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ON GENERALIZATIONS OF FUZZY METRIC SPACES

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The aim of the paper is to present three-variable generalizations of fuzzy metric spaces in sense of George and Veeramani from functional and topological points of view, respectively. From the viewpoint of functional generalization, we introduce a notion of generalized fuzzy 2-metric spaces, study their topological properties, and point out that it is also a common generalization of both tripled fuzzy metric spaces proposed by Tian et al. and $\mathcal{M}$-fuzzy metric spaces proposed by Sedghi and Shobe. Since the ordinary tripled norm is the same as the ordinary norm up to the induced topology, we keep our spirit on fuzzy normed structures and introduce a concept of generalized fuzzy 2-normed spaces from the viewpoint of topological generalization. It is proved that generalized fuzzy 2-normed spaces always induces a Hausdorff topology.

Keywords: generalized fuzzy 2-metric space, generalized fuzzy 2-normed space, tripled fuzzy metric space, Hausdorff topology

Classification: 03B52, 03G27, 52A01

1. INTRODUCTION

The concept of metrics (or distance functions in a narrow sense) plays a fundamental role in both pure and applied mathematics, especially in analysis, topology as well as clustering theory. In the real word, distance functions always possess the nature of quantitative relations between points and include some feature of fuzziness, uncertainty and probability. Under a metric, the distance between points is a nonnegative number, while it is not always fixed for some practical problems. For example, in airline problem, when traveling between two cities, we often encounter the flight delay problem. Every flight has a punctuality rate and then the distance function of time consume is no longer a usual metric. That is to say, the time consume of a flight should have a degree or possibility such that the flight will arrive at the destination within a presupposed time schedule. The probabilistic metrics [23] and some types of fuzzy metrics [7, 20] provide us tools to deal with these problems.

The concept of fuzzy metric spaces was initiated by Kramosil and Michálek [20] in 1975, which is now called KM fuzzy metric spaces and can be considered as a modification of the concept of Menger probabilistic metric spaces [23]. The Hausdorffness of a space is very important in both topology and analysis, while the KM fuzzy metric space is

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not Hausdorff. In order to overcome this disadvantage, George and Veeramani [7] in 1994 reintroduced the concept of fuzzy metric spaces by modifying the definition of KM fuzzy metric spaces as a fuzzy set on the product $X \times X \times (0, +\infty)$ satisfying certain conditions. The study of fundamental properties of fuzzy metric spaces has received a lot of attention. For instance, for the case of George and Veeramani's fuzzy metrics, Gregori and Romaguera [15] proved some topological properties; Gregori et al. [10, 12] discussed the convergence problem and gave a characterization of a class of completable spaces; Došenović et al. [6] discussed fixed point theorems in fuzzy metric spaces; Patel and Radenovic [29] gave an application to nonlinear fractional differential equation via a kind of fuzzy contractive mappings in fuzzy metric spaces. The concept of fuzzy 2-metric spaces, a generalization of fuzzy metric spaces, was introduced by Sharma and Sharma in 1997, which is extended to the concept of fuzzy (semi) n-metric spaces by Merghadi and Aliouche [24] and Vijayabalaji and Thillaigovindan [38].

However, research work done by Sharma [35] and Ha et al. [17] has shown that there are some properties of n-metric spaces which are not so similar to that of classical metric spaces. In the regard, the fuzzy n-metric spaces can not be considered an extension of fuzzy metric spaces since a fuzzy metric is not a fuzzy 1-metric directly as desired. In 2006, Mustafa and Sims introduced the concept of G-metric spaces as another approach to extend metric spaces [28]. But topologically speaking, they have no big difference from metric spaces since a same topology will be induced by them. In 2013, Chaipunya and Kumam [4] introduced the concept of g-$3ps$ spaces to describe the distance between three points, which is no longer topologically equivalent to metric spaces in general. In 2019, George et al. [13] extended partial metric spaces to the fuzzy setting in the sense of Kramosil and Michalek, and showed that this space is a T_0 space.

We know that norms and metrics can be mutually induced by each other in a natural way. Inspired by the thought in the study of fuzzy metric spaces and probabilistic normed spaces, some notions of a fuzzy normed space were proposed in succession by Cheng and Mordeson [5], Bag and Samanta [3], Saadati and Vaezpour [30], Goleţ [9]. Lots of important results were obtained in this direction. For examples, Alegre and Romaguera [2] obtained characterizations of metrizable topological vector spaces in terms of fuzzy (quasi-)norms; Abrishami-Moghaddam and Sistani [1] provided some characterizations of approximation properties in fuzzy normed spaces; Xiao et al. [39] described different topological structures of fuzzy metric spaces and normed spaces with different t-norm by making use of the quasi-metric and quasi-norm families. Meenakshi and Cokilavany [22] introduced the concept of a fuzzy 2-norm space which can induce a fuzzy 2-metric space. While the 2-norm of two non-zero vectors may be zero, which is quite unexpect. For this reason, Khan [18] introduced the conept of G-normed spaces as a different generalization of normed spaces.

In a recent paper, Tian et al. [37] generalized G-metric spaces to the so-called triple fuzzy metric spaces, and pointed out that it is a new generalization of fuzzy metric spaces. While the relationship between the triple fuzzy metric topology and the fuzzy metric topology is still not very clear. To fill this gap, we herein introduce a concept of triple fuzzy normed spaces which will induce triple fuzzy metric spaces, and show that every triple fuzzy normed space is normable and vice versa. Therefore, in the topological sense, triple fuzzy normed spaces cannot be treated as a generalization of fuzzy normed spaces. Recently, Yan equipped Morsi's fuzzy pseudo-norm with many-valued topological structures [40]. We will develop a concept of generalized fuzzy 2-normed spaces as a new way to generalize fuzzy normed spaces.

The rest of this paper is divided into four sections. Section 2 consists of some definitions and results which are important and related to this work. In Section 3, we introduce a notion of generalized fuzzy 2-metric spaces, study their topological properties. In Section 4, we first introduce the concept of fuzzy g-$3ps$ spaces and study its topological properties. The topology of fuzzy g-$3ps$ spaces is T_1 but is not necessarily Hausdorff. Then we introduce and study a new kind of spaces called generalized fuzzy 2-normed spaces, which induces a Hausdorff topology. A brief summary is given in Section 5.

2. PRELIMINARIES

In this section, we recall some basic notions and results which will be used throughout this paper. The letters $\mathbb{R}$ and $\mathbb{N}$ always denotes the set of real numbers and the set of positive integer numbers, respectively.

Definition 2.1. (Klement et al. [19]) A t-norm $*$ on $[0,1]$ is a binary operation on $[0,1]$ which is commutative (i. e., $a*b=b*a$ whenever $a,b\in[0,1]$), associative (i. e., $a*(b*c)=(a*b)*c$ whenever $a,b,c\in[0,1]$), monotone (i. e., $a*c\le b*d$ whenever $a\le b$ and $c\le d$ for $a,b,c,d\in[0,1]$) and has the top element 1 as the unit (i. e., $b*1=b$ whenever $b\in[0,1]$). A t-norm is said to be continuous if $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is a continuous function.

Example 2.2. (Klement et al. [19]) Three basic continuous t-norms:

(1) the minimum t-norm $*_m : a*_m b = a\wedge b$;

(2) the product t-norm $*_p : a*_p b = ab$;

(3) the Łukasiewicz t-norm $*_L : a*_L b = 0\vee(a+b-1)$.

Definition 2.3. Let $*$ be a t-norm. If $a*b>0$ whenever $a,b\in(0,1]$, then $*$ is called positive.

Clearly, the t-norms $*_m$ and $*_p$ are positive.

Lemma 2.4. (George and Veeramani [7], Klement et al. [19]) Let $*$ be a continuous t-norm.

(1) For any $r_1,r_2\in(0,1]$, if $r_1>r_2$, then there exists $r_3\in(0,1)$ such that $r_1*r_3\ge r_2$.

(2) If $r\in[0,1)$, then there exists $s\in(r,1)$ such that $s*s\ge r$.

Definition 2.5. (Fuzzy metric space) (George and Veeramani [7]) A triplet $(X,M,*)$ is called a fuzzy metric space in sense of George and Veeramani if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,+\infty)$ such that the following conditions are valid:

(FM1) $M(x, y, t) > 0$ for all $x, y \in X$;

(FM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

$M(x, z, r + s) \geq M(x, y, r) * M(y, z, s)$ for all $x, y, z \in X$ and $r, s > 0$;

(FM4) $M(x, y, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous.

Definition 2.6. (Tripled fuzzy metric space) (Tian et al. [37]) Let X is a nonempty set and $*$ is a continuous t-norm. A triplet $(X, F, *)$ is called a tripled fuzzy metric space if F is a fuzzy set on $X \times X \times X \times (0, +\infty)$ such that the following conditions are satisfied:

(TFM1) $F(x, y, z, t) > 0$ for all $x, y, z \in X$ and $t > 0$;

(TFM2) $F(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$;

(TFM3) $F(x, x, y, t) \geq F(x, y, z, t)$ for $y \neq z$ and $t > 0$;

(TFM4) $F(x, y, z, t)$ is invariant under all permutations of $(x, y, z) \in X^3$, for all $t > 0$;

(TFM5) $F(x, y, z, r + s) \geq F(x, u, u, r) * F(u, y, z, s)$ for all $u, x, y, z \in X$ and $r, s > 0$;

(TFM5) $F(x, y, z, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

Definition 2.7. (Fuzzy D^*-metric space) (Sedghi et al. [34]) Let X be a nonempty set and $*$ be a continuous t-norm. A triplet $(X, D, *)$ is called a fuzzy D^*-metric space in sense of George and Veeramani if D^* is a fuzzy set on $X \times X \times X \times (0, +\infty)$ such that the following conditions are satisfied:

(FDM1) $D^*(x, y, z, t) > 0$ for all $x \in X$;

(FDM2) $D^*(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$;

(FDM3) $D^*(x, y, z, t)$ is invariant under all permutations of $(x, y, z) \in X^3$, for all $t > 0$;

(FDM4) $D^*(x, y, u, r) * D^*(u, z, z, s) \leq D^*(x, y, z, r+s)$ for all $x, y, z, u \in X$ and $r, s > 0$;

(FDM5) $D^*(x, y, z, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous.

Remark 2.8. Every triple fuzzy metric space is a fuzzy D^*-metric space. Indeed, it suffices to verify that the condition (FDM4) holds: Let $x, y, z, u \in X$ and $r, s > 0$,

$$\begin{aligned} D^*(x, y, z, r + s) &= D^*(z, y, x, r + s) && \text{(TFM4)} \\ &\geq D^*(z, u, u, s) * D^*(u, y, x, r) && \text{(TFM5)} \\ &= D^*(x, y, u, r) * D^*(u, u, z, s) && \text{(TFM4)} \\ &\geq D^*(x, y, u, r) * D^*(u, z, z, s). && \text{(TFM3)} \end{aligned}$$

But in general the converse does not hold. The related example follows from the classical setting [33, Example 1.4].

3. A FUNCTIONAL GENERALIZATION OF FUZZY METRIC SPACES

3.1. On basic properties

In this subsection, we shall introduce a notion of generalized fuzzy 2-metric spaces, which is a fuzzy version of the ordinary S-metric spaces [33].

Definition 3.1. (Generalized fuzzy 2-metric space) Let X be a nonempty set and let $*$ be a continuous t-norm. We say that a triplet $(X, S, *)$ is a generalized fuzzy 2-metric space if S is a fuzzy set on $X \times X \times X \times (0, +\infty)$ such that the following conditions are satisfied:

(FS1) $S(x, y, z, t) > 0$ for all $x, y, z \in X$ and $t > 0$;

(FS2) $S(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$;

(FS3) $S(x, x, u, r) * S(y, y, u, s) * S(z, z, u, t) \leq S(x, y, z, r + s + t)$ for all $u, x, y, z \in X$ and $r, s, t > 0$;

(FS4) $S(x, y, z, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous.

Remark 3.2. Every fuzzy D^*-metric is a generalized fuzzy 2-metric . Indeed, it suffices to verify that the condition (FS3) holds: Let $x, y, z, u \in X$ and $r, s, t > 0$,

$$\begin{aligned} S(x, y, z, r + s + t) &\geq S(x, y, u, r + s) * S(u, z, z, t) && \text{(FDM4)} \\ &= S(u, x, y, r + s) * S(z, z, u, t) && \text{(FDM3)} \\ &\geq S(u, x, x, r) * S(u, y, y, s) * S(z, z, u, t) && \text{(FDM4)} \\ &= S(x, x, u, r) * S(y, y, u, s) * S(z, z, u, t). && \text{(FDM3)} \end{aligned}$$

But in general the converse does not hold. The related example can be obtained by the classical example [33, Example 2.3] (More details see Example 3.3).

Based on Remarks 2.8 and 3.2, from now on, we investigate properties, examples and fixed point results of generalized triple fuzzy metric spaces, which are inspired by some well-known conclusions in [7, 8, 33].

Example 3.3. Let $X = \mathbb{R}$. Define $S : X \times X \times X \times (0, +\infty) \longrightarrow (0, 1]$ by

$$S(x, y, z, t) = \left[\exp\left(\frac{|y + z - 2x| + |y - z|}{t} \right) \right]^{-1}$$

for all $x, y, z \in X$ and all $t > 0$. For $*_p$, the triplet $(X, S, *_p)$ is a generalized fuzzy 2-metric space, but not a fuzzy D^*-metric space. Hence, it is not a triple fuzzy metric space.

P r o o f. It is not difficult to verify that S satisfies (FS1), (FS2) and (FS4). Now we prove that for all $u, x, y, z \in X$ and $r, s, t > 0$,

$$S(x, y, z, r + s + t) \geq S(x, x, u, r) * S(y, y, u, s) * S(z, z, u, t).$$

Indeed, since

$$\begin{aligned}\frac{|y+z-2x|+|y-z|}{r+s+t} &\leq \frac{|y-x|+|z-x|+|y-z|}{r+s+t}\\ &\leq \frac{|y-u|+|u-x|+|z-u|+|u-x|+|y-u|+|u-z|}{r+s+t}\\ &= \frac{|u-x|+|x-u|}{r+s+t}+\frac{|u-y|+|y-u|}{r+s+t}+\frac{|u-z|+|z-u|}{r+s+t}\\ &\leq \frac{|u-x|+|x-u|}{r}+\frac{|u-y|+|y-u|}{s}+\frac{|u-z|+|z-u|}{t},\end{aligned}$$

we have

$$\begin{aligned}&S(x,y,z,r+s+t)\\ &= \left[\exp\left(\frac{|y+z-2x|+|y-z|}{t}\right)\right]^{-1}\\ &\geq \left[\exp\left(\frac{|u-x|+|x-u|}{r}+\frac{|u-y|+|y-u|}{s}+\frac{|u-z|+|z-u|}{t}\right)\right]^{-1}\\ &= \left[\exp\left(\frac{|x+u-2x|+|x-u|}{r}+\frac{|y+u-2y|+|y-u|}{s}+\frac{|z+u-2z|+|z-u|}{t}\right)\right]^{-1}\\ &= S(x,x,u)*_p S(y,y,u,s)*_p S(z,z,u,t).\end{aligned}$$

Hence $(X,S,*_p)$ is a generalized fuzzy 2-metric space. Clearly, S is not a fuzzy D^*-metric since it does not satisfies (FDM3). □

Example 3.4. Let $(X,M,*_m)$ be a fuzzy metric space. Define a map $S: X\times X\times X\times (0,+\infty)\longrightarrow (0,1]$ by

$$S(x,y,z,t)=M\left(x,y,\frac{t}{3}\right)\wedge M\left(x,z,\frac{t}{3}\right)\wedge M\left(y,z,\frac{t}{3}\right)$$

for all $x,y,z\in X$ and all $t>0$. Then the triplet $(X,S,*_m)$ is a generalized fuzzy 2-metric space.

P r o o f. It is not difficult to obtain that S satisfies (FS1), (FS2) and (FS4). It remains to verify that S satisfies (FS3). Let $x,y,z\in X$ and $r,s,t>0$. Then

$$\begin{aligned}&S(x,y,z,r+s+t)\\ &= M\left(x,y,\frac{r+s+t}{3}\right)\wedge M\left(x,z,\frac{r+s+t}{3}\right)\wedge M\left(y,z,\frac{r+s+t}{3}\right)\\ &\geq M\left(x,u,\frac{r}{3}\right)\wedge M\left(u,y,\frac{s+t}{3}\right)\wedge M\left(x,u,\frac{r}{3}\right)\wedge M\left(u,z,\frac{s+t}{3}\right)\wedge\\ &\quad\wedge M\left(y,u,\frac{r}{3}\right)\wedge M\left(u,z,\frac{s+t}{3}\right)\end{aligned}$$

$$\geq M\left(x,x,\frac{r}{3}\right) \wedge M\left(x,u,\frac{r}{3}\right) \wedge M\left(x,u,\frac{r}{3}\right) \wedge M\left(y,y,\frac{s}{3}\right) \wedge M\left(y,u,\frac{s}{3}\right) \wedge M\left(y,u,\frac{s}{3}\right) \wedge$$
$$\wedge M\left(z,z,\frac{t}{3}\right) \wedge M\left(z,u,\frac{t}{3}\right) \wedge M\left(z,u,\frac{t}{3}\right)$$
$$= S(x,x,u,r) \wedge S(y,y,u,s) \wedge S(z,z,u,t).$$

Hence, S satisfies (FS3) and so $(X, S, *_m)$ is a generalized fuzzy 2-metric space. □

Lemma 3.5. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. Then $S(x,x,y,t) = S(y,y,x,t)$ for all $x, y \in X$ and $t > 0$.

Proof. Let $x, y \in X$ and $t > 0$. By (FS3), we get

$$\begin{aligned} S(x,x,y,t) &= \bigvee_{t_1+t_2+t_3=t} S(x,x,y,t_1+t_2+t_3) \\ &\geq \bigvee_{t_1+t_2+t_3=t} S(x,x,x,t_1) * S(x,x,x,t_2) * S(y,y,x,t_3) \\ &= \bigvee_{t_1+t_2+t_3=t} 1 * 1 * S(y,y,x,t_3) \\ &= S(y,y,x,t). \end{aligned}$$

Similarly,

$$S(y,y,x,t) \geq \bigvee_{t_1+t_2+t_3=t} S(y,y,y,t_1) * S(y,y,y,t_2) * S(x,x,y,t_3) = S(x,x,y,t).$$

Hence, we obtain $S(x,x,y,t) = S(y,y,x,t)$. □

Proposition 3.6. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. Then for all $x, y \in X$, the map $S(x,x,y,\cdot)$ is nondecreasing.

Proof. For some $t > s > 0$, assume $S(x,x,y,s) > S(x,x,y,t)$. Then

$$\begin{aligned} S(x,x,y,s) > S(x,x,y,t) &= S(x,x,y,t-s+s) \\ &\geq S(x,x,x,t-s) * \left(\bigvee_{s_1+s_2=s} S(x,x,x,s_1) * S(y,y,x,s_2)\right) \\ &= 1 * \left(\bigvee_{s_1+s_2=s} 1 * S(x,x,y,s_2)\right) \qquad \text{(Lemma 3.5)} \\ &= S(x,x,y,s), \end{aligned}$$

a contradiction. □

3.2. On topological properties

Definition 3.7. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. For $x \in X$, $\lambda \in (0,1)$ and $t > 0$, we define the open ball $B_S(x, \lambda, t)$ and closed ball $B_S[x, \lambda, t]$ with a center x and a radius λ with respect to t as follows:

$$B_S(x, \lambda, t) = \{y \in X : S(y, y, x, t) > 1 - \lambda\}, \ B_S[x, \lambda, t] = \{y \in X : S(y, y, x, t) \geq 1 - \lambda\}.$$

Lemma 3.8. Let $(X, S, *)$ be a generalized triple fuzzy metric space. Then the following statements hold:

(1) For each $\lambda \in (0,1)$, if $0 < r \leq s$, then $B_S(x, \lambda, r) \subseteq B_S(x, \lambda, s)$;

(2) For each $t > 0$, if $0 < \lambda \leq \mu < 1$, then $B_S(x, \lambda, t) \subseteq B_S(x, \mu, t)$.

Proof. (1) Let $\lambda \in (0,1)$ and $0 < r \leq s$. Then

$$\begin{aligned} z \in B_S(x, \lambda, r) &\text{ implies } S(z, z, x, r) > 1 - \lambda \\ &\text{ implies } S(z, z, x, s) \geq G(z, z, x, r) > 1 - \lambda \qquad \text{(Proposition 3.6)} \\ &\text{ implies } z \in B_S(x, \lambda, s). \end{aligned}$$

Hence, $B_S(x, y, \lambda, r) \subseteq B_S(x, y, \lambda, s)$.

(2) If $0 < \lambda \leq \mu < 1$ and $z \in B_S(x, \lambda, t)$, then $S(z, z, x, t) > 1 - \lambda \geq 1 - \mu$, i. e., $z \in B_S(x, \mu, t)$. □

Remark 3.9. For any $x \in X$, $\lambda \in (0,1)$ and $r > 0$, since $S(x, x, x, r) = 1 > 1 - \lambda$, we have $x \in B_S(x, \lambda, r)$, which implies that $B_S(x, \lambda, r)$ is a nonempty set. Let $(X, S, *)$ be a generalized fuzzy 2-metric space and $A \subseteq X$. If for every $x \in A$, there exist $\lambda \in (0,1)$ and $t > 0$ such that $B_S(x, \lambda, t) \subseteq A$, then the subset A is called an open subset of X.

Theorem 3.10. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. Every open ball $B_S(x, \lambda, t)$ is an open subset of X.

Proof. Let $z \in B_S(x, \lambda, r)$. Then $S(z, z, x, r) > 1 - \lambda$. Since $S(z, z, x, \cdot)$ is continuous at r, there exists $r_0 \in (0, r)$ such that $S(z, z, x, r_0) > 1 - \lambda$. Let $\lambda_0 = S(z, z, x, r_0) > 1 - \lambda$. Since $\lambda_0 > 1 - \lambda$, there exists $\mu_0 \in (0,1)$ such that $\lambda_0 > 1 - \mu_0 > 1 - \lambda$. Now for a given μ_0 with $\lambda_0 > 1 - \mu_0$ there exists $\lambda_1 \in (0,1)$ such that $\lambda_0 * \lambda_1 * \lambda_1 \geq 1 - \mu_0$. For $B_S\left(z, 1 - \lambda_1, \frac{r - r_0}{2}\right)$, we claim that

$$B_S\left(z, 1 - \lambda_1, \frac{r - r_0}{2}\right) \subseteq B_S(x, \lambda, r).$$

For any $w \in B_S\left(z, 1 - \lambda_1, \frac{r - r_0}{2}\right)$, we get $S\left(w, w, z, \frac{r - r_0}{2}\right) > \lambda_1$. Therefore,

$$\begin{aligned} S(w, w, x, r) &\geq S\left(w, w, z, \frac{r - r_0}{2}\right) * S\left(w, w, z, \frac{r - r_0}{2}\right) * S(x, x, z, r_0) \\ &\geq \lambda_1 * \lambda_1 * \lambda_0 \end{aligned}$$

$$\geq 1 - \mu_0$$
$$> 1 - \lambda.$$

Thus, $w \in B_S(x, \lambda, r)$. Hence, $B_S\left(z, 1-\lambda_1, \frac{r-r_0}{2}\right) \subseteq B_S(x, \lambda, r)$. Put $\mu = 1 - \lambda_1$ and $s = \frac{r-r_0}{2}$. Then $\mu \in (0,1)$, $s > 0$, and $B_S(z, \mu, s) \subseteq B_S(x, \lambda, r)$. This shows that $B_S(x, \lambda, r)$ is open. □

Remark 3.11. Let $(X, S, *)$ be a generalized fuzzy 2-metric space and define

$$\tau_S = \{A \subseteq X : \text{for all } x \in A, \text{ there exist } \lambda \in (0,1) \text{ and } r > 0 \text{ such that } B_S(x, \lambda, r) \subseteq A\}.$$

Then τ_S is a topology of on X. Moreover, since $\left\{B_S\left(x, \frac{1}{n}, \frac{1}{n}\right)\right\}$ is a local base at x, the topology τ_S is first countable.

Theorem 3.12. The topology τ_S of a generalized fuzzy 2-metric space $(X, S, *)$ is Hausdorff.

Proof. Let x and y two distinct elements in X. Then $S(x, x, y, r) = \lambda \in (0,1)$. For each $\lambda_0 \in (\lambda, 1)$, there exists $\lambda_1 \in (0,1)$ such that $\lambda_1 * \lambda_1 * \lambda_1 \geq \lambda_0$. Considering the open balls $B_S\left(x, 1-\lambda_1, \frac{r}{3}\right)$ and $B_S\left(y, 1-\lambda_1, \frac{r}{3}\right)$, we claim

$$B_S\left(x, 1-\lambda_1, \frac{r}{3}\right) \cap B_S\left(y, 1-\lambda_1, \frac{r}{3}\right) = \emptyset.$$

If not, there exists $z \in X$ such that

$$z \in B_S\left(x, 1-\lambda_1, \frac{r}{3}\right) \cap B_S\left(y, 1-\lambda_1, \frac{r}{3}\right).$$

Then

$$\begin{aligned}
\lambda = S(x, x, y, r) &\geq S\left(x, x, z, \frac{r}{3}\right) * S\left(x, x, z, \frac{r}{3}\right) * S\left(y, y, z, \frac{r}{3}\right) \\
&= S\left(z, z, x, \frac{r}{3}\right) * S\left(z, z, x, \frac{r}{3}\right) * S\left(z, z, y, \frac{r}{3}\right) \qquad \text{(Lemma 3.5)} \\
&\geq \lambda_1 * \lambda_1 * \lambda_1 \geq \lambda_0 > \lambda,
\end{aligned}$$

a contradiction. Therefore, $(X, S, *)$ is Hausdorff. □

Definition 3.13. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. We say that a subset A of X is S-bounded if there exist $\lambda \in (0,1)$ and $r > 0$ such that $S(x, x, y, r) > 1 - \lambda$ for all $x, y \in A$.

Theorem 3.14. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. Then for every compact subset A of X is S-bounded.

Proof. Let A be a compact subset of X. Fixing $\lambda_1 \in (0,1)$ and $r_1 \in (0,1)$ and considering an open cover $\{B_S(x, \lambda_1, r_1) : x \in A\}$. By the compactness of A, there exist $x_1, x_2, \cdots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B_S(x_i, \lambda_1, r_1)$. If $x, y \in A$, then for some

i, j, $x \in B_S(x_i, \lambda_1, r_1)$ and $y \in B_S(x_j, \lambda_1, r_1)$. Hence $S(x, x, x_i, r_1) > 1 - \lambda_1$ and $S(y, y, x_j, r_1) > 1 - \lambda_1$. Let

$$\mu_1 = \min\{S(y, y, x_i, r_1) : 1 \le i \le n\}$$

and

$$\mu_2 = \min\{S(x, x, x_j, r_1) : 1 \le j \le n\}$$

Then $\mu_1, \mu_2 > 0$. On one hand, we have

$$\begin{aligned} S(x, x, y, 3r_1) &\ge S(x, x, x_i, r_1) * S(x, x, x_i, r_1) * S(y, y, x_i, r_1) \\ &\ge (1 - \lambda_1) * (1 - \lambda_1) * \mu_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} S(x, x, y, 3r_1) &\ge S(x, x, x_j, r_1) * S(x, x, x_j, r_1) * S(y, y, x_j, r_1) \\ &\ge \mu_2 * \mu_2 * (1 - \lambda_1). \end{aligned}$$

Putting $r = 3r_1$ and $\lambda_0 = \max\{(1 - \lambda_1) * (1 - \lambda_1) * \mu_1, (1 - \lambda_1) * \mu_2 * \mu_2\} > 1 - \lambda$, we have that for all $x, y \in A$, $S(x, x, y, r) > 1 - \lambda$. Therefore, A is S-bounded. □

In what follows, we study convergence of sequences, Cauchy sequences and completeness in generalized fuzzy 2-metric space.

Definition 3.15. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. We say that

(1) a sequence $\{x_n\}$ in X converges to $x \in X$ (write $x_n \to x$) if for all $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x, t) > 1 - \lambda$ for all $n \ge n_0$.

(2) a sequence $\{x_n\}$ in X is an S-Cauchy sequence (simply Cauchy sequence) if for all $\lambda \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m, t) > 1 - \lambda$ for all $n, m \ge n_0$.

(3) the space $(X, S, *)$ is complete if every Cauchy sequence is convergent.

Remark 3.16. Let $(X, S, *)$ be a generalized fuzzy 2-metric space and let $\{x_n\}$ in X be a sequence. Then the sequence $\{x_n\}$ is convergent to $x \in X$ iff $\lim_{n\to+\infty} S(x_n, x_n, x, t) = 1$ for all $t > 0$.

Proposition 3.17. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. If a sequence $\{x_n\}$ in X converges $x \in X$, then x is unique.

P r o o f. Suppose that there exists $y \in X$ with $y \ne x$ such that $\{x_n\}$ in X converges y. Then for all $\lambda \in (0, 1)$ and $t > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \ge n_1 \text{ implies } S\left(x_n, x_n, x, \frac{t}{3}\right) > 1 - \lambda_1$$

and

$$n \geq n_2 \text{ implies } S\left(x_n, x_n, y, \frac{t}{3}\right) > 1 - \lambda_1,$$

where for each $1 - \lambda_0 \in (1 - \lambda, 1)$, there exists $\lambda_1 \in (0, 1)$ such that $(1 - \lambda_1) * (1 - \lambda_1) * (1 - \lambda_1) \geq 1 - \lambda_0 > 1 - \lambda$. Put $n_0 = \max\{n_1, n_2\}$. Then for all $n \geq n_0$, we get

$$\begin{aligned} S(x, x, y, t) &\geq S\left(x, x, x_n, \frac{t}{3}\right) * S\left(x, x, x_n, \frac{t}{3}\right) * S\left(y, y, x_n, \frac{t}{3}\right) \\ &= S\left(x_n, x_n, x, \frac{t}{3}\right) * S\left(x_n, x_n, x, \frac{t}{3}\right) * S\left(x_n, x_n, y, \frac{t}{3}\right) \quad \text{(Lemma 3.5)} \\ &\geq (1 - \lambda_1) * (1 - \lambda_1) * (1 - \lambda_1) \\ &\geq 1 - \lambda_0 > 1 - \lambda. \end{aligned}$$

Hence $S(x, x, y, t) = 1$ and so $x = y$. □

Proposition 3.18. Let $(X, S, *)$ be a generalized fuzzy 2-metric space. If a sequence $\{x_n\}$ in X converges $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

Proof. Since $\{x_n\}$ is convergent to x, we have that for all $\lambda \in (0, 1)$ and $t > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \text{ implies } S\left(x_n, x_n, x, \frac{t}{3}\right) > 1 - \lambda_1$$

and

$$m \geq n_2 \text{ implies } S\left(x_m, x_m, x, \frac{t}{3}\right) > 1 - \lambda_1,$$

where for each $1 - \lambda_0 \in (1 - \lambda, 1)$, there exist $\lambda_1, n_2 \in (0, 1)$ such that $(1 - \lambda_1) * (1 - \lambda_1) * (1 - \lambda_1) \geq 1 - \lambda_0 > 1 - \lambda$. Put $n_0 = \max\{n_1, n_2\}$. Then for all $n, m \geq n_0$, we get

$$\begin{aligned} S(x_n, x_n, x_m, t) &\geq S\left(x_n, x_n, x, \frac{t}{3}\right) * S\left(x_n, x_n, x, \frac{t}{3}\right) * S\left(x_m, x_m, x, \frac{t}{3}\right) \\ &\geq (1 - \lambda_1) * (1 - \lambda_1) * (1 - \lambda_1) \\ &\geq 1 - \lambda_0 > 1 - \lambda. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. □

Proposition 3.19. Let $(X, S, *)$ be a generalized fuzzy 2-metric space and $x, y \in X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. If $x_n \to x$ and $y_n \to y$ as $n \to +\infty$, then for all $t > 0$, $S(x_n, x_n, y_n, t) \to S(x, x, y, t)$ as $n \to +\infty$.

P r o o f. For all $t > 0$, there exists $r > 0$ such that $t > 2r$. Then

$$\begin{aligned} & S(x_n, x_n, y_n, t) \\ &= S(x_n, x_n, y_n, r + t - r) \\ &\geq S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(y_n, y_n, x, t - r\right) \\ &= S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(y_n, y_n, x, r + t - 2r\right) \\ &\geq S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(x_n, x_n, x, \frac{r}{2}\right) * S\left(y_n, y_n, y, \frac{r}{2}\right) * S\left(y_n, y_n, y, \frac{r}{2}\right) * S\left(x, x, y, t - 2r\right) \end{aligned} \tag{1}$$

and

$$\begin{aligned} & S(x, x, y, t) \\ &= S(x, x, y, r + t - r) \\ &\geq S\left(x, x, x_n, \frac{r}{2}\right) * S\left(x, x, x_n, \frac{r}{2}\right) * S\left(y, y, x_n, t - r\right) \\ &= S\left(x, x, x_n, \frac{r}{2}\right) * S\left(x, x, x_n, \frac{r}{2}\right) * S\left(y, y, x_n, r + t - 2r\right) \\ &\geq S\left(x, x, x_n, \frac{r}{2}\right) * S\left(x, x, x_n, \frac{r}{2}\right) * S\left(y, y, y_n, \frac{r}{2}\right) * S\left(y, y, y_n, \frac{r}{2}\right) * \\ & S\left(x_n, x_n, y_n, t - 2r\right). \end{aligned} \tag{2}$$

Letting $n \to +\infty$ in the inequalities (1) and (2), it follows from the continuity of $*$ that

$$\lim_{n \to +\infty} S(x_n, x_n, y_n, t) \geq S(x, x, y, t - 2r) \tag{3}$$

and

$$S(x, x, y, t) \geq \lim_{n \to +\infty} S(x_n, x_n, y_n, t - 2r). \tag{4}$$

Taking the limit as $r \to 0$ in (3) and (4), and by the continuity of S, we have

$$\lim_{n \to +\infty} S(x_n, x_n, y_n, t) = S(x, x, y, t).$$

The proof is completed. □

4. A TOPOLOGICAL GENERALIZATION OF FUZZY METRIC SPACES

From Remarks 2.8 and 3.2, it is easy to see that generalized fuzzy 2-metric generalizes both triple fuzzy metrics and fuzzy D^*-metrics only in the case when they are symmetric. In this section, we further introduce a notion of fuzzy g-3ps spaces, which covers a generalized fuzzy 2-metric space and also a tripled fuzzy metric space in which the symmetric is absent. But it will be shown that the topology of fuzzy g-3ps spaces just is T_1 in general. Furthermore, the equivalence has been established between the ordinary tripled normed spaces and the oridinary normed spaces by using some topological approaches [21].

Based on the above works, we shall further introduce a notion of generalized fuzzy 2-normed space, which can induce a Hausdorff topology. Moreover, the ordirnary generalized 2-normed spaces are not topologically equivalent to the ordinary normed spaces [21].

Definition 4.1. (Fuzzy g-3ps spaces) Let X be a nonempty set and let $*$ be a continuous t-norm. We say that a triplet $(X, g, *)$ is a fuzzy g-3ps space if g is a fuzzy set on $X \times X \times X \times (0, +\infty)$ such that the following conditions are satisfied:

(Fg-1) $g(x, y, z, t) > 0$ for all $x, y, z \in X$ and $t > 0$;

(Fg-2) $g(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$;

(Fg-3) there are some constants $t, t_1, t_2, t_3 > 0$ such that

$$g(x, x, u, t_1) * g(x, x, v, t_2) * g(x, x, w, t_3) \leq g(u, v, w, t)$$

for all $x, u, v, w \in X$.

(Fg-4) $g(x, y, z, \cdot) : (0, +\infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, g, *)$ be a fuzzy g-3ps space. For all $x \in X$, $\lambda \in (0, 1)$ and $t > 0$, define an open ball in the usual sense:

$$B_g(x, \lambda, t) = \{y \in X : g(x, x, y, t) > 1 - \lambda\}.$$

We call a subset $A \subseteq X$ fuzzy bounded if

$$\beta(A) = \sup_{t>0} \inf_{x,y,z \in A} g(x, y, z, t) = 1.$$

Lemma 4.2. (A characterization of (Fg-3)) Let X be a nonempty set and let $g : X \times X \times X \times (0, +\infty) \longrightarrow (0, 1]$ be a map satisfying (Fg-1). Then the following statements are equivalent:

(1) g satisfies (Fg-3).

(2) There exist $r_0 > 0$ such that for each $x \in X$ and each $\lambda \in (0, 1)$, the set $B_g(x, \lambda, r_0) = \{y \in X : g(x, x, y, r_0) > 1 - \lambda\}$ is fuzzy bounded, i. e.,

$$\beta(B_g(x, \lambda, r_0)) = \sup_{t>0} \inf_{a,b,c \in B_g(x,\lambda,r_0)} g(a, b, c, t) = 1.$$

Proof.

(1) implies (2). Assume that (i) holds. Let $x \in X$, $\lambda \in (0, 1)$ and $t > 0$ such that $u, v, w \in B_g(x, \lambda, t/3)$. Then

$$g(x, x, u, t/3) * g(x, x, v, t/3) * g(x, x, w, t/3) \geq 1 - \lambda,$$

From (i) we obtain that $g(u, v, w, r) > 1 - \lambda$ for some $r > 0$. Thus,

$$\sup_{t>0} \inf_{a,b,c \in B_g(x,\lambda,t/3)} g(a, b, c, t) \geq 1 - \lambda.$$

Since $\lambda \in (0,1)$ are arbitrary, the ball $B_g(x,\lambda,t/3)$ is fuzzy bounded.

(2) implies (1). Assume that (ii) holds. Let $\lambda \in (0,1)$, $t_1,t_2,t_3 > 0$ and $x,u,v,w \in X$ such that

$$1-\lambda < g(x,x,u,t_1) * g(x,x,v,t_2) * g(x,x,w,t_3).$$

Put $r_0 = \max\{t_1,t_2,t_3\}$. Then $u,v,w \in B_g(x,\lambda,r_0)$. Thus, we have

$$\sup_{s>0} g(u,v,w,s) \geq \sup_{s>0} \inf_{a,b,c\in B_g(x,\lambda,r_0)} g(a,b,c,s) = 1 > 1-\lambda.$$

Hence, there exists $t>0$ such that $g(u,v,w,t) > 1-\lambda$. □

Example 4.3. Every triple fuzzy metric space $(X,F,*)$ is a fuzzy g-3*ps* space.

P r o o f. In fact, let $x,u,v,w \in X$. Then for any $\delta_1,\delta_2 > 0$ with $\delta_1+\delta_2=\delta$, we have

$$\begin{aligned} & F(u,v,w,\delta) \\ \geq\ & F(u,x,x,\delta_1) * F(x,v,w,\delta_2) \\ =\ & F(x,x,u,\delta_1) * F(v,x,w,\delta_2) \\ \geq\ & F(x,x,u,k_1\delta) * F(v,x,x,k_2\delta) * F(x,x,w,k_3\delta) \quad (\text{put } k_1\delta=\delta_1,\ (k_2+k_3)\delta=\delta_2) \\ =\ & F(x,x,u,k_1\delta) * F(x,x,v,k_2\delta) * F(x,x,w,k_3\delta). \end{aligned}$$

Thus, we can choose any $\delta > 0$ and let $\eta = \delta$ to fulfill the assumption in Lemma 4.2. Hence, every triple fuzzy metric is a fuzzy g-3*ps* space. □

Denote $\mathcal{B}_g$ the set of all open balls of $(X,g,*)$, i. e.,

$$\mathcal{B}_g = \{B_g(x,\lambda,r) : x\in X, \lambda\in(0,1), r>0\}.$$

Definition 4.4. [Topology of fuzzy g-3*ps* space] Let $(X,g,*)$ be a fuzzy g-3*ps* space. The topology of $(X,g,*)$, denoted by τ_g, has $\mathcal{B}_g$ as a subbase.

Theorem 4.5. (T_1 property) The topology τ_g for a fuzzy g-3*ps* space $(X,g,*)$ is T_1.

P r o o f. Consider any two elements x and y in X so that $x\neq y$. Then there are $r>0$ and $\lambda',\lambda'' \in (0,1)$ satisfying $g(x,x,y,r)=\lambda'$ and $g(y,y,x,r)=\lambda''$. For each $\lambda_0\in(\lambda,1)$, where $\lambda=\max\{\lambda',\lambda''\}$, we can find $\lambda_1\in(0,1)$ such that $\lambda_1*\lambda_1\geq\lambda_0$, Observe that $B_g(x,1-\lambda_1,r/2)$ and $B_g(y,1-\lambda_1,r/2)$. Clearly,

$$y\notin B_g(x,1-\lambda_1,r/2) \text{ and } x\notin B_g(y,1-\lambda_1,r/2).$$

If not, then

$$y\in B_g(x,1-\lambda_1,r/2) \text{ or } x\in B_g(y,1-\lambda_1,r/2).$$

When $y\in B_g(x,1-\lambda_1,r/2)$, we have

$$g(x,x,y,r/2) > \lambda_1 \geq \lambda_1*\lambda_1 \geq \lambda_0 > \lambda \geq \lambda',$$

which contradicts to $\lambda' \geq g(x,x,y,r/2)$. Similarly, we can prove that $x\in B_g(y,1-\lambda_1,r/2)$ is also invalid. This completes the proof. □

By the next example, we shall explicate a result that a fuzzy g-3*ps* space is not necessarily to be Hausdorff.

Example 4.6. Let $X = [0,1]$ and $g : X \times X \times X \times (0,+\infty) \longrightarrow (0,1]$ be a function given by

$$g(x,y,z,t) = \frac{t}{t+\rho(x,y,z)}$$

for all $x,y,z \in X$ and all $t > 0$, where

$$\rho(x,y,z) = \begin{cases} 0, & \text{if } x=y=z; \\ z, & \text{if } x=y\neq z, z\neq 0; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that every subset in this space is always fuzzy bounded. Hence, $(X,g,*)$ is a fuzzy g-3ps space. Observe that for $x \in X$, $\lambda \in (0,1)$ and $r > 0$, we have

$$B_g(x,\lambda,r) = \begin{cases} X \cap \left[\{x\} \cup \left(0, \frac{\lambda}{1-\lambda} r\right)\right], & \text{if } r \leq 1; \\ X, & \text{if } r > 1. \end{cases}$$

Therefore, any two balls intersect one another, which indicates that X is not Hausdorff.

In what follows, we shall introduce a new idea of generalized fuzzy normed space and study the topology of this space.

Definition 4.7. (Generalized fuzzy 2-normed spaces) Let X be a real vector space (with the null vector θ) and let $*$ be a continuous t-norm. We say that a triplet $(X,Q,*)$ is a generalized fuzzy 2-normed space (briefly, GF2NS) if Q is a fuzzy set on $X \times X \times (0,+\infty)$ such that the following conditions are satisfied:

(GFN-1) $Q(x,y,t) > 0$ for all $x,y \in X$ and $t > 0$;

(GFN-2) $Q(x,y,t) = 1$ for all $t > 0$ if and only if $x = y = \theta$;

(GFN-3) $Q(cx,cy,t) = Q(x,y,t/|c|)$ for all $x,y \in X$, $c \neq 0$ and $t > 0$;

(GFN-4) there are $r,s > 0$ such that $Q(x,x,r)*Q(y,y,r)*Q(z,z,r) \leq Q(x-z,y-z,s)$ for all $x,y,z \in X$;

(GFN-5) $Q(x,y,\cdot) : (0,+\infty) \longrightarrow [0,1]$ is continuous.

Example 4.8. Let $X = \mathbb{R}$ and define a map $Q : X \times X \times (0,+\infty) \longrightarrow [0,1]$ by

$$Q(x,y,t) = \frac{t}{t+\frac{|x|^2+|y|^2}{|x|+|y|}}$$

for all $x,y \in X$ and all $t > 0$. Then $(X,Q,*_m)$ is a generalized fuzzy 2-normed space.

Proof. It is not difficult to prove that Q satisfies (GFN-1)–(GFN-3) and (GFN-5). Now we prove that it satisfies (GFN-4). Let $x,y \in X$ with $(x,y) \neq (0,0)$, and let $r = \frac{1}{2}$, $s = 1$. Then

$$Q(x,x,r) *_m Q(y,y,r) *_m Q(z,z,r) = \frac{1}{1+2|x|} \wedge \frac{1}{1+2|y|} \wedge \frac{1}{1+2|z|}.$$

and

$$Q(x-z, y-z, s) = \frac{1}{1 + \frac{|x-z|^2+|y-z|^2}{|x-z|+|y-z|}}.$$

To show that $Q(x,x,r) *_m Q(y,y,r) *_m Q(z,z,r) \leq Q(x-z, y-z, 2)$, we take $\epsilon \in (0,1)$ (i. e., $1/2 < 1/(\epsilon+1) < 1$) such that

$$\frac{1}{\epsilon+1} \leq \frac{1}{1+2|x|} \wedge \frac{1}{1+2|y|} \wedge \frac{1}{1+2|z|},$$

which implies that

$$0 < |x| \leq \frac{\epsilon}{2}, 0 < |y| \leq \frac{\epsilon}{2} \text{ and } 0 < |z| \leq \frac{\epsilon}{2}.$$

Thus, we get

$$0 \leq |x-z| \leq \epsilon, 0 \leq |y-z| \leq \epsilon.$$

Further, we have $0 \leq |x-z|^2 + |y-z|^2 \leq |x-z| + |y-z|$, and so

$$0 < \frac{|x-z|^2+|y-z|^2}{|x-z|+|y-z|} \leq \epsilon < 1.$$

Therefore, $Q(x-z, y-z, s) > \frac{1}{\epsilon+1}$. By the arbitrariness of $\epsilon \in (0,1)$, we conclude that

$$Q(x,x,r) *_m Q(y,y,r) *_m Q(z,z,r) \leq Q(x-z, y-z, 2).$$

This completes the proof. □

Theorem 4.9. (Construction of fuzzy g-3ps from generalized fuzzy 2-norm) Let $(X, Q, *_m)$ be a GF2NS and define a function $g : X \times X \times X \times (0, +\infty) \longrightarrow [0,1]$ by

$$g(x,y,z,t) = Q(x-z, y-z, t)$$

for all $x, y, z \in X$ and all $t > 0$. Then $(X, g, *_m)$ is a fuzzy g-3ps space.

P r o o f. It is not difficult to obtain that g satisfies (Fg-1) and (Fg-4). Next, we verify that g satisfies (Fg-2) and (Fg-3).

(Fg-2) For all $x, y, z \in X$ and $t > 0$, it holds that

$$\begin{aligned} g(x,y,z,t) = 1 &\text{ if and only if } Q(x-z, y-z, t) = 1 \\ &\text{ if and only if } x-z = \theta = y-z \\ &\text{ if and only if } x = y = z. \end{aligned}$$

(Fg-3) Since $(X, Q, *_m)$ is a GF2NS, there are $r, s > 0$ such that $Q(x,x,r) \wedge Q(y,y,r) \wedge Q(z,z,r) \leq Q(x-z, y-z, s)$ for all $x, y, z \in X$. For any $x \in X$, $\lambda \in (0,1)$ and $r > 0$, we have

$$B_g(x, \lambda, r) = \{y \in X : g(x,x,y,r) > 1-\lambda\} = \{y \in X : Q(x-y, x-y, r) > 1-\lambda\}.$$

Taking any $a, b, c \in B_g(x, \lambda, r)$, we have

$$Q(x-a, x-a, r) \wedge Q(x-b, x-b, r) \wedge Q(x-c, x-c, r) > 1-\lambda.$$

Now, we have

$$\begin{aligned} g(a, b, c, s) &= Q(a-c, b-c, s) \\ &= Q(c-a, c-b, s) \\ &= Q((x-a)-(x-c), (x-b)-(x-c), s) \\ &\geq Q(x-a, x-a, r) \wedge Q(x-b, x-b, r) \wedge Q(x-c, x-c, r) \\ &> 1-\lambda. \end{aligned}$$

Hence, it holds that

$$\sup_{t>0} \inf_{a,b,c \in B_g(x,\lambda,r)} g(a, b, c, t) = 1.$$

This completes the proof. □

In what follows, we show that a fuzzy g-$3ps$ space induced by a generalized fuzzy 2-norm fulfills some interesting properties.

Proposition 4.10. (Construction of inducibility) Let X be a real vector space. A fuzzy g-$3ps$ g on X can be induced by a GF2NS under the minimum t-norm if and only if g satisfies the following equality: for all $x, y, z, a \in X$, all $t > 0$ and all $\alpha \neq 0$,

$$g(\alpha x + a, \alpha y + a, \alpha z + a, t) = g(x, y, z, t/|c|).$$

Proof. Let g be induced by a GF2NS $(X, Q, *)$. Then

$$\begin{aligned} &g(\alpha x + a, \alpha y + a, \alpha z + a, t) \\ &= Q(\alpha x + a - \alpha z - a, \alpha y + a - \alpha z - a, t) \\ &= Q(\alpha(x-z), \alpha(y-z), t) \\ &= Q(x-z, y-z, y/|\alpha|) \\ &= g(x, y, z, t/|\alpha|). \end{aligned}$$

Conversely, let g satisfy: for all $x, y, z, a \in X$, all $t > 0$ and all $\alpha \neq 0$,

$$g(\alpha x + a, \alpha y + a, \alpha z + a, t) = g(x, y, z, t/|c|).$$

Define $Q : X \times X \times (0, +\infty) \longrightarrow [0, 1]$ by

$$Q(x, y, t) = g(x, y, \theta, t)$$

for all $x, y \in X$ and all $t > 0$. Then, Q obviously fulfills (GFN-1) and (GFN-5). Next, we prove that Q satisfies (GFN-2)–(GFN-4).

(GFN-2) For each $x, y \in X$ and $t > 0$ we have

$$Q(x, y, t) = 1 \text{ if and only if } g(x, y, \theta, t) = 1 \text{ if and only if } x = y = \theta.$$

(GFN-3) For every $x, y, \in X$, $c \neq 0$ and $t > 0$,

$$\begin{aligned} Q(cx, cy, t) &= g(cx, cy, \theta, t) \\ &= g(x, y, \theta, t/|c|) \\ &= Q(x, y, \theta, t/|c|). \end{aligned}$$

(GFN-4) Since g is a fuzzy g-3ps, there exists r_0 such that $B_g(\theta, \lambda, r_0)$ is fuzzy bounded. Note that

$$\begin{aligned} B_g(\theta, \lambda, r_0) &= \{y \in X : g(\theta, \theta, x, r_0) > 1 - \lambda\} \\ &= \{y \in X : g(-x, -x, \theta, r_0) > 1 - \lambda\} \\ &= \{y \in X : g(x, x, \theta, r_0) > 1 - \lambda\} \\ &= \{y \in X : Q(x, x, r_0) > 1 - \lambda\}. \end{aligned}$$

Take any $x, y, z \in X$ satisfying

$$\begin{aligned} 1 - \lambda &< Q(x, x, r_0) * Q(y, y, r_0) * Q(z, z, r_0) \\ &\leq Q(x, x, r_0) \wedge Q(y, y, r_0) \wedge Q(z, z, r_0). \end{aligned}$$

Then $x, y, z \in B_g(\theta, \lambda, r_0)$. Since $B_g(\theta, \lambda, r_0)$ is fuzzy bounded, there exists $s > 0$ such that $g(x, y, z, s) > 1 - \lambda$, i. e., $Q(x - z, y - z, s) > 1 - \lambda$. Thus, $Q(x, x, r_0) * Q(y, y, r_0) * Q(z, z, r_0) \leq Q(x - z, y - z, s)$.

Therefore, $(X, Q, *)$ is a GF2NS. Moreover, the triplet $(X, Q, *_m)$ is a GF2NS under the minimum t-norm such that $g(x, y, z, t) = Q(x - z, y - z, t)$ and so g is induced by Q. □

Theorem 4.9 shows that every GF2NS can induce a fuzzy g-3ps under the minimum t-norm. The open ball $B_g(x, \lambda, r)$ has already been defined for fuzzy g-3ps spaces. So, we can define open balls in GF2NS using the idea of open balls in fuzzy g-3ps spaces as follows: for all $x \in X$, all $\lambda \in (0, 1)$ and all $r > 0$,

$$\begin{aligned} B_Q(x, \lambda, r) &= B_g(x, \lambda, r) \\ &= \{y \in X : g(x, x, y, r) > 1 - \lambda\} \\ &= \{y \in X : Q(x - y, x - y, r) > 1 - \lambda\}. \end{aligned}$$

This open ball will be used frequently in the rest of this paper.

Proposition 4.11. (Effect of translation and rescaling on open balls) If $(X, Q, *)$ is a GF2NS, then $B_Q(x, \lambda, r) = x + rB_Q(\theta, \lambda, 1)$ for each $x \in X$, $\lambda \in (0, 1)$ and $r > 0$.

Proof. For any $x \in X$, $\lambda \in (0, 1)$ and $r > 0$ we have

$$B_Q(x, \lambda, r)$$

$$
\begin{aligned}
&= \{y \in X : Q(x-y, x-y, r) > 1-\lambda\} \\
&= \{x+y' \in X : Q(x-x-y', x-x-y', r) > 1-\lambda\} \\
&= x + \{y' \in X : Q(-y', -y', r) > 1-\lambda\} \\
&= x + \{y' \in X : Q(y', y', r) > 1-\lambda\} \\
&= x + \{ry'' \in X : Q(ry'', ry'', r) > 1-\lambda\} \\
&= x + r\{y'' \in X : Q(y'', y'', 1) > 1-\lambda\} \\
&= x + rB_Q(\theta, \lambda, 1),
\end{aligned}
$$

as desired. □

Definition 4.12. (The defective perimeter of a set) Let $(X, Q, *)$ be a GF2NS. For any $A \subseteq X$, we say that the value of

$$\mathcal{M}(A) = \sup_{t>0} \inf_{x,y,z \in A} Q(x-z, y-z, t)$$

is the defective perimeter of A. If $\mathcal{M}(A) = 1$, then A is said to be fuzzy bounded.

Remark 4.13. By using (GFN-4), Theorem 4.9 and Definition 4.12, we can say that, there exists r_0 such that $\mathcal{M}(B_Q(x, \lambda, r_0)) = 1$ for every $x \in X$ and $\lambda \in (0,1)$. Consequently, $\mathcal{M}(B_Q(x, \lambda, r)) = 1$ for each $x \in X$, $\lambda \in (0,1)$ and $r \le r_0$.

Theorem 4.14. For a GF2NS $(X, Q, *)$, we have:

(1) $\mathcal{M}(B_Q(\theta, \lambda, 1)) = 1$;

(2) for all $x \in X$, all $\lambda \in (0,1)$ and all $r > 0$,

$$\mathcal{M}(B_Q(x, \lambda, r)) = \mathcal{M}(B_Q(\theta, \lambda, r)) = \mathcal{M}(rB_Q(\theta, \lambda, 1)).$$

Proof. (1) Remark 4.13 assures that there exists $r_0 > 0$ such that

$$\mathcal{M}(B_Q(\theta, \lambda, r_0)) = 1.$$

Now for any $x, y, z \in B_Q(\theta, \lambda, 1)$, it holds that

$$
\begin{aligned}
&Q(x,x,1), Q(y,y,1), Q(z,z,1) > 1-\lambda \\
&\text{implies } Q(r_0x, r_0x, r_0), Q(r_0y, r_0y, r_0), Q(r_0z, r_0z, , r_0) > 1-\lambda \\
&\text{implies } r_0x, r_0y, r_0z \in B_Q(\theta, \lambda, r_0) \\
&\text{implies } 1 \ge Q(r_0x - r_0z, r_0y - r_0z, t) \ge \inf_{a,b,c \in B_Q(\theta,\lambda,r_0)} Q(a-c, b-c, t) \\
&\text{implies } 1 \ge Q(x-z, y-z, t/|r_0|) \ge \inf_{a,b,c \in B_Q(\theta,\lambda,r_0)} Q(a-c, b-c, t) \\
&\text{implies } 1 \ge Q(x-z, y-z, t') \ge \inf_{a,b,c \in B_Q(\theta,\lambda,1)} Q(a-c, b-c, t') \\
&\text{implies } 1 = \sup_{t'>0} \inf_{a,b,c \in B_Q(\theta,\lambda,1)} Q(a-c, b-c, t').
\end{aligned}
$$

Hence $\mathcal{M}(B_Q(\theta,\lambda,1))=1$.

(2) Firstly, for any $r>0$ and $\lambda\in(0,1)$, we have

$$\begin{aligned}
&x,y,z\in B_Q(\theta,\lambda,r)\\
&\text{implies } x,y,z\in rB_Q(\theta,\lambda,1) \quad \text{(Proposition 4.11)}\\
&\text{implies } \frac{1}{r}x,\frac{1}{r}y,\frac{1}{r}z\in B_Q(\theta,\lambda,1)\\
&\text{implies } 1\geq Q\left(\frac{1}{r}x-\frac{1}{r}z,\frac{1}{r}y-\frac{1}{r}z,t\right)\geq \inf_{a,b,c\in B_Q(\theta,\lambda,1)} Q(a-c,b-c,t)\\
&\text{implies } 1\geq Q\left(x-z,y-z,rt\right)\geq \inf_{a,b,c\in B_Q(\theta,\lambda,1)} Q(a-c,b-c,t)\\
&\text{implies } 1\geq Q\left(x-z,y-z,t'\right)\geq \inf_{a,b,c\in rB_Q(\theta,\lambda,1)} Q(a-c,b-c,t').
\end{aligned}$$

So, we have

$$\mathcal{M}(B_Q(\theta,\lambda,r))\geq \mathcal{M}(rB_Q(\theta,\lambda,1)). \tag{5}$$

Now, the equality (5) assures that $\mathcal{M}(B_Q(\theta,\lambda,r))=1$. Following the same process of (i) it can be shown that

$$Q(x-z,y-z,t)\geq \inf_{a,b,c\in B_Q(\theta,\lambda,r)} Q(a-c,b-c,t)$$

for each $x,y,z\in B(\theta,\lambda,r)=rB_Q(\theta,\lambda,1)$ and $t>0$. Thus,

$$\mathcal{M}(rB_Q(\theta,\lambda,1))\geq \mathcal{M}\left(B_Q(\theta,\lambda,r)\right). \tag{6}$$

By using equalities (5) and (6) together, we conclude that

$$\mathcal{M}\left(B_Q(\theta,\lambda,r)\right)=\mathcal{M}(rB_Q(\theta,\lambda,1)).$$

Next, considering any open ball $B_Q(x,\lambda,r)$, we have that

$$\begin{aligned}
&a,b,c\in B_Q(x,\lambda,r)\\
&\text{implies } a-x,b-x,c-x\in B_Q(\theta,\lambda,r)\\
&\text{implies } Q(a-c,b-c,t)\geq \inf_{(a'-x'),(b'-x'),(c'-x')\in B_Q(\theta,\lambda,r)} Q(a'-c',b'-c',t).
\end{aligned}$$

Thus, it holds that

$$\mathcal{M}(B_Q(x,\lambda,r))\geq \mathcal{M}(B_Q(\theta,\lambda,r)). \tag{7}$$

Similarly it can be shown that

$$\mathcal{M}(B_Q(x,\lambda,r))\leq \mathcal{M}(B_Q(\theta,\lambda,r)). \tag{8}$$

Combining the equalities (7) and (8) we get

$$\mathcal{M}(B_Q(x,\lambda,r))=\mathcal{M}(B_Q(\theta,\lambda,r)),$$

completing the proof. □

Remark 4.15. Theorem 4.14 establishes a result that $\mathcal{M}(B(x,\lambda,r))=1$ for every $x\in X$, $\lambda\in(0,1)$ and $r>0$, i. e., every open ball is fuzzy bounded. Moreover, $\mathcal{M}(B(x,\lambda,r))$ is independent of x. So we introduce a new notation as follows:

$$\mathcal{M}_{\lambda,r}=\mathcal{M}(B_Q(x,\lambda,r))$$

for all $\lambda\in(0,1)$ and all $r>0$. It is not hard to see that

$$\lim_{(\lambda,r)\to(1,+\infty)}\mathcal{M}_{\lambda,r}=1.$$

Since $(X,Q,*)$ induces a fuzzy g-3*ps* space, we construct the topology of GF2NS following the footsteps for fuzzy g-3*ps* space by considering the set $\mathcal{B}_Q$ of all open balls of $(X,Q,*)$, i. e.,

$$\begin{aligned}\mathcal{B}_Q&=\{B_Q(x,\lambda,r):x\in X,\lambda\in(0,1),r>0\}\\&=\{x+rB_Q(\theta,\lambda,1):x\in X,\lambda\in(0,1),r>0\}.\end{aligned}$$

Definition 4.16. (Topology of GF2NS) Let $(X,Q,*)$ be a GF2NS. The topology of $(X,Q,*)$, denoted by τ_Q, has $\mathcal{B}_Q$ as a subbase.

Theorem 4.5 has shown that, the topology of a fuzzy g-3*ps* is T_1 but not Hausdorff in general. While for GF2NS, the Hausdorffness works for its topology.

Theorem 4.17. (Hausdorff property) The topology τ_Q of a GF2NS $(X,Q,*)$ is Hausdorff.

Proof. Considering any two distinct elements x and y in X, there are $r_0>0$ and $\lambda_0\in(0,1)$ such that

$$B_Q(x,1-\lambda_0,r_0)\cap B_Q(y,1-\lambda_0,r_0)=\emptyset.$$

If not, then for all $r>0$ and $\lambda\in(0,1)$,

$$B_Q(x,1-\lambda,r)\cap B_Q(y,1-\lambda,r)\neq\emptyset.$$

Take $a_r\in B_Q(x,1-\lambda,r)\cap B_Q(y,1-\lambda,r)$ for all $r>0$ and $\lambda\in(0,1)$. Then

$$\begin{aligned}&x-a_r\in B_Q(\theta,\lambda,r),y-a_r\in B_Q(\theta,\lambda,r)\\&\text{implies } Q((x-a_r)-(y-a_r),(y-a_r)-(y-a_r),\mathcal{M}_{\lambda,r})\geq\\&\quad Q(x-a_r,x-a_r,r)*Q(y-a_r,y-a_r,r)*Q(y-a_r,y-a_r,r)\geq\lambda\\&\text{implies } Q(x-y,\theta,\mathcal{M}_{\lambda,r})\geq\lambda.\end{aligned}$$

But $\mathcal{M}_{\lambda,r}\to1$ as $(\lambda,r)\to(1,+\infty)$. Thus, $Q(x-y,\theta,1)=1$. This contradicts to the fact of $x\neq y$. Hence, there are $r_0>0$ and $\lambda_0\in(0,1)$ such that

$$B_Q(x,1-\lambda_0,r_0)\cap B_Q(y,1-\lambda_0,r_0)=\emptyset.$$

This shows that $(X,Q,*)$ is Hausdorff. □

5. CONCLUSIONS

In this paper, we introduced a notion of generalized fuzzy 2-metric spaces (see Definition 3.1), studied their topological properties (see Subsection 3.2). Also, we have introduced a concept of a fuzzy g-$3ps$ space (see Definition 4.1) and have shown that every fuzzy g-$3ps$ space is T_1 (see Theorem 4.5). More importantly, we have introduced and studied a new space called generalized fuzzy 2-normed space (see Definition 4.7). This new generalized fuzzy normed space induces a fuzzy g-$3ps$ space under the minimum t-norm (see Theorem 4.9). Finally, we have proved that the topology of a generalized fuzzy normed space is Hausdorff (see Theorem 4.17).

For the cases of generalized probabilistic 2-normed spaces, our results and methods presented in this paper have many potential applications. By using them, wecan develop a theory of bounded linear operators, and establish some fuzzy versions of the open mapping theorem and the uniform boundedness principle; we can also discuss some problems in best approximation, optima, and equilibria. Furthermore, we can consider their applications in a theory of fixed points and other nonlinear problems.

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