Am. Math. Soc. *113* (1991), 135–141. [zbl](#) [MR](#) [doi](#)
- [2] *J. S. Bradley:* Hardy inequalities with mixed norms. Can. Math. Bull. *21* (1978), 405–408. [zbl](#) [MR](#) [doi](#)
- [3] *P. Drábek, K. Kuliev, M. Marletta:* Some criteria for discreteness of spectrum of half-linear fourth order Sturm-Liouville problem. NoDEA, Nonlinear Differ. Equ. Appl. *24* (2017), Article ID 11, 39 pages. [zbl](#) [MR](#) [doi](#)
- [4] *A. Gogatishvili, A. Kufner, L.-E. Persson:* Some new scales of weight characterizations of the class  $B_p$ . Acta Math. Hung. *123* (2009), 365–377. [zbl](#) [MR](#) [doi](#)
- [5] *A. Gogatishvili, A. Kufner, L.-E. Persson, A. Wedestig:* An equivalence theorem for integral conditions related to Hardy's inequality. Real Anal. Exch. *29* (2004), 867–880. [zbl](#) [MR](#) [doi](#)
- [6] *A. A. Kalybay, A. O. Baiarystanov:* Exact estimate of norm of integral operator with Oinarov condition. Mat. Zh. *21* (2021), 6–14. [zbl](#)
- [7] *V. M. Kokilashvili:* On Hardy's inequalities in weighted spaces. Soobshch. Akad. Nauk Gruzin. SSR *96* (1979), 37–40. (In Russian.) [zbl](#) [MR](#)
- [8] *A. Kufner, K. Kuliev, R. Oinarov:* Some criteria for boundedness and compactness of the Hardy operator with some special kernels. J. Inequal. Appl. *2013* (2013), Article ID 310, 15 pages. [zbl](#) [MR](#) [doi](#)
- [9] *A. Kufner, K. Kuliev, L.-E. Persson:* Some higher order Hardy inequalities. J. Inequal. Appl. *2012* (2012), Article ID 69, 14 pages. [zbl](#) [MR](#) [doi](#)
- [10] *A. Kufner, L. Maligranda, L.-E. Persson:* The Hardy Inequality: About Its History and Some Related Results. Vydavatelský servis, Pilsen, 2007. [zbl](#)
- [11] *A. Kufner, L.-E. Persson:* Weighted Inequalities of Hardy Type. World Scientific, Singapore, 2003. [zbl](#) [MR](#) [doi](#)
- [12] *A. Kufner, L.-E. Persson, A. Wedestig:* A study of some constants characterizing the weighted Hardy inequality. Banach Center Publ. *64* (2004), 135–146. [zbl](#) [MR](#) [doi](#)

- [13] *K. Kuliev, G. Kulieva, M. Eshimova*: On estimates for norm of an integral operator with Oinarov kernel. *Uzb. Math. J.* *65* (2021), 117–127. [zbl](#) [MR](#) [doi](#)
- [14] *F. J. Martin-Reyes, E. Sawyer*: Weighted inequalities for Riemann-Liouville fractional integrals of order one and greater. *Proc. Am. Math. Soc.* *106* (1989), 727–733. [zbl](#) [MR](#) [doi](#)
- [15] *V. G. Maz'ja*: Sobolev Spaces. Springer Series in Soviet Mathematics. Springer, Berlin, 1979. [zbl](#) [MR](#) [doi](#)
- [16] *B. Muckenhoupt*: Hardy's inequality with weights. *Stud. Math.* *44* (1972), 31–38. [zbl](#) [MR](#) [doi](#)
- [17] *R. Oinarov*: Two-sided norm estimates for certain classes of integral operators. *Proc. Steklov Inst. Math.* *204* (1994), 205–214; translation from *Tr. Mat. Inst. Steklova* *204* (1993), 240–250. [zbl](#) [MR](#)
- [18] *B. Opic, A. Kufner*: Hardy-Type Inequalities. Pitman Research Notes in Mathematics 219. Longman Scientific & Technical, Harlow, 1990. [zbl](#) [MR](#)
- [19] *L.-E. Persson, V. D. Stepanov*: Weighted integral inequalities with the geometric mean. *Dokl. Math.* *63* (2001), 201–202; translation from *Dokl. Akad. Nauk, Ross. Akad. Nauk* *377* (2001), 439–440. [zbl](#) [MR](#)
- [20] *D. V. Prokhorov*: On the boundedness and compactness of a class of integral operators. *J. Lond. Math. Soc., II. Ser.* *61* (2000), 617–628. [zbl](#) [MR](#) [doi](#)
- [21] *D. V. Prokhorov, V. D. Stepanov*: Weighted estimates for the Riemann-Liouville operators and applications. *Proc. Steklov Inst. Math.* *243* (2003), 278–301; translation from *Tr. Mat. Inst. Steklova* *243* (2003), 289–312. [zbl](#) [MR](#)
- [22] *G. Talenti*: Osservazioni sopra una classe di disuguaglianze. *Rend. Sem. Mat. Fis. Milano* *39* (1969), 171–185. (In Italian.) [zbl](#) [MR](#) [doi](#)
- [23] *G. Tomaselli*: A class of inequalities. *Boll. Unione Mat. Ital., IV. Ser.* *2* (1969), 622–631. [zbl](#) [MR](#)
- [24] *A. Wedestig*: Weighted Inequalities of Hardy-Type and their Limiting Inequalities: Doctoral Thesis. Luleå University of Technology, Luleå, 2003.

*Authors' addresses:* *Alois Kufner*, Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic, e-mail: [kufner@math.cas.cz](mailto:kufner@math.cas.cz); *Komil Kuliev* (corresponding author), Uzbek-Finnish Pedagogical Institute of Samarkand State University, Samarkand 140104, Uzbekistan and Institute of Mathematics named after V. I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan, Tashkent 100174, Uzbekistan, e-mail: [komilkuliev@gmail.com](mailto:komilkuliev@gmail.com); *Gulchehra Kulieva*, Samarkand State University, Samarkand 140104, Uzbekistan, e-mail: [gkulieva@mail.ru](mailto:gkulieva@mail.ru); *Mohlaroyim Eshimova*, Institute of Mathematics named after V. I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan, Tashkent 100174, Uzbekistan, e-mail: [eshimova\\_math@mail.ru](mailto:eshimova_math@mail.ru).